*Chapter* 5 - Numerical Methods

*Solution Section* 5.1 - Euler's Method

***Exercise***

Calculate the first five iterations of Euler's method with step  of



*Solution*

|  |  |
| --- | --- |
| *t* | *y* |
| 0.1 | 1.00000000 |
| 0.2 | 1.01000000 |
| 0.3 | 1.03020000 |
| 0.4 | 1.06110600 |
| 0.5 | 1.10355024 |



***Exercise***

Calculate the first five iterations of Euler's method with step  of



*Solution*

|  |  |
| --- | --- |
| *x* | *z* |
| 0.0 | 1.00000000 |
| 0.1 | 0.80000000 |
| 0.2 | 0. 65000000 |
| 0.3 | 0.54000000 |
| 0.4 | 0.46200000 |
| 0.5 | 0.40960000 |

***Exercise***



1. Use a computer and Euler's method to calculate three separate approximate solutions on the interval , one with step size , a second with step size , a second with step size .
2. Use the appropriate analytic to compute the exact solution
3. Plot the exact solution and approximate solutions as discrete points.

*Solution*

|  |  |
| --- | --- |
| *x* | *y* |
| 0.0 | 8.00000000 |
| 0.2 | 8.00000000 |
| 0.4 | 7.40000000 |
| 0.6 | 6.29600000 |
| 0.8 | 4.90496000 |
| 1.0 | 3.49537280 |

|  |  |
| --- | --- |
| *x* | *y* |
| 0.0 | 8.00000000 |
| 0.1 | 8.00000000 |
| 0.2 | 7.85000000 |
| 0.3 | 7.55600000 |
| 0.4 | 7.13264000 |
| 0.5 | 6.60202880 |
| 0.6 | 5.99182592 |
| 0.7 | 5.33280681 |
| 0.8 | 4.65621386 |
| 0.9 | 3.99121964 |
| 1.0 | 3.36280010 |

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | *y* | *x* | *y* |
| 0.0 | 8.00000000 |  |  |
| 0.05 | 8.00000000 | 0.55 | 6.16870319 |
| 0.10 | 7.96250000 | 0.60 | 5.85692451 |
| 0.15 | 7.88787500 | 0.65 | 5.53550904 |
| 0.20 | 7.77705688 | 0.70 | 5.20820096 |
| 0.25 | 7.63151574 | 0.75 | 4.87862689 |
| 0.30 | 7.45322784 | 0.80 | 4.55022987 |
| 0.35 | 7.24463101 | 0.85 | 4.22621148 |
| 0.40 | 7.00856892 | 0.90 | 3.90948351 |
| 0.45 | 6.74822617 | 0.95 | 3.60262999 |
| 0.50 | 6.46705599 | 1.00 | 3.30788014 |



t Approx. Exact Difference

--------------------------------------------------------

0.00 | 8.00000000 | 8.00000000 | 0.00000000

0.20 | 8.00000000 | 7.70592079 | -0.29407921

0.40 | 7.40000000 | 6.89107842 | -0.50892158

0.60 | 6.29600000 | 5.73257245 | -0.56342755

0.80 | 4.90496000 | 4.45469318 | -0.45026682

1.00 | 3.49537280 | 3.25909581 | -0.23627699

t Approx. Exact Difference

--------------------------------------------------------

0.00 | 8.00000000 | 8.00000000 | 0.00000000

0.10 | 8.00000000 | 7.92537375 | -0.07462625

0.20 | 7.85000000 | 7.70592079 | -0.14407921

0.30 | 7.55600000 | 7.35448389 | -0.20151611

0.40 | 7.13264000 | 6.89107842 | -0.24156158

0.50 | 6.60202880 | 6.34100587 | -0.26102293

0.60 | 5.99182592 | 5.73257245 | -0.25925347

0.70 | 5.33280681 | 5.09469796 | -0.23810885

0.80 | 4.65621386 | 4.45469318 | -0.20152068

0.90 | 3.99121964 | 3.83643550 | -0.15478414

1.00 | 3.36280010 | 3.25909581 | -0.10370430

t Approx. Exact Difference

--------------------------------------------------------

0.00 | 8.00000000 | 8.00000000 | 0.00000000

0.05 | 8.00000000 | 7.98127342 | -0.01872658

0.10 | 7.96250000 | 7.92537375 | -0.03712625

0.15 | 7.88787500 | 7.83313428 | -0.05474072

0.20 | 7.77705688 | 7.70592079 | -0.07113608

0.25 | 7.63151574 | 7.54559797 | -0.08591777

0.30 | 7.45322784 | 7.35448389 | -0.09874395

0.35 | 7.24463101 | 7.13529429 | -0.10933672

0.40 | 7.00856892 | 6.89107842 | -0.11749051

0.45 | 6.74822617 | 6.62514862 | -0.12307755

0.50 | 6.46705599 | 6.34100587 | -0.12605012

0.55 | 6.16870319 | 6.04226366 | -0.12643953

0.60 | 5.85692451 | 5.73257245 | -0.12435207

0.65 | 5.53550904 | 5.41554691 | -0.11996214

0.70 | 5.20820096 | 5.09469796 | -0.11350300

0.75 | 4.87862689 | 4.77337119 | -0.10525570

0.80 | 4.55022987 | 4.45469318 | -0.09553669

0.85 | 4.22621148 | 4.14152671 | -0.08468477

0.90 | 3.90948351 | 3.83643550 | -0.07304801

0.95 | 3.60262999 | 3.54165879 | -0.06097120

1.00 | 3.30788014 | 3.25909581 | -0.04878433

***Exercise***

Consider the initial value problem 

Use Euler's method with step size  to sketch solution on the interval 

*Solution*



t Approx. Exact Difference

--------------------------------------------------------

0.00 | 1.00000000 | 1.00000000 | 0.00000000

0.04 | 2.44000000 | 2.77812333 | 0.33812333

0.08 | 4.26707200 | 3.75770045 | -0.50937155

0.12 | 3.72005658 | 3.96254078 | 0.24248419

0.16 | 4.21993115 | 3.99446397 | -0.22546718

0.20 | 3.77444588 | 3.99918742 | 0.22474154

0.24 | 4.18308995 | 3.99988085 | -0.18320910

0.28 | 3.81546672 | 3.99998253 | 0.18451582

0.32 | 4.15342541 | 3.99999744 | -0.15342797

0.36 | 3.84754974 | 3.99999962 | 0.15244989

0.40 | 4.12909852 | 3.99999994 | -0.12909858

0.44 | 3.87322947 | 3.99999999 | 0.12677052

0.48 | 4.10891492 | 4.00000000 | -0.10891492

0.52 | 3.89410430 | 4.00000000 | 0.10589570

0.56 | 4.09204138 | 4.00000000 | -0.09204138

0.60 | 3.91125556 | 4.00000000 | 0.08874444

0.64 | 4.07786461 | 4.00000000 | -0.07786461

0.68 | 3.92545437 | 4.00000000 | 0.07454563

0.72 | 4.06591460 | 4.00000000 | -0.06591460

0.76 | 3.93727310 | 4.00000000 | 0.06272690

0.80 | 4.05582011 | 4.00000000 | -0.05582011

0.84 | 3.94714987 | 4.00000000 | 0.05285013

0.88 | 4.04728141 | 4.00000000 | -0.04728141

0.92 | 3.95542805 | 4.00000000 | 0.04457195

0.96 | 4.04005260 | 4.00000000 | -0.04005260

1.00 | 3.96238159 | 4.00000000 | 0.03761841

1.04 | 4.03392967 | 4.00000000 | -0.03392967

1.08 | 3.96823212 | 4.00000000 | 0.03176788

1.12 | 4.02874204 | 4.00000000 | -0.02874204

1.16 | 3.97316079 | 4.00000000 | 0.02683921

1.20 | 4.02434630 | 4.00000000 | -0.02434630

1.24 | 3.97731688 | 4.00000000 | 0.02268312

1.28 | 4.02062150 | 4.00000000 | -0.02062150

1.32 | 3.98082411 | 4.00000000 | 0.01917589

1.36 | 4.01746532 | 4.00000000 | -0.01746532

1.40 | 3.98378549 | 4.00000000 | 0.01621451

1.44 | 4.01479115 | 4.00000000 | -0.01479115

1.48 | 3.98628712 | 4.00000000 | 0.01371288

1.52 | 4.01252558 | 4.00000000 | -0.01252558

1.56 | 3.98840115 | 4.00000000 | 0.01159885

1.60 | 4.01060636 | 4.00000000 | -0.01060636

1.64 | 3.99018815 | 4.00000000 | 0.00981185

1.68 | 4.00898069 | 4.00000000 | -0.00898069

1.72 | 3.99169905 | 4.00000000 | 0.00830095

1.76 | 4.00760380 | 4.00000000 | -0.00760380

1.80 | 3.99297675 | 4.00000000 | 0.00702325

1.84 | 4.00643771 | 4.00000000 | -0.00643771

1.88 | 3.99405741 | 4.00000000 | 0.00594259

1.92 | 4.00545023 | 4.00000000 | -0.00545023

1.96 | 3.99497153 | 4.00000000 | 0.00502847

2.00 | 4.00461405 | 4.00000000 | -0.00461405

***Exercise***

You've seen that the error in Euler's method varies directly as the first power of the step size . This makes Euler's method an order to halve the error? How does this affect the number of required iterations?

*Solution*

Because  halving the step size should halve the error.



The number of iterations is given by: , therefore halving the step size should double the number of iterations.



*Solution Section* 5.2 - Runge-Kutta Methods

***Exercise***



1. Use a computer and Runge-Kutta method to calculate three separate approximate solutions on the interval , one with step size , a second with step size , a second with step size .
2. Use the appropriate analytic to compute the exact solution
3. Plot the exact solution and approximate solutions as discrete points.

*Solution*



Runge-Kutta 2nd Order

t Approx. Exact Difference

--------------------------------------------------------

0.00 | 1.00000000 | 1.00000000 | 0.00000000

0.20 | 0.99800666 | 0.99873333 | 0.00072667

0.40 | 0.98887689 | 0.99039969 | 0.00152281

0.60 | 0.96709749 | 0.96939486 | 0.00229738

0.80 | 0.92871746 | 0.93169588 | 0.00297842

1.00 | 0.87131508 | 0.87482637 | 0.00351128

Runge-Kutta 4th Order

t Approx. Exact Difference

--------------------------------------------------------

0.00 | 1.00000000 | 1.00000000 | 0.00000000

0.20 | 0.99873272 | 0.99873333 | 0.00000061

0.40 | 0.99039822 | 0.99039969 | 0.00000147

0.60 | 0.96939245 | 0.96939486 | 0.00000241

0.80 | 0.93169258 | 0.93169588 | 0.00000330

1.00 | 0.87482232 | 0.87482637 | 0.00000405

Runge-Kutta 2nd Order

t Approx. Exact Difference

--------------------------------------------------------

0.00 | 1.00000000 | 1.00000000 | 0.00000000

0.10 | 0.99975021 | 0.99983750 | 0.00008729

0.20 | 0.99855245 | 0.99873333 | 0.00018088

0.30 | 0.99555979 | 0.99583746 | 0.00027767

0.40 | 0.99002480 | 0.99039969 | 0.00037489

0.50 | 0.98129932 | 0.98176938 | 0.00047006

0.60 | 0.96883388 | 0.96939486 | 0.00056098

0.70 | 0.95217687 | 0.95282259 | 0.00064572

0.80 | 0.93097330 | 0.93169588 | 0.00072258

0.90 | 0.90496314 | 0.90575327 | 0.00079013

1.00 | 0.87397921 | 0.87482637 | 0.00084716

Runge-Kutta 4th Order

t Approx. Exact Difference

--------------------------------------------------------

0.00 | 1.00000000 | 1.00000000 | 0.00000000

0.10 | 0.99983748 | 0.99983750 | 0.00000002

0.20 | 0.99873329 | 0.99873333 | 0.00000004

0.30 | 0.99583739 | 0.99583746 | 0.00000007

0.40 | 0.99039960 | 0.99039969 | 0.00000009

0.50 | 0.98176926 | 0.98176938 | 0.00000012

0.60 | 0.96939471 | 0.96939486 | 0.00000015

0.70 | 0.95282241 | 0.95282259 | 0.00000018

0.80 | 0.93169568 | 0.93169588 | 0.00000020

0.90 | 0.90575304 | 0.90575327 | 0.00000023

1.00 | 0.87482612 | 0.87482637 | 0.00000025

Runge-Kutta 2nd Order

t Approx. Exact Difference

--------------------------------------------------------

0.00 | 1.00000000 | 1.00000000 | 0.00000000

0.05 | 0.99996876 | 0.99997943 | 0.00001067

0.10 | 0.99981570 | 0.99983750 | 0.00002180

0.15 | 0.99942531 | 0.99945859 | 0.00003328

0.20 | 0.99868831 | 0.99873333 | 0.00004502

0.25 | 0.99750164 | 0.99755858 | 0.00005694

0.30 | 0.99576852 | 0.99583746 | 0.00006894

0.35 | 0.99339836 | 0.99347931 | 0.00008094

0.40 | 0.99030682 | 0.99039969 | 0.00009287

0.45 | 0.98641574 | 0.98652039 | 0.00010465

0.50 | 0.98165315 | 0.98176938 | 0.00011623

0.55 | 0.97595326 | 0.97608078 | 0.00012752

0.60 | 0.96925639 | 0.96939486 | 0.00013847

0.65 | 0.96150896 | 0.96165799 | 0.00014903

0.70 | 0.95266344 | 0.95282259 | 0.00015915

0.75 | 0.94267832 | 0.94284709 | 0.00016877

0.80 | 0.93151803 | 0.93169588 | 0.00017785

0.85 | 0.91915289 | 0.91933924 | 0.00018635

0.90 | 0.90555903 | 0.90575327 | 0.00019423

0.95 | 0.89071835 | 0.89091981 | 0.00020146

1.00 | 0.87461836 | 0.87482637 | 0.00020801

Runge-Kutta 4th Order

t y y(t) Difference

--------------------------------------------------------

0.00 | 1.00000000 | 1.00000000 | 0.00000000

0.05 | 0.99997943 | 0.99997943 | 0.00000000

0.10 | 0.99983750 | 0.99983750 | 0.00000000

0.15 | 0.99945859 | 0.99945859 | 0.00000000

0.20 | 0.99873333 | 0.99873333 | 0.00000000

0.25 | 0.99755858 | 0.99755858 | 0.00000000

0.30 | 0.99583745 | 0.99583746 | 0.00000000

0.35 | 0.99347930 | 0.99347931 | 0.00000001

0.40 | 0.99039969 | 0.99039969 | 0.00000001

0.45 | 0.98652039 | 0.98652039 | 0.00000001

0.50 | 0.98176937 | 0.98176938 | 0.00000001

0.55 | 0.97608077 | 0.97608078 | 0.00000001

0.60 | 0.96939485 | 0.96939486 | 0.00000001

0.65 | 0.96165798 | 0.96165799 | 0.00000001

0.70 | 0.95282258 | 0.95282259 | 0.00000001

0.75 | 0.94284708 | 0.94284709 | 0.00000001

0.80 | 0.93169587 | 0.93169588 | 0.00000001

0.85 | 0.91933923 | 0.91933924 | 0.00000001

0.90 | 0.90575325 | 0.90575327 | 0.00000001

0.95 | 0.89091979 | 0.89091981 | 0.00000001

1.00 | 0.87482635 | 0.87482637 | 0.00000002



***Exercise***

Consider the initial value problem 

Use Runge-Kutta method with step size  to sketch solution on the interval 

*Solution*



Runge-Kutta 4th Order

t Approx. Exact Difference

--------------------------------------------------------

0.00 | 1.00000000 | 1.00000000 | 0.00000000

0.04 | 1.00079936 | 1.00079936 | -0.00000000

0.08 | 1.00318981 | 1.00318981 | -0.00000000

0.12 | 1.00714877 | 1.00714877 | -0.00000000

0.16 | 1.01263957 | 1.01263957 | -0.00000000

0.20 | 1.01961283 | 1.01961282 | -0.00000000

0.24 | 1.02800822 | 1.02800822 | -0.00000000

0.28 | 1.03775651 | 1.03775651 | -0.00000000

0.32 | 1.04878166 | 1.04878166 | -0.00000001

0.36 | 1.06100297 | 1.06100297 | -0.00000001

0.40 | 1.07433708 | 1.07433707 | -0.00000001

0.44 | 1.08869975 | 1.08869974 | -0.00000001

0.48 | 1.10400743 | 1.10400742 | -0.00000001

0.52 | 1.12017855 | 1.12017854 | -0.00000001

0.56 | 1.13713450 | 1.13713449 | -0.00000001

0.60 | 1.15480036 | 1.15480035 | -0.00000001

0.64 | 1.17310545 | 1.17310544 | -0.00000001

0.68 | 1.19198361 | 1.19198360 | -0.00000001

0.72 | 1.21137336 | 1.21137335 | -0.00000001

0.76 | 1.23121787 | 1.23121787 | -0.00000001

0.80 | 1.25146496 | 1.25146495 | -0.00000001

0.84 | 1.27206683 | 1.27206682 | -0.00000001

0.88 | 1.29297992 | 1.29297991 | -0.00000001

0.92 | 1.31416464 | 1.31416463 | -0.00000001

0.96 | 1.33558509 | 1.33558508 | -0.00000001

1.00 | 1.35720882 | 1.35720881 | -0.00000001

1.04 | 1.37900650 | 1.37900650 | -0.00000001

1.08 | 1.40095174 | 1.40095173 | -0.00000001

1.12 | 1.42302075 | 1.42302075 | -0.00000001

1.16 | 1.44519217 | 1.44519216 | -0.00000001

1.20 | 1.46744679 | 1.46744678 | -0.00000001

1.24 | 1.48976740 | 1.48976739 | -0.00000001

1.28 | 1.51213855 | 1.51213854 | -0.00000001

1.32 | 1.53454641 | 1.53454640 | -0.00000001

1.36 | 1.55697860 | 1.55697859 | -0.00000001

1.40 | 1.57942403 | 1.57942403 | -0.00000001

1.44 | 1.60187281 | 1.60187281 | -0.00000001

1.48 | 1.62431609 | 1.62431608 | -0.00000001

1.52 | 1.64674596 | 1.64674596 | -0.00000001

1.56 | 1.66915540 | 1.66915539 | -0.00000001

1.60 | 1.69153812 | 1.69153811 | -0.00000001

1.64 | 1.71388854 | 1.71388853 | -0.00000001

1.68 | 1.73620170 | 1.73620169 | -0.00000001

1.72 | 1.75847320 | 1.75847319 | -0.00000001

1.76 | 1.78069914 | 1.78069913 | -0.00000001

1.80 | 1.80287607 | 1.80287606 | -0.00000000

1.84 | 1.82500094 | 1.82500094 | -0.00000000

1.88 | 1.84707109 | 1.84707109 | -0.00000000

1.92 | 1.86908417 | 1.86908417 | -0.00000000

1.96 | 1.89103813 | 1.89103813 | -0.00000000

2.00 | 1.91293119 | 1.91293118 | -0.00000000