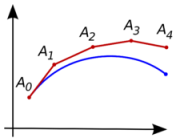
***Chapter* 5 - Numerical Methods**

***Section* 5.1 - Euler's Method**

***Euler's method*** named after *Leonhard Euler* is an example of a ***fixed-step***solver.

Euler's method is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. It is the most basic kind of explicit method for numerical integration of ordinary differential equations.

[](http://en.wikipedia.org/wiki/File:Euler_method.png)

The setting size: 

Then, 





*Last point* 

By the definition of the derivative:





The tangent line at the point  is:





This method is known as *Euler's Method* with step size *h*.

***Example***

Compute the first four step in the Euler's method approximation to the solution of  with, using the step size . Compare the result with the actual solution to the initial value problem.

*Solution*



The *first* step:









The *second* step:









The *third* step:









The *fourth* step:









The exact solution to is

|  |  |  |  |
| --- | --- | --- | --- |
|  | : Euler's | - exact | *Error* |
| 1.0 | 1.0 | 1.0 | 0 |
| 1.1 | 1.0 | 0.9948 | -0.0052 |
| 1.2 | 0.990 | 0.9786 | -0.0114 |
| 1.3 | 0.969 | 0.9501 | -0.0189 |
| 1.4 | 0.9359 | 0.9082 | -0.0277 |

***Exercises Section* 5.1 - Euler's Method**

Calculate the first five iterations of Euler's method with step of

1. 
2. 
3. Given: 
4. Use a computer and Euler's method to calculate three separate approximate solutions on the interval , one with step size , a second with step size , a second with step size .
5. Use the appropriate analytic to compute the exact solution
6. Plot the exact solution and approximate solutions as discrete points.
7. Consider the initial value problem 

Use Euler's method with step size  to sketch solution on the interval 

1. You've seen that the error in Euler's method varies directly as the first power of the step size . This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

***Section* 5.2 - Runge-Kutta Methods**

Like Euler's method, the Runge-Kutta methods are fixed-step solvers.

**The second-Order Runge-Kutta Method**

The second-Order Runge-Kutta method is also known as the improved Euler's method.

Starting from the initial value point , we compute two slopes:







But an analysis using Taylor's theorem reveals that there is an improvement in the estimate for the truncation error.

For the second-Order Runge-Kutta method, we have



The constant M depends on the function .

The second-Order Runge-Kutta method is controlled by the cube of the step size instead of the square.

Input 

For k = 1 to N









***Example***

Compute the first four step in the second-Order Runge-Kutta method approximation to the solution of  with, using the step size . Compare the result with the actual solution to the initial value problem.

*Solution*



The *first* step:































The *second* step:









The *third* step:









The *fourth* step:









|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | : Runge-Kutta | - Exact | ***Runge-Kutta Error*** | ***Euler'sError*** |
| 1.0 | 1.0 | 1.0 | **0** | **0** |
| 1.1 | 0.9950000 | 0.994829081 | **-0.000170918** | **-0.0052** |
| 1.2 | 0.9789750 | 0.978597241 | **-0.000377758** | **-0.0114** |
| 1.3 | 0.9507673 | 0.950141192 | **-0.000626182** | **-0.0189** |
| 1.4 | 0.9090979 | 0.908175302 | **-0.000922647** | **-0.0277** |

***Fourth-Order*  Runge-Kutta Method**

This method is the most commonly used solution algorithm. For most equations and systems it is suitably fast and accurate.

Starting from the initial value point , we compute two slopes:











***Example***

Compute the first four step in the second-Order Runge-Kutta method approximation to the solution of  with, using the step size . Compare the result with the actual solution to the initial value problem.

*Solution*



The *first* step:









































|  |  |  |  |
| --- | --- | --- | --- |
|  | : Runge-Kutta | - Exact | ***Runge-Kutta Error*** |
| 1.0 | 1.0 | 1.0 | **0** |
| 1.1 |  | 0.994829081 | **-0.000000086** |
| 1.2 |  | 0.978597241 | **0.000000295** |
| 1.3 |  | 0.950141192 | **-0.000000310** |
| 1.4 |  | 0.908175302 | **-0.0000004.57** |

***Exercises Section* 5.2 - Runge-Kutta Methods**

1. 
2. Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval , one with step size , a second with step size , a second with step size .
3. Use the appropriate analytic to compute the exact solution
4. Plot the exact solution and approximate solutions as discrete points.
5. Consider the initial value problem 

Use Runge-Kutta method with step size  to sketch solution on the interval 