***Lecture One* – First Order Equations**

***Section* 1.1 – Differential Equations & Solutions**

**Ordinary Differential Equations**

Involve an unknown function of a single variable with one or more of its derivatives.



*y*: is unknown function

*t*: independent variable

Some other example:











∴ The order of a differential equation is the order of the highest derivative that occurs in the equation.

 *second order*

 is not an ODE (*ω* is dependent on *x* and *t*)

This equation is called a ***partial differential equation***.

***Definition***

A first-order differential equation of the form  is said to be in normal form.

 is said to be in normal form.

*f*: is a given function of 2 variables *t* & *y* (***rate function***)

***Solutions***

A solution of the first-order, ordinary differential equation  is a differentiable function  such that  for all *t* in the interval where  is defined.

1. Can be found in explicit and implicit form by applying manipulation (integration)
2. No real solution.

***Example***

Show that  is a solution of the 1st order equation 

***Solution***





 True; it is a solution

is called the ***general solution***.

The solutions from the graph are called ***solution curves***.

***Example***

Is the function  a solution to the differential equation

***Solution***





 False; it is not a solution.

***Exercises Section* 1.1 – Differential Equations & Solutions**

1. Show that  is a solution of the 1st order equation 
2. Show that  is a solution of the 1st order equation 
3. A general solution may fail to produce all solutions of a differential equation . Show that  is a solution of the differential equation, but no value of *C* in the given general solution will produce this solution.
4. Use the given general solution to find a solution of the differential equation having the given initial condition.
5. Show that  is a solution of the 1st order equation 
6. Use the given general solution to find a solution of the differential equation having the given initial condition. 
7. Use the given general solution to find a solution of the differential equation having the given initial condition. 
8. Use the given general solution to find a solution of the differential equation having the given initial condition. 
9. Find the values of ***m*** so that the function  is a solution of the given differential equation

|  |  |
| --- | --- |
|  |  |

1. Let  is 2-parameter family solutions of the second order differential equation of . Find a solution of the second-order consisting of this differential equation and the given initial conditions.

|  |  |
| --- | --- |
|  |  |

***Section* 1.2 – Solutions to Separable Equations**

***Separable* Equation**

Separable equation is an equation that can be written with its variables separated and then easily solved.

If is independent of *y* ⇒





***Definition***

A 1st order differential equation of the form is said to be separable or to have separable variables.





 *not separable*

***Example***

At time *t* the sample contains radioactive nuclei and is given by the differential equation:



This is called the ***exponential equation***.



 ***Separable equation***













***Example***

Solve the differential equation 

***Solution***









 *Cross multiplication*





***General Method***

1. Separate the variables
2. Integrate both sides
3. Solve for the solution , if possible

**Newton's Law of Cooling**

Newton's Law of Cooling states that the rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***) (i.e. the temperature of its surroundings).



***Example***

A can of beer at 40° *F* is placed into a room when the temperature is 70° *F*. After 10 *minutes* the temperature of the beer is 50° *F*. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 *minutes* after the beer was placed into the room?

***Solution***

By Newton's law of cooling: The rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***).











 *Quotient Rule*

 ⇒ 

*Given*: 













***Losing a solution***

When we use separate variables, the variable divisors could be zero at a point.

***Example***

Find a general solution to 

***Solution***



  C*ritical points*



















If 



 ⇒ 

If  







***Implicitly Defined Solutions***

***Example***

Find the solutions of the equation , having initial conditions  and 

***Solution***











 *Quadratic Formula*

 *Implicit*























 , but it never it will be.

*Explicit Solutions*: 

*Implicit solutions*: 

***Example***

Find the solutions to the differential equation , having 

***Solution***











For 







We can't solve for 

⇒ This solution is defined as implicit.

For 







Since the initial condition < 0, then:



For 

 *True statement*

 is a solution

***Exercises Section* 1.2 – Solutions to Separable Equations**

Find the general solution of the differential equation.

|  |  |
| --- | --- |
|  |  |

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Find the exact solution of the initial value problem. Indicate the interval of existence.

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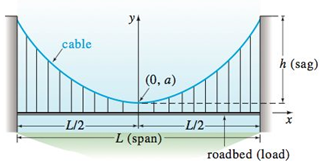
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1. A thermometer reading 100°*F* is placed in a medium having a constant temperature of 70°*F*. After 6 *min*, the thermometer reads 80°*F*. What is the reading after 20 *min*?
2. Blood plasma is stored at 40°*F*. Before the plasma can be used, it must be at 90°*F*. When the plasma is placed in an oven at 120°*F*, it takes 45 *min* for the plasma to warm to 90°*F*. How long will it take for the plasma to warm to 90°*F* if the oven temperature is set at:
3. 100°*F*.
4. 140°*F*.
5. 80°*F*.
6. A pot of boiling water at 100°*C* is removed from a stove at time  and left to cool in the kitchen. After 5 *min*, the water temperature has decreased to 80°*C*, and another 5 *min* later it has dropped to 65°*C*. Assuming Newton’s law for cooling, determine the (constant) temperature of the kitchen.
7. A murder victim is discovered at midnight and the temperature of the body is recorded at 31°*C*. One hour later, the temperature of the body is 29°*C*. Assume that the surrounding air temperature remains constant at 21°*C*. Use Newton’s law of cooling to calculate the victim’s time of death. *Note*: The normal temperature of a living human being is approximately 37°*C*.
8. Suppose a cold beer at 40°*F* is placed into a warn room at 70°*F*. suppose 10 *minutes* later, the temperature of the beer is 48°*F*. Use Newton’s law of cooling to find the temperature 25 *minutes* after the beer was placed into the room.
9. A thermometer is removed from a room where the temperature is  and is taken outside, where the air temperature is . After one-half minute the thermometer reads .
10. What is the reading of the thermometer at ?
11. How long will it take for the thermometer to reach ?
12. A thermometer is taken from an inside room to the outside, where the air temperature is . After 1 *minute* the thermometer reads , and after 5 *minutes* the thermometer reads . What is the initial temperature of the inside room?
13. A small metal bar, whose initial temperature was , is dropped into a large container of boiling water.
14. How long will it take the bar to reach  if it is known that its temperature increases  in 1 *second*?
15. How long will it take the bar to reach 
16. Two large containers ***A*** and ***B*** of the same size are filled with different fluids. The fluids in containers ***A*** and ***B*** are maintained at  and , respectively. A small metal bar, whose initial temperature is , is lowered into container ***A***. After 1 *minute* the temperature of the bar is . After 2 *minutes* the bar is removed and instantly transferred to the other container. After 1 *minute* in container ***B*** the temperature of the bar rises . How long, measured from the start of the entire process, will it take the bar to reach ?
17. A thermometer reading  is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads  after  *minute* and  after 1 *minute*. How hot is the oven?
18. At  a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is 80° *F*. the liquid bath has a controlled temperature given by , , where *t* is measured in *minutes*.
19. Assume that , describe in words what you expect the temperature  of the chemical to be like in the short term. In the long term.
20. Solve the initial-value problem.
21. Graph .
22. The mathematical model for the shape of a flexible cable strung between two vertical supports is given by



Where *W* denotes the portion of the total vertical load between the points  and 

The model is separable under the following conditions that describe a suspension bridge.



Let assume that the *x-*axis runs along the horizontal roadbed, and the *y-*axis passes through , which is the lowest point on one cable over the span of the bridge, coinciding with the interval .

In the case of a suspension bridge, the usual assumption is that the vertical load in the given equation is only a uniform roadbed, distributed along the horizontal axis. In other words, it is assumed that the weight of all cables is negligible in comparison to the weight of the roadbed and that the weight per unit length of the roadbed  is a constant *ρ*. Use this information to set up and solve an appropriate initial-value problem from which the shape (a curve with equation ) of each of the two cables in a suspension bridge is determined.

Express the solution of the IVP in terms of the sag *h* and span *L*.

1. The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if *R* represents the reaction to an amount *S* of stimulus, then the relative rates of increase are proportional:



Where *k* is a positive constant. Find *R* as a function of *S*.

1. Barbara weighs 60 *kg* and is on a diet of 1600 *calories* per day, of which 850 are used automatically by basal metabolism. She spends about 15 *cal/kg/day* times her weight doing exercises. If 1 *kg* of fat contains 10,000 *cal*. and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find her weight as a function of time. Does her weight ultimately approach an equilibrium weight?
2. When a chicken is removed from an oven, its temperature is measured at 300° *F*. Three minutes later its temperature is 200° *F*. How long will it take for the chicken to cool off to a room temperature of 70° *F*.

***Section* 1.3 – Models of Motions**

In mathematics, the rate at which a quantity changes is the derivative of that quantity.

The 2nd way of computing the rate of change comes from the application itself and is different from on application to another.

***Mechanics***

**Law of mechanics – Newton’s 2nd Law** (1665-1671)

*The force acting on a mass is equal to the rate of change of momentum with respect to time. Momentum is defined as the product of mass and velocity (m.v).*

The force is equal to the derivative of the momentum



***Position***: 

***Universal Law of gravitation***

Any body with mass M attacks any other body with mass *m* directly toward the mass M, with a magnitude proportional to the product of the 2 masses and inversely proportional to the square of the distance separating them.









Motion ball: 



***Air Resistance***



***R***: resistance force (*has sign opposite of the velocity*)

***r***: is a function that is always nonnegative

* *when a ball is falling from a high altitude, the density of the air has to be taken into account.*





















When  (*Terminal Velocity*)

 (*A*: is a constant)

***Example***

Suppose you drop a brick from the top of a building that is 250 *m* high. The brick has a mass of 2 *kg*, and the resistance force is given by . How long will it take the brick to reach the ground? what will be its velocity at that time?

***Solution***





























 (Using software to solve it)





***Finding the displacement***









***Example***

A ball of mass is protected from the surface of the earth, with velocity . Assume that the force of air resistance is given by , where . What is the maximum height reached by the ball?

***Solution***











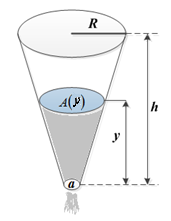










***Torricelli’s Law***

Suppose that a water tank has a ***hole with area*** ***a*** at its bottom, from which water is leaking. Denote by  the depth of water in the tank at time *t*, and by  the volume of water in the tank then. It is plausible – and true, under ideal conditions – that the velocity of water exiting through the hole is



Which is the velocity a drop of water would acquire in falling freely from the surface of the water to the hole. One can derive this formula beginning with the assumption that the sum of the kinetic and potential energy of the system remains constant. Under real conditions, taking into account the construction of a water jet from an orifice, , where *c* is an empirical constant between 0 and 1 (usually about 0.6 for a small continuous stream of water). For simplicity we take  in the following discussion.





This is a statement of *Torricelli’s* law for a draining tank.

Let  denote the horizontal cross-sectional area of the tank at height *y*. Then, applied to a thin horizontal slice of water at height  with area  and thickness , the integral method of cross sections gives



The fundamental theorem of calculus therefore implies that  and hence that





 (An alternative form of *Torricelli’s* law)



Where  and  are the cross-sectional areas of the water and the hole,

***Example***

A tank is shaped like a vertical cylinder; it initially contains water to a depth of 9 *ft*, and a bottom plug is removed at time *t* = 0 (*hours*). After 1 *hr*. the depth of the water has dropped to 4 *ft*. how long does it take for all the water to drain from the tank?

***Solution***









With initial condition 



















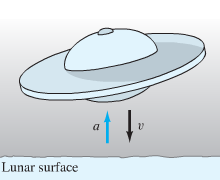
It will take 3 hours for the tank to empty.

***Exercises Section* 1.3 – Models of Motions**

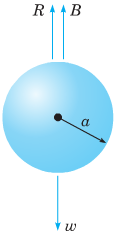
1. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.
2. A rocket is fired vertically and ascends with constant acceleration  for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.
3. A ball having mass  falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of the force due to the resistance of the medium is −1 *N*. Find the terminal velocity of the ball.

1 *N* is the force required to accelerate a 1 kg mass at a rate of : 

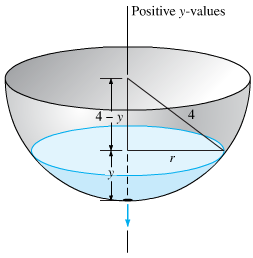
1. A ball is projected vertically upward with initial velocity from ground level. Ignore air resistance.
2. What is the maximum height acquired by the ball?
3. How long does it take the ball to reach its maximum height? How long does it take the ball to return to the ground? Are these times identical?
4. What is the speed of the ball when it impacts with the ground on its return?
5. An object having mass 70 *kg* falls from rest under the influence of gravity. The terminal velocity of the object is . Assume that the air resistance is proportional to the velocity.
6. Find the velocity and distance traveled at the end of 2 seconds.
7. How long does it take the object to reach 80% of its terminal velocity?
8. A lunar lander is falling freely toward the surface of the moon at a speed of 450 *m/s*. Its retrorockets, when fired, provide a constant deceleration of 2.5  (the gravitational acceleration produced by the moon is assumed to be included in the given acceleration). At What height above the lunar surface should the retrorockets be activated to ensure a “soft touchdown? (*v* = 0 at impact)?



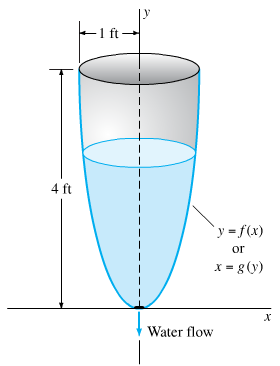
1. A body falling in a relatively dense fluid, oil for example, is acted on by three forces: a resistance force *R,* a buoyant force *B,* and its weight *w* due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius a, the resistive force is given by Stokes’s law , where *v* is the velocity of the body, and *μ* is the coefficient of viscosity of the surrounding fluid?



1. Find the limiting velocity of a solid sphere of radius *a* and density *ρ* falling freely in a medium of density  and coefficient of viscosity μ.
2. In 1910 R. A. Millikan studied the motion of tiny droplets of oil falling in an electric field. A field of strength *E* exerts a force on a droplet with charge *e*. Assume that *E* has been adjusted so the droplet is held stationary  and that *w* and *B* are as given. Find an expression for *e*.
3. A hemispherical bowl has top radius of 4 *ft*. and at time *t* = 0 is full of water. At that moment a circular hole with diameter 1 *in*. is opened in the bottom of the tank. How long will it take for all the water to drain from the tank?



1. Suppose that the tank has a radius of 3 *ft*. and that its bottom hole is circular with radius 1 *in*. How long will it take the water (initially 9 *ft*. deep) to drain completely?
2. At time *t* = 0 the bottom plug (at the vertex) of a full conical water tank 16 *feet* high is removed. After 1 *hr* the water in the tank is 9 *feet* deep. When will the tank be empty?
3. Suppose that a cylindrical tank initially containing  gallons of water drains (through a bottom hole) in *T* minutes. Use Torricelli’s law to show that the volume of water in the tank after  minutes is 
4. The clepsydra, or water clock – A 12*-hr* water clock is to be designed with the dimensions, shaped like the surface obtained by revolving the curve  around the *y-*axis. What should be this curve, and what should be the radius of the circular bottom hole, in order that the water level will fall at the constant rate of 4 inches per hour?



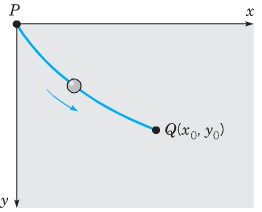
1. One of the famous problems in the history of mathematics is the brachistochrone problem: to find the curve along which a particle will slide without friction in the minimum time from one given point *P* to another point *Q*, the second point being lower than the first but not directly beneath it.

This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L’Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point *P* and to orient the axes as shown. The lower point Q has coordinates . It is then possible to show that the curve of minimum time is given by a function  that satisfies the differential equation



Where  is a certain positive constant to be determined later



1. Solve the equation  for  . Why is it necessary to choose the positive square root?
2. Introduce the new variable t by the relation



Show that the equation found in part (a) then takes the form



1. Letting , show that the solution of  for which *x* = 0 when *y* = 0 is given by



Equations (*iv*) are parametric equations of the solution of (eq. *i*) that passes through (0, 0). The graph of Eqs. (*iv*) is called a cycloid.

1. If we make a proper choice of the constant *k*, then the cycloid also passes through the point  and is the solution of the brachistochrone problem. Find *k* if  and 
2. Many chemical reactions are the result of the interaction of 2 molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If *a* is the amount of substance *A* and *b* is the substance *B* at time *t* = 0, and if *x* is the amount of product at time *t*, then the rate of formation of *x* may be given by the differential equation

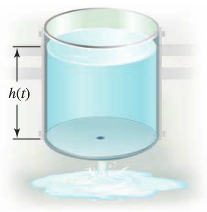


Where *k* is a constant for the reaction. Integrate both sides of this equation to obtain a relation between *x* and *t*.

1. If 
2. If 

Assume in each case that  when 

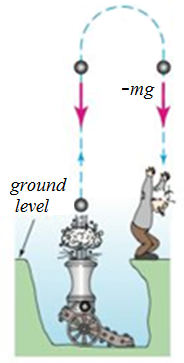
1. An open cylindrical tank initially filled with water drains through a hole in the bottom of the tank according to Torricelli’s Law. If  is the depth of water in the tank for , then Torricelli’s Law implies  , where *k* is a constant that includes the acceleration due to gravity, the radius of the tank, and the radius of the drain. Assume that the initial depth of the water is 



1. Find the solution of the initial value problem.
2. Find the solution in the case that  and .
3. In general, how long does it take the tank to drain in terms of *k* and *H*?
4. An object in free fall may be modeled by assuming that the only forces at work are the gravitational force and resistance (friction due to the medium in which the objects falls). By Newton’s second law (mass × acceleration = the sum of the external forces), the velocity of the object satisfies the differential equation

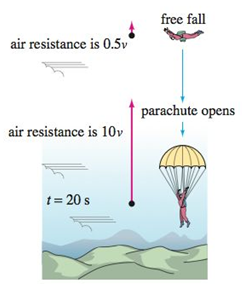


Where  is a function that models the resistance and the positive direction is downward. One common assumption (often used for motion in air) is that , where  is a drag coefficient.

1. Show that the equation can be written in the form  where 
2. For what (positive) value of *v* is  (This equilibrium solution is called the ***terminal velocity***.)
3. Find the solution of this separable equation assuming  for 
4. Graph the solution found in part (***c***) with , and verify the terminal velocity agrees with the value found in part (***b***).
5. Suppose a small cannonball weighing 16 *pounds* is shot vertically upward, with an initial velocity 

The answer to the question “How high does the cannonball go?” depends on whether we take air resistance into account.

1. Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by . Since  the last differential equation is the same as , where we take . Find the velocity  of the cannonball at time *t*.
2. Use the result in part (*a*) to determine the height  of the cannonball measured from ground level. Find the maximum height attained by the cannonball.
3. Two chemicals *A* and *B* are combined to form a chemical *C*. The resulting reaction between the two chemicals is such that for each *gram* of *A*, 4 *grams* of *B* is used. It is observed that 30 grams of the compound *C* is formed in 10 *minutes*.
4. Determine the amount of *C* at time *t* if the rate of the reaction is proportional to the amounts of *A* and *B* remaining and if initially there are 50 *grams* of *A* and 32 *grams* of *B*.
5. How much of the compound *C* is present at 15 minutes.
6. Interpret the solution as 
7. Two chemicals *A* and *B* are combined to form a chemical *C*. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of *A* and *B* not converted to chemical *C*. Initially, there are 40 *grams* of *A* and 50 *grams* of *B*, and each gram of *B*, 2 *grams* of *A* is used. It is observed that 10 *grams* of *C* is formed in 5 *minutes*.
8. How much is formed in 20 *minutes*?
9. What is the limiting amount of *C* after a long time?
10. How much of chemicals *A* and *B* remains after a long time?
11. If 100 *grams* of chemical *A* is present initially, at what time is chemical *C* half-formed?
12. A skydiver weighs 125 *pounds*, and her parachute and equipment combined weigh another 35 *pounds*. After exiting from a plane at an altitude of 15,000 *feet*, she waits 15 *seconds* and opens her parachute. Assume that the constant of proportionality has the value  during free fall and  after the parachute is opened.



Assume that her initial velocity on leaving the plane is *zero*.

1. What is her velocity and how far has she traveled 20 *seconds* after leaving the plane?
2. How does her velocity at 20 *seconds* compare with her terminal velocity?
3. How long does it take her to reach the ground?
4. A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. When friction and contraction of water at the hole are ignored, the height *h* of water in the tank is described by



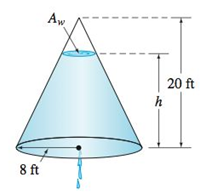
Where  and  are the cross-sectional areas of the water and the hole, respectively.

1. Find  if the initial height of the water is *H*.
2. Sketch the graph  and give the interval *I* of definition in terms of the symbols , , and *H*.
3. Suppose the tank is 10 *feet* high and has radius 2 *feet* and the circular hole has radius  *inch*. If the tank is initially full, how long will it take to empty?
4. A tank in the form of a right-circular cylinder cone standing on end, vertex down, is leaking water through a circular hole in its bottom.
5. Suppose the tank is 20 *feet* high and has radius 8 *inches*. Show that the differential equation governing the height *h* of water leaking from a tank is



In this model, friction and contraction of the water at the hole were taken into account with  and . If the tank is initially full, how long will it take the tank to empty?

1. Suppose the tank has a vertex angle of 60° and the circular hole has radius 2 *inches*. Determine the differential equation governing the height *h* of water. Use  and .
2. If the height of the water is initially 9 *feet*, how long will it take the tank to empty?
3. Suppose that the conical tank is inverted and that water leaks out a circular hole of radius 2 *inches* in the center of its circular base. Is the time it takes to empty a full tank the same as for the tank with vertex down?



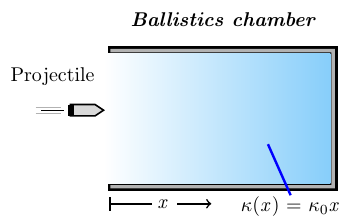
Take the friction/.contraction coefficient to be  and 

1. A differential equation for the velocity *v* of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is



Where  is a constant of proportionality. The positive direction is downward.

1. Solve the equation subject to the initial condition .
2. Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.
3. If the distance *s*, measured from the point where the mass was released above the ground, is related to velocity *v* by , find an explicit expression for  if 
4. An object is dropped from altitude 
5. Determine the impact velocity if the drag force is proportional to the square of velocity, with drag coefficient .
6. If the terminal velocity is known to −120 *mph* and the impact velocity was −90 *mph*, what was the initial altitude ?
7. An object is dropped from altitude 
8. Assume that the drag force is proportional to the velocity, with drag coefficient . Obtain an implicit solution relating velocity and altitude.
9. If the terminal velocity is known to −120 *mph* and the impact velocity was −90 *mph*, what was the initial altitude ?
10. An object of mass 3 *kg* is released from rest 500 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with , and the force due to air resistance is proportional to the velocity of the object with proportionality constant . Determine when the object will hit the ground.
11. A parachutist whose mass is 75 *kg* drops from helicopter hovering 4000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant  when the chute is closed and with constant  when the chute is open. If the chute does not open until 1 *min* after the parachutist leaves the helicopter, after how many *seconds* will he reach the ground?
12. A parachutist whose mass is 75 *kg* drops from helicopter hovering 2000 *m* above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant. Assume also that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant  when the chute is closed and with constant  when the chute is open. If the chute does not open until the velocity of the parachutist reaches , after how many seconds will he reach the ground?
13. An object of mass 5 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with , and the force due to air resistance is proportional to the velocity of the object with proportionality constant . Determine when the object will hit the ground.
14. An object of mass 500 *kg* is released from rest 1000 *m* above the ground and allowed to fall under the influence of gravity. Assume the gravitational force constant, with , and the force due to air resistance is proportional to the velocity of the object with proportionality constant . Determine when the object will hit the ground.
15. A 400-*lb* object is released from rest 500 *ft* above the ground and allowed to fall under the influence of gravity. Assuming that the force in pounds due to air resistance is , where *v* is the velocity of the object in , determine the equation of motion of the object. When will the object hit the ground?
16. An object of mass 8 *kg* is given an upward initial velocity of  and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is , where *v* is the velocity of the object in .
17. Determine the equation of motion of the object.
18. If the object is initially 100 *m* above the ground, determine when the object will hit the ground.
19. An object of mass 5 *kg* is given an downward initial velocity of  and then allowed to fall under the influence of gravity. Assume that the force in Newton due to air resistance is , where *v* is the velocity of the object in .
20. Determine the equation of motion of the object.
21. If the object is initially 100 *m* above the ground, determine when the object will hit the ground.
22. A shell of mass 2 *kg* is shot upward with an initial velocity of . The magnitude of the force on the shell due to air resistance is .
23. When will the shell reach its maximum height above the ground?
24. What is the maximum height?
25. We need to design a ballistics chamber to decelerate test projectiles fired into it. Assume the resistive force encountered by the projectile is proportional to the square of its velocity and neglect gravity.



The chamber is to be constructed so that the coefficient  associated with this resistive force is not constant but is, in fact, a linearly increasing function of distance into the chamber:

Let , where  is a constant; the resistive force then has the form .

If we use time *t* as the independent variable, Newton’s law of motion leads us to the differential equation

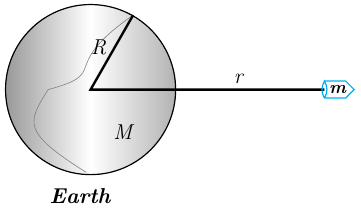


1. Adopt distance *x* into the chamber as the new independent variable and rewrite the given differential equation as a first order equation in terms of the new independent variable.
2. Determine the value  needed if the chamber is to reduce projectile velocity to 1% of its incoming value within *d* units of distance.
3. When the velocity *v* of an object is very large, the magnitude of the force due to air resistance is proportional to  with the force acting in opposition to the motion of the object. A shell of mass 3 *kg* is hot upward from the ground with an initial velocity of 500 . If the magnitude of the force due to air resistance is .
4. When will the shell reach its maximum height above the ground?
5. What is the maximum height?
6. A sailboat has been running (on a straight course) under a light wind at . Suddenly the wind picks up, blowing hard enough to apply a constant force of 600 *N* to the sailboat. The only other force acting on the boat is water resistance that is proportional to the velocity of the boat. If the proportionality constant for water resistance is  and the mass of the sailboat is 50 *kg*.
7. Find the equation of motion of the sailboat.
8. What is the limiting velocity of the sailboat under this wind?
9. When the velocity of the sailboat reaches , the boat begins to rise out of the water and plane. When this happens, the proportionality constant for the water resistance drop to . Find the equation of motion of the sailboat.
10. What is the limiting velocity of the sailboat under this wind as it is planning?
11. According to Newton’s law of gravitation, the attractive force between two objects varies inversely as the square of the distances between them. That is, 

Where  and  are the masses of the objects, *r* is the distance between them (center to center), is the attractive force, and *G* is the constant of proportionality.

Consider ta projectile of constant mass *m* being fired vertically from Earth.

Let *t* represent time and *v* the velocity of the projectile.



1. Show that the motion of the projectile, under Earth’s gravitational force, is governed by the equation



Where *r* is the distance between the projectile and the center of Earth, *R* is the radius of Earth, *M* is the mass of Earth, and .

1. Use the fact the  to obtain 
2. If the projectile leaves Earth’s surface with velocity , show that



1. Use the result of part (*c*) to how that the velocity of the projectile remains positive if and only if . The velocity  is called the escape velocity?
2. If  and  for Earth, what is Earth’s escape velocity?
3. If the acceleration due to gravity for the Moon is  and the radius of the Moon is , what is the escape velocity of the Moon?
4. A 180*-lb* skydiver drops from a hot-air balloon. After 10 *sec* of free fall, a parachute is opened. The parachute immediately introduces a drag force proportional to velocity. After an additional 4 *sec*, the parachutist reaches the ground. Assume that air resistance is negligible during free fall and that the parachute is designed so that a 200-*lb* person will reach a terminal velocity of −10 *mph*.
5. What is the speed of the skydiver immediately before the parachute is opened?
6. What is the parachutist’s impact velocity?
7. At what altitude was the parachute opened?
8. What is the balloon’s altitude?

***Section* 1.4 – Linear Equations**

A first order linear equation is given by the form:



If . This linear equation is said to be ***homogeneous***. (Otherwise it is ***nonhomogeneous or inhomogeneous***).

are called the coefficients

|  |  |
| --- | --- |
| ***Linear*** | ***Non-linear*** |
|  |  |
|  |  |
|  |  |

***Solution of the homogenous equation***





 *Convert to exponential form*



***Example***

Solve: 

***Solution***





**Solving a linear first-order Equation (*Properties*)**

1. Put a linear equation into a standard form 
2. Identify  then find 
3. Multiply the standard form by 
4. Integrate both sides

***Solution of the Inhomogeneous Equation ***









***Example***Find the general solution to: 

***Solution***















***Solution of the Nonhomogeneous Equation ***

Let assume:  

The homogeneous equation is given by 













 *Since* 



















***Example***

Find the general solution of  and the particular solution that satisfies.

***Solution***













***Example***

Find the general solution of  and the particular solution that satisfies.

***Solution***















***Notes***

1. Integrating an expression that is not the differential of any elementary function is called non-elementary.

1. In math some important functions are defined in terms of non-elementary integrals. Two such functions are the error function and the complementary error function.

***Exercises Section* 1.4 – Linear Equations**

Find the general solution of the first-order, linear equation.

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
|  |  |

Find the solution of the initial value problem

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
|  |  |

Find a solution to the initial value problem that is continuous on the given interval 

1.   
2.   
3.   
4.   

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution

|  |  |
| --- | --- |
|  |  |

1. The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements

Where  are constants.

Discuss how to solve this system subject to 

1. Let  be the performance level of someone learning a skill as a function of the training time *t*. The graph of *P* is called a ***learning curve***. We proposed the differential equation



As a reasonable model for learning, where *k* is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

1. A differential equation describing the velocity *v* of a falling mass subject to air resistance proportional to the instantaneous velocity is



Where  is a constant of proportionality. The positive direction is downward.

1. Solve the equation subject to the initial condition 
2. Use the solution in part (*a*) to determine the limiting, or terminal, velocity of the mass.
3. If the distance *s*, measured from the point where the mass was released above ground, is related to velocity *v* by , find an explicit expression for  if 
4. As a raindrop falls, it evaporates while retaining its spherical shape. If we make the further assumptions that the rate at which the raindrop evaporates is proportional to its surface and that air resistance is negligible, then a model for the velocity  of the raindrop is



Here  is the density of water,  is the radius of the raindrop at ,  is the constant of proportionality, and downward direction is taken to be the positive direction.

1. Solve for  if the raindrop falls from rest.
2. Show that the radius of the raindrop at time *t* is .
3. If  and  10 *seconds* after the raindrop falls from a cloud, determine the time at which the raindrop has evaporated completely.
4. A model that describes the population of a fishery in which harvesting takes place at a constant rate is given by



Where *k* and *h* are positive constants.

1. Solve  given the initial value 
2. Describe the behavior of the population  for increasing time in three cases , , and 
3. Use the results from part (*b*) to determine whether the fish population will ever go extinct in finite time, that is, whether there exists a time  such that . If the population goes extinct then find *T*.

***Section* 1.5 – Mixing Problems**

The physical representation of the rate of change:

*rate of change = rate in* - *rate out*

This is referred to as a ***balance law***.

Rate = Volume Rate (*gal/min*) *x* Concentration (*lb/gal*)

***Example***

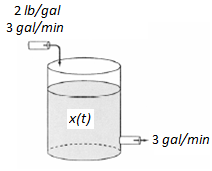
The tank initially holds 100 *gal* of pure water. At time , a solution containing 2 *lb* of salt per *gallon* begins to enter the tank at the rate of 3 *gallons* per *minute*. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant.

How much salt is in the tank after 60 *min*?

What will be the eventual salt content in the tank?

***Solution***

number of pounds of salt in the tank after *t* min.

Volume: 

Concentration at time *t*: 

Rate in = Volume Rate *x* Concentration





Rate out = Volume Rate *x* Concentration





rate of change

= rate in − rate out













Since there was no salt present in the tank initially, the initial condition is 









After 60 min:





As  then 

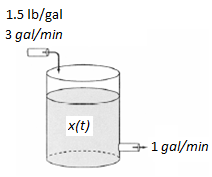


***Example***

The 600-*gal* tank is filled with 300 *gal* of pure water. A spigot is opened above the tank and a salt solution containing 1.5 *lb*. of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

***Solution***





Rate in 



Rate out 































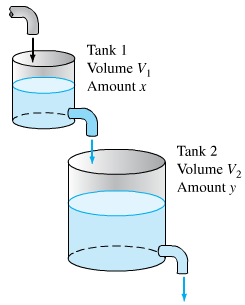







***Exercises Section* 1.5 – Mixing Problems**

1. Consider two tanks, label tank ***A*** and tank ***B*** for reference. Tank ***A*** contains 100 *gal* of solution in which is dissolved 20 *lb* of salt. Tank ***B*** contains 200 *gal* of solution which is dissolved 40 *lb* of salt. Pure water flows into the tank ***A*** at rate of 5 *gal/s*. There is a drain at the bottom of tank ***A***. The solution leaves tank ***A*** via the drain at a rate of 5 *gal/s* and flows immediately into tank ***B*** at the same rate. A drain at the bottom of tank ***B*** allows the solution to leave tank ***B*** at a rate of 2.5 *gal/s*. What is the salt content in tank ***B*** at the precise moment that tank ***B*** contains 250 *gal* of solution?
2. A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb* of sugar per gal enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly mixed at all times.
3. What will be the sugar content in the tank after 20 *minutes*?
4. How long will it take the sugar content in the tank to reach 15 *lb*?
5. What will be the eventual sugar content in the tank?
6. A tank initially contains 50 *gal* of sugar water having a concentration of 2 *lb*. of sugar for each gal of water. At time zero, pure water begins pouring into the tank at a rate of 2 *gal* per *minute*. Simultaneously, a drain is opened at the bottom of the tank so that the volume of sugar-water solution in the tank remains constant.
7. How much sugar is in the tank after 10 minutes?
8. How long will it take the sugar content in the tank to dip below 20 *lb*.?
9. What will be the eventual sugar content in the tank?
10. A 50-*gal* tank initially contains 20 *gal* of pure water. Salt-water solution containing 0.5 *lb.* of salt for each gallon of water begins entering the tank at a rate of 4 *gal/min.* simultaneously; a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 *gal/min*. What is the salt content (*lb*) in the tank at the precise moment that the tank is full of salt-water solution?
11. A tank contains 500 *gal* of a salt-water solution containing 0.05 *lb* of salt per gallon of water. Pure water is poured into the tank and a drain at the bottom of the tank is adjusted so as to keep the volume of solution in the tank constant. At what rate (gal/min) should the water be poured into the tank to lower the salt concentration to 0.01 lb/gal of water in less than one hour?
12. A tank contains 100 *gal* of fresh water. A solution containing 1 *lb./gal* of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at a rate of 3 *gal/min*. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.
13. A 200-*gal* tank is half full of distilled water. At time *t* = 0, a solution containing 0.5 *lb./gal* of concentrate enters the tank at the rate of 5 *gal/min*, and the well-stirred mixture is withdrawn at the rate of 3 *gal/min*.
14. At what time will the tank be full?
15. At the time the tank is full, how many pounds of concentrate will it contain?
16. Suppose that an Iowa class battleship has mass 51,000 metric tons (51,000,000 *kg*) and . Assume that the ship loses power when it is moving at a speed of 9 *m/sec*.
17. About how far will the ship coast before it is dead in the water?
18. About how long will it take the ship’s speed to drop to 1 m/sec?
19. A 66-*kg* cyclist on a 7-*kg* bicycle starts coasting on level ground at 9 m/sec. The 
20. About how far will the cyclist coast before reaching a complete stop?
21. How long will it take the cyclist’s speed to drop to 1 *m/sec*?
22. An Executive conference room of a corporation contains 4500  of air initially free of carbon monoxide. Starting at time *t* = 0, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 . A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 . Find the time when the concentration of carbon monoxide in the room reaches 0.01%.



1. Consider the cascade of 2 tanks with  and  the volumes of brine in the 2 tanks. Each tank also initially contains 50 lb. of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank.
2. Find the amount  of salt in tank 1 at time *t*.
3. Suppose that  is the amount of salt in tank 2 at time *t*. Show first that 

And then solve for , using the function  found in part (*a*).

1. Finally, find the maximum amount of salt ever in tank 2.
2. Suppose that in the cascade tank 1 initially 100 *gal* of pure ethanol and tank 2 initially contains 100 *gal* of pure water. Pure water flows into tank 1 at 10 *gal/min*, and the other two flow rates are also 10 *gal/min*.
3. Find the amounts  and  of ethanol in the two tanks at time  .
4. Find the maximum amount of ethanol ever in tank 2.
5. A multiple cascade is shown in the figure. At time *t* = 0, tank 0 contains 1 *gal* of ethanol and 1 *gal* of water; all the remaining tanks contain 2 *gal* of pure water each. Pure water is pumped into tank 0 at 1 *gal/.min*, and the varying mixture in each tank is pumped into the one below it at the same rate. Assume, as usual, that the mixtures are kept perfectly uniform by stirring. Let  denote the amount of ethanol in tank *n* at time *t*.
6. Show that 
7. Show that the maximum value of  for *n* > 0 is 
8. Assume that Lake Erie has a volume of 480  and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350  per year. Suppose that at the time *t* = 0 (*years*), the pollutant concentration of Lake Erie – caused by past industrial pollution that has now been ordered to cease – is 5 times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?
9. A 120 *gal* tank initially contains 90 *lb*. of salt dissolved in 90 *gal* of water. Brine containing 2 *lb./gal* of salt flows into the tank at rate of 4 *gal/min*, and the well-stirred mixture flows out the tank at the rate of 3 *gal/min*. How much salt does the tank contain when it is full?
10. A tank contains 50 *gallons* of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of . As the second solution is being added, the tank is being drained at a rate of . The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 *minutes*?



1. A 200-*gallon* tank is half full of distilled water. At time , a concentrate solution containing  enters the tank at the rate of , and well-stirred mixture is withdrawn at the rate of .
2. At what time will the tank be full?
3. At the time the tank is full, how many pounds of concentrate will it contain?



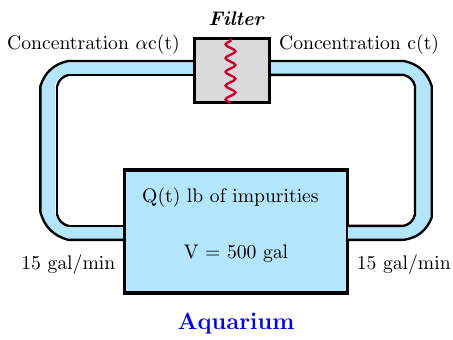
1. A 200-*gallon* tank is half full of distilled water. At time , a concentrate solution containing  enters the tank at the rate of , and well-stirred mixture is withdrawn at the rate of .
2. At what time will the tank be full?
3. At the time the tank is full, how many pounds of concentrate will it contain?



1. A 200-*gallon* tank is full of a concentrate solution containing . Starting at time , distilled water is admitted to the tank at the rate of , and well-stirred mixture is withdrawn at the same rate.
2. Find the amount of concentrate in the solution as a function of *t*.
3. Find the time at which the amount of concentrate in the tank reaches 15 *pounds*.
4. Find the quantity of the concentrate in the solution as .

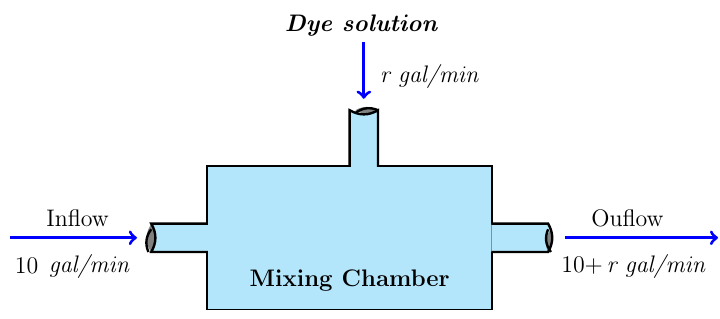


1. A tank contains 200 *liters* of fluid in which 30 *grams* of salt is dissolved. Brine containing 1 *gram* of salt per liter is then pumped into the tank at a rate of 4 *L/min*; the well-mixed solution is pumped out at the same rate.
2. Find the number  of grams of salt in the tank at time *t*.
3. Solve by assuming that pure water is pumped into the tank.
4. A large tank is filled to capacity with 500 *gallons* of pure water. Brine containing 2 *pounds* of salt per gallon is pumped into the tank at a rate of 5 *gal/min*. The well-mixed solution is pumped out at the same rate.
5. Find the number  of grams of salt in the tank at time *t*.
6. What is the concentration  of the salt in the tank at time *t*? At time ?
7. What is the concentration of the salt in the tank after a long time, that is, as ?
8. What is the concentration of the salt in the tank equal to one-half this limiting value?
9. Solve under assumption that the solution is pumped out at a faster rate of 10 *gal/min*. when tis the tank empty?
10. A large tank is filled to capacity with 100 *gallons* of fluid in which 10 *pounds* of salt is dissolved. Brine containing  *pound* of salt per gallon is pumped into the tank at a rate of 6 *gal/min*. The well-mixed solution is pumped out at the slower rate of 4 *gal/min*. Find the number of pounds of salt in the tank after 30 *minutes*.
11. A 5000-*gal* tank is maintained with a pumping system that passes 100 *gal* of water per minute through the tank. To treat a certain fish malady, a soluble antibiotic is introduced into the inflow system. Assume that the inflow concentration of medicine is  *mg/gal*, where *t* is measured in *minutes*. The well-stirred mixture flows out of the tank at the same rate.
12. Solve for the amount of medicine in the tank as function of time.
13. What is the maximum concentration of medicine achieved by this dosing and when does it occur?
14. For the antibiotic to be effective, its concentration must exceed 100 *mg/gal* for a minimum of 60 *min*. was the dosing effective?
15. A tank initially contains 400 *gal* of fresh water. At time , a brine solution with a concentration of 0.1 *lb*. of salt per gallon enters the tank at a rate of 1 *gal/min* and the well-stirred mixture flows out at a rate of 2 *gal/min*.
16. How long does it take for the tank to become empty?
17. How much salt is present when the tank contains 100 *gal* of brine?
18. What is the maximum amount of salt present in the tank during the time interval found in part (*a*)?
19. When is the maximum achieved?
20. A tank, having a capacity of 700 *gal*, initially contains 10 *lb*. of salt dissolved in 100 *gal* of water. At time , a solution containing 0.5 *lb*. of salt per gallon flows into the tank at a rate of 3 *gal/min* and the well-stirred mixture flows out of the tank at a rate of 2 *gal/min*.
21. How much time will elapse before the tank is filled to capacity?
22. What is the salt concentration in the tank when it contains 400 *gal* of solution?
23. What is the salt concentration at the instant the tank is filled to capacity?
24. A 500-*gal* aquarium is cleansed by the recirculating filter system schematically shown in the figure.

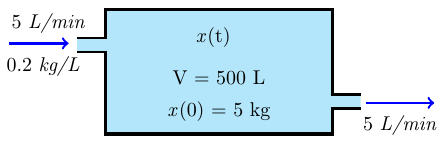


Water containing impurities is pumped out at a rate of 15 *gal/min*, filtered, and returned to the aquarium at the same rate. Assume that passing through the filter reduces the concentration of impurities by a fractional amount *α*. In the other words, if the impurity concentration upon entering the filter is , the exit concentration is , where .

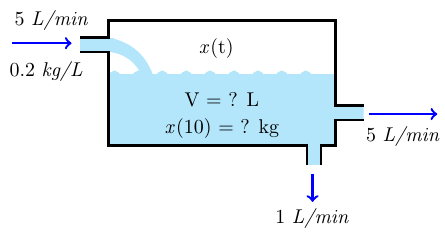
1. Apply the basic conservation principle  to obtain a differential equation for the amount of impurities present in the aquarium at time *t*. Assume that filtering occurs instantaneously. If the outflow concentration at any time is , assume that the inflow concentration at that same instant is .
2. What value of filtering constant *α*. will reduce impurity levels to 1% of their original values in a period of 3 *hr.*?
3. A mixing chamber initially contains 2 *gal* of a clear fluid. Clear fluid flows into the chamber at a rate of 10 *gal/min*. A dye solution having a concentration of 4  is injected into the mixing chamber at a rate of *r* *gal/min*. When the mixing process is started, the well-mixed mixture is pumped from the chamber at a rate .



1. Develop a mathematical model for the mixing process.
2. The objective is to obtain a dye concentration in the outflow mixture of 1 . What injection rate *r* is required to achieve this equilibrium solution? Would this equilibrium value of *r* be different if the fluid in the chamber at time  contained some dye?
3. Assume the mixing chamber contains 2 *gal* of clear fluid at time . How long will it take for the outflow concentration to rise to within 1% of the desired concentration?
4. Suppose a brine containing 0.2 *kg* of salt per liter runs into a tank initially filled with 500 *L* of water containing 5 *kg* of salt The brine enter the tank at a rate of . The mixture, kept uniform by stirring, is flowing out at the rate at the same rate.



1. Find the concentration, in , of salt in the tank after 10 *min*.
2. After 10 *min*, a leak develops in the tank and an additional liter per minute of mixture flows out of the tank. What will be the concentration, in , of salt in the tank 20 *min* after the leak develops?



***Section* 1.6 – Exact Differential Equations**

A class of equations known as exact equations for which there is also a well-defined method of solution

***Theorem***

Let the function *M*, *N*, , where are partial derivatives, be continuous in the rectangular region  then



Is an exact differential equation in *R*, *iff* 

At each point in *R*. That is, there exists a function  satisfying

 *Iff* 



***Example***

Solve the differential equation: 

***Solution***

















***Example***

Solve the differential equation: 

***Solution***

















***Example***

Solve the differential equation: 

***Solution***







Can be solved by this procedure.

**Integrating Factors**

It is sometimes possible to convert a differential equation that is not exact equation by multiplying the equation by a suitable integrating factor.

***Definition***

An integrating factor for the differential equation  is a function  such that the form  is exact.





Assuming that *µ* is a function of *x* only, we have















***Example***

Find an integrating factor for the equation , and then solve the equation.

***Solution***



















***Bernoulli* Equations**

An equation of the form  is called a ***Bernoulli equation***.

If  First−order linear differential equation

If   Separable equation.

For , the Bernoulli equation can be written as 

Let 





 Which is 1st−order linear differential equation.

***Example***

Find the general solution 

***Solution***



Let 





 ***Divide by*** 













***Example***

Find the general solution 

***Solution***



Let 



 ***Multiply both sides by*** 













**Homogeneous Equations **

The form of a homogeneous equation suggests that it may be simplified by using a variable denoted by , to represent the ratio of *y* to *x*. This



Let assume that *v* is a function of *x*, then



The most significant fact about this equation is that the variables *x* & *v* can always be separated, regardless of the form of the function F.



Solving this equation and then replacing *v* by  gives the solution of the original equation.

***Example***

Solve the differential equation 

***Solution***

















***Example***

Find the general solution 

***Solution***

Let 



















**Equations with Linear Coefficients**

For equations with linear coefficients in the form:

The general case: 

Let consider: 

If  



In this case by letting 

If , we let 

 has a solution



***Example***

Solve 

***Solution***











Let 























***Exercises Section* 1.6 – Exact Differential Equations**

Solve the differential equation

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
|  |  |

1. 
2. 
3. 
4. 
5. 

The given equation is not exact. However, if you multiply by the given integrating factor, then it becomes exact. Then solve the equation

1. 
2. 
3. 
4. 
5. 
6. 
7. 

Find the general solution of each homogenous equation

|  |  |
| --- | --- |
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Find an integrating factor and solve the given equation

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Solve the given initial-value problem

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Find an integrating factor of the form  and solve the equation

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Find the general solution by using Bernoulli

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Find the general solution by using homogeneous equations.

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Find the general solution by using Equation with Linear Coefficients

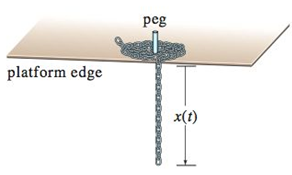
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1. Prove that  has an integrating factor that depends only on the sum  if and only if the expression

 depends only on 

Use the prove to solve the equation 

1. A portion of a uniform chain of length 8 *feet* is loosely coiled around a peg at the edge of a high horizontal platform, and the remaining portion of the chain hangs at rest over the edge of the platform.



Suppose the length of the overhanging chain is 3 *feet*, that the chain weighs 2 , and that the positive direction is downward. Starting at  seconds, the weight of the overhanging portion causes the chain on the table to uncoil smoothly and to fall to the floor. If  denotes the length of the chain overhanging the table at time , then  is its velocity. When all resistive forces are ignored, it can be shown that a mathematical model relating *v* to *x* is given by



1. Rewrite this model in differential form and solve the *DE* for *v* in terms of *x* by finding am appropriate integrating factor. Find an explicit solution .
2. Determine the velocity with which the chain leaves the platform.

***Section* 1.7 - Modeling Population Growth**

**Modeling Population Growth**

The mathematical model of the growth of a population is given by:



Where ***r***: reproductive rate.

The natural of the predictions of the model depend on the nature of the reproductive rate *r*.

**Malthusian Method**

Since *r* is a constant because the birth or death rates do not depend on time or on the size.

Therefore the solution to  is given by:





The population at time  is .

***Example***

A biologist starts with 10 cells in a culture. Exactly 24 *hrs* later he counts 25. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 10 days?

***Solution***





 **24 *hrs =* 1 *day P =* 25**















***Example***

A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 *grams* of the material was present initially and after 2 *hours* the sample lost 10% of its mass, find:

1. An expression for the mass of the material remaining at any time.
2. The mass of the material after 4 *hours*.
3. How long will it take for 75% of the material to decay?
4. The half-life of the material.

***Solution***

***Given***:  

1. 



 ***Convert to logarithm***









1. 
2. 









1. 

***Note***

We can this formula to solve most of the questions 

**Logistic Model of Growth**

Suppose an environment is capable of sustaining no more than a fixed number *K* of individuals in its populations. The quantity *K* is called the ***carrying capacity*** of the environment. In reality this model in unrealistic because environments impose limitations to population growth.

The logistic equation is given by:

The logistic equation can be solved by separation of variables





























***Example***

Suppose we start at time  with a sample of 1000 *cells*. One day later we see that the population has doubled, and sometime later we notice that the population has stabilized at 100,000.

***Solution***

***Given***:  





















**Pollution**

Consider a lake that has a volume of , it is fed by an input river, and there is another river which is fed by the lake at a rate that keeps the volume of the lake constant.

The input rate: 

The maximum flow into the lake occurs when 

In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of 2 .

Let  denote the total amount of pollution in the lake at time *t*. If we make the assumption that the pollutant is rapidly mixed throughout the lake, then



***Exercises Section* 1.7 - Modeling Population Growth**

1. The rate of growth of bacteria in a petri dish is proportional to the number of bacteria in the dish.
2. The rate of growth of a population of field mice is inversely proportional to the square root of the population.
3. A biologist starts with 100 *cells* in a culture. After 24 *hrs,* he counts 300. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 5 *days*?
4. A biologist prepares a culture. After 1 *day* of growth, the biologist counts 1000 *cells*. After 2*days,* he counts 3000. Assuming a Malthusian model, what the reproductive rate and how many cells were present initially?
5. A population of bacteria is growing according to the Malthusian model. If the population is triples in 10 *hrs*, what is the reproduction rate? How often does the population double itself?
6. Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation



where time is measured in days and *P* in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

1. Modify the logistic model to account for the fishing.
2. Find and classify the equilibrium points for your model.
3. Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? What will happen to an initial population having 2000 fish?
4. Suppose that in 1885 the population of a certain country was 50 million and was growing at the rate of 750,000 people per year at that time. Suppose also that in 1940 its population was 100 million and was then growing at the rate of 1 million per year. Assume that this population satisfies the logistic equation. Determine both the limiting population *M* and the predicted population for the year 2000.
5. The time rate of change of a rabbit population *P* is proportional to the square root of *P*. At time  (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?
6. Suppose that the fish population  in a lake is attacked by a disease at time , with the result that the fish cease to reproduce (so that the birth rate is ) and the death rate *δ* (deaths per week per fish) is thereafter proportional to . If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?
7. Suppose that when a certain lake is stocked with fish, the birth and death rates *β* and *δ* are both inversely proportional to 
8. Show that , where *k* is a constant.
9. If  and after 6 months there are 169 fish in the lake, how many will there be after 1 year?
10. The time rate of change of an alligator population *P* in a swamp is proportional to the square of *P*. The swamp contained a dozen alligators in 1988, two dozen in 1998.
11. When will there be four dozen alligators in the swamp?
12. What happens thereafter?
13. Consider a prolific breed of rabbits whose birth and death rates, *β* and *δ*, are each proportional to the rabbit population , with 
14. Show that 

Note that . This is doomsday

1. Suppose that  and that there are nine rabbits after ten months. When does doomsday occur?
2. With , repeat part (*a*)
3. What now happens to the rabbit population in the long run?
4. Consider a population  satisfying the logistic equation , where  is the time rate at which births occur and  is the rate at which deaths occur.
5. If the initial population is , and  births per month and  deaths per month are occurring at time , show that the limiting population is .
6. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time , how many months does it take for  to reach 95% of the limiting population *M*?
7. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time , how many months does it take for  to reach 105% of the limiting population *M*?
8. The amount of drug in the blood of a patient (in *mg*) due to an intravenous line is governed by the initial value problem



Where *t* is measured in hours

1. Find and graph the solution of the initial value problem.
2. What is the steady-state level of the drug?
3. When does the drug level reach 90% of the steady-state value?
4. A fish hatchery has 500 *fish* at time , when harvesting begins at a rate of *b* *fish/yr*. where . The fish population is modeled by the initial value problem.



Where *t* is measured in years.

1. Find the fish population for  in terms of the harvesting rate *b*.
2. Graph the solution in the case that . Describe the solution.
3. Graph the solution in the case that . Describe the solution.
4. A community of hares on an island has a population of 50 when observations begin at . The population for  is modeled by the initial value problem.



1. Find the solution of the initial value problem.
2. What is the steady-state population?
3. When an infected person is introduced into a closed and otherwise healthy community, the number of people who become infected with the disease (in the absence of any intervention) may be modeled by the logistic equation



Where *k* is a positive infection rate, *A* is the number of people in the community, and  is the number of infected people at . The model assumes no recovery or intervention.

1. Find the solution of the initial value problem in terms of *k*, *A*, and .
2. Graph the solution in the case that  .
3. For fixed values of *k* and *A*, describe the long-term behavior of the solutions for any  with 
4. The reaction of chemical compounds can often be modeled by differential equations. Let  be the concentration of a substance in reaction for  (typical units of *y* are *moles/L*). The change in the concentration of a substance, under appropriate conditions, is , where  is a rate constant and the positive integer *n* is the order of the reaction.
5. Show that for a first-order reaction , the concentration obeys an exponential decay law.
6. Solve the initial value problem for a second-order reaction  assuming 
7. Graph and compare the concentration for a first-order and second-order reaction with  and 
8. The growth of cancer turmors may be modeled by the Gomperts growth equation. Let  be the mass of the tumor for . The relevant intial value problem is



Where *a* and *K* are positive constants and 

1. Graph the growth rate function  assuming  and . For what values of *M* is the growth rate positive? For what values of *M* is maximum?
2. Solve the initial value problem and graph the solution for , , and . Describe the growth pattern of the tumor. Is the growth unbounded? If not, what is the limiting size of the tumor?
3. In the general equation, what is the meaning of *K*?
4. The halibut fishery has been modeled by the differential equation



Where  is the biomass (the total mass of the members of the population) in kilograms at time *t* (measured in years), the carrying capacity is estimated to be  and .

1. If , find the biomass a year later.
2. How long will it take for the biomass to reach .
3. Suppose a population  satisfies , where *t* is measured in years.
4. What is the carrying capacity?
5. What is ?
6. When will the population reach 50% of the carrying capacity?
7. The board of directors of a corporation is calculating the price to pay for a business that is forecast to yield a continuous flow of profit of $500,000 per *year*. The money will earn a nominal rate of 5% per year compounded continuously. What is the present value of the business?
8. For 20 *years*?
9. Forever (in perpetuity)?
10. The population of a community is known to increase at a rate proportional to the number of people present at a time *t*. If the population has doubled in 6 *years*, how long it will take to triple?
11. Let population of country be decreasing at the rate proportional to its population. If the population has decreased to 25% in 10 *years*, how long will it take to be half?
12. Suppose that we have an artifact, say a piece of fossilized wood, and measurements show that the ratio of *C*−14 to carbon in the sample is 37% of the current ratio. Let us assume that the wood died at time 0, then compute the time *T* it would take for one gram of the radioactive carbon to decay this amount.

***Section* 1.8 – Basic Electrical Circuit**

***Resistor*:** (**O**hm’s **L**aw)

A ***resistor*** is a component of a circuit that resists the flow of electrical current. It has two terminals across which electricity must pass, and it is designed to drop the voltage of the current as it flows from one terminal to the other. Resistors are primarily used to create and maintain known safe currents within electrical components.

A voltage  across the terminals of a resistor is proportional to the current  in it. The constant proportional ***R*** is called the resistance of the resistor in **V**olt/**A**mpere or **O**hms (Ω), and is given by the equation:





For series resistors, the equivalent resistor is:





Then: 

For resistors in parallels:



Then: 

***Inductor*:** (**F**araday’s **L**aw)

When a current in a circuit is changing, then the magnetic flux is linking the same circuit changes. This change in flux causes an *emf* ***v*** to be induced in the circuit.

***Inductance*** is symbolized by letter ***L***, is measured in ***h*enrys** (***H***), and is represented graphically as a coiled wire – a reminder that inductance is a consequence of a conductor linking a magnetic field.



The voltage  is proportional to the time rate of change of the current, and is given by:

 and 

For series inductors, the equivalent inductor is:





For inductors in parallels:





***Capacitance*:** (**C**oulomb’s **L**aw)

The circuit parameter of ***capacitance*** is represented by letter ***C***, is measured in ***f****arads* (***F***), and is symbolized graphically by two short parallel conductive plates.



The farad is an extremely large quantity of capacitance, practical capacitor values usually lie in the picofarad (*pF*) to microfarad (*μF*) range.

The graphic symbol for a capacitor is a reminder that capacitance occurs whenever electrical conductors are separated by a dielectric, or insulating, material. This condition implies that electric charge is not transported through the capacitor. Although applying a voltage to the terminals of the capacitor cannot move a charge through the dielectric, it can displace a charge within the dielectric. As the voltage varies with time, the displacement of charge also varies with time, causing what is known as the displacement current.

The potential ***v*** between the terminals of a capacitor is proportional to the charge *q* on it.





 *C* is **C**oulombs/**V**olts or farads.

For capacitances in series, the equivalent capacitance is given by:

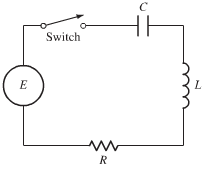
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For capacitances in parallels:

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***RLC circuit***

*RLC* circuit is a basic building block in electrical circuits and networks. A second order linear differential equations with constant coefficients is their use as a model of the flow of electric current in the simple series circuit



|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| The current *I*, measured in amperes (***A***), is a function of time *t*.  A ***resistor*** with a resistance of *R* *ohms* (**Ω**)  An ***inductor*** with an inductance of *L henries* (***H***)  A ***capacitor*** with a capacitance of *C farads* (***F***)  The impressed ***voltage*** *E* in volts (***V***) is a given function of time. | |  |  | | --- | --- | | ***Circuit Element*** | ***Voltage Drop*** | | Inductor |  | | Resistor |  | | Capacitor |  | |

In series with a source of electromotive force (such as a battery or a generator) that supplies a voltage of  volts at time *t*. If the switch shown in the circuit is closed, this results in a current of  *amperes* in the circuit and a charge of  *coulombs* on the capacitor at time *t*. The relation between the functions *Q* and the current *I* is



We use ***mks*** electric units, in which time is measured in seconds.

According to elementary principles of electricity, the voltage drops across the three circuit elements.

Kirchhoff's Current Law (***KCL***) (also known as Kirchhoff's ***First Law***)

***The algebraic sum of all the currents at any node in a circuit equals to zero.***

***Current is distributed when it reaches a junction: the amount of current entering a junction must equal the amount of current leaving that junction.***

Kirchhoff's Voltage Law (***KVL***) (also known as Kirchhoff's ***Second Law***)

***The algebraic sum of all the voltages around any closed path in a circuit equals to zero.***

***In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.***

According to the elementary laws of electricity, we know that

The voltage drop across the resistor is *IR*.

The voltage drop across the capacitor is .

The voltage drop across the inductor is .

* When the switch is open, no current flows in the circuit; when the switch is closed, there is a current  and a charge  on the capacitor.

The current and charge in the simple RLC circuit satisfy the basic electrical equation



The units for voltage, resistance, current, charge, capacitance, inductance, and time are all related:





Since , we can get the second-order linear differential equation



For the charge , under the assumption that the voltage is known.

It is the current, in most problems, rather than the charge *Q* that us of primary interest, so we differentiate both sides and substitute *I* for  to obtain



With initial conditions are











Hence  is also determined by the initial charge and current, which are physically measurable quantities.

* The most important conclusion is that the flow of current in the circuit is precisely the same form as the one that describes the motion of a spring-mass system.

***Summary***

In RLC circuit:



In terms of current: 



***Without capacitor*** 

Where  is the change on the capacitor and  is the applied voltage.

***Example***

Suppose the electrical circuit has a resistor of  and a capacitor of . Assume the voltage source is .

If the initial current is 0 *A*, find the resulting current.

***Solution***

 → 













## *Example*

The electrical analog of a carriage on wheels, coupled to the wall through a spring.



***a*)** Mechanical system. ***b*)** Electrical analog.

***A mechanical system with a one coordinates movement.***

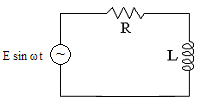
In the case of the electrical network, the equation was obtained by applying Kirchhoff's current law at the node ***v***, and is seen to be identical to the equation that would have been obtained by applying D'Alembert's principle to the mechanical system.

The differential equation for both systems is:



In particular, if one uses the force current analogy (or force-torque for a rational system). The topology of the electrical analog is very similar to that of the mechanical system.

## *Example*: Alternating Circuit



*Alternating Circuit*

Translating the circuit into differential equation:















At



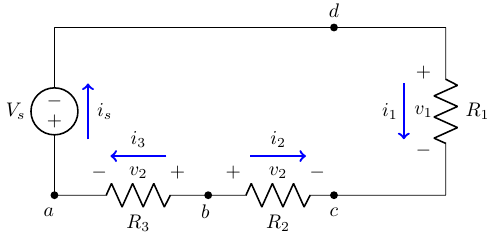




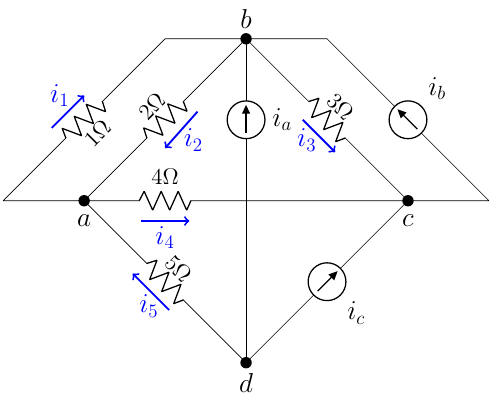


***Exercises Section* 1.8 - Basic Electrical Circuit**

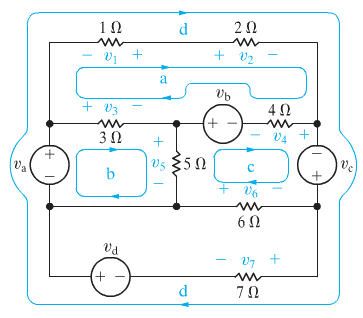
1. Sum the currents at each node in the circuit



1. Sum the currents at each node in the circuit



1. Sum the voltges around rach designated path in the circuit



A resistor and a capacitor of are joined in series with an electronic force (*emf*)  and no charge on the capacitor at . Find the ensuing charge on the capacitor at time *t* for the given:

|  |  |
| --- | --- |
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An inductor and a resistor are joined in series with an electronic force (*emf*)  and no charge on the capacitor at . Find the ensuing current in the current at time *t* for the given:

|  |  |  |
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1. An *RL* circuit with a  resistor and a  inductor is driven by a voltage . If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.
2. An *RL* circuit with a  resistor and a  inductor is driven by a voltage . If the initial inductor current is zero, determine the subsequence resistor and inductor current and the voltages.
3. An *RL* circuit with a  resistor and a  inductor is driven by a voltage . If the initial inductor current is 1 *A*, determine the subsequence resistor and inductor current and the voltages.
4. An *RC* circuit with a  resistor and a  capacitor is driven by a voltage . If the initial capacitor current is zero, determine the subsequence resistor and capacitor current and the voltages.
5. Solve the general initial value problem modeling the *RC* circuit

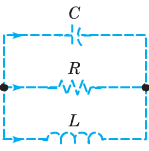


Where *E* is a constant source of *emf*

1. Solve the general initial value problem modeling the *LR* circuit



Where *E* is a constant source of *emf*

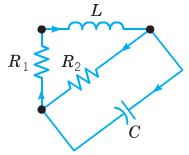
1. ******Consider the circuit shown and let , , and  be the currents through the capacitor, resistor, and inductor, respectively. Let , , and  be the corresponding voltage drops. The arrows denote the arbitrary chosen directions in which currents and voltage drops will be taken to be positive.
2. Applying Kirchhoff’s second law to the upper loop in the circuit, show that 
3. Applying Kirchhoff’s first law to either node in the circuit, show that



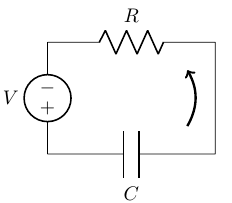
1. Use the current-voltage relation through each element in the circuit to obtain the equations



1. Eliminate , ,  and to obtain 
2. Consider the circuit. Use the method outlined to show that the current *I* through the inductor and the voltage *V* across the capacitor satisfy the system of differential equations.



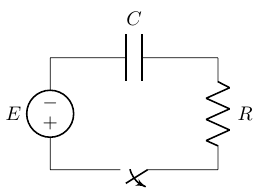


1. Consider an electric circuit containing a capacitor, resistor, and battery.

The charge  on the capacitor satisfies the equation



Where *R* is the resistance, *C* is the capacitance, and *V* is the constant voltage supplied by the battery.

1. If , find  at time *t*.
2. Find the limiting value  that approaches after a long time.
3. Suppose that  and that at time  the battery is removed and the circuit is closed again. Find  for .
4. A circuit containing an electromotive force, a capacitor with a capacitance of *C* farads (*F*), and a resistor with a resistance of *R* ohms . The voltage drop across the capacitor is , where *Q* is the charge (in coulombs), so in this case ***Kirchhoff’s Law*** gives



But , so we have 

Find the charge and the current at time *t*

1. Suppose the resistance is , the capacitance is 0.05 *F*, a battery gives voltage of 60 *V* and initial charge is 
2. Suppose the resistance is , the capacitance is 0.01 *F*,  and initial charge is 
3. A heart pacemaker consists of a switch, a battery voltage , a capacitor with constant capacitance *C*, and the heart as a resistor with constant resistance *R*. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage *E* across the heart satisfies the linear differential equation



Solve the *DE*, subject to 

1. A 30−volt electromotive force is applied to an *LR*-series circuit in which the inductance is 0.1 *henry* and the resistance is 50 *ohms*.
2. Find the current  if 
3. Determine the current as 
4. Solve the equation when  and 
5. A 100−volt electromotive force is applied to an *RC*-series circuit in which the resistance is 200 *ohms* and the capacitance is  *farad*.
6. Find the charge  if 
7. Find the current as 
8. A 200−volt electromotive force is applied to an *RC*-series circuit in which the resistance is 1000 *ohms* and the capacitance is  *farad*.
9. Find the charge  if 
10. Determine the charge as 
11. An electromotive force



Is applied to an *LR*-series circuit in which the inductance is 20 *henries* and resistance is 2 *ohms*. Find the current  if 

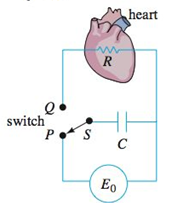
1. Suppose an *RC*-series circuit has a variable resistor. If the resistance at time *t* is given by  where  and  are known positive constants, then



If  and , where  and  are constants, show that



1. A heart pacemaker, consists of a switch, a battery, a capacitor, and the heart as a resistor.



When the switch *S* is at *P*, the capacitor charges; when *S* is at *Q*, the capacitor discharges, sending an electrical stimulus to the heart. The electrical stimulus is being applied to the heart, the voltage *E* across the heart satisfies the linear DE.



1. Let assume that over the time interval of length , , the switch *S* is at position *P* and the capacitor is being charges. When the switch is moved to position *Q* at time  the capacitor discharges, sending an impulse to the heart over the time interval of length : . Thus over the initial charging/discharging interval the voltage to the heart is actually modeled by the piecewise-defined differential equation



By moving *S* between *P* and *Q*, the charging and discharging over time intervals of lengths  and  is repeated indefinitely. Suppose , . , and , , , , , and so on.

Solve for  for 

1. Suppose for the sake of illustration that . Graph the solution in part (*a*) for 

***Section* 1.9 - Existence and Uniqueness of Solutions**

The questions of existence and uniqueness

* When can we be sure that a solution exists**?**
* How many different solutions are there

**Existence of Solutions**

***Example***

Consider the initial value problem:  with 

***Solution***



There is ***no solution*** to the given initial value















***Theorem:* Existence of Solutions**

Suppose the function is defined and continuous on the rectangle ***R*** in the *tx*-plane. Then given any point , the initial value problem



has a solution  defined in an interval containing . Furthermore, the solution will be defined at least until the solution curve  leaves the rectangle ***R***.

**Interval of Existence of a Solution**

***Example***

Consider the initial value problem  with . Find the solution and its interval of existence.

***Solution***

The right-hand side is  which is continuous on the entire *tx*-plane.

The solution to the initial value problem is:











 is discontinuous at . Hence the solution to the initial value problem is defined only for .

The interval: 

***Theorem*: Existence of a Unique Solution**

Let R be a rectangular region in the *xy-*plane defined by  that contains the point  in its interior. If  and  are continuous on R, then there exists some interval , contained in [*a, b*], and a unique function , defined on  that is a solution of the initial*-*value problem (IVP)



**Mathematics & Theorems**

Any theorem is a logical statement which has hypotheses (when it's true) and conclusions (true)

***The Hypotheses of the Uniqueness of Solutions Theorem***

1. The equation is in normal form 
2. The right-hand side  and its derivative  are both continuous in the rectangle ***R***.
3. The initial point  is in the rectangle ***R***.

For the uniqueness theorem the conclusions are as follows:

1. There is one and only one solution to the initial value problem.
2. The solution exists until the solution curve  leaves the rectangle ***R***.

***Example***

Consider the initial value problem . Is there a solution to this equation with initial condition ? If so, is the solution unique?

***Solution***



The right-hand side:  is continuous except where .

We can take ***R*** to be any rectangle which contains the point  to avoid , we can choose  and 

Then *f* is continuous everywhere in ***R*** ⇒ hypotheses of the existence theorem are satisfied.

Since  is also continuous in ***R***.

There is only one solution.

***Exercises Section* 1.9 - Existence and Uniqueness of Solutions**

Which of the initial value problems are guaranteed a unique solution

|  |  |
| --- | --- |
|  |  |

1. Show that and  are both solutions of the initial value problem , where . Explain why this fact doesn't contradict Theorem
2. Use a numerical solver to sketch the solution of the given initial value problem



1. Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.
2. Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).

***Section* 1.10 - Autonomous Equations and Stability**

A first-order autonomous equation is an equation of the form





***Definition***

The valuewhere the functionassigns to the point represent the slope of a line (line segment) call ***a lineal element***.

***Example***: Given and consider the point

The slope of the lineal element is 

**The Direction Fields**

What we draw a lineal element at each point  with slope  then the collection of these lineal elements is called a ***direction field*** or a ***slope field*** of the differential equation.

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***Example***

Sketch the direction field for the following differential equation. Sketch the set of integral curves for this differential equation, how the solutions behave as  and if this behavior depends on the value of  describe this dependency



***Solution***







This divided into 4 regions.

For , assume  

, the slopes will flatten out while staying positive



For , assume  

Therefore, tangent lines in this region will have negative slopes and apparently not very steep.



 (Steeper)



For , assume   Not to steep



For , assume  

Start out fairly flat neary = 2, then will get fairly steep.



|  |  |
| --- | --- |
| Value of |  |
|  |  |
|  |  |
|  |  |
|  |  |

**Autonomous 1st order DE**

A system, which does not explicitly contain the independent variable *t* is called an ***autonomous system***. Otherwise, the system is called *non-autonomous system*.

|  |  |
| --- | --- |
| ***Autonomous*** | ***Not- Autonomous*** |
|  |  |
|  |  |
|  |  |

***Equilibrium* Points & Solutions**

 and also called a ***critical point***.



From these equilibrium points, we can determine the stability of the system.

* An equilibrium point is ***stable*** if all nearby solutions stay nearby.



* An equilibrium point is ***asymptotically stable*** if all nearby solutions not only stay nearby, but also tend to the equilibrium point. An equilibrium point is stable if all nearby solutions.



1. If , then *f* is ***decreasing*** at and is asymptotically stable.
2. If , then *f* is ***increasing*** at and is unstable.
3. If  no conclusion can be drawn.

The family of all solution curves without the presence of the independent variable is called the ***phase portrait***.

When an independent variable ***t*** is interpreted as time and the solution curve could be thought of as the path of a particle moving in the solution space, then the system is considered as a ***dynamical system***, where the solution curves are called *t****rajectories*** or ***orbits***.

***Example***

Discover the behavior as of all solutions to the differential equation



***Solution***

The equilibrium points: 



 are equilibrium.



 ***unstable***

 is ***asymptotically stable***

 ***unstable***



These are constant functions, the position of the point the phase line modeled by them is also constant

-1 1 2

***Phase Portrait***

***Exercises Section* 1.10 - Autonomous Equations and Stability**

The graph of the right-hand side  is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

**1.**



**2.**



**3.**



**4.** Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



An autonomous differential equation is given. Perform each of the following to exercises **5 − 8**

1. Sketch a graph of 
2. Use the graph of *f* to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.
3. Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

|  |  |  |
| --- | --- | --- |
|  |  |  |

Determine the stability of the equilibrium solutions

1. 
2. 
3. A tank contains 100 *gal* of pure water. A salt solution with concentration 3 *lb/gal* enters the tank at a rate of 2 *gal/min*. Solution drains from the tank at a rate of 2 *gal/min*. Use the qualitative analysis to find the eventual concentration of the salt in the tank.
4. A mathematical model for rate at which a drug disseminates into the bloodstream at time *t*.



Where *r* and *k* are positive constants. The function  describes the concentration of the drug in the bloodstream at time *t*.

1. Since the DE is autonomous, use the phase portrait concept to find the limiting value of  as 
2. Solve  subject to . Sketch the graph of  and verify your prediction in part (*a*). At what time is the concentration one-half this limiting value?
3. When forgetfulness is taken into account, the rate of memorization of a subject is given by



Where , ,  is the amount memorized in time *t*, *M* is the total amount to be memorized, and  is the amount remaining to be memorized.

1. Since the DE is autonomous, use the phase portrait concept to find the limiting value of  as . Interpret the result
2. Solve  subject to . Sketch the graph of  and verify your prediction in part (*a*).
3. The number  of supermarkets throughout the country that are using a computerized checkout system is described by the initial-value problem



1. Use the phase portrait concept to predict how many supermarkets are expected to adopt the new procedure over a long period of time. Sketch a solution curve of the given initial-value problem.
2. Solve the initial-value problem and then graph it to verify the solution in part (*a*)
3. How many companies are expected to adopt the new technology when ?