***Solution Section* 1.1 – Introduction to System of Linear Equations**

***Exercise***

Find a solution for *x, y, z* to the system of equations



***Solution***

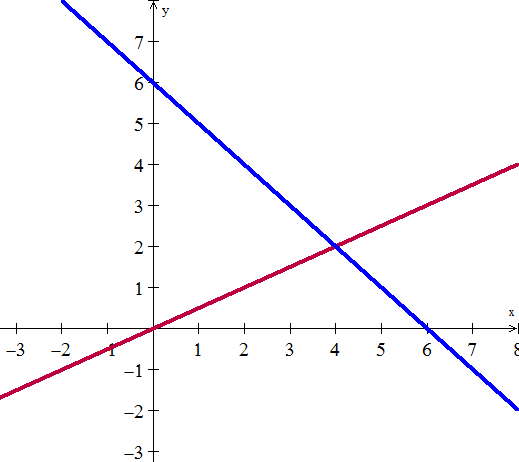




Solution: 

***Exercise***

Draw the two pictures in two planes for the equations: 

***Solution***

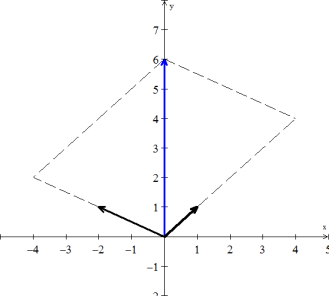
The matrix form of the 2 equations:



***Row picture*** is the 2 lines from the given equations and their intersection is the point

(4, 2) which is the solution for the system.

***Column Picture*** is the column vectors  and 





The parallelogram show how the solution vector  can be written as the linear combination of the column vectors.

***Exercise***

Normally 4 planes in 4-dimensional space meet at a \_\_\_\_\_\_\_\_. Normally 4 column vectors in 4-deimensional space can combine to produce *b*. what combinations of  produces ?

What 4 equations for  are you solving?

***Solution***

Normally 4 planes in 4-dimensional space meet at a ***point***.

The combination of the vectors producing *b* is:





The system of equations that satisfies the given vectors is:



***Exercise***

What 2 by 2 matrix *A* rotates every vector through 45° ?

The vector (1, 0) goes to . The vector (0, 1) goes to .

Those determine the matrix. Draw these particular vectors is the *xy*-plane and find *A*.

***Solution***





***Exercise***

What two vectors are obtained by rotating the plane vectors  and  by 30° (*cw*) ?

Write a matrix *A* such that for every vector *v* in the plane, *Av* is the vector obtained by rotating *v* clockwise by 30°.

Find a matrix *B* such that for every 3-dimensional vector *v*, the vector *Bv* is the reflection of *v* through the plane . 

***Solution***

Rotating the vectors by 30° (*cw*) yields:

For the vector  yields to 

And for the vector  yields to 

The desired matrix is: 

To get 1 from  is to multiply by 

The unit vector to the plane  is 















The solution: 

***Exercise***

Find a system of linear equation corresponding to the given augmented matrix



***Solution***



***Exercise***

Find a system of linear equation corresponding to the given augmented matrix



***Solution***



***Exercise***

Find the augmented matrix for the given system of linear equations.



***Solution***



***Exercise***

Find the augmented matrix for the given system of linear equations.



***Solution***



***Exercise***

Find the augmented matrix for the given system of linear equations.



***Solution***



***Solution Section* 1.2 – Gaussian Elimination**

***Exercise***

When elimination is applied to the matrix 

1. What are the first and second pivots?
2. What is the multiplier in the first step ( times row 1 is subtracted from row 2)?
3. What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3?
4. What is the multiplier , subtracting 0 times row 1 from row 3?

***Solution***

1. The first pivot is 3 and when 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7.



1. The multiplier in the first step is .
2. If we reduce the entry 9 to 2, that drop of 7 in the position would force a row exchange.



1. The multiplier  is already zero because  and no needs row elimination.

***Exercise***

Use elimination to reach upper triangular matrices ***U***. Solve by back substitution or explain why this impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the  in equation (3).

***Solution***

For the *first* system:





The solutions are:  and the pivots are 1, -2, -2.

For the *second* system:





The three planes don’t meet. But if we change ‘3’ in the last equation to ‘-5’



 There are unique infinite many solutions!

The three planes now meet along a whole line.

***Exercise***

For which numbers *a* does the elimination break down (1) permanently (2) temporarily



Solve for *x* and *y* after fixing the second breakdown by a row change.

***Solution***

The matrix form is: 

If , the elimination brakes down temporarily.



The system is in upper triangular form and entry row 2 column 2 is not equal to zero, therefore the system has a solution.

If ,









If ,

, the system will fail and has no solution.

If ;

, the system has a unique solution.

***Exercise***

Find the pivots and the solution for these four equations:



***Solution***











The pivots are diagonal entries and the solution is: 

***Exercise***

Look for a matrix that has row sums 4 and 8, and column sums 2 and *s*.



The four equations are solvable only if *s* = \_\_\_\_. Then find two different matrices that have the correct row and column sums.

***Solution***







***Exercise***

Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of *A* is a \_\_\_\_\_\_\_ of the first two rows. Find a third equation that can’t be solved together with  and 

***Solution***

The system is singular if row 3 of *A* is a ***linear combination*** of the first two rows.

There are many possible of a third equation that can’t be solved together with  and .



***Exercise***

Solve the linear system by Gauss-Jordan elimination.



***Solution***







***Solution***: 

***Exercise***

Solve the linear system by Gauss-Jordan elimination.



***Solution***

***Solution***: 

***Exercise***

Solve the given linear system by any method



***Solution***

***Solution***: 

***Exercise***

Solve the given linear system by any method



***Solution***

***Solution***: 

***Solution Section* 1.3 – Matrices and Matrix operations**

***Exercise***

For the matrices:  and , when does 

***Solution***















***Exercise***

Find a combination  that gives the zero vector:



Those vectors are independent or dependent?

The vectors lie in a \_\_\_\_\_\_.

The matrix W with those columns is not invertible.

***Solution***

; Therefore those vectors are dependent

The vectors lie in a plane

***Exercise***

The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations . Find a combination of left sides that gives zero. What combination of must be zero?

***Solution***

The 5 by 5 centered difference matrix is



The five equations  are:



Observe that the sum of the first





***Exercise***

A direct graph starts with *n* nodes. There are possible edges, each edge leaves one of the *n* nodes and enters one of the *n* nodes (possibly itself). The *n* by *n* adjacency matrix has  when edge leaves node *i* and enter node *j*; if no edge then . Here are directed graphs and their adjacency matrices:

The *i*, *j* entry of is .

Why does that sum count the two-step paths from *i* to any node to *j*?

The *i*, *j* entry of  counts *k*-steps paths:



List all 3-step paths between each pair of nodes and compare with . When  has ***no zeros***, that number *k* is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

***Solution***

The number  will be “**1**” if there is an edge from node *i* to *k* and an edge from *k* to *j*.

This is a 2-step path. The number  will be “**0**” if either of those edge (from node *i* to *k* and from *k* to *j*) is missing.

The sum of  is the number of 2-step paths leaving *i* and entering *j*.

Matrix multiplication is right for this count.

The 3-step paths are counted by ; we look at paths to node 2:



The  contain Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ….

Fibonacci’s rule  show up in 



There are ***13 six-step*** paths from node one to node 1.

***Exercise***

*A* is 3 by 5, B is 5 by 3, *C* is 5 by 1, and *D* is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

1. 
2. 
3. 
4. 
5. 
6. 
7. 

***Solution***

1. 



1. 



1. 



1. 
2. 
3. 
4. 

Matrices B and C are not the same size.

***Exercise***

What rows or columns or matrices do you multiply to find.

1. The third column of *AB*?
2. The second column of *AB*?
3. The first row of *AB*?
4. The second row of *AB*?
5. The entry in row 3, column 4 of *AB*?
6. The entry in row 2, column 3 of *AB*?

***Solution***

1. *A* (column 3 of *B*)
2. *A* (column 2 of *B*)
3. (Row 1 of *A*) *B*
4. (Row 2 of *A*) *B*
5. (Row 3 of *A*) (Column 4 of *B*)
6. (Row 2 of *A*) (Column 3 of *B*)

***Exercise***

Add *AB* to *AC* and compare with :



***Solution***

















***Exercise***

True or False

1. If  is defined then *A* is necessarily square.
2. If  and  are defined then *A* and *B* are square.
3. If  and  are defined then and  are square.
4. If , then 

***Solution***

1. True
2. False, if *A* has an order *m* by *n* and *B* *n* by *m*: 
3. True; 
4. False, if *B* is the matrix of all zeros.

***Exercise***

*a*) Find a nonzero matrix *A* such that 

*b*) Find a matrix that has  but 

***Solution***

1. A nonzero matrix *A* such that 



1. A matrix that has  but 







***Exercise***

Suppose you solve  for three special right sides *b*:



If the three solutions  are the columns of a matrix *X*, what is *A* times *X*?

***Solution***



Therefore, 

***Exercise***

Show that  is different from , when



Write down the correct rule for 

***Solution***





























***Exercise***

Find the product of the 2 matrices by rows or by columns: 

***Solution***

By rows: 

By columns: 

***Exercise***

Find the product of the 2 matrices by rows or by columns: 

***Solution***

By rows: 



By columns: 



***Exercise***

Find the product of the 2 matrices by rows or by columns: 

***Solution***

By rows: 





By columns: 



***Exercise***

Find the product of the 2 matrices by rows or by columns: 

***Solution***

By rows: 



By columns: 



***Solution Section* 1.4 – Inverse Matrices - Finding **

***Exercise***

Apply Gauss-Jordan method to find the inverse of this triangular “Pascal matrix”

***Triangular Pascal matrix*** 

***Solution***











* The inverse matrix  looks like *A*, except odd-numbered diagonals are multiplied by -1.

***Exercise***

If *A* is invertible and , prove that 

***Solution***

 ***Multiply by  both sides***.

 ***Multiplication is associative***





***Exercise***

If , find two matrices  such that 

***Solution***

Let 







***Exercise***

If *A* has ***row*** 1 + ***row*** 2 = ***row*** 3, show that *A* is not invertible

1. Explain why  can’t have a solution.
2. Which right sides  might allow a solution to
3. What happens to ***row*** 3 in elimination?

***Solution***

1. Let be the row vectors of *A* and *x* is a solution to .

Then .

Since 

Means 

Implies  a contradiction

1. If 

Since 





1. In the elimination matrix, the third row will be zero.

***Exercise***

True or false (with a counterexample if false and a reason if true):

1. A 4 by 4 matrix with a row of zeros is not invertible.
2. A matrix with 1’s down the main diagonal is invertible.
3. If *A* is invertible then  is invertible.
4. If *A* is invertible then  is invertible.

***Solution***

1. True, because it can have at most 3 pivots.
2. False, if the matrix:  and only has 2 pivots, thus is not invertible.
3. True, If *A* is invertible then necessarily is invertible.
4. True,  where *x* is nonzero matrix.



Since A is invertible, this can only be true if x was zero to begin with. Thus  must also be invertible.

***Exercise***

Do there exist 2 by 2 matrices *A* and *B* with real entries such that , where *I* is the identity matrix?

***Solution***

Let 













Therefore,  for any 2 by 2 matrices.

***Exercise***

If *B* is the inverse of , show that  is the inverse of *A*.

***Solution***

Since *B* is the inverse of  that implies: 

Show that  is the inverse of *A*











Therefore,  is the inverse of *A*.

***Exercise***

Find and check the inverses (assuming they exist) of these block matrices.



***Solution***

































***Exercise***

For which three numbers *c* is this matrix not invertible, and why not?



***Solution***

,  (zero column 2 / row 2)

,  (equal rows)

,  (equal columns)

***Exercise***

Find  and  (if they exist) by elimination.



***Solution***

























 doesn’t exist, and if we add the columns in *B*, the result is zero.

***Exercise***

Find  using the Gauss-Jordan method, which has a remarkable inverse.



***Solution***











***Exercise***

Find the inverse. 

***Solution***



***Exercise***

Find the inverse. 

***Solution***



***Exercise***

Find the inverse. 

***Solution***



***Exercise***

Find the inverse. 

***Solution***



This matrix is singular

***Exercise***

Show that *A* is not invertible for any values of the entries



***Solution***

Since the matrix *A* had zero’s on its diagonals, therefore *A* is not invertible.

***Exercise***

Prove that if *A* is an invertible matrix and *B* is row equivalent to *A*, then *B* is also invertible.

***Solution***

***Solution Section* 1.5 – Diagonal, Triangular, and Symmetric Matrices**

***Exercise***

Solve  to find***c***. Then solve  to find ***x***. What was *A*?



***Solution***













***Exercise***

Find *L* and *U* for the symmetric matrix



Find four conditions on *a, b, c, d* to get  with four pivots

***Solution***



***Exercise***

Determine whether the given matrix is invertible



***Solution***

The matrix is a diagonal matrix with nonzero entries on the diagonal, so it is invertible.



***Exercise***

Find  by inspection 

***Solution***







***Exercise***

Find  by inspection 

***Solution***







***Exercise***

Find  by inspection 

***Solution***







***Exercise***

Decide whether the given matrix is symmetric 

***Solution***

Not symmetric, since 

***Exercise***

Decide whether the given matrix is symmetric 

***Solution***

Symmetric

***Exercise***

Decide whether the given matrix is symmetric 

***Solution***

Not symmetric, since 

***Exercise***

Find all values of the unknown constant(s) in order for *A* to be symmetric



***Solution***

***Exercise***

Find a diagonal matrix *A* that satisfies the given condition 

***Solution***





***Exercise***

Let *A* be an  symmetric matrix

1. Show that  is symmetric
2. Show that  is symmetric

***Solution***

1. The property of the transpose states that 





 ***A is symmetric***



1. 

 ***A and I are symmetric***

 ***Symmetric***

***Exercise***

Prove if , then *A* is symmetric and 

***Solution***

If , then









So *A* is symmetric.

Since 



***Exercise***

A square matrix *A* is called ***skew-symmetric*** if . Prove

1. If *A* is an invertible skew-symmetric matrix, then  is skew-symmetric.
2. If *A* and *B* are skew-symmetric matrices, then so are  for any scalar *k*.
3. Every square matrix *A* can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.



***Solution***

1. 

 ***skew-symmetric***



∴  is also skew-symmetric

1. Let *A* and *B* are skew-symmetric matrices









1. We need to prove from the hint that is symmetric and  is skew-symmetric



 *Thus*  ***is symmetric***





 *Thus*  ***is skew-symmetric***

***Exercise***

Suppose *R* is rectangular (*m* by *n*) and *A* is symmetric (*m* by *m*)

1. Transpose  to show its symmetric
2. Show why  has no negative numbers on its diagonal.

***Solution***

1. 







1. 

= ***Product of the diagonal entry by itself.***

= length squared of column j.

***Exercise***

If *L* is a lower-triangular matrix, then  is \_\_\_\_\_\_\_Triangular

***Solution***

 is ***upper*** triangular.

 is a lower-triangular because *L* is.

The transpose carries the lower-triangular matrices to the upper-triangular (and vice versa).

***Exercise***

True or False

1. The block matrix  is automatically symmetric
2. If *A* and *B* are symmetric then their product is symmetric
3. If *A* is not symmetric then  is not symmetric
4. When *A, B, C* are symmetric, the transpose of *ABC* is *CBA*.

***Solution***

1. False: 
2. False 
3. True by definition.
4. True  Since 

***Exercise***

Find 2 by 2 symmetric matrices  with these properties

1.  is not invertible
2. *A* is invertible but cannot be factored into *LU* (row exchanges needed)
3. *A* can be factored into  but not into  (because of negative *D*)

***Solution***

1. 
2.  only need a *zero* in the diagonal.
3. 









***Exercise***

A group of matrices includes *AB* and  if it includes *A* and *B* . “Products and inverses stay in the group.” Which of these sets are groups?

Lower triangular matrices *L* with 1’s on the diagonal, symmetric matrices *S*, positive matrices *M*, diagonal invertible matrices *D*, permutation matrices *P*, matrices with . ***Invent two more matrix groups***.

***Solution***

The lower triangular matrices *L* with 1’s on the diagonal form a group.

Clearly the product of two is a third. The Gauss-Jordan method shows that the inverse of one is another.

The symmetric matrices don’t form a group. An example of the 2 symmetric matrices *A* and *B* whose product is not symmetric



The positive matrices do not form a group.

, the inverse is not symmetric.

The diagonal invertible matrices form a group.

The permutation matrices form a group.

The matrices with  form a group. If *A* and *B* are two matrices, then so are *AB* and , as





There are many more matrix groups. For example, given two, the block matrices  form a third as *A* ranges over the first group and *B* ranges over the second.

Another example is the set of all products *cP* where *c* is a nonzero scalar and *P* is a permutation matrix of given size.

***Exercise***

Write  as the product *EH* of an elementary row operation matrix *E* and a symmetric matrix *H*.

***Solution***







An elementary row operation matrix has the form 

The inverse is: 



Since matrix *H* is symmetric, therefore:





***Exercise***

When is the product of two symmetric matrices symmetric? Explain your answer.

***Solution***

 is symmetric iff 



 ***A and B are symmetric***



 is symmetric iff *A* and *B* commute

***Exercise***

Express  in terms of  and 

***Solution***





***Solution Section* 1.6 – The Properties of Determinants**

***Exercise***

Verify that  when: 

***Solution***











***Exercise***

For which value(s) of ***k*** does *A* fail to be invertible? 

***Solution***

For ***A*** to have an invertible the determinant cannot be equal to zero. To ***fail*** det(A) = 0.









***Exercise***

Without directly evaluating, show that 

***Solution***



It is equal to zero, since first row and third row are proportional.



***Exercise***

If the entries in every row of *A* add to zero, solve ***Ax*** = 0 to prove det *A* = 0. If those entries add to one, show that det (*A – I*) = 0. Does this mean det *A = I*?

***Solution***

If ***x*** = (1, 1, … , 1), then ***Ax*** = the sums of the rows of ***A***. Since every row of *A* add to zero, that implies ***Ax*** = 0. Since A has non-zero nullspace, it is not invertible and det *A* = 0. If the entries in every row of *A* sum to one, then the entries in every row of *A –* I sum to zero. A – I has a non-zero nullspace and det (*A –* I) =0. This does not mean that det *A* = I.

***Example***:  every row of *A* add to zero 

***Exercise***

Does  in general?

1. True or false if ***A*** and ***B*** are square *n* x *n* matrices?
2. True or false if ***A*** is *m* x *n* and B is *n* x *m* with ?

***Solution***

1. Matrices *A* and *B* are square matrices, then by the property:



Therefore it is true for any ***A*** and ***B*** square matrices.

1. False, example if 





***Exercise***

True or false, with a reason if true or a counterexample if false:

1. The determinant of  is 1 + det ***A***.
2. The determinant of ABC is .
3. The determinant of 4*A* is 
4. The determinant of *AB – BA* is zero. (try an example)
5. If *A* is not invertible then *AB* is not invertible.
6. The determinant of *A – B* equals to det *A* – det *B*.

***Solution***

1. ***False***, if 



1. ***True***, .
2. ***False***, in general  if A is *n* x *n*.
3. ***False***, 









1. False, any matrix is invertible, iff its determinant is nonzero. So det *A* = 0 which

. Therefore, AB can’t be invertible.

1. 



***Exercise***

Use row operations to show the 3 by 3 “Vandermonde determinant” is



***Solution***







 ***Multiply the main diagonal by* (*b - a*)**



***Exercise***

The inverse of a 2 by 2 matrix seems to have determinant = 1:



What is wrong with this calculation? What is the correct 

***Solution***

The  (*ad – bc*) it is part of the determinant and it is not the solution.







***Exercise***

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci’s rule . The same rule will continue for all sizes . Which Fibonacci number is ?



***Solution***



The cofactor  for  is the determinant .



The cofactor 











The actual number: .

Since  follows Fibonacci’s rule , it must be .

***Exercise***

Evaluate the determinant: 

***Solution***



***Exercise***

Evaluate the determinant: 

***Solution***







***Exercise***

Evaluate the determinant: 

***Solution***





***Exercise***

Evaluate the determinant: 

***Solution***







***Exercise***

Find all the values of λ for which det(***A***) = 0: 

***Solution***





 ***Solve for λ.***



***Exercise***

Find all the values of λ for which det(***A***) = 0: 

***Solution***











***Exercise***

Prove that if a square matrix ***A*** has a column of zeros, then det(***A***) = 0

***Solution***

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:



By definition, the determinant of ***A*** using the cofactor:







***Exercise***

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:



1. Why is the first statement true? Somehow *B* doesn’t enter.
2. Show by example that equality fails (as shown) when *C* enters.
3. Show by example that the answer  is also wrong.

***Solution***

1. If we don’t pick any 0 entries, then the first two columns are picked from ***A*** and the last two rows are from D. We can’t pick any columns or rows from B, because there aren’t any left.
2. .

and 

1. Use the example from part (*b*): 



***Exercise***

Show that the value of the following determinant is independent of *θ*.



***Solution***









***Exercise***

Show that the matrices  commute if and only if 

***Solution***







Iff 



 **√**

***Exercise***

Show that  for every  matrix A.

***Solution***

Let 

















***Exercise***

What is the maximum number of zeros that a  matrix can have without a zero determinant? Explain your reasoning.

***Solution***

The maximum number of zeros that a  matrix can have without a zero determinant is 12 zeros.

If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

***Solution Section* 1.7 – Properties of Determinants: Cramer’s Rule**

***Exercise***

Use Cramer’s Rule with ratios  to solve *A****x*** *= b*. Also find the inverse matrix . Why is the solution ***x*** is the first part the same as column 3 of ? Which cofactors are involved in computing that column ***x***?



Find the volumes of the boxes whose edges are columns of ***A*** and then rows of .

***Solution***





The solution is: 











The solution ***x*** is the third column of  because ***b*** = (0, 0, 1) is the third column of *I*.

The volume of the boxes whose edges are columns of ***A*** = det(***A***) = 2.

Since . The box from rows of  has volume 

***Exercise***

Verify that  and determine whether the equality  holds



***Solution***

Thus, 









***Exercise***

Verify that  

***Solution***













***Exercise***

Verify that  

***Solution***











***Exercise***

Verify that  

***Solution***











***Exercise***

Solve by using Cramer’s rule 

***Solution***









***Solution***: 

***Exercise***

Solve by using Cramer’s rule 

***Solution***











***Solution***: 

***Exercise***

Solve by using Cramer’s rule 

***Solution***





***Solution***: 

***Exercise***

Solve by using Cramer’s rule 

***Solution***











***Solution***: 

***Exercise***

Show that the matrix *A* is invertible for all values of θ, then find  using 



***Solution***

 ⇒ ***A*** is invertible













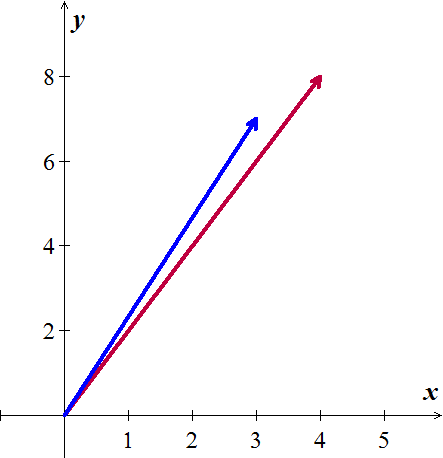


***Solution Section* 1.8 – Vectors in 2-Space, 3-Space, and n-Space**

***Exercise***

Sketch the following vectors with initial points located at the origin 

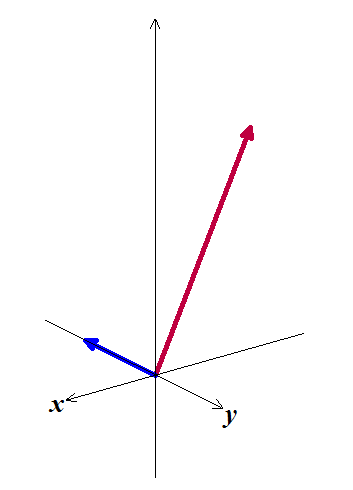
***Solution***



***Exercise***

Sketch the following vectors with initial points located at the origin 

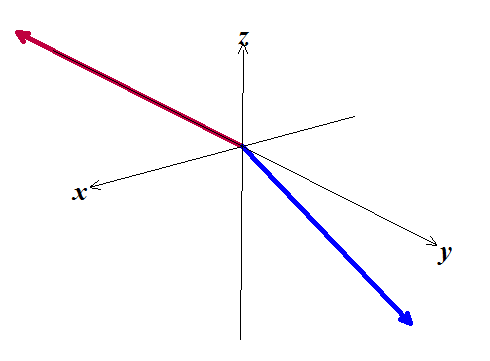
***Solution***



***Exercise***

Sketch the following vectors with initial points located at the origin 

***Solution***



***Exercise***

Find the components of the vector  

***Solution***





***Exercise***

Find the components of the vector  

***Solution***





***Exercise***

Find the components of the vector  

***Solution***





***Exercise***

Find the terminal point of the vector that is equivalent to ***u*** = (1, 2) and whose initial point is 

***Solution***

The terminal point: 





The terminal point: 

***Exercise***

Find the initial point of the vector that is equivalent to ***u*** = (1, 1, 3) and whose terminal point is 

***Solution***

The initial point: 





The initial point: 

***Exercise***

Find a nonzero vector ***u*** with initial point *P*(−1, 3 , −5) such that

1. ***u*** has the same direction as ***v*** = (6, 7, −3)
2. ***u*** is oppositely directed as ***v*** = (6, 7, −3)

***Solution***

1. ***u*** has the same direction as ***v*** ⇒ ***u*** = ***v*** = (6, 7, −3)

The initial point *P*(−1, 3 , −5) then the terminal point : 

1. ***u*** is oppositely as ***v*** ⇒ ***u*** = −***v*** = (−6, −7, 3)

The initial point *P*(−1, 3 , −5) then the terminal point : 

***Exercise***

Let ***u*** = (−3, 1, 2), ***v*** = (4, 0, −8), and ***w*** = (6, −1, −4). Find the components

1. 
2. 
3. 
4. 
5. 
6. 

***Solution***

1. 
2. 
3. 
4. 
5. 





1. 





***Exercise***

Let ***u*** = (2, 1, 0, 1, −1) and ***v*** = (−2, 3, 1, 0, 2). Find scalars *a* and *b* so that *a****u*** + *b****v*** = (−8, 8, 3, −1, 7)

***Solution***







  ***Unique solution***

***Exercise***

Find all scalars  such that 

***Solution***









***Solution Section* 1.9 – Norm, Dot product, and distance in *Rn***

***Exercise***

If  and , what are the smallest and largest possible values of  and ?

***Solution***













The minimum value occurs when the dot product is a small as possible, *v* and *w* are parallel, but point in opposite directions. Thus the smallest value is -15.

The maximum value occurs when the dot product is a large as possible, *v* and *w* are parallel and point in same direction. Thus the largest value is 15.

***Exercise***

If  and , what are the smallest and largest possible values of  and ?

***Solution***













The minimum value occurs when the dot product is a small as possible, *v* and *w* are parallel, but point in opposite directions. Thus the smallest value is -21.  and 

The maximum value occurs when the dot product is a large as possible, *v* and *w* are parallel and point in same direction. Thus the largest value is 21.  and 

***Exercise***

Given that  and . Similarly, and . The angle  is . Substitute into the trigonometry formula  for  to find 

***Solution***













***Exercise***

Can three vectors in the *xy* plane have  and  and ?

***Solution***

Let consider: , , 











Yes, it is.

***Exercise***

Find the norm of *v*, a unit vector that has the same direction as *v*, and a unit vector that is oppositely directed.

1. *v* = (4, −3)
2. *v* = (1, −1, 2)
3. *v* = (−2, 3, 3, −1)

***Solution***

1. 

***Same direction unit vector***: 

***Opposite direction unit vector***: 

1. 

***Same direction unit vector***:



***Opposite direction unit vector***:



1. 

***Same direction unit vector***:



***Opposite direction unit vector***:



***Exercise***

Evaluate the given expression with ***u*** = (2, −2, 3), ***v*** = (1, −3, 4), and ***w*** = (3, 6, −4)

1. 
2. 
3. 
4. 
5. 

***Solution***

1. 







1. 









1. 







1.  









1. 





***Exercise***

Let ***v*** = (1, 1, 2, −3, 1). Find all scalars *k* such that 

***Solution***













***Exercise***

Find 

1. *u* = (3, 1, 4), *v* = (2, 2, −4)
2. *u* = (1, 1, 4, 6), *v* = (2, −2, 3, −2)
3. *u* = (2, −1, 1, 0, −2), *v* = (1, 2, 2, 2, 1)

***Solution***

1. 





1. 





1. 





***Exercise***

Find the Euclidean distance between ***u*** and ***v***, then find the angle between them

1. *u* = (3, 3, 3), *v* = (1, 0, 4)
2. *u* = (1, 2, −3, 0), *v* = (5, 1, 2, −2)
3. *u* = (0, 1, 1, 1, 2), *v* = (2, 1, 0, −1, 3)

***Solution***

1. 













1. 













1. 











***Exercise***

Find a unit vector that has the same direction as the given vector

1. (−4, −3)
2. 
3. (1, 2, 3, 4, 5)

***Solution***

1. 





1. 





1. 





***Exercise***

Find a unit vector that is oppositely to the given vector

1. (−12, −5)
2. (3, −3, 3)
3. 

***Solution***

1. 





1. 







1. 





***Exercise***

Verify that the Cauchy-Schwarz inequality holds

1. *u* = (−3, 1, 0), *v* = (2, −1, 3)
2. *u* = (0, 2, 2, 1), *v* = (1, 1, 1, 1)
3. *u* = (1, 3, 5, 2, 0, 1), *v* = (0, 2, 4, 1, 3, 5)

***Solution***

1. 













 Cauchy-Schwarz inequality holds

1. 











 Cauchy-Schwarz inequality holds

1. 











 Cauchy-Schwarz inequality holds

***Solution Section* 1.10 – Orthogonality**

***Exercise***

Determine whether ***u*** and ***v*** are orthogonal 

***Solution***







∴ ***u*** and ***v*** are not orthogonal

***Exercise***

Determine whether ***u*** and ***v*** are orthogonal 

***Solution***





∴ ***u*** and ***v*** are orthogonal

***Exercise***

Determine whether ***u*** and ***v*** are orthogonal 

***Solution***





∴ ***u*** and ***v*** are orthogonal

***Exercise***

Determine whether ***u*** and ***v*** are orthogonal 

***Solution***





∴ ***u*** and ***v*** are orthogonal

***Exercise***

Determine whether the vectors form an orthogonal set 

***Solution***





∴ Vectors don’t form an orthogonal set

***Exercise***

Determine whether the vectors form an orthogonal set 

***Solution***





∴ Vectors don’t form an orthogonal set

***Exercise***

Determine whether the vectors form an orthogonal set 

***Solution***







∴ Vectors form an orthogonal set

***Exercise***

Determine whether the vectors form an orthogonal set 

***Solution***



∴ Vectors don’t form an orthogonal set

***Exercise***

Determine whether the vectors form an orthogonal set 

***Solution***







∴ Vectors form an orthogonal set

***Exercise***

Find a unit vector that is orthogonal to both ***u*** = (1, 0, 1) and ***v*** = (0, 1, 1)

***Solution***

Let  be the unit vector that is orthogonal to both ***u*** and ***v***.











The orthogonal vector to both ***u*** and ***v*** is , therefore the unit vector is







The possible vectors are: 

***Exercise***

*a*) Show that ***v*** = (*a, b*) and ***w*** = (−*b, a*) are orthogonal vectors.

*b*) Use the result to find two vectors that are orthogonal to ***v*** = (2, −3).

*c*) Find two unit vectors that are orthogonal to (−3, 4)

***Solution***

1.  are orthogonal vectors.
2. (2, 3) and (−2, 3).
3. 



***Exercise***

Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal to ***a***.



***Solution***











***Exercise***

Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal to ***a***.



***Solution***









***Exercise***

Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal to ***a***.



***Solution***











***Exercise***

Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal to ***a***.



***Solution***











***Exercise***

Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal to ***a***.



***Solution***













***Exercise***

Find the vector component of ***u*** along ***a*** and the vector component of ***u*** orthogonal to ***a***.



***Solution***













***Exercise***

Project the vector ***v*** onto the line through ***a***, Check that ***e*** is perpendicular to ***a***:



***Solution***



















***e*** is perpendicular to ***a***

***Exercise***

Project the vector ***v*** onto the line through ***a***, Check that ***e*** is perpendicular to ***a***:



***Solution***

















***e*** is perpendicular to ***a***

***Exercise***

Project the vector ***v*** onto the line through ***a***, Check that ***e*** is perpendicular to ***a***:



***Solution***



















***e*** is perpendicular to ***a***

***Exercise***

Draw the projection of ***b*** onto ***a*** and also compute it 

***Solution***









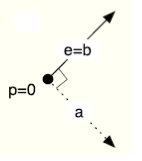




***Exercise***

Draw the projection of ***b*** onto ***a*** and also compute it 

***Solution***













***Exercise***

Find the projection matrix  onto the line through 

***Solution***





***Exercise***

Show that if ***v*** is orthogonal to both  and , then ***v*** is orthogonal to for all scalars and .

***Solution***



 ***If v is orthogonal to*** & 







***Exercise***

1. Project the vector ***v*** = (3, 4, 4) onto the line through ***a*** = (2, 2, 1) and then onto the plane that also contains .
2. Check that the first error vector ***v – p*** is perpendicular to ***a***, and the second error vector ***v – p***\* is also perpendicular to ***a***\*.

***Solution***

1. 







The plane contains the vectors ***a*** and ***a***\* and is the column space of ***A***.









1. The error vector: 

.

Therefore ***e*** is perpendicular to ***a***.



The error vector: 

.

Therefore ***e\**** is perpendicular to ***a\****.

***Exercise***

Compute the projection matrices  onto the lines through  and . Multiply those projection matrices and explain why their product  is what it is.

Project  onto the lines , , and also onto .Add up the three projections .

***Solution***

For 







For 













This because are perpendicular.

For 















The reason is that  is perpendicular to .

Hence, when you compute the three projections of a vector and add them up you get back to the vector you start with.

***Exercise***

If  show that . When *P* projects onto the column space of *A*, *I – P* projects onto the \_\_\_\_.

***Solution***







  ***By definition***







When *P* projects onto the column space of *A*, then *I – P* projects onto the left nullspace.

Because ; if *P****v*** is in the column space of *A*, then is a vector perpendicular to C(A).

***Exercise***

What linear combination of  and  is closest to ?

***Solution***



So this ***v*** is actually in the span of the two given vectors.

***Solution Section* 1.11 – Cross Product**

***Exercise***

Prove when the cross product  is perpendicular to ***u***, then 

***Solution***

Let  and 









***Exercise***

Find ***u*** × ***v***, where  and show that ***u*** × ***v***  is perpendicular to ***u*** and to ***v***.

***Solution***



= (2, −7, −6)



= 2 − 14 + 12





= 6 − 0 − 6



***u*** × ***v***  is orthogonal to both ***u*** and ***v***.

***Exercise***

Given  Compute the vectors

1. ***u*** × ***v***
2. ***v*** × ***w***
3. ***u*** × **(*v*** × ***w*)**
4. **(*u*** × ***v*)** × ***w***
5. ***u*** × **(*v*** −2 ***w*)**

***Solution***

1. 



1. 



1. 





1. 





1. 







***Exercise***

Use the cross product to find a vector that is orthogonal to both

1. 
2. 
3. 

***Solution***

1. 

****

****

1. 

****

****

1. 

****

****

***Exercise***

Find the area of the parallelogram determined by the given vectors

1. 
2. 
3. 

***Solution***

1. 









1. 







1. 









***Exercise***

Find the area of the parallelogram with the given vertices ****

***Solution***



****

****

****

****

****

****

****

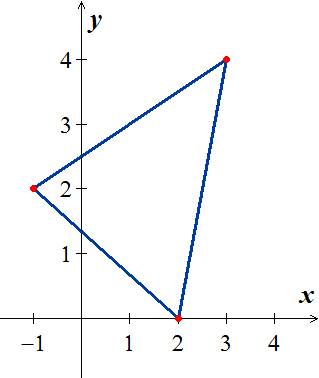
The area of the parallelogram is

****

***Exercise***

Find the area of the triangle with the given vertices:

1. 
2. 

***Solution***

1. 

****

****

****

****

The area of the triangle is

****

1. 

****

****

****

****

The area of the triangle is

****

***Exercise***

1. Find the area of the parallelogram with edges  and 
2. Find the area of the triangle with sides ***v, w***, and ***v + w***. Draw it.
3. Find the area of the triangle with sides ***v, w***, and ***v*** – ***w***. Draw it.

***Solution***



1.  (which is the parallelogram *OABC*)
2. The area of the triangle with sides ***v, w*,** and ***v + w*** is the triangle OCB or OAB which it is half the parallelogram (by definition).







1. The area of the triangle with sides ***v, w*,** and ***v - w*** is equivalent to the triangle OAC which it is half the parallelogram (by definition).





***Exercise***

Find the volume of the parallelepiped with sides ***u***, ***v***, and ***w***.

1. 
2. 

***Solution***

1. 

The volume of the parallelepiped is 

1. 

The volume of the parallelepiped is 

***Exercise***

Compute the scalar triple product 

1. 
2. 
3. 
4. 

***Solution***

1. 
2. 
3. 
4. 

***Exercise***

Use the cross product to find the sine of the angle between the vectors 

***Solution***



****

****











***Exercise***

Simplify 

***Solution***















***Exercise***

Prove Lagrange’s identity: 

***Solution***

Let  and 



















***Exercise***

Polar coordinates satisfy . Polar area  includes *J*:



The two columns are orthogonal. Their lengths are \_\_\_\_\_\_. Thus *J* = \_\_\_\_\_\_.

***Solution***

The length of the first column is: 

The length of the second column is: 







So *J* is the product 1. *r* = *r*.





