***Chapter Two* – First Order Equations**

***Section* 2.1 – Differential Equations & Solutions**

**Ordinary Differential Equations**

Involve an unknown function of a single variable with one or more of its derivatives.



*y*: is unknown function

*t*: independent variable

Some other example:











∴ The order of a differential equation is the order of the highest derivative that occurs in the equation.

 *second order*

 is not an ODE (*ω* is dependent on *x* and *t*)

This equation is called a ***partial differential equation***.

**Definition**

A first-order differential equation of the form  is said to be in normal form.

 is said to be in normal form.

*f*: is a given function of 2 variables *t* & *y* (***rate function***)

***Solutions***

A solution of the first-order, ordinary differential equation  is a differentiable function  such that  for all *t* in the interval where  is defined.

1. Can be found in explicit and implicit form by applying manipulation (integration)
2. No real solution.

***Example***

Show that  is a solution of the 1st order equation 

*Solution*





 True; it is a solution

is called the ***general solution***.

The solutions from the graph are called ***solution curves***.

***Example***

Is the function  a solution to the differential equation

*Solution*





 False; it is not a solution.

***Exercises Section* 2.1 – Differential Equations & Solutions**

1. Show that  is a solution of the 1st order equation 
2. Show that  is a solution of the 1st order equation 
3. A general solution may fail to produce all solutions of a differential equation. In exercise 6, show that  is a solution of the differential equation, but no value of C in the given general solution will produce this solution.
4. Use the given general solution to find a solution of the differential equation having the given initial condition.
5. Show that  is a solution of the 1st order equation 
6. Use the given general solution to find a solution of the differential equation having the given initial condition. 

***Section* 2.2 – Solutions to Separable Equations**

**Separable Equation**

Separable equation is an equation that can be written with its variables separated and then easily solved.

If is independent of *y*⇒





***Definition***

A 1st order differential equation of the form is said to be separable or to have separable variables.





 *not separable*

***Example***

At time *t* the sample contains radioactive nuclei and is given by the differential equation:



This is called the ***exponential equation***.



 ***Separable equation***













***Example***

Solve the differential equation 

*Solution*









 *Cross multiplication*





 ∴ Equilibrium point

***General Method***

1. Separate the variables
2. Integrate both sides
3. Solve for the solution , if possible

***Using definite Integration***

***Example***

A can of beer at 40° F is placed into a room when the temperature is 70° F. After 10 minutes the temperature of the beer is 50° F. What is the temperature of the beer as a function of time? What is the temperature of the beer 30 minutes after the beer was placed into the room?

*Solution*

By Newton's law of cooling: The rate of change of an object's temperature (***T***) is proportional to the difference between its temperature and the ambient temperature (***A***).











 *Quotient Rule*







*Given*: 

























***Losing a solution***

When we use separate variables, the variable divisors could be zero at a point.

***Example***

Find a general solution to 

*Solution*



 *critical points*



















If 







If  









***Implicitly Defined Solutions***

***Example***

Find the solutions of the equation , having initial conditions  and 

*Solution*











 *Quadratic Formula*

 *Implicit*























 , but it never it will be.

*Explicit Solutions*: 

*Implicit solutions*: 

***Example***

Find the solutions to the differential equation , having 

*Solution*











For 







We can't solve for 

⇒ This solution is defined as implicit.

For 







Since the initial condition < 0, then:



For 

 *True statement*

 is a solution

***Exercises Section* 2.2 – Solutions to Separable Equations**

Find the general solution of the differential equation. If possible, find an explicit solution.

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 

Find the exact solution of the initial value problem. Indicate the interval of existence.

1. 
2. 
3. 
4. A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume that the surrounding air temperature remains constant at 21°C. Use Newton’s law of cooling to calculate the victim’s time of death. *Note*: The normal temperature of a living human being is approximately 37°C

***Section* 2.3 – Models of Motions**

***Newton's 2nd Law***

*The force acting on a mass is equal to the rate of change of momentum with respect to time. Momentum is defined as the product of mass and velocity (m.v).The force is equal to the derivative of the momentum*



***Position***: 

**Air Resistance**



***R***: resistance force (*has sign opposite of the velocity*)

***r***: is a function that is always nonnegative

* *when a ball is falling from a high altitude, the density of the air has to be taken into account.*















When  (*Terminal Velocity*)

 (*A*: is a constant)

***Example***

Suppose you drop a brick from the top of a building that is 250 m high. The brick has a mass of 2 kg, and the resistance force is given by . How long will it take the brick to reach the ground? what will be its velocity at that time?

*Solution*





























 (Using software to solve it)





***Finding the displacement***













***Example***

A ball of mass is protected from the surface of the earth, with velocity . Assume that the force of air resistance is given by , where . What is the maximum height reached by the ball?

*Solution*























***Exercises Section* 2.3 – Models of Motions**

1. A stone is released from rest and dropped into a deep well. Eight seconds later, the sound of the stone splashing into the water at the bottom of the well returns to the ear of the person who released the stone. How long does it take the stone to drop to the bottom of the well? How deep is the well? Ignore air resistance. The speed of sound is 340 *m/s*.
2. A rocket is fired vertically and ascends with constant acceleration  for 1.0 *min*. At that point, the rocket motor shuts off and the rocket continues upward under the influence of gravity. Find the maximum altitude acquired by the rocket and the total time elapsed from the take-off until the rocket returns to the earth. *Ignore air resistance*.
3. A ball having mass  falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of the force due to the resistance.

***Section* 2.4 – Linear Equations**

A first order linear equation is given by the form:



If 

The linear equation is said to be ***homogeneous***. (Otherwise it is ***nonhomogeneous***).

are called the coefficients

The general form: 

|  |  |
| --- | --- |
| ***Linear*** | ***Non-linear*** |
|  |  |
|  |  |
|  |  |

***Solution of the homogenous equation***







 *Convert to exponential form*







***Example***

Solve: 

*Solution*





***Solution of the Nonhomogeneous Equation*** 

Let assume:  

The homogeneous equation is given by 













 *Since* 



















***Example***Find the general solution to: 

*Solution*

















**Solving a linear first-order Equation (Properties)**

1. Put a linear equation into a standard form 
2. Identify  then find 
3. Multiply the standard form by 
4. Integrate both sides

***Solution of the Inhomogeneous Equation***











***Example***

Find the general solution of  and the particular solution that satisfies .

*Solution*

























***Example***

Find the general solution of  and the particular solution that satisfies .

*Solution*





























***Notes***

1. Integrating an expression that is not the differential of any elementary function is called non-elementary.

1. In math some important functions are defined in terms of non-elementary integrals. Two suxh functions are the error function and the complementary error function.

***Exercises Section* 2.4 – Linear Equations**

Find the general solution of the first-order, linear equation.

1. 
2. 
3. 
4. 

Find the solution of the initial value problem

1. 
2. 
3. 

Find the solution of the initial value problem. Discuss the interval of existence and provide a sketch of your solution

1. 
2. 

Find the general solution of the given differential equation. Then find the particular solution satisfying the given initial condition of

1. 
2. 
3. 

***Section* 2.5 – Mixing Problems**

The physical representation of the rate of change:

*rate of change = rate in* - *rate out*

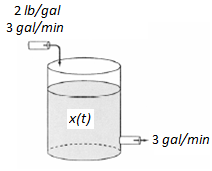
This is referred to as a ***balance law***.

Rate = Volume Rate (*gal/min*) *x* Concentration (*lb/gal*)

***Example***

The tank initially holds 100 gal of pure water. At time , a solution containing 2 lb of salt per gallon begins to enter the tank at the rate of 3 gallons per minute. At the same time a drain is opened at the bottom of the tank so that the volume of solution in the tank remains constant. How much salt is in the tank after 60 min?

*Solution*

number of pounds of salt in the tank after *t* min.

Volume: 

Concentration at time *t*: 

Rate in = Volume Rate *x* Concentration





Rate out = Volume Rate *x* Concentration





rate of change

= rate in - rate out



























After 60 min:

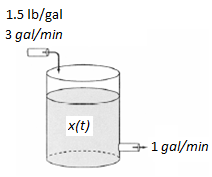




***Example***

The 600-gal tank is filled with 300 gal of pure water. A spigot is opened above the tank and a salt solution containing 1.5 lb of salt per gallon of solution begins flowing into the tank at the rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave tank at a rate of 1 *gal/min*. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 *gal*)?

*Solution*







Rate in 



Rate out 





















































***Exercises Section* 2.5 – Mixing Problems**

1. A tank contains 100 *gal* of pure water. At time zero, a sugar-water solution containing 0.2 *lb* of sugar per gal enters the tank at a rate of 3 *gal/min*. Simultaneously, a drain is opened at the bottom of the tank allowing the sugar solution to leave the tank at 3 *gal/min*. Assume that the solution in the tank is kept perfectly mixed at all times.
2. What will be the sugar content in the tank after 20 minutes?
3. How long will it take the sugar content in the tank to reach 15 lb?
4. What will be the eventual sugar content in the tank?
5. A 50-gal tank initially contains 20 *gal* of pure water. Salt-water solution containing 0.5 *lb.* of salt for each gallon of water begins entering the tank at a rate of 4 *gal/min*. Simultaneously; a drain is opened at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 *gal/min*. What is the salt content (*lb*) in the tank at the precise moment that the tank is full of salt-water solution?
6. Consider two tanks, label tank A and tank B for reference. Tank A contains 100 *gal* of solution in which is dissolved 20 *lb* of salt. Tank B contains 200 *gal* of solution which is dissolved 40 *lb* of salt. Pure water flows into the tank A at rate of 5 *gal/s*. There is a drain at the bottom of tank A. The solution leaves tank A via the drain at a rate of 5 *gal/s* and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 *gal/s*. What is the salt content in tank B at the precise moment that tank B contains 250 *gal* of solution?
7. A tank contains 500 *gal* of a salt-water solution containing 0.05 *lb* of salt per gallon of water. Pure water is poured into the tank and a drain at the bottom of the tank is adjusted so as to keep the volume of solution in the tank constant. At what rate (gal/min) should the water be poured into the tank to lower the salt concentration to 0.01 lb/gal of water in less than one hour?

***Section* 2.6 - Existence and Uniqueness of Solutions**

The questions of existence and uniqueness

* When can we be sure that a solution exists**?**
* How many different solutions are there

**Existence of Solutions**

***Example***

Consider the initial value problem:  with 

*Solution*



There is ***no solution*** to the given initial value















***Theorem:*Existence of Solutions**

Suppose the function is defined and continuous on the rectangle ***R*** in the *tx*-plane. Then given any point , the initial value problem



has a solution  defined in an interval containing . Furthermore, the solution will be defined at least until the solution curve  leaves the rectangle ***R***.

**Interval of Existence of a Solution**

***Example***

Consider the initial value problem  with . Find the solution and its interval of existence.

*Solution*

The right-hand side is  which is continuous on the entire *tx*-plane.

The solution to the initial value problem is:











 is discontinuous at . Hence the solution to the initial value problem is defined only for .

The interval: 

**Mathematics & Theorems**

Any theorem is a logical statement which has hypotheses (when it's true) and conclusions (true)

***The Hypotheses of the Uniqueness of Solutions Theorem***

1. The equation is in normal form 
2. The right-hand side  and its derivative  are both continuous in the rectangle ***R***.
3. The initial point  is in the rectangle ***R***.

For the uniqueness theorem the conclusions are as follows:

1. There is one and only one solution to the initial value problem.
2. The solution exists until the solution curve  leaves the rectangle ***R***.

***Example***

Consider the initial value problem . Is there a solution to this equation with initial condition ? If so, is the solution unique?

*Solution*



The right-hand side:  is continuous except where .

We can take ***R*** to be any rectangle which contains the point  to avoid , we can choose  and 

Then *f* is continuous everywhere in ***R*** ⇒ hypotheses of the existence theorem are satisfied.

Since  is also continuous in ***R***.

There is only one solution.

It is important to determine and prove a theorem concerning the existence and uniqueness of solutions of an O.D.E.

* Are the  solutions to exist?



⇒ Solutions exist for the system.

* ***Uniqueness***: Assume is another solution. We want to prove is actually  i.e.









So that, , then multiply both sides by to obtain: 

***Exercises Section* 2.6 - Existence and Uniqueness of Solutions**

Which of the initial value problems are guaranteed a unique solution.

1. 
2. 
3. 
4. Show that and  are both solutions of the initial value problem , where . Explain why this fact doesn't contradict Theorem
5. Use a numerical solver to sketch the solution of the given initial value problem



1. Where does your solver experience difficulty? why? Use the image of your solution to estimate the interval of existence.
2. Find an explicit solution; then use your formula to determine the interval of existence. How does it compare with the approximation found in part (*a*).

***Section* 2.7 - Autonomous Equations and Stability**

A first-order autonomous equation is an equation of the form





***Definition***

The valuewhere the functionassigns to the point represent the slope of a line (line segment) call ***a lineal element***.

***Example***: Given and consider the point

The slope of the lineal element is 

**The Direction Fields**

What we draw a lineal element at each point  with slope  then the collection of these lineal elements is called a direction field or a slope field of the differential equation.

**Autonomous 1st order DE**

A system, which does not explicitly contain the independent variable*t* is called an ***autonomous system***. Otherwise, the system is called *non-autonomous system*.

|  |  |
| --- | --- |
| ***Autonomous*** | ***Not- Autonomous*** |
|  |  |
|  |  |
|  |  |
|  |  |

***Equilibrium* Points & Solutions**

 and also called a ***critical point***.



From these equilibrium points, we can determine the stability of the system.

* An equilibrium point is ***stable*** if all nearby solutions stay nearby.



* An equilibrium point is ***asymptotically stable*** if all nearby solutions not only stay nearby, but also tend to the equilibrium point. An equilibrium point is stable if all nearby solutions.



1. If , then *f* is ***decreasing*** at and is asymptotically stable.
2. If , then *f* is ***increasing*** at and is unstable.
3. If  no conclusion can be drawn.

The family of all solution curves without the presence of the independent variable is called the ***phase portrait***.

When an independent variable***t*** is interpreted as time and the solution curve could be thought of as the path of a particle moving in the solution space, then the system is considered as a ***dynamical system***, where the solution curves are called *t****rajectories*** or ***orbits***.

***Example***

Discover the behavior as of all solutions to the differential equation



*Solution*

The equilibrium points: 



are equilibrium.



 ***unstable***

 is ***asymptotically stable***

 ***unstable***



These are constant functions, the position of the point the phase line modeled by them is also constant

-1 1 2

***Phase Portrait***

***Exercises Section* 2.7 - Autonomous Equations and Stability**

The graph of the right-hand side  is shown. Identify the equilibrium points and sketch the equilibrium solutions in the *ty*-plane. Classify each equilibrium point as either unstable or asymptotically stable.

**1.**



**2.**



**3.**



**4.** Impose the equilibrium solution(s), classifying each as either unstable or asymptotically stable



An autonomous differential equation is given. Perform each of the following to exercises 5 - 8

*i*) Sketch a graph of 

*ii*) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

*iii*) Sketch the equilibrium solutions in the *ty*-plane into regions. Sketch at least one solution trajectory in each of these regions.

1. 
2. 
3. 
4. 

Determine the stability of the equilibrium solutions

1. 
2. 
3. A tank contains 100 gal of pure water. A salt solution with concentration 3 lb/gal enters the tank at a rate of 2 gal/min. Solution drains from the tank at a rate of 2 gal/min. Use the qualitative analysis to find the eventual concentration of the salt in the tank.

***Section* 2.8 - Modeling Population Growth**

**Modeling Population Growth**

The mathematical model of the growth of a population is given by:



Where ***r***: reproductive rate.

The natural of the predictions of the model depend on the nature of the reproductive rate *r*.

**Malthusian Method**

Since *r* is a constant because the birth or death rates do not depend on time or on the size.

Therefore the solution to  is given by:





The population at time  is .

***Example***

A biologist starts with 10 cells in a culture. Exactly 24 *hrs* later he counts 25. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 10 days?

*Solution*





 *24 hrs = 1 day P = 25*















**Logistic Model of Growth**

The logistic equation is given by:





***Example***

Suppose we start at time  with a sample of 1000 cells. One day later we see that the population has doubled, and sometime later we notice that the population has stabilized at 100,000.

*Solution*





























***Exercises Section* 2.8 - Modeling Population Growth**

1. A biologist starts with 100 cells in a culture. After 24 *hrs,* he counts 300. Assuming a Malthusian model, what the reproductive rate? What will be the number of cells of the end of 5 *days*?
2. A biologist prepares a culture. After 1 *day* of growth, the biologist counts 1000 cells. After 2*days,* he counts 3000. Assuming a Malthusian model, what the reproductive rate and how many cells were present initially?
3. A population of bacteria is growing according to the Malthusian model. If the population is triples in 10 *hrs*, what is the reproduction rate? How often does the population double itself?
4. Consider a lake that is stocked with walleye pike and that the population of pike is governed by the logistic equation



where time is measured in days and *P* in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

1. Modify the logistic model to account for the fishing.
2. Find and classify the equilibrium points for your model.
3. Use qualitative analysis to completely discuss the fate of the fish population with this model. In particular, if the initial fish population is 1000, what happens to the fish as time passes? What will happen to an initial population having 2000 fish?