***Solution Section* 3.1 − Mathematical Induction**

***Exercise***

Prove that  whenever *n* is a nonnegative integer.

***Solution***

Since *n* is a nonnegative integer that implies to 

1. For ***n* = 0** ⇒ 

 ***√***

Hence  is true.

1. Assume that  is true













***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that  whenever *n* is a positive integer.

***Solution***

Since *n* is a positive integer that implies to 

1. For ***n* = 1**



 ***√***

Hence  is true.

1. Assume that  is true











 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that  whenever *n* is a nonnegative integer.

***Solution***

1. For ***n* = 0** ⇒ 

 ***√***

Hence  is true.

1. Assume that  is true









***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that  whenever *n* is a nonnegative integer.

***Solution***

1. For ***n* = 0** ⇒ 

 ***√***

Hence  is true.

1. Assume that  is true

We need to prove that  is also true













***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Find a formula for the sum of the first *n* even positive integers. Prove the formula.

***Solution***





1. For ***n* = 1**



 ***√***

Hence  is true.

1. Assume that  is true

We need to prove that  is also true 





***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

*a*) Find a formula for  by examining the values of this expression for values of this expression for small values of *n*.

*b*) Prove the formula.

***Solution***

1. 
2. For 

 ***√***

Hence  is true.

Assume that  is true

We need to prove that  is also true











 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that  whenever *n* is a positive integer.

***Solution***

1. For ***n* = 1**



 ***√***

Hence  is true.

1. Assume that  is true

We need to prove that  is also true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that for very positive integer *n*  

***Solution***

For ***n* = 1** ⇒ 

 ***√***

Hence  is true

Assume that  is true

We need to prove that  is also true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that for very positive integer *n*  .

***Solution***

For ***n* = 1**



 ***√***

Hence  is true

Assume that  is true

We need to prove that  is also true







***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that for very positive integer *n *

***Solution***

For ***n* = 1**



 ***√***

Hence  is true

Assume that  is true







 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Let  be the statement that  where *n* is an integer greater than 1.

1. Show is the statement ?
2. Show that  is true, completing the basis step of the proof.
3. What is the inductive hypothesis?
4. What do you need to prove in the inductive step?
5. Complete the inductive step.
6. Explain why these steps show that this inequality is true whenever *n* is an integer greater than 1.

***Solution***

1. 
2. 



***√***

***Exercise***

Prove that  if *n* is an integer greater than 6.

***Solution***

For ***n* = 7** ⇒ ; Hence  is true

Assume that  is true, we need to prove that 



 Since 



***√***

**∴** The statement  is true

***Exercise***

Prove that for every positive integer *n*: 

***Solution***

For ***n* = 1**





Hence  is true

Assume that  is true.

We need to prove that 



















Which is clearly true since 

***Exercise***

Use mathematical induction to prove that 2 divides whenever *n* is a positive integer.

***Solution***

For ***n* = 1**



since 2 divides 2;

Hence  is true

Assume that 2 divides  is true, we need to prove that 2 divides  is true





 ***√***

2 divides  and certainly 2 divides , so 2 divides their sum.

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Use mathematical induction to prove that 3 divides whenever *n* is a positive integer.

***Solution***

For ***n* = 1**



since 3 divides 3 ***√***

Hence  is true

Assume that 3 divides  is true.

We need to prove that 3 divides  is also true





 ***√***

By the inductive hypothesis, 3 divides  and certainly 3 divides , so 3 divides their sum.

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Use mathematical induction to prove that 5 divides whenever *n* is a positive integer.

***Solution***

For ***n* = 1**

, which is divisible by 5

Hence  is true

Assume that 5 divides  is true.

We need to prove that 5 divides  is also true





 ***√***

By the inductive hypothesis, 5 divides  and certainly 5 divides , so 5 divides their sum.

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Use mathematical induction to prove that is divisible by 8 whenever *n* is an odd positive integer.

***Solution***

For ***n* = 1**

, which is divisible by 8

Hence  is true

Assume that 8 divides  is true, than implies to .

We need to prove that 8 divides  is also true





By the inductive hypothesis, 8 divides  and certainly 8 divides , so 8 divides their sum.

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Use mathematical induction to prove that 21 divides whenever *n* is a positive integer.

***Solution***

For ***n* = 1**

, which is divisible by 21

Hence  is true.

Assume that 21 divides  is true.

We need to prove that 21 divides  is also true













By the inductive hypothesis, 21 divides  and certainly 21 divides , so 21 divides their sum.

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true for every positive integer *n*. 

***Solution***

For ***n* = 1**



Hence  is true.

 is true













 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true for every positive integer *n*. 

***Solution***

For ***n* = 1**

***√***

Hence  is true.

 is true

















 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true for every positive integer *n*. 

***Solution***

For ***n* = 1**

***√***

Hence  is true.

 is true













 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true













 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence, is true.

 is true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true













 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true for every positive integer *n*. 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**





 ***√***

Hence,  is true.

 is true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true











 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true







 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true: 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

 is true











 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true by mathematical induction:



***Solution***

For ***n* = 1**



 ***√***

Hence,  is true.

For ***k***: 





 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true by mathematical induction:



***Solution***

For ***n* = 1**





 ***√***

Hence,  is true.

For ***k***: 

Is : 















 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true by mathematical induction:



***Solution***

For ***n* = 1**



 ***√***

Hence,  is true.

For ***k***: 

Is : 











 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true by mathematical induction:



***Solution***

For ***n* = 1**

***√***

Hence,  is true.

For ***k***: 

Is : 











 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true by mathematical induction: 

***Solution***

For ***n* = 1**





 *Square both sides*

 ***√***

Hence,  is true.

For ***k***: 



Is : 







 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true by mathematical induction: 

***Solution***

For ***n* = 2**



 ***√***

Hence,  is true.

For ***k***: 

Is : 











 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true by mathematical induction:  (*a* and *m* are constant)

***Solution***

For ***n* = 1**



 ***√***

Hence,  is true.

 is true



****

****

****

** *√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true for every positive integer *n*. 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

Assume that  is true 

We need to prove that  is true, that is 





 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true for every positive integer *n*. 3 is a factor of 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

Assume that  is true 3 is a factor of 

We need to prove that  is true, that is 





 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement is true for every positive integer *n*. 4 is a factor of 

***Solution***

For ***n* = 1**

 ***√***

Hence,  is true.

Assume that  is true 4 is a factor of 

We need to prove that  is true, that is 







By the induction hypothesis, 4 is a factor of  and 4 is a factor of 4, so 4 is a factor of the  term. ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement by mathematical induction: 

***Solution***

For ***n* = 3**



 ***√***

Hence,  is true.

Assume that  is true: 

We need to prove that :  is true







***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement by mathematical induction: If , then 

***Solution***

For ***n* = 1**



 ***√***

since ****** ⇒  is true.

Assume that  is true: 

We need to prove that :  is true



 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement by mathematical induction: If , then 

***Solution***

For ***n* = 4**



***√***

Hence,  is true.

Assume that  is true: 

We need to prove that :  is true





 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement by mathematical induction: 

***Solution***

For ***n* = 2**



***√***

Hence,  is true.

Assume that  is true: ;

We need to prove that :  is true







 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement by mathematical induction: 

***Solution***

For ***n* = 5**



 ***√***

Hence,  is true.

Assume that  is true: 

We need to prove that :  is true









 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

Prove that the statement by mathematical induction: 

***Solution***

For ***n* = 5**



 ***√***

Hence,  is true.

Assume that  is true: 

We need to prove that :  is true











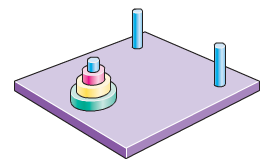


 ***√***

Hence  is true.

∴ By the mathematical induction, the given statement is true.

***Exercise***

A pile of *n* rings, each smaller than the one below it, is on a peg on board. Two other pegs are attached to the board. In the game called the Tower of Hanoi puzzle, all the rings must be moved, one at a time, to a different peg with no ring ever placed on top of a smaller ring.

Find the least number of moves that would be required.

Prove your result by mathematical induction.

***Solution***

With 1 ring, 1 move is required.

With 2 rings, 3 moves are required 

With 3 rings, 7 moves are required 

With *n* rings,  moves are required

For ***n* = 1**

***√***

Hence,  is true.

Assume that  is true: 







 ***√***

∴ By the mathematical induction, the given statement is true.

***Solution Section* 3.2 − Recursive Definitions and Structural Induction**

***Exercise***

Find ,,, and  if  is defined recursively by  and for 

|  |  |
| --- | --- |
|  |  |
|  |  |

***Solution***

1. 























1. 

























1. 

























1. 

























***Exercise***

Find ,,,  and  if  is defined recursively by  and for 

|  |  |
| --- | --- |
|  |  |
|  |  |

***Solution***

1. 































1. 





























1. 







































1. 





























***Exercise***

Find ,,  and  if  is defined recursively by  and for 

|  |  |
| --- | --- |
|  |  |
|  |  |

***Solution***

1. 























1. 























1. 























1. 























***Exercise***

Determine whether each of these proposed definitions is a valid recursive definition of a function *f* from the set of nonnegative integers to the set of integers. If *f* is well defined, fund a formula for  when *n* is nonnegative integer and prove that your formula is valid.

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9.  if *n* is odd and  and  if *n* is even and 
10.  if *n* is odd and  and  if *n* is even and 

***Solution***

1. This is invalid, since ,  is not defined.
2. , this is a valid, since  is provided and each subsequent value is determined by the previous one. , this is true for .

Assume it is true for , then



1. , this is a valid.



The sequence:  

By induction:

The basis step: 

If 

Then 

1. 



Given: 

Then the sequence: 2, 3, 2, 1, 0, … 

By induction: Basis step:  and 

If 

Then 

1.  

Then the sequence: 1, 2, 2, 4, 4, 8, 8, … 

By induction: Basis step:  and

If 

Then



1.   

This is valid, since the values *n* = 0, 1, 2 are given. The sequence: 1, 0, 2, 2, 0, 4, 4, 0, 8, …

|  |  |
| --- | --- |
| We conjecture the formula | ***Prove*** |
| when |  |
| when |  |
| when |  |

Assume the inductive hypothesis that the formula is valid for smaller inputs. Then

For  we have  as desired

For  we have  as desired

For  we have  as desired

1.  This is not valid, since  has not been defined
2.  

This is ***invalid***, because the value at  is defined in 2 conflicting ways, first as  and then as 

1.  

This is ***invalid***, since we have a conflict for odd .

On one hand , but the other hand .

However,  and , so these apparently conflicting rules tell us that  on the other hand. We got the same answer either way.

The sequence: 2, 2, 4, 4, 8, 8, …

1.  

The sequence: 1, 3, 9, 27, 81, …

This is a valid, since we conjecture the formula 

By induction: Basis step: 

If 

Then 

***Exercise***

Give a recursive definition of the sequence  if

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |

***Solution***

1. 







1. 







1. 







1. 







1. 







1. ,





The sequence alternate: 0,2, 0, 2, …



1. 





The sequence alternate: 2,6, 12, 20, 30, …

The difference between successive terms are 4, 6, 8, 10, ….



1. 





The sequence alternate: 1, 4, 9, 16, 25, …

The difference between successive terms are 3, 5, 7, 9, ….



***Exercise***

Prove that  when *n* is a positive integer and  is the *n*th Fibonacci number.

***Solution***

For   is true since both values are 1

Assume the inductive hypothesis. Then







***Exercise***

Prove that  when *n* is a positive integer and  is the *n*th Fibonacci number.

***Solution***

Using the principle of mathematical induction

For   is true since both values are 1

Let assume that 

We need to prove that 



 **(by the *definition* of the Fibonacci numbers)**

***Exercise***

Give a recursive definition of

1. The set of odd positive integers
2. The set of positive integers powers of 3
3. The set of polynomial with integer coefficients
4. The set of even integers
5. The set of positive integers congruent to 2 modulo 3.
6. The set of positive integers not divisible by 5

***Solution***

1. Off integers are obtained from other odd integers by adding 2.

Thus, we can define this set *S* as follows ; and if .

1. Powers of 3 are obtained from other powers of 3 by multiplying by 3.

Thus, we can define this set *S* as follows ; and if .

1. There are several ways to do this. One that is suggested by Horner’s method is as follows. We assume that the variable for these polynomials is the letter *x*. All integers are in *S*; if  and *n* is any integer, then  is in *S*.

Another method constructs the polynomials term by term. Its base case is to let 0 be in *S*; and its inductive step is to say that if , *c* is an integer, and *n* is a nonnegative integer, then  is in *S*.

1. Off integers are obtained from other even integers by adding 2.

Thus, we can define this set *S* as follows ; and if .

1. The smallest positive integer congruent to 2 modulo 3 is 2, so . All the others can be obtained by adding multiples of 3, so the inductive step is that 
2. The positive integers no divisible by 5 are the ones congruent to 1, 2, 3, or 4 *modulo* 5.

Thus, we can define this set *S* as follows ; and if 

***Exercise***

Let *S* be the subset of the set of ordered pairs of integers defined recursively by

*Basis step*: (0, 0) ∊ *S*.

*Recursive step*: If (*a, b*) ∊ *S*, then  *and *

1. List the elements of *S* produced by the first five applications of the recursive definition.
2. Use strong induction on the number of applications of the recursive step of the definition to show that  when (*a, b*) ∊ *S*.
3. Use structural induction to show that  when (*a, b*) ∊ *S*.

***Solution***

1. Apply each recursive step rules to the only element given in the basis step, we see that (2, 3) and (3, 2) are in *S*.

If we apply the recursive step to these we add (4, 6), (5, 5) and (6, 4).

The next round gives us (6, 9), (7, 8), and (9, 6). Add (8, 12), (9, 11), (10, 10), (11, 9), and (12, 8); and a fifth set of applications adds (10, 15), (11, 4), (12, 13), (13, 12), (14, 1), and (15, 10).

1. Let  be the statement that  when (*a, b*) ∊ *S* is obtained by *n* applications to the recursive step.

For ,  is true, since the only element of S obtained with no applications of the recursive step is (0, 0), and ***√***

Assume the inductive hypothesis that  whenever (*a, b*) ∊ *S* is obtained by *k* or fewer applications of the recursive step, and consider an element obtained with  applications of the recursive step. Since the final application of the recursive step to an element  must applied to an element, that .

We need to check that this inequality implies  and .

This is clear, since each is equivalent to  and 5 divides both  and 5.

1. This holds for the basis step, since 

If this holds for , then it also holds for the elements obtained from  in the recursive step by the same argument as in part (*b*).

***Exercise***

Let *S* be the subset of the set of ordered pairs of integers defined recursively by

*Basis step*: (0, 0) ∊ *S*.

*Recursive step*: If (*a, b*) ∊ *S*, then ,  *and *

1. List the elements of *S* produced by the first five applications of the recursive definition.
2. Use strong induction on the number of applications of the recursive step of the definition to show that  whenever (*a, b*) ∊ *S*.
3. Use structural induction to show that  whenever (*a, b*) ∊ *S*.

***Solution***

1. Apply each recursive step rules to the only element given in the basis step, we see that (0, 1), (1, 1) and (2, 1) are in *S*.

2nd step: (0, 2), (1, 2), (2, 2), (3, 2) and (4, 2).

3rd step: (0,3), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3) and (6, 3).

4th step: (0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (7, 4) and (8,4)

5th step: (0, 5), (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), (7, 5), (8, 5), (9, 5), and (10, 5)

1. Let  be the statement that  whenever (*a, b*) ∊ *S* is obtained with no applications of the recursive step.

For the basis step, the only element of *S* obtained with no applications of the recursive step is , then is true. Therefore  is true.

Assume that  whenever (*a, b*) ∊ *S* is obtained by *k* or fewer applications of the recursive step. Consider an element obtained with *k* + 1 applications of the recursive step.

We know that , we need to check this inequality implies , , and .

Thus is clear that , respectively, to  to obtain these inequalities.

1. This holds for the basis step, since .

If this holds for , then it also holds for the elements obtained from in the recursive step, since adding , respectively, to  yiekds , , and .

***Solution Section* 3.3 − The Basics of Counting**

***Exercise***

There are 18 mathematics majors and 325 computer science majors at a college

1. In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?
2. In how many ways can one representative be picked who either a mathematics major or a computer science major?

***Solution***

1. 
2. 

***Exercise***

An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

***Solution***

Using the product rule: there are 

***Exercise***

A multiple-choice test contains 10 questions. There are four possible answers for each question

1. In how many ways can a student answer the questions on the test if the student answers every question?
2. In how many ways can a student answer the questions on the test if the student can leave answers blank?

***Solution***

1. 
2. There are 5 ways to answer each question 0 give any if the 4 answers or give no answer at all



***Exercise***

A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of the shirts are made?

***Solution***

 different types of shirt.

***Exercise***

How many different three-letter initials can people have?

***Solution***

 different initials.

***Exercise***

How many different three-letter initials with none of the letters repeated can people have?

***Solution***



***Exercise***

How many different three-letter initials are there that begin with an A?

***Solution***



***Exercise***

How many bit strings are there of length eight?

***Solution***



***Exercise***

How many bit strings of length ten both begin and end with a 1?

***Solution***



***Exercise***

How many bit strings of length *n*, where *n* is a positive integer, start and end with 1s?

***Solution***

***Exercise***

How many strings are there of lowercase letters of length four or less, not counting the empty string?

***Solution***

The number of strings of length 4 or less by counting the number of the strings of length 

There are 26 letters to choose from, and a string of length *i* is specified by choosing its characters, one after another.

The product rules there are 





***Exercise***

How many strings are there of four lowercase letters that have the letter *x* in them?

***Solution***

Number of strings of length of 4 lowercase: 

Number of strings of length of 4 lowercase other than ***x***: 



***Exercise***

How many positive integers between 50 and 100

1. Are divisible by 7? Which integers are these
2. Are divisible by 11? Which integers are these
3. Are divisible by 7 and 11? Which integers are these

***Solution***

1. Neither 50 nor 100 is divisible by 7

There are  integers less than 50 that are divisible by 7

There are  integers less than 100 that are divisible by 7

This leaves  numbers between 50 and 100 that are divisible by 7.

They are 56, 63, 70, 77, 84, 91, and 98.

1. Neither 50 nor 100 is divisible by 11

There are  integers less than 50 that are divisible by 11

There are  integers less than 100 that are divisible by 1

This leaves  numbers between 50 and 100 that are divisible by 11

They are 55, 66, 77, 88, and 99.

1. A number is divisible by 7 and 11 which is 77. There is only one such number between 50 and 100, namely 77.

***Exercise***

How many positive integers less than 100

1. Are divisible by 7?
2. Are divisible by 7 but not by 11?
3. Are divisible by both 7 and 11?
4. Are divisible by either 7 or 11?
5. Are divisible by exactly one of 7 and 11?
6. Are divisible by neither 7 nor 11?

***Solution***

7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98 *are divisible by* 7

11, 22, 33, 44, 55, 66, 77, 88, 99 *are divisible by* 11

1. Every 7th number is divisible by 7. Therefore,  such numbers. The  multiple of 7 does not occur until the number 7*k* has been reached.
2. There are 13 such numbers since 77 is the only one divisible by 11.
3. There is only **1** number (77) divisible by both 7 and 11
4.  such numbers and 14 such numbers divisible by 7 and only 1 is divisible by 11. Therefore, there are  divisible by either 7 or 11
5. The number of numbers divisible of them:  (*subtract* (*d*) *from* (*c*))
6. Subtract part (d) from the total number of positive integers less than 100.



***Exercise***

How many positive integers less than 1000

1. Are divisible by 7?
2. Are divisible by 7 but not by 11?
3. Are divisible by both 7 and 11?
4. Are divisible by either 7 or 11?
5. Are divisible by exactly one of 7 and 11?
6. Are divisible by neither 7 nor 11?
7. Have distinct digits?
8. Have distinct digits and are even?

***Solution***

1. Every 7th number is divisible by 7. Therefore,  such numbers. The  multiple of 7 does not occur until the number 7*k* has been reached.
2. Every 11th number is divisible by 11. Therefore,  numbers.

Since 77 is the first number that is divisible by 7 and 11, and there are  numbers divisible by 77.

There are  numbers divisible by 7 but not by 11.

1. There are **12** numbers divisible by both 7 and 11 (from part ***b***)
2. There are  divisible by either 7 or 11
3. The number of numbers divisible of them:  (*subtract* (***d***) *from* (***c***))
4. Subtract part (**d**) from the total number of positive integers less than 1000.



1. If we assume that numbers are written without leading 0’s, then we can break down this part in three cases: one-digit numbers, two-digit numbers and three-digit numbers.

There are 9 one-digit numbers, and each of them has distinct digits.

There are 90 two-digit numbers (10 – 99), and all but 9 of them have distinct digits, so there are 81 two-digit numbers with distinct digits. Or the first digit 1 through 9 (9 choices), using the product rule:  choices in all.

For three-digit numbers there are  distinct digits

Therefore  total distinct digits.

1. If we use to count the odd numbers with distinct digits and subtract from part (***g***), we can get the numbers distinct digits and are even.

There are 5 odd one-digit numbers.

For two-digit numbers; first the ones digits (5 choices), then the tens digit (8 choices) – neither the ones digit value nor 0 is available, therefore there are 40 such two-digit numbers (half of 81).

For three-digit numbers, first the ones digits (5 choices), the hundreds digit (8 choices), then the tens digit (8 choices). There are  distinct digits

So  total odd numbers with distinct digits.

Therefore  total distinct digits.

***Exercise***

A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

***Solution***

There are 50 choices to make each of which can be done in 3 ways, namely by choosing the governor, choosing the senior senator, or choosing the junior senator.



***Exercise***

How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

***Solution***



***Exercise***

How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | ***Letters*** | | ***Digits*** | | | | | L | L | D | D | D | D | | 26 | 26 | 10 | 10 | 10 | 10 | | |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | ***Digits*** | | ***Letters*** | | | | | D | D | L | L | L | L | | 10 | 10 | 26 | 26 | 26 | 26 | |

Therefore: 

***Exercise***

How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

***Solution***



***Exercise***

How many strings of eight English letter are there.

1. That contains no vowels, if letters can be repeated?
2. That contains no vowels, if letters cannot be repeated?
3. That starts with a vowel, if letters can be repeated?
4. That starts with a vowel, if letters cannot be repeated?
5. That contains at least one vowel, if letters can be repeated?
6. That contains at least one vowel, if letters cannot be repeated?

***Solution***

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
|  | **NV** | **NV** | **NV** | **NV** | **NV** | **NV** | **NV** | **NV** |
| ***a*** | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 |
| ***b*** | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 |
|  | **V** | **L** | **L** | **L** | **L** | **L** | **L** | **L** |
| ***c*** | 5 | 26 | 26 | 26 | 26 | 26 | 26 | 26 |
| ***d*** | 5 | 25 | 24 | 23 | 22 | 21 | 20 | 19 |

1. There are 8 slots which can be filled with  non-vowels.

By the product rule: 

1. 
2. 
3. 
4. By the product rule: 
5. 

***Exercise***

How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

***Solution***

The count ordered arrangements of length 4 from the 10 people, then we get  arrangements.

However, we can rotate the people around the table in 4 ways and get the same seating arrangement, so the overcounts by a factor of 4.

Therefore, there are 

***Exercise***

In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

1. The bride must be in the picture?
2. Both the bride and groom must be in the picture?
3. Exactly one of the bride and the groom is in the picture?

***Solution***

1. The bride is in any of the 6 positions.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **1** | **2** | **3** | **4** | **5** | **6** |
| ***B*** | ***P*** | ***P*** | ***P*** | ***P*** | ***P*** |
| 1 | 9 | 8 | 7 | 6 | 5 |

Then, it will leave us with 5 remaining positions.

This can be done in  ways.

Therefore 

1. The bride is in any of the 6 positions.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **1** | **2** | **3** | **4** | **5** | **6** |
| ***B*** | ***G*** | ***P*** | ***P*** | ***P*** | ***P*** |
| 1 | 1 | 8 | 7 | 6 | 5 |

Then place the groom in any of the 5 remaining positions.

Then, it will leave us with 4 remaining positions in the picture.

This can be done in  ways.

Therefore 

1. For the just the bride to be in the picture:  ways.

There are 40,320 ways for just the groom to be in the picture.

Therefore,

***Exercise***

How many different types of homes are available if a builder offers a choice of 6 basic plans, 3 roof styles, and 2 exterior finishes?

***Solution***

 *different homes types*

***Exercise***

A menu offers a choice of 3 salads, 8 main dishes, and 7 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

***Solution***

 ***different meals***

***Exercise***

A couple has narrowed down the choice of a name for their new baby to 4 first names and 5 middle names. How many different first- and middle-name arrangements are possible?

***Solution***

 *possible arrangements*

***Exercise***

An automobile manufacturer produces 8 models, each available in 7 different exterior colors, with 4 different upholstery fabrics and 5 interior colors. How many varieties of automobile are available?

***Solution***



***Exercise***

A biologist is attempting to classify 52,000 species of insects by assigning 3 initials to each species. Is it possible to classify all the species in this way? If not, how many initials should be used?

***Solution***

 *This would not be enough*.

 *Which is more than enough*

***Exercise***

How many 4-letter code words are possible using the first 10 letters of the alphabet under:

1. No letter can be repeated
2. Letters can be repeated
3. Adjacent can’t be alike

***Solution***

1. 10.9.8.7 = 5040
2. 10.10.10.10 = 10,000
3. 10.9.9.9 = 7290

***Exercise***

How many 3 letters license plate without repeats

***Solution***

26.25.24 = 15600 possible

***Exercise***

How many ways can 2 coins turn up heads, H, or tails, T – if the combined outcome (H, T) is to be distinguished from the outcome (T, H)?

***Solution***



***Exercise***

|  |  |  |
| --- | --- | --- |
| *a* | *b* |  |
| *c* |  |
| *b* | *a* |  |
| *c* |  |
| *c* | *a* |  |
| *b* |  |

How many 2-letter code words can be formed from the first 3 letters of the alphabet if no letter can be used more than once?

***Solution***



***Exercise***

A coin is tossed with possible outcomes heads, H, or tails, T. Then a single die is tossed with possible outcomes 1, 2, 3, 4, 5, or 6. How many combined outcomes are there?

***Solution***



***Exercise***

In how many ways can 3 coins turn up heads, H, or tails, T – if combined outcomes such as (H,T,H), (H, H, T), and (T, H, H) are to be considered different?

|  |  |  |
| --- | --- | --- |
| ***H*** | ***H*** |  |
|  |  |
|  | ***T*** |  |
|  |  |
| ***T*** | ***H*** |  |
|  |  |
|  | ***T*** |  |

***Solution***



***Exercise***

An entertainment guide recommends 6 restaurants and 3 plays that appeal to a couple.

1. If the couple goes to dinner or to a play, how many selections are possible?
2. If the couple goes to dinner and then to a play, how many combined selections are possible?

***Solution***

***a*)** 3 + 6 = 9

***b*)** 6.3 = 18

***Solution Section* 3.4 − Permutations and Combinations**

***Exercise***

Decide whether the situation involves ***permutations*** or ***combinations***

1. A batting order for 9 players for a baseball game
2. An arrangement of 8 people for a picture
3. A committee of 7 delegates chosen from a class of 30 students to bring a petition to the administration
4. A selection of a chairman and a secretary from a committee of 14 people
5. A sample of 5 items taken from 71 items on an assembly line
6. A blend of 3 spices taken from 7 spices on a spice rack
7. From the 7 male and 10 female sales representatives for an insurance company, team of 8 will be selected to attend a national conference on insurance fraud.
8. Marbles are being drawn without replacement from a bag containing 15 marbles.
9. The new university president named 3 new officers a vice-president of finance, a vice-president of academic affairs, and a vice-president of student affairs.
10. A student checked out 4 novels from the library to read over the holiday.
11. A father ordered an ice cream cone (chocolate, vanilla, or strawberry) for each of his 4 children.

***Solution***

1. Permutation
2. Permutation
3. Combination
4. Permutation
5. Combination
6. Combination
7. Combination
8. Combination
9. Permutation
10. Combination
11. Neither

***Exercise***

How many different permutations are the of the set ?

***Solution***



***Exercise***

How many permutations of end with *a*?

***Solution***

To find the permutation to with *a*, then we may forget about the *a*, and leave us 



***Exercise***

Find the number of 5-permutations of a set with nine elements

***Solution***

 ***by Theorem***

***Exercise***

In how many different orders can five runners finish a race if no ties are allowed?

***Solution***



***Exercise***

A coin flipped eight times where each flip comes up either heads or tails. How many possible outcomes

1. Are there in total?
2. Contain exactly three heads?
3. Contain at least three heads?
4. Contain the same number of heads and tails?

***Solution***

1. Each flip can be either heads or tails: There are 
2. 
3. At least three heads means: 3, 4, 5, 6, 7, 8 heads.



OR



1. To have an equal number of heads and tails means 4 heads and 4 tails.

Therefore; 

***Exercise***

A coin flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

1. Are there in total?
2. Contain exactly two heads?
3. Contain at most three tails?
4. Contain the same number of heads and tails?

***Solution***

1. Each flip can be either heads or tails: There are 
2. 
3. At most three tails means: 3, 2, 1, 0 tails.



1. To have an equal number of heads and tails means 5 heads and 5 tails.

Therefore; 

***Exercise***

How many bit strings of length 12 contain?

1. Exactly three 1s?
2. At most three 1s?
3. At least three 1s?
4. An equal number of 0s and 1s?

***Solution***

1. We need to choose the 3 positions that contains the 1’s



1. At most three 1’s means to contains 3, 2, 1, 0 −1’s:



1. At least three 1’s means to contains 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 −1’s:



OR



1. To have an equal number of 0’s and 1’s means 6 1’s.

Therefore; 

***Exercise***

A group contains *n* men and *n* women. How many ways are there to arrange these people in a row if the men and women alternate?

***Solution***

Consider the order in which the men appear relative to each other. There are *n* men  arrangements is allowed.

Consider the order in which the women appear relative to each other. There are *n* women  arrangements is allowed.

Men and women must alternate, and there are the same number of men and women; therefore there are exactly 2 possibilities: either the row with a man ends with a woman ***or*** it starts with a woman ends with a man.

By the product rule there are 

***Exercise***

In how many ways can a set of two positive integers less than 100 be chosen?

***Solution***



***Exercise***

In how many ways can a set of five letters be selected from the English alphabet?

***Solution***



***Exercise***

How many subsets with an odd number of elements does a set with 10 elements have?

***Solution***



***Exercise***

How many subsets with more than two elements does a set with 100 elements have?

***Solution***

There are  subsets of a set with 100 elements. All of them have more than 2 subsets except the empty set, the 100 subsets consisting of one element each, and with 2 elements.

Therefore; 

***Exercise***

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?

***Solution***

First position the men relative to each other. Since there are 8 men, there are  ways to do this.

This creates 9 slots where a woman may stand: in front of the first man, between the first and second men, …, between the 7th and 8 men, and behind the 8th man.

We need to choose 5 of these positions, in order, for the first through 5th woman to occupy.

Therefore, 

***Exercise***

How many ways are there for six men and 10 women to stand in a line so that no two men stand next to each other?

***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **16** | **15** | **14** | **13** | **12** | **11** | **10** | **9** | **8** | **7** | **6** | **5** | **4** | **3** | **2** | **1** |
| W | W | W | W | W | M | W | M | W | M | W | M | W | M | W | M |

Since there are 10 women, there are 

This creates 11 slots where a man may stand.

This can be done is 

Therefore 

***Exercise***

A professor writes 40 discrete mathematics true/false questions of the statements in these questions. 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

***Solution***



***Exercise***

Thirteen people on a softball team show up for a game.

1. How many ways are there to choose 10 players to take the field?
2. How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
3. Of the 13 people who show up, there are three women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

***Solution***

1. 
2. The order in which the choices are made: 
3. There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men.

Therefore, there are  to choose the players if at least one of them must be a woman.

***Exercise***

A club has 25 members

1. How many ways are there to choose four members of the club to serve on an executive committee?
2. How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

***Solution***

1. Since the order of choosing the members is not relevant, we need to use a combination



1. Since the order of choosing the members is matter, we need to use a permutation.



***Exercise***

How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers, *k*, *k* + 1, *k* + 2, in the order

1. Where these consecutive integers can perhaps be separated by other integers in the permutation?
2. Where they are in consecutive positions in the permutation?

***Solution***

1. The consecutive numbers are 5, 6, 7, since can be separate by other integers the permutation can be written as 5, 6, 32, 7.

In order to specify such 4-permutation, we need to choose 3 consecutive integers. They can be {1, 2, 3} to (98, 99, 100}; thus, there are 98 possibilities. There are 4 possibilities, we need to decide which 97 other positive integers not exceeding 100 is to fill this slot, and there are 97 choices.

In fact, every 4-permutation consisting of 4 consecutive numbers, in order, has been double counted.

Therefore, we need to subtract the number of such 4-permutations. Clearly there are 97 of them.

Further thought shows that every other 4-permutation in our collection arises in a unique way. Therefore 

1. The consecutive numbers be consecutive in the 4-permutation.

There are only 2 places to put the fourth number in slot 1 and slot 4.

Therefore, 

***Exercise***

The English alphabet contains 21 constants and five vowels. How many strings of six lowercase letters of the English alphabet contain?

1. Exactly one vowel?
2. Exactly two vowels?
3. At least one vowel?
4. At least two vowels?

***Solution***

1. This can be done 6 ways. We need to choose the vowel and this can be done in 5 ways. Each other 5 positions can contain any of the 21 consonants, so there are  ways to fill the rest of the string.

Therefore, the answer is 

1. The position of the vowels can be done in . We need to choose the 2 vowels in  ways. Each other 4 positions can contain any of the 21 consonants, so there are  ways to fill the rest of the string.

Therefore, the answer is 

1. Count the number of strings with no vowels and subtract this from the total number of stings. 
2. Subtracting the total number of strings from the number of strings with no vowels and the number of strings with one vowel. Answer: 

***Exercise***

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have

1. The same number of men and women?
2. More women than men?

***Solution***

1. 
2. There are  to choose the committee to be composed only of women

 if there are to be five women and one man, and  if there are to be four women and two men.

Therefore, 

***Exercise***

How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 0 1 | | 0 1 | | 0 1 | | 0 1 | | 0 1 | | 0 1 | | 0 1 | | 0 1 | | 1 | 1 |

There are 8 blocks consisting of the string 01



***Exercise***

How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

***Solution***

Glue 2 1’s to the right of each 0, giving a collection of 9 tokens: five 001’s and four 1’s.



***Exercise***

A concert to raise money for an economics prize is to consist of 5 works; 2 overtures, 2 sonatas, and a piano concerto.

1. In how many ways can the program be arranged?
2. In how many ways can the program be arranged if an overture must come first?

***Solution***

1. 
2. 

***Exercise***

A zydeco band from Louisiana will play 5 traditional and 3 original Cajun compositions at a concert. In how many ways can they arrange the program if

1. The begin with a traditional piece?
2. An original piece will be played last?

***Solution***

1. 
2. 

***Exercise***

In an election with 3 candidates for one office and 6 candidates for another office, how many different ballots may be printed?

***Solution***

*Office* **1**: 

*Office* **2**: 

Multiplication principle: 

***Exercise***

A business school gives courses in typing, shorthand, transcription, business English, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken? assume that the order in which courses are schedules matters.

***Solution***



***Exercise***

If your college offers 400 courses, 25 of which are in mathematics, and your counselor arranges your schedule of 4 courses by random selection, how many schedules are possible that do not include a math course? Assume that the order in which courses are scheduled matters.

***Solution***



***Exercise***

A baseball team has 19 players. How many 9-player batting orders are possible?

***Solution***



***Exercise***

A chapter of union Local 715 has 35 members. In how many different ways can the chapter select a president, a vice-president, a treasurer, and a secretary?

***Solution***



***Exercise***

An economics club has 31 members.

1. If a committee of 4 is to be selected, in how many ways can the selection be made?
2. In how many ways can a committee of at least 1 and at most 3 be selected?

***Solution***

1. 
2. 





***Exercise***

In a club with 9 male and 11 female members, how many 5-member committees can be chosen that have

1. All men?
2. All women?
3. 3 men and 2 women?

***Solution***

1. 
2. 
3. 

***Exercise***

In a club with 9 male and 11 female members, how many 5-member committees can be selected that have

1. At least 4 women?
2. No more than 2 men?

***Solution***

1. 
2. 

***Exercise***

In a game of musical chairs, 12 children will sit in 11 chairs arranged in a row (one will be left out). In how many ways can this happen, if we count rearrangements of the children in the chairs as different outcomes?

***Solution***



***Exercise***

A group of 3 students is to be selected from a group of 14 students to take part in a class in cell biology.

1. In how many ways can this be done?
2. In how many ways can the group who will not take part be chosen?

***Solution***

1. 
2. 

***Exercise***

Marbles are being drawn without replacement from a bag containing 16 marbles.

1. How many samples of 2 marbles can be drawn?
2. How many samples of 2 marbles can be drawn?
3. If the bag contains 3 yellow, 4 white, and 9 blue marbles, how many samples of 2 marbles can be drawn in which both marbles are blue?

***Solution***

1. 
2. 
3. 

***Exercise***

A bag contains 5 black, 1 red, and 3 yellow jelly beans; you take 3 at random. How many samples are possible in which the jelly beans are

1. All black?
2. All red?
3. All yellow?
4. 2 black and 1 red?
5. 2 black and 1 yellow?
6. 2 yellow and 1 black?
7. 2 red and 1 yellow?

***Solution***

1. 
2. No 3 read. 
3. 
4. 
5. 
6. 
7. *There is only* **1*****red***.

***Solution Section* 3.5 − Applications of Recurrence Relations**

***Exercise***

*a*) Find a recurrence relation for the number of permutation of a set with *n* elements

*b*) Use the recurrence relation to find the number of permutations of a set with *n* elements using iteration.

***Solution***

1. A permutation of a set with *n* elements of a choice of a first element, followed by a permutation of a set of  elements. Therefore 
2. 







***Exercise***

A vending machine dispensing books of stamps accepts only one-dollar coins, $1 bills, and $5 bills.

1. Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matter.
2. What are the initial conditions?
3. How many ways are there to deposit $10 for a book of stamps?

***Solution***

1. Let  be the number of ways to deposit *n* dollars in the vending machine. We must express in terms of earlier terms in the sequence. If we want to deposit *n* dollars, we may start with a dollar coin and then deposit *n* – 1 dollars. This gives us  ways to deposit *n* dollars.

We can start with a dollar bill and then deposit *n* – 1 dollars. This gives us  more ways to deposit *n* dollars.

Finally, we can deposit a five-dollar bill and follow that with *n* – 5 dollars; there are ways to do this, Therefore the recurrence relation is 

1. We need initial conditions for all *n* from 0 to 4. Clearly,  (deposit nothing) and  (deposit either the dollar coin or the dollar bill)



1. 











Therefore, there are 1217 ways to deposit $10.

***Exercise***

*a*) Find a recurrence relation for the number of bit strings of length *n* that contain three consecutive 0s.

*b*) What are the initial conditions?

*c*) How many bit strings of length seven contain three consecutive 0s?

***Solution***

1. Let  be the number of bit strings of length *n* containing three consecutive 0’s. In order to construct a bit string of length *n* containing three consecutive 0’s we could start with 1 and follow with a string of length *n* – 1 three consecutive 0’s, or we could start with a 01 and follow with a string of length *n* – 2 three consecutive 0’s, or we could start with a 001 and follow with a string of length *n* – 3 three consecutive 0’s, or we could start with a 000 and follow with a string of length *n* – 3.

These 4 cases are mutually exclusive and exhaust the possibilities for how the string might start. We can write down the recurrence relation, valid for all 

1. There are no bit strings of length 0, 1, or 2 containing 3 consecutive 0’s, so the initial conditions are 
2. 





























Therefore, there are 47 bits of length 7 containing three consecutive 0’s.

***Exercise***

*a*) Find a recurrence relation for the number of bit strings of length *n* that do not contain three consecutive 0s.

*b*) What are the initial conditions?

*c*) How many bit strings of length seven do not contain three consecutive 0s?

***Solution***

1. Let  be the number of bit strings of length *n* do not contain three consecutive 0’s. In order to construct a bit string of length *n* of this type we could start with 1 and follow with a string of length *n* – 1 not containing three consecutive 0’s, or we could start with 01 and follow with a string of length *n* – 2 not containing three consecutive 0’s, or we could start with a 001 and follow with a string of length *n* – 3 not containing three consecutive 0’s.

These 3 cases are mutually exclusive and exhaust the possibilities for how the string might start since it cannot start 000.

We can write down the recurrence relation, valid for all 

1. There are no bit strings of length 0, 1, or 2 containing 3 consecutive 0’s, so the initial conditions are 
2. 





























Therefore, there are 81 bits of length 7 that do not contain three consecutive 0’s.

***Exercise***

*a*) Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one stair or two stairs at a time.

*b*) What are the initial conditions?

*c*) In how many can this person climb a flight of eight stairs

***Solution***

1. Let  be the number of ways to climb *n* stairs. In order to climb *n* stairs, a person must either start with a step of one stair and climb *n* – 1 stairs  or else start with a step of two stairs and then climb *n* – 2 stairs  or else start with a step of two stairs and then climb *n* – 3 stairs . From this analysis we can immediately write down the recurrence relation, valid for all 
2. The initial conditions are , since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb stairs (one step twice or two steps at once).
3. Each term in our sequence  is the sum of the previous three terms, so the sequence begins 

Thus, a person can climb a flight of 8 stairs in 81 ways under the restrictions in this problem.

***Solution Section* 3.6 − Solving Linear Recurrence Relations**

***Exercise***

Determine which of these are linear and homogeneous recurrence relations with constant coefficients. Also find the degree of those that are

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 

***Solution***

1. Linear (terms  all to the first power), has constant coefficients (3, 4 and 5), and is homogeneous (no terms are functions of just *n*); has degree 3
2. Linear (terms  all to the first power), doesn’t have constant coefficients (2*n*), and is homogeneous
3. Linear, homogeneous, with constant coefficients; degree 4
4. Linear with constant coefficients, not homogeneous because of 2
5. Not linear since 
6. Linear, homogeneous, with constant coefficients; degree 2
7. Linear but not homogeneous because of the *n*.
8. Linear, homogeneous, with constant coefficients; degree 2
9. Linear with constant coefficients, not homogeneous because of 3
10. Not linear since 
11. Linear, homogeneous, with constant coefficients; degree 3
12. Linear with constant coefficients, not homogeneous

***Exercise***

Solve these recurrence relations together with the initial conditions given

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 

***Solution***

1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 

1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 

1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 

1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 

1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 



1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 

1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 

1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 

1. The characteristic polynomial is 

The general solution: 



Therefore, the solution is 

***Exercise***

How many different messages can be transmitted in *n* microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

***Solution***

The model is the recurrence relation  with 

The characteristic polynomial is 

So, the roots are −1, and 2

The general solution: 

Plugging in initial conditions gives



Therefore, the solution is in *n* microseconds  messages can be transmitted.

***Exercise***

In how many ways can a  rectangular checkerboard be tiled using  and  pieces?

***Solution***

Let  be the number of ways like to tile a  board with  and  pieces. To obtain the recurrence relation, imagine what tiles are placed at the left-hand end of the board. We can place a  tile there, leaving a board to be tiled, which of course can be done in ways.

We can place a  tile at the edge, oriented vertically, leaving  board to be tiled, which of course can be done in ways.

Finally, we can place two  tiles horizontally, one above the other, leaving a  board to be tiled, which of course can be done in ways. These 3 possibilities are disjoint.

Therefore, our recurrence relation is 

The initial conditions are , since there is only one way to tile as  board and  board. This recurrence relation has characteristic roots −1 and 2.

So, the general solution is 

Plugging in initial conditions gives



Therefore, the solution is 



***Exercise***

Find the solution to 

***Solution***



The characteristic polynomial is 

That implies to: 

So, the roots are 1, −1, and 2

The general solution:





Plugging in initial conditions gives









Therefore, the solution is 

***Exercise***

Find the solution to 

***Solution***

This is a third-degree recurrence relation.

The characteristic polynomial is 

By the rational root test, the possible rational roots are 

We find that  (using calculator).



So, the roots are −2, −1, and 3.

The general solution:



Plugging in initial conditions gives







The solution to the system of equations is 

Therefore, the specific solution is 

***Exercise***

Find the solution to 

***Solution***

This is a fourth-degree recurrence relation.

The characteristic polynomial is 

That implies to: 

So, the roots are 1, −1, 2, −2

The general solution: 

Plugging in initial conditions gives









The solution to the system of equations is 

Therefore, the solution is 

***Exercise***

Find the recurrence relation 

***Solution***

This is a third-degree recurrence relation.

The characteristic polynomial is 

By the rational root test, the possible rational roots are 

We find that  (using calculator).



Hence the only root is 2, with multiplicity 3.

The general solution: 

Plugging in initial conditions gives





Therefore, the solution: 



***Exercise***

Find the recurrence relation 

***Solution***

This is a third-degree recurrence relation.

The characteristic polynomial is 



Hence the only root is −1, with multiplicity 3.

The general solution: 

Plugging in initial conditions gives









Therefore, the specific solution is 



***Exercise***

Find the general form of the solutions of the recurrence relation 

***Solution***

This is a fourth-degree recurrence relation.

The characteristic polynomial is 



The roots are −2 and 2, each with multiplicity 2.

The general solution: 

***Exercise***

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, −2, −2, −2, 3, 3, −4?

***Solution***

There are 4 distinct roots, so . The multiplicities are 4, 3, 2, and 1.

The general solution:



***Exercise***

What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots −1, −1, −1, 2, 2, 5, 5, 7?

***Solution***

There are 4 distinct roots, so . The multiplicities are 3, 2, 2, and 1.

The general solution: