***Solution Section* 2.1 − Sequences and Summations**

***Exercise***

Find these terms of the sequence , where 



***Solution***

1. 



1. 





1. 





1. 





***Exercise***

What is the term  of the sequence , if  equals



***Solution***

1. 



1. 
2. 



1. 



***Exercise***

What are the terms  of the sequence , if  equals





***Solution***

1. 







1. 







1. 







1. 







1. 







1. 







1. 







1. 







***Exercise***

Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

***Solution***

1. 
2. The second pattern, , as we see the difference to the previous increasing by value of 1.

So, the next term .

Therefore; the sequence is 1, 2, 4, 7, 11, 16, …

1. 1, 2, 4, 1, 2, 4, … Repeating the terms

***Exercise***

Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

***Solution***

One rule should be that each term is greater than the previous term by 2; the sequence would be 3, 5, 7, 9, 11, 13, . . .

Another rule could be that the  old prime.

The sequence would be 3, 5, 7, 11, 13, 17, …

The sequence: 3, 5, 7, 12, 23, 43, 75, 122, 187, 273 from an equation 

***Exercise***

Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

1. 
2. 
3. 
4. 
5. 

***Solution***

1. 









1. 









1. 









1. 









1. 









***Exercise***

Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

1. 
2. 
3. 
4. 
5. 

***Solution***

1. 











1. 









1. 











1. 









1. 







***Exercise***

Let 

1. Find 
2. Show that , , and 
3. Show that  for all integers *n* with 

***Solution***

1. 









|  |
| --- |
| ***√***  ***Or***            ***√*** |
| ***√***  ***Or***            ***√*** |
| ***√***  ***Or***          ***√*** |

***Exercise***

Is the sequence  a solution of the recurrence relation  if

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 

***Solution***

1. Let  We get  which is a true statement.

∴  is a solution of the recurrence relation.

1. Let 

We get  which is a false statement.

∴  is not a solution.

1. Let 

We get 

which is a false statement.

∴  is not a solution.

1. Let  We get









 which is a true statement

∴  is a solution of the recurrence relation.

1. Let  We get







 which is a true statement

∴  is a solution of the recurrence relation.

1. Let  We get











 which is a true statement

∴  is a solution of the recurrence relation.

1. Let  We get











 which is a false statement

∴  is not a solution.

1. Let  We get









 which is a false statement

∴  is a solution of the recurrence relation.

***Exercise***

Is the sequence  a solution of the recurrence relation  if

1. 
2. 
3. 
4. 

***Solution***

1. 







1. 

















1.  











1. 









***Exercise***

A person deposits $1,000.00 in an account that yields 9% interest compounded annually.

1. Set up a recurrence relation for the amount in the account at the end of *n* years.
2. Find an explicit formula for the amount in the account at the end of *n* years.
3. How much money will the account contain after 100 years?

***Solution***

1. The amount after  years multiplied by 1.09 to give the amount after n years, since 9% of the value must be added to account for the interest. Therefore, we have . The initial condition is .
2. Since multiplying by 1.09 for each year, the solution is .
3. 



***Exercise***

Suppose that the number of bacteria in a colony triples every hour.

1. Set up a recurrence relation for the number of bacteria after *n* hours have elapsed.
2. If 100 bacteria are used to begin new colony, how many bacteria will be in the colony in 10 hours?

***Solution***

1. Since the number of bacteria triples every hour, the recurrence relation should say that the number of bacteria after *n* hours is 3 times the number of bacteria after *n* – 1 hours.

Let  denote the number of bacteria after *n* hours, this statement translates into the recurrence relation 

1. The initial condition is .





 









***Exercise***

A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with *n* cars made in the *n*th month.

1. Set up a recurrence relation for the number of cars produced in the first *n* months by this factory.
2. How many cars are produced in the first year?
3. Find an explicit formula for the number of cars produced in the first *n* months by this factory

***Solution***

1. Let  be the number of cars produced in the first *n* months.

The initial condition is .

Since *n* cars are made in the *n*th month then , where  is the first *n*–1 months

1. The number of cars produced in the first year is .

Plug in , we get









 









1. 

***Exercise***

For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

1. 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …
2. 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …
3. 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …
4. 3, 6, 12, 24, 48, 96, 192, …
5. 15, 8, 1, −6, −13, −20, −27, …
6. 3, 5, 8, 12, 17, 23, 30, 38, 47, …
7. 2, 16, 54, 128, 250, 432, 686, …
8. 2, 3, 7, 25, 121, 721, 5041, 40321, …
9. 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, …
10. 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, …
11. 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, …

***Solution***

1. We have one 1 and one 0, then two 1 and two 0, then three of each, and so on increasing the repetition by one each time. Since we have only one at the end, then we need three 1 and four 0 to continue the sequence.
2. A pattern is that the positive integers are increasing order, with odd number showing once and each even number repeated.

Thus, the next terms are 9, 10, 10.

1. The terms in the odd locations are the successive terms in the geometric sequence that starts with 1 and has ratio 2, and the terms in the even locations are all 0. The *n*th term is 0 if *n* even and is  if *n* is odd.

Thus, the next three terms are 32, 0, 64.

1. The first term is 3 and each successive term is twice the predecessor. The *n*th term is .

Thus, the next three terms are 384, 768, 1536.

1. The first term is 15 and each successive term is 7 less than its predecessor. The *n*th term is  .

Thus, the next three terms are −34, −41, −48.

1. The first term is 3 and each successive term by adding *n* to its predecessor. 



The *n*th term is .

Thus, the next three terms are 57, 68, 80.

1. Since all numbers are even, then if we divide by 2 the sequence becomes: 1, 8, 27, 64, 125, 216, 343, …. This sequence appears to be , therefore the *n*th term is .

Thus, the next three terms are 1024, 1458, 2000.

1. The *n*th term appears to be .

Thus, the next three terms are 362881, 3628801, 39916801.

1. The first term is 3 then by adding 3 to the predecessor, then 5, then 7, and so on.



Then the *n*th term is .

Thus, the next three terms are 123, 146, 171.

1. This an arithmetic sequence whose difference is 4. Thus, the *n*th term is .

Thus, the next three terms are 47, 51, 55.

1. This is a binary expansion of *n*. Thus, the next three terms are 1100, 1101, 1110.

***Solution Section* 2.2 − Algorithms**

***Exercise***

List all the steps used by the Algorithm 1 to find the maximum of the list

1, 8, 12, 9, 11, 2, 14, 5, 10, 4.

***Solution***

The ***for*** loop then begins, with *i* set equal from 2 to (number of the sequence).

The statement of the loop is executed since 2 < 10. This is an ***if*** … ***then*** statement.

*max***:=** 1

***for*** *i* := 2 to 10

***if max***  **then** 

, since 1 < 8, then 

, since 8 < 12, then 

, since 12 < 9 ***is not true***, then 

, since 12 < 11 ***is not true***, then 

, since 12 < 2 ***is not true***, then 

, since 12 < 14, then 

, since 14 < 5 ***is not true***, then 

, since 14 < 10 ***is not true***, then 

, since 14 < 4 ***is not true***, then 

Therefore ***max*** has the value 

***Exercise***

Devise an algorithm that finds the sum of all the integers in a list.

***Solution***

***Procedure*** *sum* 

*sum***:=** *a*1

***for*** *i* := 2 to *n*



***return*** *sum***{**is the sum of all the elements in the list}

***Exercise***

Describe an algorithm that takes as an input a list of *n* integers and produces as output the largest difference obtained by subtracting an integer in the list from the one following it.

***Solution***

For *i* going from 1 through *n* − 1, compute the value of the  element in the list minus the  element in the list. If this is larger than the answer, reset the answer to be this value.

***Exercise***

Describe an algorithm that takes as an input a list of *n* integers in non-decreasing order and produces the list of all values that occur more than once.

***Solution***

***Procedure*** *negatives* 

*k***:=** *0*

***for*** *i* := 1 to *n*

***if***  ***then*** 

***return*** *k* **{**the number of negative integers in the list}

***Exercise***

Describe an algorithm that takes as an input a list of *n* integers and finds the location of the last even integer in the list or returns 0 if there are no even integers in the list.

***Solution***

***Procedure*** *last even loction* 

*k***:=** *0*

***for*** *i* := 1 to *n*

***if***  ***then*** 

***return*** *k* **{** *is the desired location* (*or* 0 *if there are no evens*)}

***Exercise***

Describe an algorithm that interchanges the values of the variables *x* and *y*, using only assignments. What is the minimum number of assignment statements needed to do this?

***Solution***

We cannot simply write  followed by .







***Exercise***

List all the steps used to search for 9 in the sequence 1, 3, 4, 5, 6, 7, 9, 11 using

*a*) a linear search *b*) a binary search

***Solution***

1. Note that *n* = 8 and *x* = 9.

***procedure*** linear\_search (*x*: integer; : integers)

*i* := 1

***while*** ( *i* ≤ *8* and 



The ***while*** loop is executed as long as  and the  element is not equal to 9.















Therefore the body of the loop is not executed (so *i* is still equal to 7), and control passes beyond the loop.

***if*** *i* ≤ *n* ***then*** *location* := *i*

***else*** *location* := 0

The else clause is not executed. This completes the procedure, so location has the correct value, namely 7, which indicates the location of the element *x* in the list: 9 is the seventh element.

1. ***procedure*** linear\_search (*x*: integer; : increasing integers)

*i* := 1

*j* := 8

***while*** *i <* *j*

The while step is executed, first 

Then since *x* (= 9) is greater than , the statement  is executed, so *i* has the value 5.



 fails thus , so 

At this point , the condition  is true, location is set to 7, as it should be, and the algorithm is finished.

***Exercise***

Describe an algorithm that inserts an integer *x* in the appropriate position into the list  of integers that are in increasing order.

***Solution***

***procedure*** insert 



*i* := 1

***while ***

*i* := *i* +1 {***The loop ends when i is the index for x***}

***for***  {***Shove the rest of the list to the right***}

******

**

{*x* has been inserted into the correct spot in the list, now of length *n* + 1}

***Solution Section* 2.3 – Divisibility and Modular Arithmetic**

***Exercise***

Does 17 divide each of these numbers?



***Solution***

1.  *Yes*
2.  *No*, remainder 16
3.  *Yes*
4.  *No*, remainder 15

***Exercise***

Prove that if *a* is an integer other than 0, then



***Solution***

1. 
2. 

***Exercise***

Show that if  and , where *a* and *b* are integers, then *a* = *b* or *a* = −*b*.

***Solution***

Let *s* and *t* are integers such that and .

. Since , we conclude that .

The only way for this to happen, since *s* and *t* are integers, is for .

Therefore, either *a* = *b* or *a* = −*b*.

***Exercise***

Show that if *a*, *b*, and *c* are integers, where  and , such that , then 

***Solution***

Since   for some integers *t*

Since , divide both sides by *c* to obtain  and this result to   ***√***

***Exercise***

What are the quotient and remainder when

1. 19 is divided by 7?
2. −111 is divided by 11?
3. 789 is divided by 23?
4. 1001 is divided by 13?
5. 0 is divided by 19?
6. 3 is divided by 5?
7. −1 is divided by 3?
8. 4 is divided by 1?

***Solution***

1.  
2.  
3.  
4.  
5.  
6.  
7.  
8.  

***Exercise***

What time does a 12-hour clock read

1. 80 hours after it reads 11:00?
2. 40 hours before it reads 12:00?
3. 100 hours after it reads 6:00?

***Solution***

1. , the clock reads 7:00.
2.  (12 − 40 = −28)



= 8

The clock reads 8:00.

1. , the clock reads 10:00.

***Exercise***

What time does a 24-hour clock read

1. 100 hours after it reads 2:00?
2. 45 hours before it reads 12:00?
3. 168 hours after it reads 19:00?

***Solution***

1. , the clock reads 6:00
2. , the clock reads 15:00
3. , the clock reads 19:00

***Exercise***

Suppose *a* and *b* are integers, , and . Find the integer *c* with  such that

1. 
2. 
3. 
4. 
5. 
6. 

***Solution***

1. 
2. 
3. 
4. 
5. 
6.  

***Exercise***

Suppose *a* and *b* are integers, , and . Find the integer *c* with  such that

1. 
2. 
3. 
4. 

***Solution***

1. 
2.  
3. 
4. 

***Exercise***

Let *m* be a positive integer. Show that  if 

***Solution***

Given  means that *a* and *b* have the same remainder  and  for some integer and *r*.





Which says that *m* divides (is a factor). This precisely the definition of 

***Exercise***

Let *m* be a positive integer. Show that  if 

***Solution***

Assume that . This means that , .

Computing , we know that  for some nonnegative *r* less than *m* (namely, ). Therefore . By definition this means that *r* must also equal   ***√***

***Exercise***

Show that if *n* and *k* are positive integers, then 

***Solution***

The quotient  lies between 2 consecutive integers, let say  possibly equal to b. There exists a positive integer *b* such that . In particular . Also since  we have 

 so , therefore 

***Exercise***

Evaluate these quantities

1. −17 ***mod*** 2
2. 144 ***mod*** 7
3. −101 ***mod*** 13
4. 199 ***mod*** 19
5. 13 ***mod*** 3
6. −97 ***mod*** 11

***Solution***

1. , the remainder is 1. That is, .

Note that we do not write  so 

1. , the remainder is 4. That is, 
2. , the remainder is 3. That is, 
3. , the remainder is 9. That is, 
4. , the remainder is 1. That is, 
5. , the remainder is 2. That is, 

***Exercise***

Find *a* **div** *m* and *a* **mod** *m* when

1. *a* = 228, *m* = 119
2. *a* = 9009, *m* = 223
3. *a* = −10101, *m* = 333
4. *a* = −765432, *m* = 38271

***Solution***

1. 

.

1. 

.

1. 

.

1.  ⇒ .

***Exercise***

Find the integer *a* such that

1. 
2. 
3. 
4. 
5. 

***Solution***

1. −15 already satisfies the inequality, the answer 
2. 24 is too large to satisfy the inequality, we subtract 31 and obtain 
3. 24 is too small to satisfy the inequality, we add 41 and obtain 
4. 
5. 

***Exercise***

Decide whether each of these integers is congruent to 5 modulo 17.



***Solution***

1. , so 
2. , so 
3. , so 
4. , so 

***Exercise***

Find each of these values.

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 

***Solution***

1. 

 

1.  
2.  
3. 





1. 







1. 







1. 







1. 







1. 







***Solution Section* 2.4 – Integer Representations and Algorithms**

***Exercise***

Convert the decimal expansion of each of these integers to a binary expansion

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 1. 321 | 1. 1023 | 1. 100632 | 1. 231 | 1. 4532 |

***Solution***

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 321 | 160 | 80 | 40 | 20 | 10 | 5 | 2 | 1 |  |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | ← |





1.  

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1023 | 511 | 255 | 127 | 63 | 31 | 15 | 7 | 3 | 1 |  |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ← |



|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 100632 | 50316 | 25158 | 12579 | 636289 | 3144 | 1572 | 786 | 393 | 196 | 98 | 49 | 24 |
|  | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 12 | 6 | 3 | 1 |  |
| 0 | 0 | 1 | 1 | ← |



|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 231 | 115 | 57 | 28 | 14 | 7 | 3 | 1 |  |
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | ← |



|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 4532 | 2266 | 1133 | 566 | 283 | 141 | 70 | 35 | 17 | 8 | 4 | 2 | 1 |  |
| 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | ← |



***Exercise***

Convert binary the expansion of each of these integers to a decimal expansion

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| *d)* |  |  |  |
| *g*) |  |  |  |

***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ***Decimal*** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| ***Binary*** | 0 | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

1. 





1. 





1. 



1. 



1. 





1. 





1. 
2. 

***Exercise***

Convert the binary expansion of each of these integers to an octal expansion

|  |  |
| --- | --- |
|  |  |
|  |  |

***Solution***

1. 
2. 
3. 
4. 

***Exercise***

Convert the octal expansion of each of these integers to a binary expansion

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |

***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ***Octal*** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| ***Binary*** | 0 | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

1.  ⇒ 
2.  ⇒ 
3.  ⇒ 
4.  ⇒ 
5.  ⇒ 

***Exercise***

Convert the hexadecimal expansion of each of these integers to a binary expansion

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ***Hexadecimal*** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |
| ***Binary*** | 0 | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 |

1.  ⇒ 
2. 

⇒ 

1.  ⇒ 
2. 

⇒ 

1. 

⇒ 

1. 

⇒ 

***Exercise***

Show that the binary expansion of a positive integer can be obtained from its hexadecimal expansion by translating each hexadecimal digit into a block of four binary digits.

***Solution***

Let  be the hexadecimal expansion of a positive integer. The value of that integer is 

If we replace each hexadecimal digit by its binary expansion , then



Therefore the value of the entire number is





Which is the value of the binary expansion 

***Exercise***

Show that the binary expansion of a positive integer can be obtained from its octal expansion by translating each octal digit into a block of three binary digits.

***Solution***

Let  be the octal expansion of a positive integer. The value of that integer is 

If we replace each octal digit by its binary expansion , then



Therefore the value of the entire number is





Which is the value of the binary expansion 

***Exercise***

Explain how to convert from binary to base 64 expansions and from base 64 expansions to binary expansions and from octal to base 64 expansions and from base 64 expansions to octal expansions

***Solution***

, in base 64 we need 64 symbols, from 0 to up to something representing 63. Corresponding to each such symbol would be a binary string of 6 digits, from 000000 for 0 to 001010 for ***a***, 100011 for ***z***, 100100 for ***A***, 111101 for ***Z***, for 111110 for @, and 111111 for $.

To translate from binary to base 64, we group the binary digits from the right in groups of 6 and use the list of correspondences to replace each 6 bits by one base-64 digits. To convert from base 64 to binary, we just replace each base-64 digit by its corresponding 6 bits.

For conversion between octal and base 64, we change the binary strings in the table to octal strings, replacing each 6-bit string by its 2-digit octal equivalent, and then follow the same procedures as above, interchanging base-64 digits and 2-digits strings of octal digits.

***Exercise***

Find the sum and product of each of these pairs of numbers. Express your answers as a base 3 expansions

|  |  |
| --- | --- |
|  |  |
|  |  |

***Solution***

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | |  |
|  |  | |  |
|  |  | |  |
|  | |  |  |

***Exercise***

Find the sum and product of each of these pairs of numbers. Express your answers as an octal expansion.

|  |  |
| --- | --- |
|  |  |
|  |  |

***Solution***

|  |  |  |
| --- | --- | --- |
|  |  | |
|  | 2 | |
|  | 1 | |
|  | |  |

***Exercise***

Find the sum and product of each of these pairs of numbers. Express your answers as a hexadecimal expansion.

|  |  |
| --- | --- |
|  |  |
|  |  |

***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ***Decimal*** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| ***Hexadecimal*** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |

1. 



|  |  |
| --- | --- |
| 14 | 1 |

1. 



|  |  |
| --- | --- |
| 2 | 1 |

1. 



|  |  |
| --- | --- |
|  | 11 |

1. 









 

 

 



 







 







 

 







10 



***Solution Section* 2.5 – Primes and Greatest Common Divisors**

***Exercise***

Determine whether each of these integers is prime.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  | 1. 93 | 1. 101 |
| 1. 107 | 1. 113 |  |  |  |

***Solution***

The numbers: 29, 71, 97, 19, 101, 107, and 113 are primes.

Not Prime:     

***Exercise***

Find the prime factorization of each these integers.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  | 1. 101 | 1. 143 |
| 1. 289 | 1. 899 |  |  |  |

***Solution***

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9.  (***Prime***)
10. 
11. 
12. 

***Exercise***

Find the prime factorization of 10!

***Solution***





***Exercise***

Show that if  is composite if *a* and *m* are integers greater than 1 and *m* is odd. [*Hint*: Show that *x* + 1 is a factor of the polynomial  if *m* is odd]

***Solution***

Since *m* is odd, then we can factor 

Because *a* and *m* are both greater than 1, we know that  . This provides a factoring of  into proper factors, so is composite.

***Exercise***

Show that if  is an odd prime, then  for some nonnegative integer *n*. [*Hint*: First show the polynomial identity  holds, where  and *t* is odd]

***Solution***

Assume , then the claimed identity is



By multiplying out the right-hand side and noticing the “telescoping” that occurs.

Let show that *m* is a power of 2 that is only prime factor is 2.

Suppose to the contrary that *m* has an odd prime factor *t* and , where *k* is a positive integer.

Letting  in the identity given in the hint, we have . Because  and the prime  can have no proper factor greater than 1, we must have , so  and  contradicting the fact that *t* is prime. This completes the proof by contradiction.

***Exercise***

Which positive integers less than 12 are relatively prime to 12?

***Solution***

By inspection with mental arithmetic, the greatest common divisors of the numbers from 1 to 11 with 12 whose ***gcd*** is 1, are 1, 5, 7, and 11. These are so few since 12 had many factors – in particular, both 2 and 3.

***Exercise***

Which positive integers less than 30 are relatively prime to 30?

***Solution***

The prime factors of 30 are 2, 3, and 5.

Thus we are looking for positive integers less than 30 that have none of these prime factors. Since the smallest prime number other than these is 7, and  is already greater than 30, in fact only primes (and the number 1) will satisfy this condition.

Therefore the answer is 1, 7, 11, 13, 17, 18, 23, and 29.

***Exercise***

Determine whether the integers in each of these sets are pairwise relatively prime.

|  |  |  |  |
| --- | --- | --- | --- |
|  | 1. 14, 17, 85 | 1. 25, 41, 49, 64 | 1. 17, 18, 19, 23 |
| 1. 11, 15, 19 | 1. 14, 15, 21 | 1. 12, 17, 31, 37 | 1. 7, 8, 9, 11 |

***Solution***

1.  These are pairwise relatively prime
2.  These are not pairwise relatively prime
3.  These are pairwise relatively prime
4.  These are pairwise relatively prime
5.  These are pairwise relatively prime
6.  These are not pairwise relatively prime
7.  These are pairwise relatively prime
8.  These are pairwise relatively prime

***Exercise***

We call a positive integer ***perfect*** if it equals the sum of its positive divisors other than itself

1. Show that 6 and 28 are perfect.
2. Show that  is a perfect number when  is prime

***Solution***

1. Since , and these three summands are the only proper divisors of 6, we conclude that 6 is perfect.

 are also the only proper divisors of 28

1. We need to find all proper divisors of . Certainly all the numbers  are proper divisors, and their sum is  (geometric series). Also each of these divisors times  is also a divisor, and all but the last is proper. Again adding up this geometric series we find a sum of . There are no other proper divisors. Therefore the sum of all the divisors is





Which is our original number. Therefore this number is perfect.

***Exercise***

Show that if  is prime, then *n* is prime. *Hint*: Use the identity 

***Solution***

We will prove the assertion by proving its contrapositive.

Suppose that *n* is not prime. Then by definition  for some integers *a* and *b* each greater than 1. Since , the first factor in the suggested identity, is greater than 1. The second factor is also greater than 1.

Thus  is the product of 2 integers each greater than 1, so it is not prime.

***Exercise***

Determine whether each of these integers is prime, verifying some of Mersenne’s claims

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

***Solution***

1. . 2, 3, 5, 7, 11 are not factors of 127, since , therefore 127 is prime.
2.  So this number is not prime.
3.  So this number is not prime.
4. . Since  then 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, and 89 are not factors of 8191, therefore 8191 is prime.

***Exercise***

What are the greatest common divisors of these pairs of integers?

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 

***Solution***

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 

***Exercise***

What is the least common multiple of each pair

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 
2. 
3. 
4. 
5. *Undefined*
6. 
7. 
8. 
9. 
10. 
11. *Undefined*

***Exercise***

Find gcd(1000, 625) and lcm(1000, 625) and verify that 

***Solution***









Therefore, 

***Exercise***

Find *gcd*(92928, 123552) and lcm(92928, 123552) and verify that 

***Solution***

















***Exercise***

Use the Euclidean algorithm to find

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

***Solution***

1. 5 = 1 ∙ 5 + 0



1. 101 = 100 ∙ 1 + 1

1 = 1 ∙ 1 + 0



1. 277 = 123 ∙ 2 + 31

123 = 31 ∙ 3 + 30

31 = 30 ∙ 1 + 1

30 = 1 ∙ 30 + 0



1. 14039 = 1529 ∙ 9 + 278

1529 = 278 ∙ 5 + 139

278 = 139 ∙ 2 + 0



1. 14038 = 1529 ∙ 9 + 277

1529 = 277 ∙ 5 + 144

277 = 144 ∙ 1 + 133

144 = 133 ∙ 1 + 11

133 = 11 ∙ 12 + 1

11 = 1 ∙ 11 + 0





1. 18 = 12 ∙ 1 + 6

12 = 6 ∙ 2 + 0



1. 201 = 111 ∙ 1 + 90

111 = 90 ∙ 1 + 21

90 = 21 ∙ 4 + 6

21 = 6 ∙ 3 + 3

6 = 3 ∙ 2 + 0



1. 1331 = 1001 ∙ 1 + 330

1001 = 330 ∙ 3 + 11

330 = 11 ∙ 30 + 0



1. 













1. 





1. 



















***Exercise***

Prove that the product of any three consecutive integers is divisible by 6.

***Solution***

Consider the product  for some integer *n*.

Since every second integer is even (divisible by 2), then this product is divisible by 2.

Since every third integer is divisible by 3, then this product is divisible by 3.

Therefore this product has both 2 and 3 in its prime factorization and is therefore divisible by 

***Exercise***

Show that if *a, b*, and *m* are integers such that  and , then 

***Solution***

From  we know that  for some integer *s*. If *d* is a common divisor of *a* and *m*, then it divides the right-hand side of this equation, so it also divides *b*. We can rewrite the equation as , and they by similar reasoning, we see that every common divisor of *b* and *m* is also a divisor of *a*.

This shows that the set of common divisors of *a* and *m* is equal to the set of common divisors of *b* and *m*, so certainly 

***Exercise***

Prove or disprove that  is prime whenever *n* is a positive integer.

***Solution***

|  |  |
| --- | --- |
|  | |
|  | 1523 |
|  | 1447 |
| *n* = 3 | 1373 |
| *n* = 4 | 1301 |
| *n* = 5 | 1231 |
| *n* = 6 | 1163 |

Using calculator or spread sheet because it is hard to get started:

All the values are prime. This may lead us to believe that the propositions is true, but it gives no clue as to how to prove it.

If we let *n* = 1601, then .

So we got a counterexample and the proposition is false.

The smallest *n* for which this expression is not prime is *n* = 80; this gives the value 

***Solution Section* 2.6 – Applications of Congurences**

***Exercise***

Find the memory locations assigned by the hashing function *h*(*k*) = *k* **mod** 97 to the records of customers with Social Security numbers?

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |

***Solution***

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 

***Exercise***

A parking lot has 31 visitor spaces, numbered from 0 to 30. Visitors are assigned parking spaces using the hashing function *h*(*k*) = *k* **mod** 31, where *k* is the number formed from the first three digits on a visitor’s license plate.

1. Which spaces are assigned by the hashing function to cars that have these first three digits on their license plates: 317, 918, 007, 100, 111, 310
2. Describe a procedure visitors should follow to find a free parking space, when the space they are assigned is occupied.

***Solution***

1. 











1. Take the next available space, where the next space is computed by adding 1 to the space number and pretending that 30 + 1 = 0.

***Exercise***

Find the sequence of pseudorandom numbers generated by the linear congruential generator

1.  with seed **.
2.  with seed **.

***Solution***

1. Given , the 





The sequence keep continue to repeat 1, 5, 4, 1, 5, 4, …

1. Given , the 





The sequence keep continue to repeat 3, 6, 4, 3, 6, 4, …

***Exercise***

Find the sequence of pseudorandom numbers generated by using the pure multiplicative generator  with seed **.

***Solution***











Since , the sequence repeats forever: 2, 6, 7, 10, 8, 2, 6, 7, 10, 8, …

***Exercise***

The first nine digits of the ISBN-10 of the European version of the fifth edition of this book are . What is the check digit for that book?

***Solution***

Let *d* be the check digit.





So 

This is equivalent to:  or 

***Exercise***

The ISBN-10 of the sixth edition of Elementary Number Theory and Its Applications is , where *Q* is a digit. Find the value of *Q*.

***Solution***





 

 Since 

Therefore 

This is equivalent to: 

***Exercise***

The USPS sells money orders identified by 11-digit number . The first ten digits identify the money order:  is a check digit that satisfies . Find the check digit for the USPS money orders that have identification number that start with these ten digits

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |

***Solution***

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 

***Exercise***

Determine which single digit errors are detected by the USPS money order code.

***Solution***

If one digit change to a value not congruent to it modulo 9, then the modular equivalence implied by the equation in the preamble will no longer hold. Therefore all single digit errors are detected except for the substitution of a 9 for a 0 or vice versa.

***Exercise***

Determine which transposition errors are detected by the USPS money order code.

***Solution***

Because the first ten digits are added, any transposition error involving them will go undetected.

The sum of the first ten digits will be the same for the transposed number as it is for the correct number.

Suppose that the last digit is transposed with another digit; without loss of generality; we can assume it’s the tenth digit and that  .

Then the correct equation will be

 

But the equation resulting from the error will read

 

Subtract equations  & 



 *Divide by* 2 *both sides since* 2 *is prime*

 Which is false

The check equation will fail.

Therefore we conclude that transposition errors involving the eleventh digits are detected.