***SOLUTION Section* 4.1 – Relations and Their Properties**

***Exercise***

List the ordered pairs in the relation *R* from *A* = {0, 1, 2, 3, 4} to *B* = {0, 1, 2, 3} where  if and only if

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

***Solution***

1. {(0, 0), (1, 1), (2, 2), (3, 3)}
2. {(4, 0), (1, 3), (3, 1), (2, 2)}
3. {(1, 0), (2, 0), (2,1), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (4, 3)}
4. {(1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 0) , (3, 3) , 4, 0)} **(**means *b* is multiple of *a* ≠ 0**)**
5. {(1, 0), (0, 1), (1,1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1), (4, 3)} **(**means *relatively prime***)**
6. {(1, 2), (2, 1), (2, 2)} **(**Mean *least common multiple* is 2**)**.

***Exercise***

1. List all the ordered pairs in the relation  on the set {1, 2, 3, 4, 5, 6}
2. Display this relation graphically.
3. Display this relation in tabular form.

***Solution***

1. {(1, 1), (1, 2), (1,3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)}

|  |  |
| --- | --- |
|  |  |

***Exercise***

For each of these relations on the set {1, 2, 3, 4}, decide whether it is reflexive, symmetric, antisymmetric and transitive

1. {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
2. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
3. {(2, 4), (4, 2)}
4. {(1, 2), (2, 3), (3, 4)}
5. {(1, 1), (2, 2), (3, 3), (4, 4)}
6. {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}

***Solution***

1. This relation is not reflexive, since (1, 1) is not included

It is not symmetric, since (2, 4) is included but not (4, 2)

It is not antisymmetric, since it includes (2, 3) and (3, 2) but 2 ≠ 3

 &  It is transitive.

1. This relation is reflexive, since (1, 1), (2, 2), (3, 3), and (4, 4)} are included

It is symmetric, since (2, 1) and (1, 2) are included

It is not antisymmetric, since it includes (2, 1) and (1, 2) but 2 ≠ 1

 It is transitive.

1. This relation is not reflexive, since (1, 1) is not included

It is symmetric, since (2, 4) and (4, 2) are included

It is not antisymmetric, since it includes (2, 4) and (4, 2) but 2 ≠ 4

It is not transitive, since it includes (2, 4) and (4, 2) but not (2, 2)

1. This relation is not reflexive, since (1, 1) is not included

It is not symmetric, since (1, 2) is included but not (2, 1)

It is antisymmetric, since no cases of (*a, b*) and (*b, a*) both being in the relation

It is not transitive, since it includes (1, 2) and (2, 3) but not (1, 3)

1. This relation is reflexive, since (1, 1), (2, 2), (3, 3), and (4, 4)} are included and it is *symmetric*

It is antisymmetric, since no cases of (*a, b*) and (*b, a*) both being in the relation

It is transitive, since the only time the hypothesis  is met is when 

1. This relation is not reflexive, since (1, 1) is not included

It is not symmetric, since (1, 4) is included but not (4, 1)

It is not antisymmetric, since it includes (1, 3) and (3, 1)

It is not transitive, since it includes (2, 3) and (3, 1) but not (2, 1)

***Exercise***

Determine whether the relation *R* on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where  if and only if

1. *a* is taller than *b*.
2. *a* and *b* were born on the same day
3. *a* has the same first name as *b*.
4. *a* and *b* have a common grandparent.

***Solution***

1. I am *not* taller than myself, therefore *being taller* is not reflexive

It is *not* symmetric, since I am taller than my kid but my kid is not

It is antisymmetric since we never have *a* taller than *b* and *b* taller than *a* even if *a* = *b*

It is transitive since if *a* taller than *b* and *b* taller than *c* that implies that *A* taller then *c*

1. The relation is reflexive since *a* is born on the same day

It is symmetric, since *a* and *b* were born on the same day

It is *not* antisymmetric since *a* and *b* were born on the same day but *a* ≠ *b*

It is transitive since if *a* and *b* were born on the same day and *b* and *c* were born on the same day that implies that *a* and *c* were born on the same day

1. The relation is reflexive since *a* has the same first name as *a*

It is symmetric, since *a* has the same first name as *b* than *b* has the same first name as *a*

It is *not* antisymmetric since *a* has the same first name as *b* but *a* ≠ *b*

It is transitive since if *a* has the same first name as *b* and *c* has the same first name as *c* that implies that *a* has the same first name as *c*

1. The relation is reflexive since *a* and *a* have a common grandparent

It is symmetric, since *a* and *b* have a common grandparent than *b* and *a* have a common grandparent

It is *not* antisymmetric since *a* and *b* have a common grandparent but *a* ≠ *b*

It is transitive since if *a* and *b* have a common grandparentand *b* and *c* have a common grandparent that implies that *a* and *c* have a common grandparent

***Exercise***

Determine whether the relation ***R*** on the set of all ***real numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where  if and only if

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |

***Solution***

1. The relation is *not* reflexive since 

It is symmetric, since  then  because 

It is *not* antisymmetric since (1, −1)  and (−1, 1)  but 

It is *not* transitive since (1, −1) and (−1, 1)  but 

1. The relation is reflexive since 

It is symmetric, since  *iff* 

It is *not* antisymmetric since (1, −1)  and (−1, 1)  but 

It is transitive since the product 1’s and −1’s is 

1. The relation is reflexive since  is a rational number

It is symmetric, since  is rational, then 

It is *not* antisymmetric since (1, −1)  but (−1, 1)  but 

It is transitive since  then  is a rational number  then  is a rational number, therefore  is rational that means that 

1. The relation is *not* reflexive since 

It is *not* symmetric, since  then  but  therefore 

It is antisymmetric since  and  that implies to  which 

It is *not* transitive since  and   so 

1. The relation is reflexive since  always positive

It is symmetric, since 

It is *not* antisymmetric since (2, 3)  and (3, 2)  but 

It is *not* transitive since   but 

1. The relation is *not* reflexive since 

It is symmetric, since 

It is antisymmetric since (2, 0)  and (0, 2)  but 

It is *not* transitive since  (2, 0)  and  (0, −2)  so 

1. The relation is *not* reflexive since 

It is *not* symmetric, since  but 

It is antisymmetric since (*x, y*)  and (*y, x*)  then *x* = 1 and *y* = 1, so *x = y*

It is transitive since (*x, y*)  and (*y, z*)  then *x* = 1 and *y* = 1, so (*x, z*) 

1. The relation is *not* reflexive since 

It is symmetric, since  and 

It is *not* antisymmetric since  and  but 

It is *not* transitive since  and but 

***Exercise***

Determine whether the relation *R* on the set of all ***integers*** ***numbers*** is reflexive, symmetric, antisymmetric, and/or transitive, where  if and only if

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | |
|  |  |  | |  | |

***Solution***

1. This relation is not reflexive, since 1 ≠ 1 for instance

It is symmetric, if 

It is *not* antisymmetric since 

It is *not* transitive since 

1. This relation is not reflexive, since (0, 0) is not included 

It is symmetric, because  (commutative property of multiplication)

It is *not* antisymmetric since (2, 3) and (3, 2) are both included

It is transitive holds between *x* and *y* if and only if either *x* and *y* are both positive or *x* and *y* are both negative

1. This relation is *not* reflexive, since (1, 1) is not included 

It is symmetric, because  is equivalent to 

It is *not* antisymmetric since (1, 2) and (2, 1) are in the relation

It is *not* transitive since (1, 2) and (2, 1) are in the relation but (1, 1) is not

1.  means that  for some *t*.

This relation is reflexive since 

It is symmetric since is  then , therefore  so 

It is *not* antisymmetric since and 

It is transitive since  means  and  means 

; therefore 

1.  means that  for some *t*.

This relation is reflexive since 

It is *not* symmetric since is  but 

It is *not* antisymmetric since 2 is multiple of −2 but 

It is transitive since  and   therefore *x* is a multiply of *z*.

1. This relation is *not* reflexive, since 

It is *not* symmetric since is  but 

It is antisymmetric since  and 







 and  when 

It is *not* transitive since  and  but 

1. This relation is *not* reflexive, since 

It is *not* symmetric since is  but 

It is antisymmetric since  and , only when .

It is transitive since  and  

***Exercise***

Show that the relation  on nonempty set *S* is symmetric and transitive, but not reflexive.

***Solution***

If , then the hypothesis of the conditional statements in the definitions of symmetric and transitive are never true, so those statements are always true by definition.

 the statement  is false for an element of *S*, so  is not true, thus *R* is not reflexive.

***Exercise***

Show that the relation  on nonempty set  is reflexive, symmetric and transitive.

***Solution***

Since the domain is empty, then the relation is vacuously reflexive, symmetric and transitive s

***Exercise***

Give an example of a relation on a set that is

1. both symmetric and antisymmetric
2. neither symmetric nor antisymmetric

***Solution***

1. The empty set on {*a*} − vacuously symmetric and antisymmetric
2. {(*a, b*), (*b, a*), (*a, c*)} on {*a, b, c*}

***Exercise***

A relation *R* is called ***asymmetric*** if implies that . Explore the notion of an asymmetric relation to the following

1. {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
2. {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
3. {(2, 4), (4, 2)}
4. {(1, 2), (2, 3), (3, 4)}
5. {(1, 1), (2, 2), (3, 3), (4, 4)}
6. {(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)}
7. *a* is taller than *b*.
8. *a* and *b* were born on the same day
9. *a* has the same first name as *b*.
10. *a* and *b* have a common grandparent.

***Solution***

The relations (***a***), (***b***), and (***c***) are not *asymmetric* since they contain pairs of the form (*x, x*)

The relation (***f***) is not *asymmetric* since both (1, 3) and (3, 1) are in the relation

The relation (***d***) is not *asymmetric*

The relation (***g***) is *asymmetric* since if *a* taller than *b*, then *b* can’t be taller than *a*.

The relation (***h***) is not *asymmetric* since *a* and *b* were born on the same day but 

The relation (***i***) is not *asymmetric* since *a* has the same first name as *b* but 

The relation (***j***) is not *asymmetric* since *a* and *b* have a common grandparent but 

***Exercise***

Let *R* be the relation  on the set of integers. Find

***a****)*  ***b)*** 

***Solution***

1. 
2. 

***Exercise***

Let *R* be the relation  on the set of positive integers. Find

***a****)*  ***b)*** 

***Solution***

1. 
2. 

***Exercise***

Let *R* be the relation on the set of all states in the U.S. consisting of pairs (*a, b*) where state *a* borders state *b*. Find

***a****)*  ***b)*** 

***Solution***

1. Since this relation is symmetric, 
2. This relation consists of all pairs (*a, b*) in which state *a* does not border state *b*.

***Exercise***

Let  and  be relation from {1, 2, 3} to {1, 2, 3, 4}. Find

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

***Solution***

1. 
2. 
3. 
4. 

***Exercise***

Let the relation R = {(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)} and the relation S = {(2, 1), (3, 1), (3, 2), (4, 2)}

Find 

***Solution***











***Exercise***

Find the following:

***Solution***

1. 





1. 





1. 





1. 



1. 





1. 







1. 



 **(From part *a*)**



1. 



 **(From part *a*)**



1. 

 (*clearly*) that means (*Transitive*).

Therefore, 

1. 

 (*clearly*) that means. Therefore, 

1. 

. Therefore, 

1. 

. Clearly this can always be done simply by choosing b to be large enough. Therefore, 

1. 

 *iff a > c*. Therefore, 

1. 

. Clearly this can always be done simply by choosing b to be large enough. Therefore, 

1. 

. Clearly this can always be done simply by choosing b to be large enough. Therefore, 

***Exercise***

Let  be the “*divides*” and “*is a multiple of*” relations on the set of all positive integers, respectively. That is  and 

Find the following:

***Solution***

1.  if and only if *a|b* or *b|a*
2.  if and only if *a|b* and *b|a* with  and 
3.  this relation holds between 2 integers if holds and does not hold.  if and only if *a|b* and 
4.  this relation holds between 2 integers if holds and does not hold.  if and only if *b|a* and 
5.  this relation holds between 2 integers if holds and does not hold and holds and does not hold. if and only if *a|b* or 

***SOLUTION Section* 4.2 – Representing Relations**

***Exercise***

Represent each of these relations on {1, 2, 3} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

1. {(1, 1), (1, 2), (1, 3)}
2. {(1, 2), (2, 1), (2, 2), (3, 3)}
3. {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
4. {(1, 3), (3, 1)}

***Solution***

|  |  |
| --- | --- |
|  |  |

***Exercise***

Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order). Then draw the directed graphs representing each relation

1. {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}
2. {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}
3. { (1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)}
4. {(2, 4), (3, 1), (3, 2), (3, 4)}

***Solution***

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |

***Exercise***

List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

***Solution***

|  |  |
| --- | --- |
| 1. {(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)} |  |
| 1. {(1, 2), (2, 2), (3, 2)} |  |
| 1. {(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)} |  |

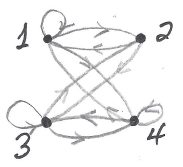
***Exercise***

List the ordered pairs in the relations on {1, 2, 3, 4} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order). Then draw the directed graphs representing each relation

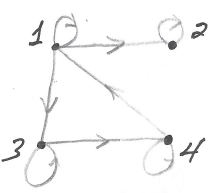
  

***Solution***

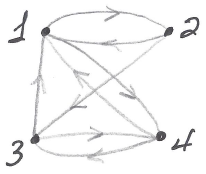
1. {(1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4)}



1. {(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1), (4, 4)}



1. {(1, 2), (1, 4), (2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)}



***Exercise***

Let *R* be the relation represented by the matrix



Find: *a*)  *b*)  *c*) 

***Solution***

1. 



1. 



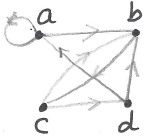
1. 



***Exercise***

Draw the directed graph that represents the relation {(*a, a*), (*a, b*), (*b, c*), (*c, b*), (*c, d*), (*d, a*), (*d, b*)}

***Solution***



***Exercise***

Determine whether the relations represented by the directed graphs are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive

|  |  |  |
| --- | --- | --- |
| ***a)*** | ***b)*** | ***c)*** |
| ***d)*** | ***e)*** | ***f)*** |

***Solution***

1. {(*a, b*), (*a, c*), (*b, c*), (*c, b*)}

It is not reflexive since (*a, a*) doesn’t exist

It is not symmetric

It is transitive since (*a, b*), (*b, c*) ⇒ (*a, c*)

1. {(*a, a*), (*a, c*), (*b, a*), (*b ,b*), (*b, c*), (*c, c*)}

It is reflexive

It is not symmetric

It is transitive since (*b, a*), (*a, c*) ⇒ (*b, c*)

1. { (*a, c*), (*b, a*), (*c, d*), (*d, b*)}

It is not reflexive; it is not symmetric, and not transitive since

1. {(*a, a*), (*a, b*), (*b, a*), (*b, b*), (*c, a*), (*c, c*), (*c, d*), (*d, d*)}

It is reflexive, not symmetric (no (*a, c*)), and not transitive (*c, a*), (*a, b*) but no (*c, b*)

1. {(*a, a*), (*a, b*) , (*a, c*), (*b, a*), (*b, b*), (*b, c*), (*c, a*) , (*c, b*), (*d, d*)}

It is not reflexive; it is symmetric and transitive

1. {(*a, a*), (*a, b*), (*b, a*), (*b, b*), (*c, c*), (*c, d*), (*d, d*), (*d, c*)}

It is reflexive; it is symmetric and transitive

***SOLUTION Section* 4.3 – Closures of Relations**

***Exercise***

Let *R* be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1), (1, 1), (1, 2), (2, 0),

(2, 2), and (3, 0). Find the

1. Reflexive closure of R.
2. Symmetric closure of R.

***Solution***

1. The reflexive closure of *R* is *R* with all (*a, a*). In this case the closure of R is

{(0, 0), (0, 1), (1, 1), (1, 2), (2,0), (2, 2), (3, 0), (3, 3)}

1. The symmetric closure of *R* is *R* with (*b, a*) for which (*a, b*) is in *R*. In this case the symmetric of *R* is

{(0, 1), (0, 2), (0, 3), (1, 0) (1, 1), (1, 2), (2,0), (2, 1) (2, 2), (3, 0) }

***Exercise***

Let *R* be the relation  on the set of integers. What is the reflexive closure of *R*?

***Solution***

When we add all the pairs (*x, x*) to the given relation we have all of Z×Z, which the relation will always holds.

***Exercise***

Let *R* be the relation  on the set of integers. What is the symmetric closure of *R*?

***Solution***

To form the symmetric closure, we need to add all the pairs (*b, a*) such that (*a, b*) is in *R*.

We need to include pairs (*b, a*) such that *a* divides *b*, which is equivalent to saying that we need to include all the pairs (*a, b*) such that *b* divides *a*.

Thus the closure is {(*a, b*) | *a* divides *b* **or** *b* divides *a*}

***Exercise***

How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

***Solution***

To form a reflexive closure, we simply need to add a loop at each vertex that does not already have one.

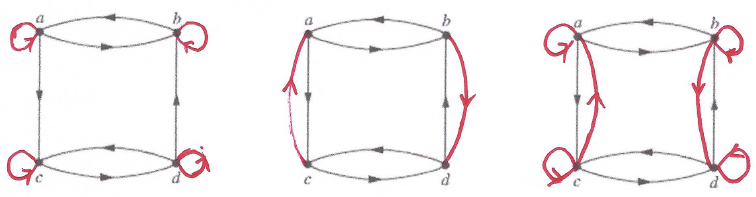
***Exercise***

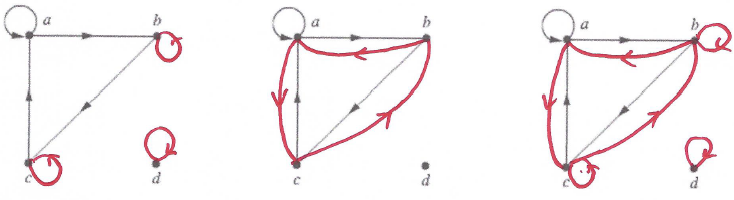
Draw the directed graph of the *reflexive*, *symmetric*, and *both reflexive and symmetric* closure of the relations with the directed graph shown

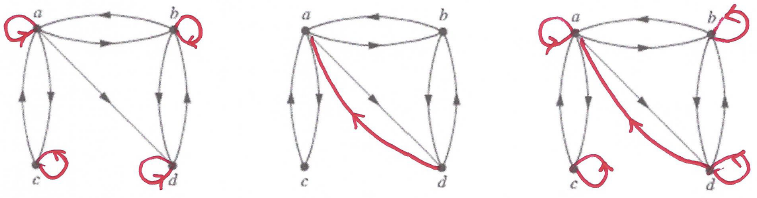
|  |  |  |
| --- | --- | --- |
| ***a)*** | ***b)*** | ***c)*** |

***Solution***

***Reflexive Symmetric Reflexive*** *and* ***Symmetric***







***Exercise***

1. Determine whether these sequences of vertices are paths in this directed graph

|  |  |
| --- | --- |
| 1. *a, b, c, e* 2. *b, e, c, b, e* 3. *a, a, b, e, d, e* 4. *b, c, e, d, a, a, b* 5. *b, c, c, b, e, d, e, d* 6. *a, a, b, b, c, ,c, b, e, d* |  |

2. Find all circuits of length three in the directed graph

***Solution***

1. This is a path
2. This is not a path (no edge from *e* to *c*)
3. This is a path
4. This is not a path (no edge from *d* to *a*)
5. This is a path
6. This is not a path (no loop at *b*)

2. A circuit of length 3 can be written as a sequence of 4 vertices.

Start @ ***b***: *bccb* and *beab*

Start @ ***c***: *ccbc* and *cbcc*

Start @ ***d***: *deed*, *eede* and *edee*

*eabe, dead, eade, abea, adea, aaaa, cccc, and eeee*

***Exercise***

Let *R* be the relation on the set {1, 2, 3, 4, 5} containing the ordered pairs (1, 3), (2, 4), (3, 1), (3, 5), (4, 3), (5, 1), and (5, 2). Find

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |

***Solution***



1. 
2. 
3. 
4. 
5. 
6. 

***Exercise***

Let *R* be the relation on the pair (*a, b*) if a and b are cities such that there is a direct non-stop airline flight from *a* to *b*. When is (*a, b*) in

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |

***Solution***

1. The pair (*a, b*) is in  precisely when there is a city *c* such that there is a direct flight from *a* to *c* and a direct flight from *c* to *b* – when it is possible to fly from *a* to *b* with a scheduled stop in some intermediate city.
2. The pair (*a, b*) is in  precisely when there are cities *c* and *d* such that there is a direct flight from *a* to *c*, a direct flight from *c* to *d,* and a direct flight from *d* to *b* – when it is possible to fly from *a* to *b* with two scheduled stops in some intermediate cities.
3. The pair (*a, b*) is in  precisely when it is possible to fly from *a* to *b*.

***Exercise***

Let *R* be the relation on the set of all students containing the ordered pair (*a, b*) if *a* and *b* are in at least one common class and *a* ≠ *b*. When is (*a, b*) in

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Solution***

1. The pair  if there is a person *c* other than *a* or *b* who is in a class with *a* and a class with *b*.  as long *a* is taking a class that has at least one other person in it, that person serves as the “*c*”.
2. The pair  if there are persons *c* different from *a* and *d* different from *b* ***and*** *c* such that *c* is in a class with *a*, *c* is in class with *d*, and *d* is in class with *b*.
3. The pair  if there is a sequence of persons , with  such that , and for each *i* from 1 to *n*,  and  is at least one class with 

***Exercise***

Suppose that the relation *R* is reflexive. Show that  is reflexive.

***Solution***

Since , clearly if , then 

***Exercise***

Suppose that the relation *R* is symmetric. Show that  is symmetric.

***Solution***

Suppose , then there is a path from *a* to *b* in *R*. Given such a path, if *R* is symmetric, then the reverse of every edge in the path is also in *R*; therefore there is a path from *b* to *a* in *R*.

This means that whenever (*a, b*) is.

***Exercise***

Suppose that the relation *R* is irreflexive. Is the relation  necessarily irreflexive.

***Solution***

It is certainly possibly for  to contain some pairs (*a, a*).

For example: *R* = {(1, 2), (2, 1)}

***SOLUTION Section* 4.4 – Equivalence Relations**

***Exercise***

Which of these relations on {0, 1, 2, 3} are equivalence relations?

Determine the properties of an equivalence relation that the others lack.

What are the equivalence classes of the equivalence relations?

1. {(0, 0), (1, 1), (2, 2), (3, 3)}
2. {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
3. {(0, 0), (1, 1), (1, 2), (2, 1), (3, 2), (3, 3)}
4. {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}
5. {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

***Solution***

1. This is an equivalence relation. It is reflexive, symmetric and transitive.

The equivalence classes all have just one element.

Each element is in an equivalence class by itself.

1. This is not reflexive since pair (1, 1) is missing. It is symmetric and it is not transitive since the pairs (0, 2) and (2, 3) are there, but not (0, 3).

This is not an equivalence relation.

1. This is an equivalence relation. It is reflexive, symmetric and transitive.

The elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class.

1. This is reflexive and symmetric and it is not transitive since the pairs (1, 3) and (3, 2) are there, but not (1, 2).

This is not an equivalence relation.

1. This is reflexive, it is not symmetric since (2, 1) is missing and it is not transitive since the pairs (2, 0) and (0, 1) are there, but not (2, 1).

This is not an equivalence relation.

***Exercise***

Which of these relations on the set of all people are equivalence relations?

Determine the properties of an equivalence relation that the others lack.

What are the equivalence classes of the equivalence relations?

1. 
2. 
3. 
4. 
5. 

***Solution***

1. This relation is reflexive, since *a* is the same person (same age).

If *a* is the same age as *b*, then *b* has to be the same age as *a*. this relation is symmetric.

If *a* is the same age as *b* and *b* is the same age as *c*, then *a* has to be the same age as *c*. this relation is transitive.

An equivalence class is the set of all people who are the same age. To really identify the equivalence class and the equivalence relation itself, one would need to specify exactly what ine meant by the “same age”. For example, we could define two people to be the same age if their official dates of birth were identical.

1. For each pair (*m, w*) of a man and a woman, the set of offspring of their union, if nonempty, is an equivalence class. In many cases, then, an equivalence class consists of all the children in a nuclear family with children.
2. Let assume the relation is biological parentage. It is possible that *a* to be the the child of *W* and *X*, *b* is the child of *X* and *Y*, and *c* is the child of *Y* and *Z*. Then *a* is related to *b*, and *b* is related to *c*, but *a* is not related to *c*. This is not an equivalence relation, since it is not transitive. Therefore, this is not an equivalence relation.
3. If *a* met *b* and *b* met *c*, then it is not neccesary that *a* met *c*. This is not an equivalence relation, since it is not transitive. Therefore, this is not an equivalence relation.
4. If *a* speaks the same language (english) as *b* and *b* speaks the same language (spanish) as *c*, then it is not neccesary that *a* can speak spanish as *c*. This is not an equivalence relation, since it is not transitive.

***Exercise***

Which of these relations on the set of all functions from Z to Z are equivalence relations?

Determine the properties of an equivalence relation that the others lack.

What are the equivalence classes of the equivalence relations?

1. 
2. 
3. 
4. 
5. 

***Solution***

1. This is an equivalence relation, one of the general form that 2 things are considered equivalent if they have the same “something” (is 1).

 This relation is reflexive.

If  then  ∴ this relation is symmetric

If  and , then  ∴ this relation is transitive.

There is one equivalence class for each  and it contains all those functions whose value at 1 is *n*.

1. Let  for all . Then *f* is related to *g* since  and *g* is related to *h* since , but , therefore *f* is related to *h* since that have no values in common. Hence, this is not an equivalence relation because it is not transitive.
2. It is not reflexive relation since .

It is not symmetric since if , then 

It is not transitive since  and ,

This is not an equivalence relation.

1. This relation is reflexive, 

, this relation is symmetric

, this relation is transitive.

This is an equivalence relation.

The set of equivalence classes is uncountable. For each function , there is the equivalence class consisting of all functions *g* for which there is a constant *C* such that  for all .

1. It is not reflexive relation since  since it not given. This is not an equivalence relation.

***Exercise***

Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

***Solution***

One relation is that *a* and *b* are related if they were born in the same state. Here the equivalence classes are the nonempty sets of students from each state.

Another example is for *a* to be related to *b* if *a* and *b* have lived the same number of complete decades. The equivalence classes are the set of all 10 to 19 years olds. The set of all 20 to 29 year olds, and so on.

A Third example is *a* to be related to *b* if 10 is a divisor of the difference between *a*’s age and *b*’s age, where “age” means the whole number of years since birth, as of the first day of class.

For each *i* = 0, 1, 2, …, 9, there is the equivalence class (if it is nonempty) of those students whose age ends with the digit *i*.

***Exercise***

Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

***Solution***

Two buildings are equivalent, if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year.

For another example, we can define 2 buildings to be equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on.

The third example, partition the set of all buildings into 2 classes – those in which you do have a class this semester and those in which you don’t. Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don’t have a class is equivalent to every building in which you don’t have a class.

***Exercise***

Let *R* be the relation on the set of all sets of real numbers such that *S R T* if and only if *S* and *T* have the same cardinality. Show that *R* is an equivalence relation. What are the equivalence classes of the sets {0, 1, 2} and ?

***Solution***

Two sets have the same cardinality if there is a bijection (1−1 and onto function) from one set to the other.

We need to prove that *R* is reflexive, symmetric, and transitive.

Every set has the same cardinality as itself because of the identity function.

If *f* is a bijection from *S* to *T*, then  is a bijection from *T* tot *S*, so *R* is symmetric.

If *f* is a bijection from *S* to *T*, and *g* is a bijection from *T* to *U*, then  is a bijection from *S* to *U*, so *R* is transitive.

The equivalence class {1, 2, 3} is the set of all 3-element sets of real numbers, including such sets as {4, 25, 1948} and .

Similarly, is the set of all infinite countable sets of real numbers, such as the set of natural numbers, the set of rational numbers, and the set of the prime numbers, but not including the set {1, 2, 3} (it’s too small) or the set of all real numbers (too big).

***Exercise***

Suppose that *A* is a nonempty set, and *f* is a function that has *A* as its domain. Let *R* be the relation on A consisting of all ordered pairs (*x, y*) such that 

1. Show that *R* is an equivalence relation on *A*.
2. What are the equivalence classes of *R*?

***Solution***

1. It is reflexive since  for all 

It is symmetric since , then 

It is transitive since  and then 

1. The equivalence class of *x* is the set of all  such that  (definition of inverse). Thus the equivalence classes are precisely the sets  for every *b* in the range of *f*.

***Exercise***

Suppose that *A* is a nonempty set, and *R* is an equivalence relation on *A*. Show that there is a function *f* with *A* as its domain such that  if and only if 

***Solution***

The function that sends each  to its equivalence class [*x*] is obviously such a function.

***Exercise***

Determine whether the relation with the directed graph shown is an equivalence relation

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Solution***

1. The relation is reflexive since there is a loop at each vertex.

It is symmetric since every edge has 2 vertices and pointing in the both direction.

It is not transitive since we have {d, a} and {a, c} but not {d, a}

1. The relation is reflexive since there is a loop at each vertex.

It is symmetric since every edge has 2 vertices and pointing in the both direction.

It is transitive since paths of length 2 are accompanied by the path of length 1, edge between the same 2 vertices in the same direction.

This relation is an equivalence relation. The equivalence classes are {*a, d*} and {*b, c*}

1. The relation is reflexive since there is a loop at each vertex.

It is symmetric since every edge has 2 vertices and pointing in the both direction.

It is not transitive (*a, b*) and (*b, c*) but not (*a, c*).

***Exercise***

Which of these collections of subsets are partitions of {1, 2, 3, 4, 5, 6}

1. {1, 2}, {2, 3, 4}, {4, 5, 6}
2. {1}, {2, 3, 6}, { 4}, {5}
3. {2, 4, 6}, {1, 3, 5}
4. {1, 4, 5}, {2, 6}

***Solution***

1. This is not a partition, since the sets are not pairwise disjoint. 2 and 4 appear in 2 of the sets.
2. This is a partition
3. This is a partition
4. This is not a partition, since element 3 is missing from the sets

***Exercise***

Which of these collections of subsets are partitions of {−3, −2, −1, 0, 1, 2, 3}

1. {−3, −1, 1, 3}, {−2, 0, 2}
2. {−3, −2, −1, 0}, {0, 1, 2, 3}
3. {−3, 3}, {−2, 2}, {−1, 1}, {0}
4. {−3, −2, 2, 3}, {−1, 1}

***Solution***

1. This a partition, since it satisfies the definition
2. This is not a partition, since the subsets are not disjoint
3. This a partition, since it satisfies the definition
4. This is not a partition, since the union of the subsets leaves out 0

***SOLUTION Section* 4.5 − Partial Orderings**

***Exercise***

Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

1. {(0, 0), (1, 1), (2, 2), (3, 3)}
2. {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
3. {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)}
4. {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
5. {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}
6. {(0, 0), (2, 2), (3, 3)}
7. {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)}
8. {(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)}
9. {(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)}
10. {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)}

***Solution***

1. This relation is reflexive because each of 0, 1, 2, 3 is related to itself.

This relation is antisymmetric because *a* to be related to *b* is for *a* to be equal to *b*

Since *a* is related to *b* and *b* related to *c* and , then *a* is related to *c*. So the relation is transitive.

The equality relation on any set satisfies all three conditions, therefore is a partial ordering.

1. It is reflexive but it is not antisymmetric since we have  and  but . Therefore this is not a partial ordering.
2. This relation is reflexive because each of 0, 1, 2, 3 is related to itself.

This relation is antisymmetric because *a* to be related to *b* is for *a* to be equal to *b*

It is transitive for the same reason and 

Therefore is a partial ordering.

1. This relation is reflexive because each of 0, 1, 2, 3 is related to itself.

This relation is antisymmetric because *a* to be related to *b* is for *a* to be equal to *b*

It is transitive for the same reason and , , and 

Therefore is a partial ordering.

1. It is reflexive but it is not antisymmetric since we have  and  but . Therefore this is not a partial ordering.
2. Since 1 is not related to itself, so this relation is not reflexive. Therefore *R* is not a partial ordering.
3. This relation is reflexive because each of 0, 1, 2, 3 is related to itself.

This relation is antisymmetric because *a* to be related to *b* is for *a* to be equal to *b*

It is transitive for the same reason and , and 

Therefore is a partial ordering.

1. Since , so this relation is not transitive. Therefore *R* is not a partial ordering.
2. Since , so this relation is not transitive. Therefore *R* is not a partial ordering.
3. Since  but , so this relation is not antisymmetric and it is not transitive because . Therefore *R* is not a partial ordering.

***Exercise***

Is (*S, R*) a poset If *S* is the set of all people in the world and , where *a* and *b* are people, if

1. *a* is a taller than *b*?
2. *a* is not taller than *b*?
3. *a* = *b* or *a* is an ancestor of *b*?
4. *a* and *b* have a common friend?
5. *a* is a shorter than *b*?
6. *a* weighs more than *b*?
7. *a* = *b* or *a* is a descendant of *b*?
8. *a* and *b* do not have a common friend?

***Solution***

1. Since nobody is taller than himself, this relation is not reflexive, so (*S, R*) is not a poset.
2. To be not a taller means exactly the same height or shorter. 2 different people *x* and *y* could have the same height, in which case , so *R* is not antisymmetric.

Therefore, this relation is not a poset.

1. The equality clause in the given of *R* guarantees that *R* is reflexive.

If *a* is ancestor to *b*, then *b* can’t be ancestor to *a*, so the relation is vacuously antisymmetric.

If *a* is ancestor to *b* and *b* is ancestor to *c*, then *a* is ancestor to *c*, thus *R* is transitive.

Therefore, this relation is a poset.

1. Let *x* and *y* be any 2 distinct friends, , so *R* is not antisymmetric. Therefore, this relation is not a poset.
2. Let 2 people can be the same height since are not the same person, so *R* is not antisymmetric. Therefore, this relation is not a poset.
3. Since nobody is weight more than himself, this relation is not reflexive, so this relation is not a poset.
4. Since , then the *R* is reflexive.

Given that  but if *a* is a descendant of *b*, then *b* cannot be a descendant of *a*. So the relation is vacuously antisymmetric.

if *a* is a descendant of *b* and *b* is a descendant of *c*, then *a* is a descendant of *c*. So the *R* is transitive.

Therefore, this relation is a poset.

1. Since anyone and himself have a common friend, then this relation is not reflexive, so this relation is not a poset.

***Exercise***

Which of these are posets?

|  |  |  |  |
| --- | --- | --- | --- |
| 1. (Z, =) |  |  |  |
|  |  |  |  |

***Solution***

1. The equality relation of any set satisfies all three conditions. Therefore a partial order.
2. This is not a poset since the relation is not reflexive 
3. The relation is reflexive since the relation involved the equality sign.
4. This is not a poset since the relation is not reflexive 
5. The equality relation of any set satisfies all three conditions. Therefore a partial order.
6. This is not a poset since the relation is not reflexive 
7. The relation is reflexive since the relation involved the equality sign.
8. This is not a poset since the relation is not reflexive 

It is not antisymmetric since 

It is not transitive 

***Exercise***

Determine whether the relations represented by these zero-one matrices are partial orders

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |

***Solution***

1. The relation is 

This is not antisymmetric because .

Therefore this matrix is not a partial order.

1. The relation is 

It is clearly reflexive. The pairs (1, 2) and (1, 3) are in the relation that neither can be part of a counterexample to antisymmetry or transitivity.

1. The relation is 

It is clearly reflexive. The pairs (1, 3) and (2, 1) are in the relation that neither can be part of a counterexample to antisymmetry.

It is not transitive since, (2, 1) and (1, 3) that will lead to (2, 3) which is not in the relation.

Therefore this matrix is not a partial order.

1. The relation is 

It is clearly reflexive. The pair (3, 1) is in the relation that can’t be part of a counterexample to antisymmetry or transitivity.

1. The relation is 

It is not transitive since, (4, 1) and (1, 3) are in the relation but not (4, 3).

Therefore this matrix is not a partial order.

1. The relation is 

It is not transitive since, (4, 1) and (1, 3) are in the relation but not (4, 3).

Therefore this matrix is not a partial order.

***Exercise***

Determine whether the relation with the directed graph shown is a partial order.

|  |  |
| --- | --- |
|  |  |
|  |  |

***Solution***

1. This is relation is not transitive since there no relation (arrow) between *a* and *d*.



1. This is relation is not transitive since there no relation (arrow) from *c* and *b*.
2. This relation is reflexive since all points have an arrow to itself.

This relation is antisymmetric since no pair of arrows going in opposite directions between 2 different points.

Therefore this relation is a partial order.

***Exercise***

Let  be a poset. Show that  is also a poset, where  is the inverse of *R*. The poset  is called the dual of .

***Solution***

Since *R* is reflexive, then  is clearly reflexive.

Suppose that  and . Then , so , so 

If  and , then  and , since *R* is transitive, so , therefore , thus  is transitive.

Therefore  is a poset

***Exercise***

Draw the Hasse diagram for the “greater than or equal to” relation on {0, 1, 2, 3, 4, 5}

***Solution***



***SOLUTION Section* 4.6 − Graphs: Definitions and Basic Properties**

***Exercise***

Determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops.

|  |  |  |
| --- | --- | --- |
| *a)* | *b)* | *c)* |
| *d)* | *e)* | *f)* |
| *g)* |  |  |

***Solution***

1. This is a simple graph; the edges are undirected, and there are no parallel edges or loops.
2. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.
3. This is a pseudograph; the edges are undirected, and there are no parallel edges or loops.
4. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.
5. This is a directed graph; the edges are directed, and there are no parallel edges.
6. This is a directed multigraph; the edges are directed, and there are parallel edges.
7. This is a directed multigraph; the edges are directed, and there is a set of parallel edges.

***Exercise***

Define each graph formally by specifying its vertex set, its edge set, and a table giving the edge-endpoint function

|  |  |
| --- | --- |
| *a)* | *b)* |

***Solution***

1. Vertex set 

Edge set 

Edge-endpoint function:

|  |  |
| --- | --- |
| ***Edge*** | ***Endpoints*** |
|  |  |
|  |  |
|  |  |

1. Vertex set 

Edge set 

Edge-endpoint function:

|  |  |
| --- | --- |
| ***Edge*** | ***Endpoints*** |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

***Exercise***

Graph *G* has vertex set  and edge set , with edge-endpoint function as follow

|  |  |
| --- | --- |
| ***Edge*** | ***Endpoints*** |
|  |  |
|  |  |
|  |  |
|  |  |

***Solution***



***Exercise***

Graph *H* has vertex set  and edge set , with edge-endpoint function as follow

|  |  |
| --- | --- |
| ***Edge*** | ***Endpoints*** |
|  |  |
|  |  |
|  |  |
|  |  |

***Solution***



***Exercise***

Show that the 2 drawings represent the same graph by labeling the vertices and edges of the right-hand drawing to correspond to those of the left-hand drawing.

|  |
| --- |
| *a)* |
| *b)* |
| *c)* |

***Solution***

1. If you just hold the vertex  turn it around to up position and stretch vertically little



1. Hold the edge  and twisted as the vertices switch position.





***Exercise***

For each of the graphs

1. Find all edges that are incident on 
2. Find all vertices that are adjacent to 
3. Find all edges that are adjacent to 
4. Find all loops
5. Find all parallel edges
6. Find all isolated vertices
7. Find the degree of 
8. Find the total degree of the graph

|  |  |
| --- | --- |
|  |  |

***Solution***

1.  are incident on 

 are adjacent to 

 are adjacent to 

 are loops.

 are parallel;  are parallel

 is an isolated vertex.

Degree of 

Total degree = 20

1.  are incident on 

 are adjacent to 

 are adjacent to 

 are loops.

 are parallel

Isolated vertex: none.

Degree of 

Total degree = 14

***Exercise***

Let *G* be a simple graph. Show that the relation *R* on the set of vertices of *G* such that  if and only if there is an edge associated to  is a symmetric, irreflexive relation on *G*.

***Solution***

In a simple graph, edges are undirected.

If , then there is edge associated with . But , so this edge is associated with  and therefore. So, *R* is symmetric.

A simple graph does not allow loops; that is if there is an edge associated with , then . Thus  never holds, and so by definition *R* is irreflexive.

***Exercise***

Let *G* be an undirected graph with a loop at every vertex. Show that the relation *R* on the set of vertices of *G* such that  if and only if there is an edge associated to  is a symmetric, reflexive relation on *G*.

***Solution***

If , then there is edge associated with , and since the graph is undirected, this is also edge joining vertices  and therefore. So, *R* is symmetric.

The relation is reflexive because the loops guarantees that  for each vertex *u*.

***Exercise***

Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed? Describe a graph that models the electronic mail sent in a network in a particular week.

***Solution***

We can have a vertex for each mailbox or e-mail address in the network, with a directed edge between two vertices if a message is sent from the tail of the edge to the head.

We use directed edge for each message sent during the week.

***Exercise***

A bipartite graph *G* is a simple graph whose vertex set can be portioned into two disjoint nonempty subsets  and  such that vertices in  may be connected to vertices in , but no vertices in  are connected to other vertices in  and no vertices in  are connected to other vertices in . For example, the graph *G* illustrated in (*i*) can be redrawn as shown in (*ii*). From the drawing in (*ii*), you can see that *G* is bipartite with mutually disjoint vertex set  and 

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| (*i*) | | (*ii*) | | |
|  | | | | |
|  | |  | |  |
|  | |  | |  |

***Solution***



1.  form a triangle, we can’t create a bipartite graph *G*.



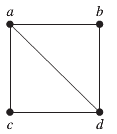
1.  form a triangle, therefore we can’t create a bipartite graph *G*.



***SOLUTION Section* 4.7 − Representing Graphs and Graph Isomorphism**

***Exercise***

Use the adjacency list to represent the given graph, then represent with an adjacency matrix

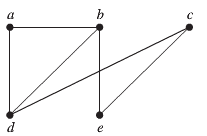


***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | ***Vertex*** | ***Adjacent Vertices*** | | *a* | *b, c* | | *b* | *a, d* | | *c* | *a, d* | | *d* | *a, b, c* | |  |

***Exercise***

Use the adjacency list to represent the given graph, then represent with an adjacency matrix

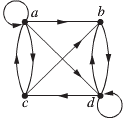


***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | ***Vertex*** | ***Adjacent Vertices*** | | *a* | *b, d* | | *b* | *a, d, e* | | *c* | *d, e* | | *d* | *a, b, c* | | *e* | *b, c* | |  |

***Exercise***

Use the adjacency list to represent the given graph, then represent with an adjacency matrix

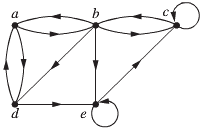


***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | ***Initial Vertex*** | ***Terminal Vertices*** | | *a* | *a, b , c, d* | | *b* | *d* | | *c* | *a, b* | | *d* | *b, c, d* | |  |

***Exercise***

Use the adjacency list to represent the given graph, then represent with an adjacency matrix



***Solution***

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| |  |  | | --- | --- | | ***Initial Vertex*** | ***Terminal Vertices*** | | *a* | *b, d* | | *b* | *a, c, d, e* | | *c* | *b, c* | | *d* | *a, e* | | *e* | *c , e* | |  |

***Exercise***

Draw a graph with the given adjacency

***Solution***

|  |  |  |
| --- | --- | --- |
|  | ***b)*** |  |

***Exercise***

Represent the given graph using adjacency matrix

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Solution***

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Exercise***

Draw an undirected graph represented by the given adjacency

***Solution***

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Exercise***

Find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

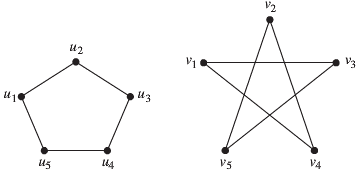
|  |  |  |
| --- | --- | --- |
|  |  |  |

***Solution***

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Exercise***

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

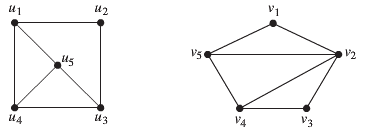


***Solution***

Both graphs have 5 vertices and 5 edges. However, each vertex in the second graph has of degree 2, whereas the first does not.

***Exercise***

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

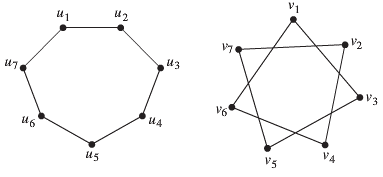


***Solution***

Both graphs have 5 vertices and 7 edges. However, the second graph has a vertex of degree 4, whereas the first does not.

***Exercise***

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



***Solution***

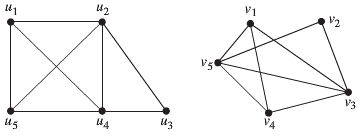
Both graphs have 7 vertices and 7 edges.

, , , , , , and 

∴ The graphs are isomorphic.

***Exercise***

Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



***Solution***

Both graphs have 5 vertices and 8 edges.

, , , , and 

∴ The graphs are isomorphic.

***SOLUTION Section* 4.8 − Connectivity**

***Exercise***

Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



***a)*** *a, e, b, c, b* ***b)*** *a, e, a, d, b, c, a* ***c)*** *e, b, a, d, b, e* ***d)*** *c, b , d, a, e, c*

***Solution***

1. This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is not a simple, since it uses an edge more than once.
2. This is not a path, since there is no edge from *c* to *a*.
3. This is not a path, since there is no edge from *b* to *a*.
4. This is a path of length 5, which is a circuit. It is simple, since no edges are repeated.

***Exercise***

Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? Which are the lengths of those that are paths?



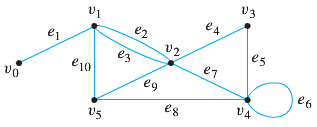
***a)*** *a, b, e, c, b* ***b)*** *a, d, a, d, a* ***c)*** *a, d, b, e, a* ***d)*** *a, b , e, c, b, d, a*

***Solution***

1. This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is simple, since no edges are repeated.
2. This is a path of length 4, which is a circuit. It is not simple, since it uses an edge more than once.
3. This is not a path, since there is no edge from *d* to *b*.
4. This is not a path, since there is no edge from *b* to *d*.

***Exercise***

Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



*a)*  *b)*  *c)* 

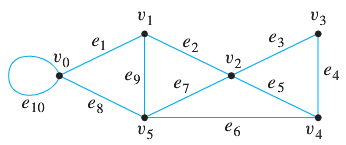
*d)*  *e)*  *f)* 

***Solution***

1. It is trail since no repeated edge. It is not a path, repeated vertex , it is not a circuit, since it ends at a vertex other than the one at which it began
2. It is a walk, it is not a trail since it has a repeated edge . It is not a circuit, since it ends at a vertex other than the one at which it began .
3. It is a closed walk, starts and ends at the same vertex . It is a trail since no repeated edge. It is not a path or a circuit, since no edge.
4. It is a path and it is circuit but not a simple circuit since it has a repeated vertex 
5. It is a closed walk, starts and ends at the same vertex . It is not a trail since it has repeated edges .
6. It is a path, it is not a circuit, since it ends at a vertex other than the one at which it began.

***Exercise***

Determine whether of the following walks are trails, paths, circuits, or simple circuits or just walk to the graph below.



*a)*  *b)*  *c)* 

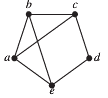
*d)*  *e)*  *f)* 

***Solution***

1. It is not a trail since it has a repeated edge . It is not a path, repeated vertex , it is not a circuit, since it ends at a vertex other than the one at which it began
2. It is a closed walk, starts and ends at the same vertex . It is a trail since no repeated edge. It is a circuit.
3. It is not a trail since it has repeated edges . It is a circuit, but not a simple circuit.
4. It is a path and it is circuit but not a simple circuit since it has a repeated vertex 
5. It is a trail since no repeated edge. It is not a circuit, since it ends at a vertex other than the one at which it began.
6. It is a path, it is not a circuit, since it ends at a vertex other than the one at which it began.

***SOLUTION Section* 4.9 − Euler and Hamilton Paths**

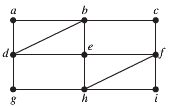
***Exercise***

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

***Solution***

The vertices *a, b, c, e* have degree 3, therefore the graph has no Euler circuit.

It is not Euler path since there is more than 2 vertices with an odd degree.

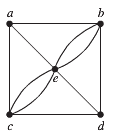
  
***Exercise***

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

***Solution***

All the vertex degree are even, so there is an Euler circuit.

Circuit form: *a, b, c, f, i, h, g, d, e, h, f, e, b, d, a*

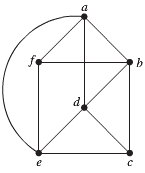
***Exercise***

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

***Solution***

The vertices *a, d* have degree 3, therefore the graph has no Euler circuit.

It has an Euler path *a, e, c, e, b, e, d, b, a, c, d*. (it has exactly 2 vertices of odd degree)

***Exercise***

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

***Solution***

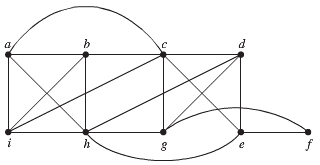
The vertices *c, f*  have degree 3, therefore the graph has no Euler circuit.

There is an Euler path between the two vertices of odd degree.

One such path is: *f, a, b, c, d, e, f, b, d, a, e, c*.

***Exercise***

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



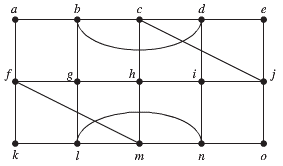
***Solution***

All the vertex degree are even, so there is an Euler circuit.

Form: *a, i, h, g, d, e, f, g, c, e, h, d, c, a, b, I, c, b, h, a*

***Exercise***

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



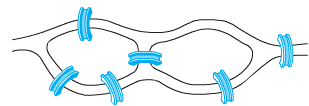
***Solution***

All the vertex degree are even, so there is an Euler circuit.

Circuit: *a, b, c, d, e, j, c, h, i, d, b, g, h, m, n, o, j, i, n, l, m, f, g, l, k, f, a*

***Exercise***

Can someone cross all the bridges shown in this map exactly once and return to the starting point?



***Solution***

Vertices *a* and *b* are the banks of the river, and vertices *c* and *d* are the islands.

Each vertex has even degree, so the graph has an Euler circuit, such as: *a, c, b, a, d, c, a*.

Therefore a walk of the type described is possible.

***Exercise***

Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

***Solution***

Yes, the path: *a, b, c, d, e, f, g, d, a*.

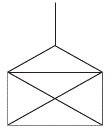
***Exercise***

Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

***Solution***

1, 2, 3, 4, 5, 6, 2, 7, 8, 9, 10, 11, 7, 12, 13, 14, 15, 16, 12, 17

***Exercise***

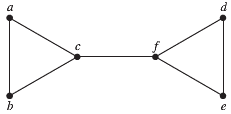
Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture

***Solution***

No

***Exercise***

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.



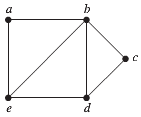
***Solution***

The graph is not a Hamilton circuit because of the cut edge {*c, f*}.

Every simple circuit must be confined to one of the 2 components obtained by deleting this edge.

***Exercise***

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

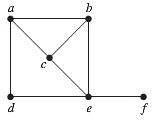


***Solution***

Hamilton circuit: *a, b, c, d, e, a*.

***Exercise***

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

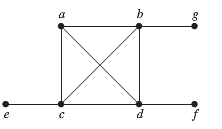


***Solution***

The graph is not a Hamilton circuit because of the cut edge {*e, f*}.

***Exercise***

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

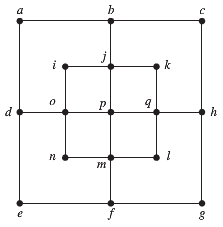


***Solution***

No Hamilton circuit exists, because once a purported circuit has reached *e* it would be nowhere to go.

***Exercise***

Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.



***Solution***

This graph has no Hamilton circuit.

If it did, then certainly the circuit would have to contain edges {*d, a*} and {*a, b*}, since these are the only edges incident to vertex *a*. By the same reasoning, the circuit would have to contain the other six edges around the outside of the figure. These 8 edges already complete a circuit, and this circuit omits the 9 vertices on the inside.

Therefore, there is no Hamilton circuit.

***Exercise***

Imagine that the drawing below is a map showing 4 cities and the distances in kilometers between them. Suppose that a salesman must travel to each city exactly once, starting and ending in city *A*. Which route from city to city will minimize the total distance that must be traveled?



***Solution***

|  |  |
| --- | --- |
| ***Route*** | ***Total Distance (Km)*** |
| *ABCDA* | 30 + 30 + 25 + 40 = 125 |
| *ABDCA* | 30 + 35 + 25 + 50 = 140 |
| *ACBDA* | 50 + 30 + 35 + 40 = 155 |
| *ACDBA* | 50 + 25 + 35 + 30 = 140 |
| *ADBCA* | 40 + 35 + 30 + 50 = 155 |
| *ADCBA* | 40 + 25 + 30 + 30 = 125 |

Thus either route *ABCDA* or *ADCBA* fives the minimum total distance of 125 km.