***Chapter* 4 - Laplace Transform**

***Section* 4.1 - The Definition of the Laplace Transform**

***Definition***

Suppose  is a function of *t* defined for . The ***Laplace transform*** of *f* is the function



The integral of the Laplace transform is an improper integral because the upper limit is ∞.



***Example***

Assume 

*Solution*









***Example***

Assume 

*Solution*















*Laplace transform to any power*



***Example***

Assume 

*Solution*

























***Exercises Section* 4.1 - The Definition of the Laplace Transform**

Use Definition of Laplace transform to find the Laplace transform of:

1. 
2. 
3. 
4. 
5. 

Use Definition of Laplace transform to show the Laplace transform of

1.  is 

***Section* 4.2 - Basic Properties of the Laplace Transform**

***The Laplace Transform of Derivatives***

***Proposition***

Suppose *y* is a piecewise differentiable function of exponential order. Suppose also that  is of the exponential order.





***Proof***









Let: 

  
; which converges to 0 for  as . Therefore,



***Proposition***





***Proposition***





***Laplace Transform Linear***



***Example***

Find the Laplace transform of 

*Solution*





***Example***

Transform the initial value problem into an algebraic equation involving . Solve the resulting equation for the Laplace transform of *y*.

*Solution*

For the right-hand side



















**Laplace Transform of the Product of an Exponential with a Function**

The result is a translation in the Laplace transform



***Example***

Compute the Laplace transform of the function 

*Solution*

Let 

With 







***Proposition*: Derivative of a Laplace Transform**





*Example*

Compute the Laplace transform of 

*Solution*











***Exercises Section* 4.2 - Basic Properties of the Laplace Transform**

Find the Laplace transform and defined the time domain of

1. 
2. 

Transform the initial value problem into an algebraic equation involving . Solve the resulting equation for the Laplace transform of *y*.

1. 
2. 
3. 
4. 

Find the Laplace transform of

1. 
2. 
3. 
4. 
5. 

Transform the initial value problem into an algebraic equation involving . Solve the resulting equation for the Laplace transform of *y*.

1. 
2. 
3. 

***Section* 4.3 - Inverse Laplace Transform**

***Definition***

If is a continuous function of exponential order and , then we call the inverse Laplace transform of F,









Note: Inverse transforms are not unique. If and are identical except at a discrete set of points, then . However, there is at most one continuous function  satisfying 

***Laplace Transform Linear***

***Proposition***





***Example***

Compute the inverse Laplace transform of 

*Solution*







***Example***

Compute the inverse Laplace transform of 

*Solution*











***Example***

Compute the inverse Laplace transform of 

*Solution*







***Example***

Find the inverse Laplace transform of 

*Solution*



















***Exercises Section* 4.3 - Inverse Laplace Transform**

Find the inverse Laplace transform of

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. 
15. 

***Section* 4.4 - Using Laplace Transform to Solve Differential Equations**

***Example***

Use Laplace transform to find the solution to the initial value problem

*Solution*





















**Homogeneous Equations**

***Example***

Use Laplace transform to find the solution to the initial value problem

*Solution*

















**Inhomogeneous Equations**

***Example***

Use Laplace transform to find the solution to the initial value problem

*Solution*































**Higher-Order Equations**

***Example***

Find the solution to the initial value problem

*Solution*























***Exercises Section* 4.4- Using Laplace Transform to Solve Differential Equations**

Solve using the Laplace transform:

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. Given: 
14. Show that the general solution is:  and find 
15. Use Laplace transform to solve the system

# *Section* 4.5- Basic Electrical Circuit

***Resistor*:** (**O**hm’s **L**aw)

A voltage  across the terminals of a resistor is proportional to the current  in it. The constant proportional **R** is called the resistance of the resistor in **V**olt/**A**mpere or **O**hms (Ω), and is given by the equation:





For series resistors, the equivalent resistor is:





Then: 

For resistors in parallels:



Then: 

***Laplace Transform***



The block diagram is shown below

*Ouput Signal*

*Input Signal*

*R*

***System***







***Inductor*:** (**F**araday’s **L**aw)

When a current in a circuit is changing, then the magnetic flux is linking the same circuit changes. This change in flux causes an *emf***v** to be induced in the circuit. The voltage  is proportional to the time rate of change of the current, and is given by:





and



For series inductors, the equivalent inductor is:

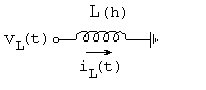




For inductors in parallels:







***Laplace Transform***



The block diagram:





**A**

*sL*

**Ω**

***Capacitance*:** (**C**oulomb’s **L**aw)

The potential **v** between the terminals of a capacitor is proportional to the charge *q* on it.



⇒

C is **C**oulombs/**V**olts or farads.

For capacitances in series, the equivalent capacitance is given by:

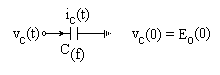




For capacitances in parallels:







***Laplace Transform***













*Cs*

***Summary***

|  |  |  |  |
| --- | --- | --- | --- |
|  | ***Abv.*** |  | ***Unit*** |
| Capacitor | C |  | Farad (**F**) |
| Current | I |  | Ampere (**A**) |
| Electric Charge | q |  | Coulomb (**C**) |
| Electromotive Force | emf |  | **Emf** |
| Inductor | L |  | Henry (**H**) |
| Resistor | R |  | Ohm (**Ω**) |
| Time | t |  | Second (**s**) |
| Voltage | V |  | Volt (**V**) |

### Simple Electrical Circuits Notations

In an electrical network, the flow of the current is consists of a finite number of closed loops, or circuits may be determined by the rules known as Kirchhoff’s laws:

*a)* Current Law: *The sum of the currents into one points is zero, and*

*b*) Voltage Law: *The sum of the voltage drops in a specified direction, around any closed loop, is zero*.



\*\* A closed loop is called to be oriented, when a positive direction has been assigned.

# *Electrical Network*(*Circuit*)

***Example***

Suppose the electrical circuit has a resistor of  and a capacitor of . Assume the voltage source is . If the initial current is 0 A, find the resulting current.



*Solution*

































***OR***























## *Example*

The electrical analog of a carriage on wheels, coupled to the wall through a spring.



***a*)** Mechanical system. ***b*)** Electrical analog.

A mechanical system with a one coordinates movement.

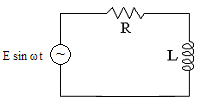
In the case of the electrical network, the equation was obtained by applying Kirchhoff's current law at the node ***v***, and is seen to be identical to the equation that would have been obtained by applying D'Alembert's principle to the mechanical system.

The differential equation for both systems is:



In particular, if one uses the force current analogy (or force-torque for a rational system). The topology of the electrical analog is very similar to that of the mechanical system.

## *Example*: Alternating Circuit



*Alternating Circuit*

Translating the circuit into differential equation:



⇒

From Appendix B, using the solution of the linear system:

 ⇒

Where 

⇒



Therefore, we can rewrite to:



That implies to:



By substituting the above, we can reformulate into:





At











Assume: ,

, and

 (Z is the impedance)

Then the value of the current can be written as:

***Summary***

In RLC circuit:



In terms of current: 



***Without capacitor*** 

Where  is the change on the capacitor and  is the applied voltage.

*Important facts that the differential equations for electrical and mechanical (Translation and Rotational) are identical in some forms.*

**TABLE A**

Relationships between the variables of the analog system components.

|  |  |  |
| --- | --- | --- |
| ***Electrical*** | ***Mechanical Translation*** | ***Mechanical Rotational*** |
| = Nφ |  | T = J |

*Engineers sometimes utilize the similarity by determining the properties of a proposed mechanical system with a simple electrical analog.*

**TABLE B**

Analogous between electrical and mechanical systems.

|  |  |  |
| --- | --- | --- |
| ***Electrical*** | ***Mechanical Translation*** | ***Mechanical Rotational*** |
| Current, ***i*** | Force , ***f*** N, lb | Torque, ***T*** N-m, lb-ft |
| Voltage, ***V*** | Velocity, ***v*** | Angular velocity, ***ω*** |
| Flux linkages | Displacement | Angular displacement,  Nφ,*xh* or *rad* |
| Capacitance. ***C*** | Mass, **M**  kg, slug | Moment of inertia,  **J** kg-m2, lb-ft/sec2**.** |
| Conductance  **G = 1/R** | Damping coefficient  (of dash pot) **D** or **B**  **N/m/sec**, lb/ft/sec | Rotational damping  Coefficient friction:  **D** or **B** |
| Inductance, ***L*** | Compliance  **= 1/k**  of spring | Torsional compliance  **= 1/k** of spring  **k** N.m/rad |

***Exercises Section* 4.5 - Basic Electrical Circuit**

A resistor and a capacitor of are joined in series with an electronic force (emf)  and no charge on the capacitor at . Find the ensuing charge on the capacitor at time *t* for the given:

1. 
2. 
3. 

An inductor and a resistor are joined in series with an electronic force (emf)  and no charge on the capacitor at . Find the ensuing current in the current at time *t* for the given:

1. 
2. 
3. 