***Solution Section* 3.1 – Introduction to Linear Systems**

***Exercise***

Find a solution for *x, y, z* to the system of equations



***Solution***

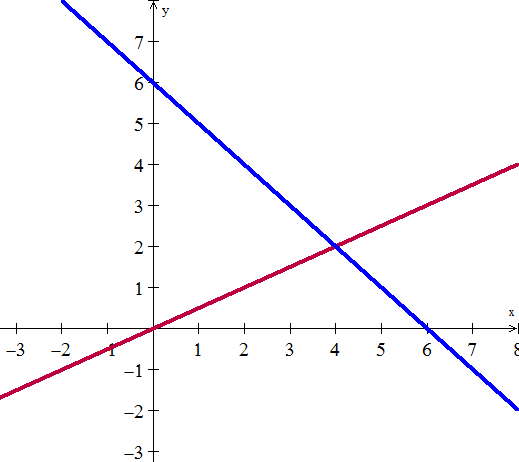




Solution: 

***Exercise***

Draw the two pictures in two planes for the equations: 

***Solution***

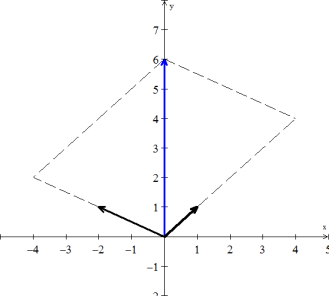
The matrix form of the 2 equations:



***Row picture*** is the 2 lines from the given equations and their intersection is the point

(4, 2) which is the solution for the system.

***Column Picture*** is the column vectors  and 





The parallelogram show how the solution vector  can be written as the linear combination of the column vectors.

***Exercise***

Normally 4 planes in 4-dimensional space meet at a \_\_\_\_\_\_\_\_. Normally 4 column vectors in 4-deimensional space can combine to produce *b*. what combinations of  produces ?

What 4 equations for  are you solving?

***Solution***

Normally 4 planes in 4-dimensional space meet at a ***point***.

The combination of the vectors producing *b* is:





The system of equations that satisfies the given vectors is:



***Exercise***

What 2 by 2 matrix *A* rotates every vector through 45° ?

The vector (1, 0) goes to . The vector (0, 1) goes to .

Those determine the matrix. Draw these particular vectors is the *xy*-plane and find *A*.

***Solution***





***Exercise***

What two vectors are obtained by rotating the plane vectors  and  by 30° (*cw*) ?

Write a matrix *A* such that for every vector *v* in the plane, *Av* is the vector obtained by rotating *v* clockwise by 30°.

Find a matrix *B* such that for every 3-dimensional vector *v*, the vector *Bv* is the reflection of *v* through the plane . 

***Solution***

Rotating the vectors by 30° (*cw*) yields:

For the vector  yields to 

And for the vector  yields to 

The desired matrix is: 

To get 1 from  is to multiply by 

The unit vector to the plane  is 











The solution: 

***Exercise***

Find a system of linear equation corresponding to the given augmented matrix



***Solution***



***Exercise***

Find a system of linear equation corresponding to the given augmented matrix



***Solution***



***Exercise***

Find the augmented matrix for the given system of linear equations.



***Solution***



***Exercise***

Find the augmented matrix for the given system of linear equations.



***Solution***



***Exercise***

Find the augmented matrix for the given system of linear equations.



***Solution***



***Solution Section* 3.2 – Gaussian Elimination**

***Exercise***

When elimination is applied to the matrix 

1. What are the first and second pivots?
2. What is the multiplier in the first step ( times row 1 is subtracted from row 2)?
3. What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?
4. What is the multiplier , subtracting 0 times row 1 from row 3?

***Solution***

1. The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.



1. The multiplier in the first step is .
2. If we reduce the entry 9 to 2, that drop of 7 in the position would force a row exchange.



1. The multiplier  is already zero because  and no needs row elimination.

***Exercise***

Use elimination to reach upper triangular matrices ***U***. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the  in equation (3).

***Solution***

For the *first* system:





The solutions are:  and the pivots are 1, -2, -2.

For the *second* system:





The three planes don’t meet. But if we change ‘3’ in the last equation to ‘-5’



 There are unique infinite many solutions!

The three planes now meet along a whole line.

***Exercise***

For which numbers *a* does the elimination break down (1) permanently (2) temporarily



Solve for *x* and *y* after fixing the second breakdown by a row change.

***Solution***

The matrix form is: 

If , the elimination brakes down temporarily.



The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If ,









If ,

, the system will fail and has no solution.

If ;

, the system has a unique solution.

***Exercise***

Find the pivots and the solution for these four equations:



***Solution***











The pivots are diagonal entries and the solution is: 

***Exercise***

Look for a matrix that has row sums 4 and 8, and column sums 2 and *s*.



The four equations are solvable only if *s* = \_\_\_\_. Then find two different matrices that have the correct row and column sums.

***Solution***







***Exercise***

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of *A* is a \_\_\_\_\_\_\_ of the first two rows. Find a third equation that can’t be solved together with  and 

***Solution***

The system is singular if row 3 of *A* is a ***linear combination*** of the first two rows.

There are many possible of a third equation that can’t be solved together with  and .



***Exercise***

Solve the linear system by Gauss-Jordan elimination.



***Solution***







***Solution***: 

***Exercise***

Solve the linear system by Gauss-Jordan elimination.



***Solution***

***Solution***: 

***Exercise***

Solve the linear system by Gauss-Jordan elimination.



***Solution***













∴ Solution: 

***Exercise***

Solve the linear system by Gauss-Jordan elimination.



***Solution***













∴ Solution: 

***Exercise***

Solve the given linear system by any method



***Solution***



***Solution***: 

***Exercise***

Solve the given linear system by any method 

***Solution***









***Solution***: 

***Exercise***

Add 3 times the second row to the first of 

***Solution***







***Exercise***

Solve the system using Gaussian elimination 

***Solution***













∴ Solution: 

***Exercise***

For what value(s) of *k*, if any, does the system  have

1. A unique solution?
2. Infinitely many solutions?
3. No solution?

***Solution***









1. Unique solution if 
2. Infinitely solution if 
3. No solution if 

***Solution Section* 3.3 – Algebra of Matrices**

***Exercise***

For the matrices:  and , when does 

***Solution***







***Exercise***

*A* is 3 by 5, *B* is 5 by 3, *C* is 5 by 1, and *D* is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

1. 
2. 
3. 
4. 
5. 
6. 
7. 

***Solution***

1. 



1. 



1. 



1. 
2. 
3. 
4. 

Matrices *B* and *C* are not the same size.

***Exercise***

What rows or columns or matrices do you multiply to find.

1. The third column of *AB*?
2. The second column of *AB*?
3. The first row of *AB*?
4. The second row of *AB*?
5. The entry in row 3, column 4 of *AB*?
6. The entry in row 2, column 3 of *AB*?

***Solution***

1. *A* (column 3 of *B*)
2. *A* (column 2 of *B*)
3. (Row 1 of *A*) *B*
4. (Row 2 of *A*) *B*
5. (Row 3 of *A*) (Column 4 of *B*)
6. (Row 2 of *A*) (Column 3 of *B*)

***Exercise***

Add *AB* to *AC* and compare with :



***Solution***

















***Exercise***

True or False

1. If  is defined then *A* is necessarily square.
2. If  and  are defined then *A* and *B* are square.
3. If  and  are defined then and  are square.
4. If , then 

***Solution***

1. True
2. False, if *A* has an order *m* by *n* and *B* *n* by *m*: 
3. True; 
4. False, if *B* is the matrix of all zeros.

***Exercise***

*a*) Find a nonzero matrix *A* such that 

*b*) Find a matrix that has  but 

***Solution***

1. A nonzero matrix *A* such that 



1. A matrix that has  but 







***Exercise***

Suppose you solve  for three special right sides *b*:



If the three solutions  are the columns of a matrix *X*, what is *A* times *X*?

***Solution***



Therefore, 

***Exercise***

Show that  is different from , when



Write down the correct rule for 

***Solution***

















 ⇒ 









***Exercise***

Find the product of the 2 matrices by rows or by columns: 

***Solution***

By rows: 

By columns: 

***Exercise***

Find the product of the 2 matrices by rows or by columns: 

***Solution***

By rows:  

By columns:  

***Exercise***

Find the product of the 2 matrices by rows or by columns: 

***Solution***

By rows: 





By columns:  

***Exercise***

Find the product of the 2 matrices by rows or by columns: 

***Solution***

By rows: 



By columns: 



***Exercise***

Given   Find 

***Solution***







***Exercise***

Given   Find 

***Solution***













***Exercise***

Given   Find 

***Solution***

***Undefined***



***Exercise***

Given   Find 

***Solution***

1. 
2. 

***Exercise***

Consider the matrices

Compute the following (where possible):

***a***)  ***b***)  ***c***)  ***d***)  ***e***)  ***g***) 

***Solution***

1. 
2. 
3. 
4. 
5. 

***g)*** 

***Solution Section* 3.4 – Inverse Matrices**

***Exercise***

Apply Gauss-Jordan method to find the inverse of this triangular “Pascal matrix”

***Triangular Pascal matrix*** 

***Solution***











* The inverse matrix  looks like *A*, except odd-numbered diagonals are multiplied by -1.

***Exercise***

If *A* is invertible and , prove that 

***Solution***

 ***Multiply by  both sides***.

 ***Multiplication is associative***





***Exercise***

If , find two matrices  such that 

***Solution***

Let 







***Exercise***

If *A* has ***row*** 1 + ***row*** 2 = ***row*** 3, show that *A* is not invertible

1. Explain why  can’t have a solution.
2. Which right sides  might allow a solution to
3. What happens to ***row*** 3 in elimination?

***Solution***

1. Let be the row vectors of *A* and *x* is a solution to .

Then .

Since 

Means 

Implies  a contradiction

1. If 

Since 





1. In the elimination matrix, the third row will be zero.

***Exercise***

True or false (with a counterexample if false and a reason if true):

1. A 4 by 4 matrix with a row of zeros is not invertible.
2. A matrix with 1’s down the main diagonal is invertible.
3. If *A* is invertible then  is invertible.
4. If *A* is invertible then  is invertible.

***Solution***

1. True, because it can have at most 3 pivots.
2. False, if the matrix:  and only has 2 pivots, thus is not invertible.
3. True, If *A* is invertible then necessarily is invertible.
4. True,  where *x* is nonzero matrix.



Since *A* is invertible, this can only be true if x was zero to begin with. Thus  must also be invertible.

***Exercise***

Do there exist 2 by 2 matrices *A* and *B* with real entries such that , where *I* is the identity matrix?

***Solution***

Let 













Therefore,  for any 2 by 2 matrices.

***Exercise***

If *B* is the inverse of , show that  is the inverse of *A*.

***Solution***

Since *B* is the inverse of  that implies: 

Show that  is the inverse of *A*











Therefore,  is the inverse of *A*.

***Exercise***

Find and check the inverses (assuming they exist) of these block matrices.



***Solution***

































***Exercise***

For which three numbers *c* is this matrix not invertible, and why not?



***Solution***

,  (zero column 2 / row 2)

,  (equal rows)

,  (equal columns)

***Exercise***

Find  and  (if they exist) by elimination.



***Solution***

























 doesn’t exist, and if we add the columns in *B*, the result is zero.

***Exercise***

Find  using the Gauss-Jordan method, which has a remarkable inverse.



***Solution***











***Exercise***

Find the inverse.

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 
2. 





1. 





1. 











1. 
2. 
3.  This matrix is ***singular***

***Exercise***

Show that *A* is not invertible for any values of the entries



***Solution***

Since the matrix *A* had zero’s on its diagonals, therefore *A* is not invertible.

***Exercise***

Prove that if *A* is an invertible matrix and *B* is row equivalent to *A*, then *B* is also invertible.

***Solution***

Since *B* is row equivalent to *A*, there exist some elementary matrices  such that . Because  and *A* are invertible, then *B* is also invertible.

***Exercise***

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying 

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 





1. 





The inverse matrix doesn’t exist

***Exercise***

Show that the inverse of  is 

***Solution***











***Solution Section* 3.5 – Determinants and Cramer’s Rule**

***Exercise***

Verify that  when: 

***Solution***









 ***√***

***Exercise***

For which value(s) of ***k*** does *A* fail to be invertible? 

***Solution***

For ***A*** to have an invertible the determinant cannot be equal to zero. To ***fail*** det(A) = 0.









***Exercise***

Without directly evaluating, show that 

***Solution***



It is equal to zero, since first row and third row are proportional.



***Exercise***

If the entries in every row of *A* add to zero, solve ***Ax*** = 0 to prove det *A* = 0. If those entries add to one, show that det (*A – I*) = 0. Does this mean det *A = I*?

***Solution***

If ***x*** = (1, 1, … , 1), then ***Ax*** = the sums of the rows of ***A***. Since every row of *A* add to zero, that implies ***Ax*** = 0. Since A has non-zero nullspace, it is not invertible and det *A* = 0. If the entries in every row of *A* sum to one, then the entries in every row of *A –* I sum to zero. A – I has a non-zero nullspace and det (*A –* I) =0. This does not mean that det *A* = I.

***Example***:  every row of *A* add to zero 

***Exercise***

Does  in general?

1. True or false if ***A*** and ***B*** are square *n* x *n* matrices?
2. True or false if ***A*** is *m* x *n* and B is *n* x *m* with ?

***Solution***

1. Matrices *A* and *B* are square matrices, then by the property:



Therefore it is true for any ***A*** and ***B*** square matrices.

1. False, example if 





***Exercise***

True or false, with a reason if true or a counterexample if false:

1. The determinant of  is 1 + det ***A***.
2. The determinant of ABC is .
3. The determinant of 4*A* is 
4. The determinant of *AB – BA* is zero. (try an example)
5. If *A* is not invertible then *AB* is not invertible.
6. The determinant of *A – B* equals to det *A* – det *B*.

***Solution***

1. ***False***, if 



1. ***True***, .
2. ***False***, in general  if *A* is *n* x *n*.
3. ***False***, 









1. False, any matrix is invertible, iff its determinant is nonzero. So det *A* = 0 which

. Therefore, AB can’t be invertible.

1. 



***Exercise***

Use row operations to show the 3 by 3 “Vandermonde determinant” is



***Solution***









 ***Multiply the main diagonal by* (*b - a*)**



***Exercise***

The inverse of a 2 by 2 matrix seems to have determinant = 1:



What is wrong with this calculation? What is the correct 

***Solution***

The  (*ad – bc*) it is part of the determinant and it is not the solution.







***Exercise***

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci’s rule . The same rule will continue for all sizes . Which Fibonacci number is ?



***Solution***



The cofactor  for  is the determinant .



The cofactor 











The actual number: .

Since  follows Fibonacci’s rule , it must be .

***Exercise***

Evaluate the determinant:

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 
2. 





1. 





1.  
2.  
3. 





1. 
2. 
3.  Since row 3 has zero.
4. 

***Exercise***

Find all the values of λ for which det(***A***) = 0: 

***Solution***





 ***Solve for λ.***



***Exercise***

Find all the values of λ for which det(***A***) = 0: 

***Solution***











***Exercise***

Prove that if a square matrix ***A*** has a column of zeros, then det(***A***) = 0

***Solution***

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:



By definition, the determinant of ***A*** using the cofactor:







***Exercise***

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:



1. Why is the first statement true? Somehow *B* doesn’t enter.
2. Show by example that equality fails (as shown) when *C* enters.
3. Show by example that the answer  is also wrong.

***Solution***

1. If we don’t pick any 0 entries, then the first two columns are picked from ***A*** and the last two rows are from D. We can’t pick any columns or rows from B, because there aren’t any left.
2. .

and 

1. Use the example from part (*b*):  

***Exercise***

Show that the value of the following determinant is independent of *θ*.



***Solution***









Therefore, the determinant is independent of *θ*.

***Exercise***

Show that the matrices  commute if and only if 

***Solution***







Iff 



 **√**

***Exercise***

Show that  for every  matrix A.

***Solution***

Let 

















***Exercise***

What is the maximum number of zeros that a  matrix can have without a zero determinant? Explain your reasoning.

***Solution***

The maximum number of zeros that a  matrix can have without a zero determinant is 12 zeros.

If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

***Exercise***

Evaluate *det* ***A***, *det* ***E***, and *det* (***AE***). Then verify that (*det* ***A***)( *det* ***E***) = *det*(***AE***)



***Solution***











 ***√***

***Exercise***

Show that  is not invertible for any values of *α, β, γ*

***Solution***













 Therefore, this matrix in not invertible.

***Exercise***

Use Cramer’s Rule with ratios  to solve *A****x*** *= b*. Also find the inverse matrix . Why is the solution ***x*** is the first part the same as column 3 of ? Which cofactors are involved in computing that column ***x***?



Find the volumes of the boxes whose edges are columns of ***A*** and then rows of .

***Solution***



The solution is: 











The solution ***x*** is the third column of  because ***b*** = (0, 0, 1) is the third column of *I*.

The volume of the boxes whose edges are columns of ***A*** = det(***A***) = 2.

Since . The box from rows of  has volume 

***Exercise***

Verify that  and determine whether the equality  holds



***Solution***

Thus, 









***Exercise***

Verify that  

***Solution***













***Exercise***

Verify that  

***Solution***











***Exercise***

Verify that  

***Solution***











***Exercise***

Solve by using Cramer’s rule

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 



***Solution***: 

1. 



***Solution***: 

1. 











Solution: 

1. 



***Solution***: 

1. 











∴ Solution: 

***Exercise***

Show that the matrix *A* is invertible for all values of θ, then find  using 



***Solution***

 ⇒ ***A*** is invertible















***Solution Section* 3.6 – Vectors in 2-Space, 3-Space, and *n*-Space**

***Exercise***

Sketch the following vectors with initial points located at the origin

1. 
2. 
3. 

***Solution***

|  |  |
| --- | --- |
|  |  |
|  |  |

***Exercise***

Find the components of the vector 

1. 
2. 
3. 

***Solution***

1.  
2.  
3.  

***Exercise***

Find the terminal point of the vector that is equivalent to ***u*** = (1, 2) and whose initial point is 

***Solution***

The terminal point: 





The terminal point: 

***Exercise***

Find the initial point of the vector that is equivalent to ***u*** = (1, 1, 3) and whose terminal point is 

***Solution***

The initial point: 



 The initial point: 

***Exercise***

Find a nonzero vector ***u*** with initial point *P*(−1, 3 , −5) such that

1. ***u*** has the same direction as ***v*** = (6, 7, −3)
2. ***u*** is oppositely directed as ***v*** = (6, 7, −3)

***Solution***

1. ***u*** has the same direction as ***v*** ⇒ ***u*** = ***v*** = (6, 7, −3)

The initial point *P*(−1, 3 , −5) then the terminal point : 

1. ***u*** is oppositely as ***v*** ⇒ ***u*** = −***v*** = (−6, −7, 3)

The initial point *P*(−1, 3 , −5) then the terminal point : 

***Exercise***

Let ***u*** = (−3, 1, 2), ***v*** = (4, 0, −8), and ***w*** = (6, −1, −4). Find the components

1. 
2. 
3. 
4. 
5. 
6. 

***Solution***

1. 
2. 
3. 
4. 
5. 





1. 





***Exercise***

Let ***u*** = (2, 1, 0, 1, −1) and ***v*** = (−2, 3, 1, 0, 2). Find scalars *a* and *b* so that *a****u*** + *b****v*** = (−8, 8, 3, −1, 7)

***Solution***







  ***Unique solution***

***Exercise***

Find all scalars  such that 

***Solution***









***Exercise***

Find the distance between the given points 

***Solution***







***Exercise***

Let *V* be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on 

1. Compute  and  for ***u*** = (0, 4), ***v*** = (1, −3), and *k* = 2.
2. Show that (0, 0) **≠ 0**.
3. Show that (−1, −1) = 0.
4. Show that  for 
5. Find two vector space axioms that fail to hold.

***Solution***

1. 



1. 





Therefore (0, 0) is not the zero vector **0** required (by Axiom).

1. 







Therefore (−1, −1) = **0** holds.

1. Let 







 holds

1. Axiom 7: 





Therefore, ; Axiom 7 fails to hold

Axiom 8: 





Therefore, ; Axiom 8 fails to hold

***Solution Section* 3.7 – Linear Dependence and Independence**

***Exercise***

Given three independent vectors . Take combinations of those vectors to produce . Write the combinations in a matrix form as 

 which is 

What is the test on a matrix **V** to see if its columns are linearly independent?

If  show that  are linearly independent.

If  show that  are linearly *dependent*.

***Solution***

The nullspace of **V** must contain only the *zero* vector. Then  is the only combination of the columns that gives **V***x* = zero vector.



If , then the matrix *M* is invertible. So if *x* is any nonzero vector we know that *Mx* is nonzero. Since ***w***’s are given as independent and *WMx* is nonzero. Since , this says that *x* is not in the nullspace of **V**. therefore;  are independent.

If , that implies 

, which means that  are linearly *dependent*.

The other way, the vector  is in that nullspace, and . Then certainly  which is the same as . So the  are dependent.

***Exercise***

Find the largest possible number of independent vectors among



***Solution***

Since , there are at most three

independent vectors among these: furthermore, applying row reduction to the matrix gives three pivots, showing that are independent.

***Exercise***

Show that are independent but  are dependent:



Solve either . The *v*’s go in the columns of ***A***.

***Solution***



This matrix has 3 pivots with rank of 3 equals to rows that implies the  are independent.

 that shows that  are dependent.

***Exercise***

Decide the dependence or independence of

1. The vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1).
2. The vectors  and  and .

***Solution***

1. These are linearly independent.  only if 
2. These are linearly dependent: 

***Exercise***

Find two independent vectors on the plane  in . Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

***Solution***

This plane is the nullspace of the matrix





The pivot is 1st column, and the rest are 3 variables.

If  . The vector is 

If  . The vector is 

If  . The vector is 

The 3 vectors  are linearly independent.

We can’t find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

***Exercise***

Determine whether the vectors are linearly dependent or linearly independent in 

|  |  |
| --- | --- |
| 1. (4, −1, 2), (−4, 10, 2) 2. (8, −1, 3), (4, 0, 1) | 1. (−3, 0, 4), (5, −1, 2), (1, 1, 3) 2. (−2, 0, 1), (3, 2, 5), (6, −1, 1), (7, 0, −2) |

***Solution***

1. The vector equation *a*(4, −1, 2) + *b*(−4, 10, 2) = (0, 0 ,0)



Therefore the system has only the trivial solution *a* = *b* = 0.

We conclude that the given set of vectors is linearly independent.

1. *a*(8, −1, 3) + *b*(4, 0, 1) = (0, 0 ,0)



Therefore, the system has only one trivial solution *a* = *b* = 0.

We conclude that the given set of vectors is linearly independent

1. The vector equation *a*(−3, 0, 4) + *b*(5, −1, 2) + *c*(1, 1, 3) = (0, 0 ,0)



Therefore the system has only the trivial solution *a* = *b* = *c* = 0.

We conclude that the given set of vectors is linearly independent.

1. The vector equation *a*(−2, 0, 1) + *b*(3, 2, 5) + *c*(6, −1, 1) + *d*(7, 0, −2) = (0, 0 ,0)



Therefore the system has nontrivial solutions 

We conclude that the given set of vectors is linearly dependent.

***Exercise***

Determine whether the vectors are linearly dependent or linearly independent in 

1. (3, 8, 7, −3), (1, 5, 3, −1), (2, −1, 2, 6), (1, 4, 0, 3)
2. (0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, −1)
3. (0, 3, −3, −6), (−2, 0, 0, −6), (0, −4, −2, −2), (0, −8, 4, −4)
4. (3, 0, −3, 6), (0, 2, 3, 1), (0, −2, −2, 0), (−2, 1, 2, 1)

***Solution***

1. 

The system has only the trivial solution and the vectors are linearly independent.

1. 





The system has only the trivial solution and the vectors are linearly independent.

1. 

The system has only the trivial solution and the vectors are linearly independent.

1. 



Therefore, the system has only one trivial solution *a* = *b* = *c* = *d* = 0.

The given set of vectors is linearly independent

***Exercise***

*a* ) Show that the three vectors  form a linearly dependent set in .

*b*) Express each vector in part (*a*) as a linear combination of the other two.

***Solution***

1. The vector equation 



The solution: 

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

1. Since  and if we let *t* = 1, then 



***Exercise***

For which real values of λ do the following vectors form a linearly dependent set in 



***Solution***





For , the determinant is zero and the vectors form a linearly dependent set.

***Exercise***

Show that if  is a linearly independent set of vectors, then so is every nonempty subset of S.

***Solution***

Let  be a nonempty subset of *S*.

If this set is linearly dependent, then there would be a nonzero solution  to . This can be expanded to a nonzero solution of  by taking all other coefficients as 0. This contradicts the linear independence of S, so the subset must be linearly independent.

***Exercise***

Show that if  is a linearly dependent set of vectors in a vector space *V*, and if  are vectors in *V* that are not in *S*, then  is also linearly dependent.

***Solution***

If *S* is linearly dependent, then there is a nonzero solution  to . Thus  is a nonzero solution to  so the set  is linearly dependent.

***Exercise***

Show that  is linearly independent and  does not lie in span , then  is a linearly independent.

***Solution***

If  are linearly dependent, there exist a nonzero solution to  with  (since  and  are linearly independent).

 which contradicts that  is not in span . Thus  is a linearly independent.

***Exercise***

By using the appropriate identities, where required, determine  are linearly dependent.

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Solution***

1. From the identity 



Therefore, the set is linearly dependent.

1. 





Therefore, the set is linearly independent.

1. 







Therefore, the set is linearly independent.

1. 







Therefore, the set is linearly dependent.

1. By using the double angle:

 are linearly dependent.

***Exercise***

 are linearly independent in  because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński’s test.

***Solution***

The Wronskian: 







 are linearly independent

***Exercise***

Use the Wronskian to show that  span a three-dimensional subspace of 

***Solution***

The Wronskian: 









Since  for all real *x* values, the vectors are linearly independent.

***Exercise***

Show by inspection that the vectors are linearly dependent.



***Solution***





***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

1. .
2. .
3. 

***Solution***

1. 



The system has only he trivial solution *a = b = c =* 0.



The system has only the trivial solution and the vectors are linearly independent

1. 

The system has only the trivial solution and the vectors are linearly independent

1.  The vectors are linearly independent

***Exercise***

Suppose that the vectors  are linearly dependent. Are the vectors , , and  also linearly dependent?

(***Hint***: Assume that , and see what the  can be.)

***Solution***

Given:  are linearly dependent, then there are scalar  such that.

Assume that 







If  and since  are linearly dependent, therefore,  are linearly dependent

***Solution Section* 3.8 – Dot product and Orthogonality**

***Exercise***

If  and , what are the smallest and largest possible values of  and ?

***Solution***













The minimum value occurs when the dot product is a small as possible, *v* and *w* are parallel, but point in opposite directions. Thus the smallest value is -15.

The maximum value occurs when the dot product is a large as possible, *v* and *w* are parallel and point in same direction. Thus the largest value is 15.

***Exercise***

If  and , what are the smallest and largest possible values of  and ?

***Solution***













The minimum value occurs when the dot product is a small as possible, *v* and *w* are parallel, but point in opposite directions. Thus the smallest value is -21.  and 

The maximum value occurs when the dot product is a large as possible, *v* and *w* are parallel and point in same direction. Thus the largest value is 21.  and 

***Exercise***

Given that  and . Similarly, and . The angle  is . Substitute into the trigonometry formula  for  to find 

***Solution***













***Exercise***

Can three vectors in the *xy* plane have  and  and ?

***Solution***

Let consider: , , 











Yes, it is.

***Exercise***

Find the norm of *v*, a unit vector that has the same direction as *v*, and a unit vector that is oppositely directed.

1. *v* = (4, −3)
2. *v* = (1, −1, 2)
3. *v* = (−2, 3, 3, −1)

***Solution***

1. 

***Same direction unit vector***: 

***Opposite direction unit vector***: 

1. 

***Same direction unit vector***:



***Opposite direction unit vector***:

******

1. 

***Same direction unit vector***:

******

***Opposite direction unit vector***:

******

***Exercise***

Evaluate the given expression with ***u*** = (2, −2, 3), ***v*** = (1, −3, 4), and ***w*** = (3, 6, −4)

1.  *b)* 

c)  *d)* 

*e)* 

***Solution***

1. 







1. 









1. 







1.  









1. 





***Exercise***

Let ***v*** = (1, 1, 2, −3, 1). Find all scalars *k* such that 

***Solution***













***Exercise***

Find 

1. *u* = (3, 1, 4), *v* = (2, 2, −4)
2. *u* = (1, 1, 4, 6), *v* = (2, −2, 3, −2)
3. *u* = (2, −1, 1, 0, −2), *v* = (1, 2, 2, 2, 1)

***Solution***

1. 





1. 





1. 





***Exercise***

Find the Euclidean distance between ***u*** and ***v***, then find the angle between them

1. *u* = (3, 3, 3), *v* = (1, 0, 4)
2. *u* = (1, 2, −3, 0), *v* = (5, 1, 2, −2)
3. *u* = (0, 1, 1, 1, 2), *v* = (2, 1, 0, −1, 3)

***Solution***

1. 













1. 













1. 











***Exercise***

Find a unit vector that has the same direction as the given vector

1. (−4, −3) *b)*  *c)* (1, 2, 3, 4, 5)

***Solution***

1. 





1. 





1. 





***Exercise***

Find a unit vector that is oppositely to the given vector

1. (−12, −5)
2. (3, −3, 3)
3. 

***Solution***

1. 





1. 







1. 





***Exercise***

Verify that the Cauchy-Schwarz inequality holds

1. *u* = (−3, 1, 0), *v* = (2, −1, 3)
2. *u* = (0, 2, 2, 1), *v* = (1, 1, 1, 1)
3. *u* = (1, 3, 5, 2, 0, 1), *v* = (0, 2, 4, 1, 3, 5)

***Solution***

1. 













 Cauchy-Schwarz inequality holds

1. 











 Cauchy-Schwarz inequality holds

1. 











 Cauchy-Schwarz inequality holds

***Exercise***

Find  and then the angle  *θ* between ***u*** and ***v*** 

***Solution***





***Exercise***

Find the norm: ,  for 

***Solution***

***Exercise***

Find all numbers *r* such that: 

***Solution***







***Exercise***

Find the distance between and 

***Solution***









***Exercise***

Given ***u*** = (1, −5, 4), ***v*** = (3, 3, 3)

1. Find 
2. Find the cosine of the angle *θ* between ***u*** and ***v***.

***Solution***

1. 
2. 

***Exercise***

Determine whether ***u*** and ***v*** are orthogonal

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 





∴ ***u*** and ***v*** are not orthogonal

1.  

∴ ***u*** and ***v*** are orthogonal

1.  

∴ ***u*** and ***v*** are orthogonal

1.  

∴ ***u*** and ***v*** are orthogonal

***Exercise***

Determine whether the vectors form an orthogonal set

1. 
2. 
3. 
4. 
5. 
6. 
7. 

***Solution***

1. 

∴ Vectors don’t form an orthogonal set

1. 

∴ Vectors don’t form an orthogonal set

1. ; These vectors are not orthogonal
2. ******; These vectors are orthogonal
3. 





∴ Vectors form an orthogonal set

1. 

∴ Vectors don’t form an orthogonal set

1. 





∴ Vectors form an orthogonal set

***Exercise***

Find a unit vector that is orthogonal to both ***u*** = (1, 0, 1) and ***v*** = (0, 1, 1)

***Solution***

Let  be the unit vector that is orthogonal to both ***u*** and ***v***.











The orthogonal vector to both ***u*** and ***v*** is , therefore the unit vector is







The possible vectors are: 

***Exercise***

*a*) Show that ***v*** = (*a, b*) and ***w*** = (−*b, a*) are orthogonal vectors.

*b*) Use the result to find two vectors that are orthogonal to ***v*** = (2, −3).

*c*) Find two unit vectors that are orthogonal to (−3, 4)

***Solution***

1.  are orthogonal vectors.
2. (2, 3) and (−2, 3).
3. 



***Exercise***

Show that if ***v*** is orthogonal to both  and , then ***v*** is orthogonal to for all scalars and .

***Solution***



 ***If v is orthogonal to ***& 





***Exercise***

Show that  is orthogonal to  if and only if 

***Solution***

Suppose that  is orthogonal to . Then











So . Therefore, .

Suppose . Then













So we can see that  is orthogonal to 

We conclude that  is orthogonal to  if and only if , as desired.

***Exercise***

Given 

1. Find 
2. Find  and then the angle  *θ* between ***u*** and ***v***.

***Solution***

1. 









1. 







***Exercise***

*a*) Show that ***v*** = (*a, b*) and ***w*** = (*−b, a*) are orthogonal vectors

*b*) Use the result in part (*a*) to find two vectors that are orthogonal to ***v*** = (2, −3)

*c*) Find two unit vectors that are orthogonal to (−3, 4)

***Solution***

1. ; 2 vectors are orthogonal vectors.
2. ***v*** = (2, −3) ⇒ ***w*** = (−3, −2) and ***w*** = (3, 2)
3. (−3, 4) ⇒ 



***Exercise***

Show that *A*(3, 0, 2), .*B*(4, 3, 0), and *C*(8, 1, −1) are vertices of a right triangle. At which vertex is the right angle?

***Solution***







The right triangle at point *B*

***Exercise***

Establish the identity: 

***Solution***

Let 























Therefore;  is true.

**2nd *method*:**











***Solution*** ***Section* 3.9 – Eigenvalues and Eigenvectors**

***Exercise***

Find the eigenvalues and eigenvectors of :



Check the trace  and the determinant  for *A* and also .

***Solution***

***For*** ***A***:







The eigenvalues of ***A*** are .

The trace of a square matrix A is the sum of the elements on the main diagonal: 2 + 2 agrees with 1+ 3. The det(***A***) = 3 agrees with the product .

The eigenvectors for ***A*** are:

:





Therefore the eigenvector 

:



Therefore the eigenvector 

***For*** :



The eigenvalues of are . ***Or*** 





:



Therefore the eigenvector 

:



Therefore the eigenvector 

***For*** :





The eigenvalues of are .

:





Therefore the eigenvector 

:





Therefore the eigenvector 

***For*** :





The eigenvalues of are .

:





Therefore the eigenvector 

:



Therefore the eigenvector 

The eigenvalues

The eigenvalues 

The eigenvalues 

***Exercise***

Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues



***Solution***



 is an eigenvectors corresponding to the eigenvalue 7.



 is an eigenvectors corresponding to the eigenvalue 0.









The eigenvalues are: 

***Exercise***

For which real numbers c does this matrix A have



1. Two real eigenvalues and eigenvectors.
2. A repeated eigenvalue with only one eigenvector
3. Two complex eigenvalues and eigenvectors.

***Solution***











1. Two real eigenvalues and eigenvectors, when 
2. A repeated eigenvalue with only one eigenvector, when 
3. Two complex eigenvalues and eigenvectors, when 

***Exercise***

Find the eigenvalues of ***A***, ***B***, ***AB***, and ***BA***:



1. The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***A*** times eigenvalues of ***B***.
2. The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***BA***.

***Solution***

Since ***A*** is a lower triangular, then 

Since ***B*** is a upper triangular, then 

1. The eigenvalues of ***AB*** are not equal to eigenvalues of ***A*** times eigenvalues of ***B***.
2. The eigenvalues of ***AB*** are equal to the eigenvalues of ***BA***.

***Exercise***

When  show that (1, 1) is an eigenvector and find both eigenvalues of



***Solution***



If 









The eigenvalues for :







The eigenvector: 

***Exercise***

The eigenvalues of *A* equal to the eigenvalues of . This is because  equals . That is true because \_\_\_\_\_. Show by an example that the eigenvectors of *A* and  are not the same.

***Solution***



Therefore, A and have the same eigenvalues.

Let consider the matrix: 



The eigenvalues are: 

For 









***Exercise***

Let . Compute the eigenvalues and eigenvectors of *A*.

***Solution***







The eigenvalues of ***A*** are:

For 





The eigenvector is: 

For 



The eigenvector is: 

***Exercise***

Let 

1. What is the characteristic polynomial for *A* (i.e. compute ?
2. Verify that 1 is an eigenvalue of *A*. What is a corresponding eigenvector?
3. What are the other eigenvalues of *A*?

***Solution***

1. 









1. 







1 is an eigenvalue of *A*.







The eigenvector for  is 

1. 

***Exercise***

For the matrix:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

1. Find the characteristic equation
2. Find the eigenvalues
3. Find the eigenvectors

***Solution***

1. 





The characteristic equation: 

1. 

The eigenvalues are 

1. 



Therefore the eigenvector 





Therefore the eigenvector 

1. For the matrix: 
2. 





⇒ The characteristic equation: 

1. 

⇒ The eigenvalues are 

1. 



Therefore the eigenvector 

1. For the matrix: 
2. 



⇒ The characteristic equation: 

1. 

The eigenvalues are 

1. For 



Therefore the eigenvector 

For 



Therefore the eigenvector 

1. For the matrix 
2. 





The characteristic equation: 

1. 
2. For  , we have: 





Therefore the eigenvector 

For  , we have: 





Therefore the eigenvector 

1. For the matrix: 
2.  



 ⇒ The characteristic equation: 

1. 
2. 

Therefore the eigenvector 

Therefore the eigenvector 

Therefore the eigenvector 

1. For the matrix: 
2. 









⇒ The characteristic equation: 

1. 
2. 



Therefore the eigenvector 



Therefore the eigenvector 





Therefore the eigenvector 

1. For the matrix: 
2. 









⇒ The characteristic equation: 

1. 



1. 



Therefore the eigenvector 

For 



Therefore the eigenvector 

For 



Therefore the eigenvector 

1. For the matrix: 
2. 







⇒ The characteristic equation: 

1. 
2. 



Therefore the eigenvector 





Therefore the eigenvector 





Therefore the eigenvector 

 Therefore the eigenvector 

1. For the matrix: 
2. 











⇒ The characteristic equation: 

1. 
2. 



Therefore the eigenvector 

 Therefore the eigenvector 







Therefore the eigenvector 





Therefore the eigenvector 

1. For the matrix 
2. 







The characteristic equation: 

1. 
2. For  , we have: 



If we let ; therefore the eigenvector 

For  , we have: 





If we let ; therefore the eigenvector 

For  , we have: 





If we let ; therefore the eigenvector 

***Exercise***

Find the eigenvalues of  for 

***Solution***

The eigenvalues are: 

The eigenvalues of  are: 

***Exercise***

Find the eigenvalues of the matrices



***Solution***

The eigenvalues for:



The eigenvalues are: 

The eigenvalues for: 

The eigenvalues for:  

The eigenvalues for:

The eigenvalues are: 

***Exercise***

Given the matrix 

1. Find the characteristic polynomial.
2. Find the eigenvalues
3. Find the bases for its eigenspaces
4. Graph the eigenspaces
5. Verify directly that , for all associated eigenvectors and eigenvalues.

***Solution***

1. 





The characteristic polynomial is 

1. 
2. For  , we have: 

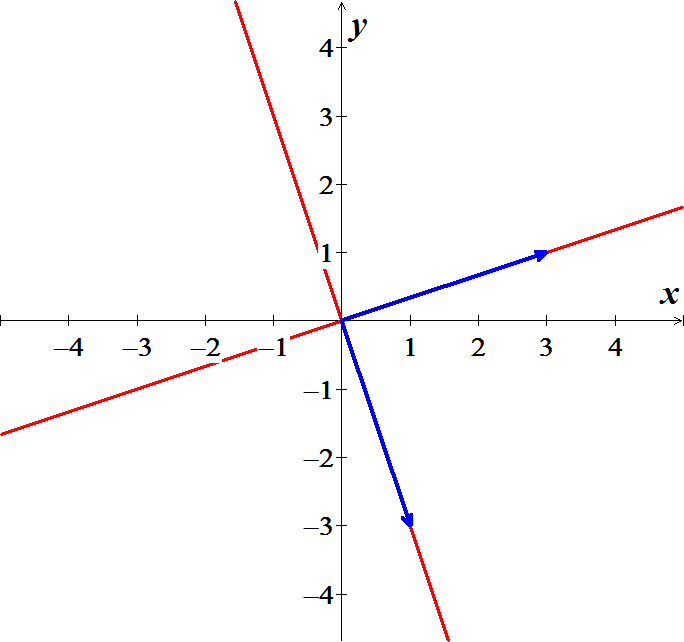
 

Therefore the eigenvector 

For  , we have: 

Therefore the eigenvector 



1. 

***√***



***√***

***Exercise***

Given the matrix 

1. Find the characteristic polynomial.
2. Find the eigenvalues
3. Find the bases for its eigenspaces
4. Graph the eigenspaces
5. Verify directly that , for all associated eigenvectors and eigenvalues.

***Solution***

1. 







The characteristic polynomial is 

1. 
2. For  , we have: 

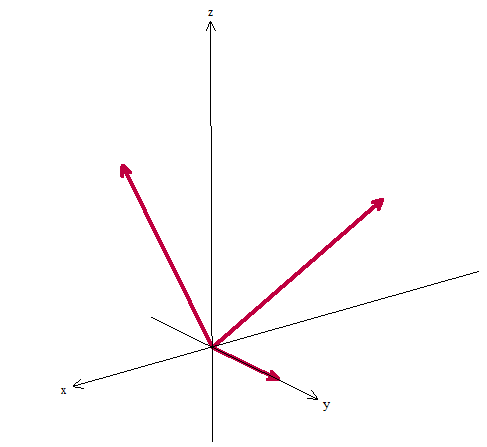


Therefore the eigenvector 

For  , we have: 



Therefore the eigenvector 



1. 

***√***



***√***



***√***

***Exercise***

Given: . Compute 

***Solution***





The eigenvalues are: 

For  , we have: 



The eigenvector 

For  , we have: 



If we let ;

The eigenvector 

For  , we have: 



The eigenvector 









