***Solution*** ***Section* 2.1 – Real Vectors Spaces**

***Exercise***

We are given three different vectors . Construct a matrix so that the equations  and  are solvable, but is not solvable.

1. How can you decide if this possible?
2. How could you construct A?

***Solution***

The equations and will be solvable.

 (solvable?)

Ifis not solvable, we have the desired matrix *A*.

Ifis solvable, then it is not possible to construct *A*.

When the column space contains and , it will have to contain their linear combinations.

So would necessarily be in that column space and would necessarily be solvable.

***Exercise***

For which vectors  do these systems have a solution?

1. 
2. 
3. 

***Solution***

1. 



Solution for every *b*.

1. 



Solvable only if 

1. 





Solvable only if 

***Exercise***

Let *V* be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operation on 

1. Compute  and  for ***u*** = (0, 4), ***v*** = (1, −3), and *k* = 2.
2. Show that (0, 0) **≠ 0**.
3. Show that (−1, −1) = 0.
4. Show that  for 
5. Find two vector space axioms that fail to hold.

***Solution***

1. 



1. 





Therefore (0, 0) is not the zero vector **0** required (by Axiom).

1. 







Therefore (−1, −1) = **0** holds.

1. Let 







 holds

1. Axiom 7: 





Therefore, ; Axiom 7 fails to hold

Axiom 8: 





Therefore, ; Axiom 8 fails to hold

***Solution Section* 2.2 – Subspaces**

***Exercise***

Suppose *S* and *T* are two subspaces of a vector space **V**.

1. The sum  contains all sums  of a vector *s* in *S* and a vector *t* in *T*. Show that  satisfies the requirements (addition and scalar multiplication) for a vector space.
2. If *S* and *T* are lines in , what is the difference between  and ? That union contains all vectors from *S* and *T* or both. Explain this statement: The span of  is .

***Solution***

1. Let  be vectors in *S*, Let  be vectors in *T*, and let *c* be a scalar. Then

 and



Thus  is closed under addition and scalar multiplication, it satisfies the two requirements for a vector space.

1. If *S* and *T* are distinct lines, then *S* and *T* is a plane, whereas  is not even closed under addition. The span of  is the set of all combinations of vectors in this union. In particular, it contains all sums  of a vector ***s*** in *S* and a vector ***t*** in *T*, and these sums form .  contains both *S* and *T*; so it contains .  is a vector space.
2. So it contains all combinations of vectors in itself; in particular, it contains the span of . Thus the span of  is .

***Exercise***

Determine which of the following are subspaces of ?

1. All vectors of the form (*a*, 0, 0)
2. All vectors of the form (*a*, 1, 1)
3. All vectors of the form (*a*, *b*, *c*), where *b* = *a* + *c*
4. All vectors of the form (*a*, *b*, *c*), where *b* = *a* + *c +* 1
5. All vectors of the form (*a*, *b*, 0)

***Solution***

1. 



This is a subspace of 

1.  which is not in the set. Therefore, this is not a subspace of 
2. 













This is a subspace of 

1.  so  is not in the set.

Therefore, this is not a subspace of 

1. 



This is a subspace of 

***Exercise***

Determine which of the following are subspaces of ?

1. All sequences ***v*** in  of the form *v* = (*v*, 0, *v*, 0, …)
2. All sequences ***v*** in  of the form *v* = (*v*, 1, *v*, 1, …)
3. All sequences ***v*** in  of the form *v* = (*v*, 2*v*, 4*v*, 8*v*, 16*v*, …)

***Solution***

1. 



This is a subspace of

1. 



 is not in the set

Since *k* ≠ 1, then is not a subspace of

1. 





This is a subspace of

***Exercise***

Which of the following are linear combinations of ***u*** = (0, −2, 2) and ***v*** = (1, 3, −1)?

1. (2, 2, 2)
2. (3, 1, 5)
3. (0, 4, 5)
4. (0, 0 ,0)

***Solution***



1. 

(2, 2, 2) = 2***u*** + 2***v*** is a linear combination of ***u*** and ***v***.

1. 

(3, 1, 5) = 4***u*** + 3***v*** is a linear combination of ***u*** and ***v***.

1. 

(0, 4, 5) is not a linear combination of ***u*** and ***v***.

1. 

(0, 0, 0) = 0***u*** + 0***v*** is a linear combination of ***u*** and ***v***.

***Exercise***

Which of the following are linear combinations of ***u*** = (2, 1, 4), ***v*** = (1, −1, 3) and ***w*** = (3, 2, 5)?

1. (−9, −7, −15)
2. (6, 11, 6)
3. (0, 0 ,0)

***Solution***



1. 

Therefore, 

1. 

Therefore, 

1. 

Therefore, 

***Exercise***

Which of the following are linear combinations of 

1. 
2. 
3. 

***Solution***



1. 

 is a linear combinations of *A*, *B*, and *C*.

1. 

 is a linear combinations of *A*, *B*, and *C*.

1. 

 is a linear combinations of *A*, *B*, and *C*.

***Exercise***

Determine whether the given vectors span 

1. 
2. 
3. 

***Solution***

1. 

The system is consistent for all values so the given vectors span .

1. 

The system is not consistent for all values so the given vectors do not span .

1. 

The system has a solution only if . But since this is a restriction that the given vectors don’t span on all of . So the given vectors do not span .

***Exercise***

Suppose that . Which of the following vectors are in span 

1. (2, 3, −7, 3)
2. (0, 0, 0, 0)
3. (1, 1, 1, 1)
4. (−4, 6, −13, 4)

***Solution***

In order to be span , there must exists scalars *a, b, c* that 



1. 

This system is consistent, it has only solution is *a* = 2, *b* = −1, *c* = −1 , therefore (2, 3, −7, 3) is in span 

1. The vector (0, 0 , 0, 0) is obviously in span since 
2. 

This system is inconsistent, therefore (1, 1, 1, 1) is not in span 

1. 

This system is consistent, it has only solution is *a* = 3, *b* = −3, *c* = 1 , therefore (−4, 6, −13, 4) is in span 

***Exercise***

Let  and . Which of the following lie in the space spanned by  and 

1. 
2. 
3. 
4. 

***Solution***

1. , therefore  is in span 
2. In order for  to be in span , there must exist scalars *a* and *b* such that



When 

Therefore  is not in span 

1. In order for  to be in span , there must exist scalars *a* and *b* such that



When 

Therefore  is not in span 

1. In order for 0 to be in span , there must exist scalars *a* and *b* such that



Therefore  is in span 

***Solution Section* 2.3 – Linear Independence**

***Exercise***

Given three independent vectors . Take combinations of those vectors to produce . Write the combinations in a matrix form as 

 which is 

What is the test on a matrix **V** to see if its columns are linearly independent?

If  show that  are linearly independent.

If  show that  are linearly *dependent*.

***Solution***

The nullspace of **V** must contain only the *zero* vector. Then  is the only combination of the columns that gives **V***x* = zero vector.



If , then the matrix *M* is invertible. So if *x* is any nonzero vector we know that *Mx* is nonzero. Since ***w***’s are given as independent and *WMx* is nonzero. Since , this says that *x* is not in the nullspace of **V**. therefore;  are independent.

If , that implies 

, which means that  are linearly *dependent*.

The other way, the vector  is in that nullspace, and . Then certainly  which is the same as . So the  are dependent.

***Exercise***

Find the largest possible number of independent vectors among



***Solution***

Since , there are at most three

independent vectors among these: furthermore, applying row reduction to the matrix gives three pivots, showing that are independent.

***Exercise***

Show that are independent but  are dependent:



Solve either . The *v*’s go in the columns of ***A***.

***Solution***



This matrix has 3 pivots with rank of 3 equals to rows that implies the  are independent.

 that shows that  are dependent.

***Exercise***

Decide the dependence or independence of

1. The vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1).
2. The vectors  and  and .

***Solution***

1. These are linearly independent.  only if 
2. These are linearly dependent: 

***Exercise***

Find two independent vectors on the plane  in . Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

***Solution***

This plane is the nullspace of the matrix





The pivot is 1st column, and the rest are 3 variables.

If  . The vector is 

If  . The vector is 

If  . The vector is 

The 3 vectors  are linearly independent.

We can’t find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

***Exercise***

Determine whether the vectors are linearly independent or linearly independent in 

1. (4, −1, 2), (−4, 10, 2)
2. (−3, 0, 4), (5, −1, 2), (1, 1, 3)
3. (−2, 0, 1), (3, 2, 5), (6, −1, 1), (7, 0, −2)

***Solution***

1. The vector equation *a*(4, −1, 2) + *b*(−4, 10, 2) = (0, 0 ,0)



Therefore the system has only the trivial solution *a* = *b* = 0.

We conclude that the given set of vectors is linearly independent.

1. The vector equation *a*(−3, 0, 4) + *b*(5, −1, 2) + *c*(1, 1, 3) = (0, 0 ,0)



Therefore the system has only the trivial solution *a* = *b* = *c* = 0.

We conclude that the given set of vectors is linearly independent.

1. The vector equation *a*(−2, 0, 1) + *b*(3, 2, 5) + *c*(6, −1, 1) + *d*(7, 0, −2) = (0, 0 ,0)



Therefore the system has nontrivial solutions 

We conclude that the given set of vectors is linearly dependent.

***Exercise***

Determine whether the vectors are linearly independent or linearly independent in 

1. (3, 8, 7, −3), (1, 5, 3, −1), (2, −1, 2, 6), (1, 4, 0, 3)
2. (0, 0, 2, 2), (3, 3, 0, 0), (1, 1, 0, −1)
3. (0, 3, −3, −6), (−2, 0, 0, −6), (0, −4, −2, −2), (0, −8, 4, −4)

***Solution***

1. 

The system has only the trivial solution and the vectors are linearly independent.

1. 





The system has only the trivial solution and the vectors are linearly independent.

1. 

The system has only the trivial solution and the vectors are linearly independent.

***Exercise***

*a* ) Show that the three vectors  form a linearly dependent set in .

*b*) Express each vector in part (*a*) as a linear combination of the other two.

***Solution***

1. The vector equation 



The solution: 

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

1. Since  and if we let *t* = 1, then 



***Exercise***

For which real values of λ do the following vectors form a linearly dependent set in 



***Solution***





For , the determinant is zero and the vectors form a linearly dependent set.

***Exercise***

Show that if  is a linearly independent set of vectors, then so is every nonempty subset of S.

***Solution***

Let  be a nonempty subset of *S*.

If this set is linearly dependent, then there would be a nonzero solution  to . This can be expanded to a nonzero solution of  by taking all other coefficients as 0. This contradicts the linear independence of S, so the subset must be linearly independent.

***Exercise***

Show that if  is a linearly dependent set of vectors in a vector space *V*, and if  are vectors in *V* that are not in *S*, then  is also linearly dependent.

***Solution***

If *S* is linearly dependent, then there is a nonzero solution  to . Thus  is a nonzero solution to  so the set  is linearly dependent.

***Exercise***

Show that  is linearly independent and  does not lie in span , then  is a linearly independent.

***Solution***

If  are linearly dependent, there exist a nonzero solution to  with  (since  and  are linearly independent).

 which contradicts that  is not in span . Thus  is a linearly independent.

***Exercise***

By using the appropriate identities, where required, determine  are linearly dependent.

1. 
2. 
3. 
4. 

***Solution***

1. From the identity 



Therefore, the set is linearly dependent.

1. 





Therefore, the set is linearly independent.

1. 







Therefore, the set is linearly independent.

1. 







Therefore, the set is linearly dependent.

***Exercise***

 are linearly independent in  because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński’s test.

***Solution***

The Wronskian: 







 are linearly independent

***Exercise***

Use the Wronskian to show that  span a three-dimensional subspace of 

***Solution***

The Wronskian: 









Since  for all real *x* values, the vectors are linearly independent.

***Solution*** ***Section* 2.4 – Coordinates and Basis**

***Exercise***

Suppose  is a basis for  and the *n* by *n* matrix *A* is invertible. Show that  is also a basis for .

***Solution***

Put the basis vectors  in the columns of an invertible matrix **V**. then  are the columns of ***A*V**. Since ***A*** is invertible, so is ***A*V** and its column give a basis.

Suppose . This is  with . Multiply by  to get . By linear independence of , all . So the  are independent.

***Exercise***

Consider the matrix 

1. Which vectors  will make the columns of ***A*** linearly dependent?
2. Which vectors  will make the columns of ***A*** a basis for ?
3. For , compute a basis for the four subspaces.

***Solution***

1. All linear combination of 
2. To satisfy *b + d* = 0. For example 



1. 





The first 2 columns span the column space *C*(***A***).

If  that implies that the nullspace *N*(***A***): 

Rank(***A***) = 2 and  is a basis for the one-dimensional *N*(***A***).

***Exercise***

Find a basis for  in .

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

***Solution***

This plane is the nullspace of the matrix



The special solutions:  give a basis for the nullspace, and for the plane.

The intersection of this plane with the *xy*-plane is a line  and the vector  lies in the *xy*-plane.

The vector  is perpendicular to both vectors : the space vectors perpendicular to a plane  is one-dimensional, it gives a basis.

***Exercise***

**U** comes from ***A*** by subtracting row 1 from row 3:



Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

***Solution***



1. The pivots are in the first two columns, so one possible basis for *C*(***A***) is  and for *C*(***U***) is .
2. Both ***A*** and ***U*** have the same nullspace *N*(***A***) = *N*(***U***), with basis 
3. Both ***A*** and ***U*** have the same row space 

***Exercise***

Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives , and check entries to prove is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

***Solution***



 and 







***Exercise***

Choose three independent columns of . Then choose a different three independent columns. Explain whether either of these choices forms a basis for .

***Solution***





Rank(***A***) = 3, the columns space is 3 which form a basis of . The variable is 

If  

*N*(*A*) is spanned by , which gives the relation of the columns. The special solution  gives a relation . If we take any two columns from the first three columns and the column 4, they will span a three dimensional space since there will be no relation among them. Hence they form a basis of .

***Exercise***

Which of the following sets of vectors are bases for ?

1. 
2. 

***Solution***

1. 





Therefore the vectors  are linearly independent and span  , so they form a basis for .

1. 





Therefore the vectors  are linearly dependent, so they don’t form a basis for .

***Exercise***

Which of the following sets of vectors are bases for ?

1. 
2. 
3. 

***Solution***

1.  Therefore the set of vectors are linearly independent.

The set form a basis for .

1.  Therefore the set of vectors are linearly independent.

The set form a basis for .

1.  Therefore the set of vectors are linearly dependent.

The set don’t form a basis for .

***Exercise***

Let *V* be the space spanned by 

1. Show that  is not a basis for *V*.
2. Find a basis for *V*.

***Solution***

1. 







If 



This shows that  is linearly dependent, therefore it is not a basis for *V*.

1. For  to hold for all real *x* values, we must have  and . Therefore the vectors  are linearly independent.





This proves that the vectors  span *V*. We can conclude that  can form a basis for *V*.

***Exercise***

Find the coordinate vector of ***w*** relative to the basis  for 

1. 
2. 
3. 
4. 
5. 

***Solution***

1. We must first express ***w*** as a linear combination of the vectors in *S;* 





Therefore, 

1. Solve 





Therefore, 

1. Solve 



Therefore, 

1. Solve 





Therefore, 

1. Solve 





Therefore, 

***Exercise***

Find the coordinate vector of *v* relative to the basis 

1. 
2. 

***Solution***

1. Solve 



Therefore, 

1. Solve 





Therefore, 

***Exercise***

Show that  is a basis for , and express *A* as a linear combination of the basis vectors

1. 
2. 

***Solution***

1. Matrices  are linearly independent if the equation





Has only the trivial solution.

For these matrices to span , it must be expressed every matrix  as





The 2 equations can be written as linear systems



, that the homogeneous system has only the trivial solution.

span 







1. Matrices  are linearly independent if the equation





Has only the trivial solution.

For these matrices to span , it must be expressed every matrix  as





The 2 equations can be written as linear systems



, that the homogeneous system has only the trivial solution.

span 





***Solution Section* 2.5 – Dimension**

***Exercise***

Consider the eight vectors



1. List all of the one-element. Linearly dependent sets formed from these.
2. What are the two-element, linearly dependent sets?
3. Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
4. Which four-element sets are linearly dependent? Explain why.

***Solution***

1.  zero vector is the only linearly dependent.
2. The set that contains zero vector and any other vector.
3. 2-dimension:



1-dimensional subspace if we allow duplicates (zero vector) 

1. All four-element sets are linearly dependent in three-dimensional space.

***Exercise***

Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

1. 
2. 
3. 
4. 

***Solution***

1. 

The solution: 

The solution space has dimension 1 and a basis 

1. 

The solution: 



The solution space has dimension 2 and a basis 

1. 

The solution: 



The solution space has dimension 2 and a basis 

1. 

The solution: 

The solution space has dimension 1 and a basis 

***Exercise***

If  for the shift matrix *S*. Show that ***A*** must have this special form:





“The subspace of matrices that commute with the shift *S* has dimension \_\_\_\_\_\_.”

***Solution***









The subspace of matrices that commute with the shift *S* has dimension 3, because the matrix has only three variables

***Exercise***

Find bases for the following subspaces of 

1. All vectors of the form (*a, b, c*, 0)
2. All vectors of the form (*a, b, c*, *d*), where *d* = *a + b* and *c = a – b*.
3. All vectors of the form (*a, b, c*, *d*), where *a = b* = *c = d*.

***Solution***

1. The subspace can be expressed as span is a set of linearly independent vectors. Therefore *S* forms a basis for the subspace, so its dimension is 3.
2. The subspace contains all vectors , the setis linearly independent vectors. Therefore *S* forms a basis for the subspace, so its dimension is 2.
3. The subspace contains all vectors , we can express the set as span *S* and it is linearly independent. Therefore *S* forms a basis for the subspace, so its dimension is 1.

***Solution*** ***Section* 2.6 – Row Space, Column Space, and Null Space**

***Exercise***

List the row vectors and column vectors of the matrix



***Solution***

Row vectors: 

Column vectors: 

***Exercise***

Express the product A***x*** as a linear combination of the column vectors of A.

1. 
2. 

***Solution***

1. 
2. 

***Exercise***

Determine whether ***b*** is in the column space of *A*, and if so, express ***b*** as a linear combination of the column vectors of *A*.

1. 
2. 
3. 

***Solution***

1. 



1. 

The system *A****x*** = ***b*** is inconsistent and ***b*** is not in the column space of *A*.

1. 



***Exercise***

Suppose that  is a solution of a nonhomogeneous linear system A***x*** = ***b*** and that the solution set of the homogeneous system A***x*** = **0** is given by the formulas



1. Find a vector form of the general solution of A***x*** = **0**
2. Find a vector form of the general solution of A***x*** = ***b***

***Solution***

1. 
2. 

***Exercise***

Find the vector form of the general solution of the given linear system A***x*** = ***b***; then use that result to find the vector form of the general solution of A***x*** = **0**.

1. 
2. 
3. 

***Solution***

1. 

The solution of A***x*** = ***b*** is  or 

The general form of the solution of A***x*** = **0** is 

1. 

The solution of A***x*** = ***b*** is  or 

The general form of the solution of A***x*** = **0** is 

1. 

The solution of A***x*** = ***b*** is 

The general form of the solution of A***x*** = **0** is 

***Exercise***

Find a basis for the null space of A.

1. 
2. 
3. 

***Solution***

1. 

The general form of the solution of A***x*** = **0** is , therefore the vector  forms a basis for the null space of *A*.

1. 

The general form of the solution of A***x*** = **0** is , therefore the vectors  form a basis for the null space of *A*.

1. 



The general form of the solution of A***x*** = **0** is , therefore the vectors  form a basis for the null space of *A*.

***Exercise***

Find a basis for the subspace of spanned by the given vectors

1. 
2. 

***Solution***

1. 

A basis for the subspace is 

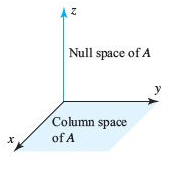
1. 

A basis for the subspace is 

***Exercise***

*a*) Let 

Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.



*b*) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.

***Solution***

1. 

The general form of the solution of A***x*** = **0** is, therefore the null space of *A* is the *z*-axis, and the column space is the span of  which is all linear combinations of *y* and *x* (*xy*-plane)

1. 

***Exercise***

Given the vectors  and 

1. Are they linearly independent?
2. Are they a basis for any space?
3. What space **V** do they span?
4. What is the dimension of that space?
5. What matrices ***A*** have **V** as their column space?
6. Which matrices have **V** as their nullspace?
7. Describe all vectors  that complete a basis for .

***Solution***

1.  are independent – the only combination to give **0** is .
2. Yes, they are a basis for whatever space **V** they span.
3. That space **V** contains all vectors . It is the *xy* plane in .
4. The dimension of **V** is 2 since the basis contains 2 vectors.
5. This **V** is the column space of any 3 by *n* matrix ***A*** of rank 2, if every column is a combination of . In particular ***A*** could just have columns .
6. This **V** is the nullspace of any *m* by 3 matrix ***B*** of rank 1, if every row is a multiple of . In particular take . Then .
7. Any third vector  will complete a basis for  provided .

***Exercise***

Which of the following subsets of are actually subspaces?

1. The plane of vectors  with 
2. The plane of vectors with .
3. The vectors with .
4. All linear combinations of  and .
5. All vectors that satisfies 
6. All vectors with .

***Solution***

1. This is subspace

* For  with and  with  the sum

 is in the same set as 

* For an element  with ,  and , thus it is in the same set.

1. This is not a subspace. For example for  and  is not in the set.
2. This is not a subspace. For example for  and  are in the set, but their sum  is not in the set.
3. This is subspace, by definition of linear combination.

* For 2 vectors  and  the sum





is still the linear combination of *v* and *w*.

* For an element  ,  is still the linear combination of *v* and *w*, thus it is the same set

1. This is subspace, these are the vectors orthogonal to 

* For  with  and  with  the sum  is in the same set as 
* For an element  with ,  and , thus it is in the same set.

1. This is not a subspace. For example for  and  is not in the set.

***Exercise***

True or False (check addition or give a counterexample)

1. The symmetric matrices in *M*  from a subspace.
2. The skew-symmetric matrices in *M*  from a subspace.
3. The un-symmetric matrices in *M*  from a subspace.
4. Invertible matrices
5. Singular matrices

***Solution***

1. True:  and  lead to 
2. True:  and  lead to 
3. False: 
4. False:  and  are invertible matrices but  is not invertible. ∴ The zero matrix is not invertible but any linear subspace should contain the zero matrix. So invertible matrices do not form a linear subspace.
5. False:  and  are singular matrices but  is not singular.

***Exercise***

If we add an extra column b to a matrix *A*, then the column space gets larger unless \_\_\_\_\_. Give an example where the column space gets larger and an example where it doesn’t. Why is  solvable exactly when the column space doesn’t get larger – it is the same for *A* and ?

***Solution***

If we add an extra column ***b*** to a matrix ***A***, then the column space gets larger unless ***it contains b*** that is a linear combination of the columns of ***A***.

Let ; then the column space gets larger if  and it doesn’t if .

The equation  is solvable exactly when ***b*** is a (nontrivial) linear combination of the column of ***A***.

The equation  is solvable exactly when ***b*** lies in the column space, when the column space doesn’t get larger.

***Exercise***

For which right sides (find a condition on ) are these solvable. (Use the column space  and the equation )

1. 
2. 

***Solution***

1. The column space consists of the vectors for  is 

They are scalar multiples of 

1. By substituting with new variable *z*, then the column space consists of the vectors 

They are linear combinations of , 

***Exercise***

Show that the matrices *A* and  (with extra columns) have the same column space. But find a square matrix with  smaller than . Important point: An *n* by *n* matrix has  exactly when A is an \_\_\_\_\_\_ matrix.

***Solution***

Each column of ***AB*** is a combination of the columns of ***A*** (the combining coefficients are the entries in the corresponding column of B). So any combination of the columns of  is a combination of the columns of ***A*** alone. Thus ***A*** and  have the same column space.

Let ; then , so .

 is the line through .

Any *n* by *n* matrix has  exactly when ***A*** is an ***invertible*** matrix, because  is solvable for any given ***b*** when ***A*** is invertible.

***Exercise***

The column of *AB* are combinations of the columns of *A*. This means: The column space of *AB* is contained in (possibly equal to) to the column space of *A*. Give an example where the column spaces *A* and *AB* are not equal.

***Solution***

The column space of ***AB*** is contained in (possibly equal to) to the column space of ***A***.

 is a case when  has a smaller column space than ***A***.

***Exercise***

Find a square matrix *A* where  (the column space of  is smaller than .

***Solution***

For example, ; then .

Thus  is generated by vector , which is of one dimensional, but  is a zero space.

Hence  is strictly smaller than .

***Exercise***

Suppose  and  have the same (complete) solutions for every ***b***. Is true that ?

***Solution***

Yes, if , let ***y*** be any vector of the correct size, and set . Then ***y*** is a solution to  and it is also a solution to ; 

***Exercise***

Apply Gauss-Jordan elimination to  and . Reach  and :

Solve  to find  (its free variable is ).

Solve  to find  (its free variable is ).

***Solution***



The free variable is , since it is the only one. We have to let 



The special solution is 



The free variable is that implies to 



The particular solution is 

***Solution*** ***Section* 2.7 – Rank, Nullity, and the Fundamental Matrix Spaces**

***Exercise***

Verify that 



***Solution***







***Exercise***

Find the rank and nullity of the matrix; then verify that the values obtained satisfy 

1. 
2. 
3. 

***Solution***

1. 



1. 



1. 



***Exercise***

If *A* is an  matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of *A*.

***Solution***

The largest possible value for the rank of an  matrix:

*  (when every column of the *rref* (A) contains a leading 1)
*  (when every row of the *rref* (A) contains a leading 1)

The smallest possible value for the nullity of an  matrix:

*  (when every column of the *rref* (A) contains a leading 1)
*  (when every row of the *rref* (A) contains a leading 1)

***Exercise***

Discuss how the rank of *A* varies with *t*.

*a*)  *b*) 

***Solution***

1. 



Solve for *t*: 

Therefore,  for 

If *t* = 1, 

If *t* = −2, 

1. 



Solve for *t*: 

Therefore,  for 

If *t* = 1, 

If , 

***Exercise***

Are there values of *r* and *s* for which



Has rank 1? Has rank 2? If so, find those values.

***Solution***

Since the third column will always have a nonzero entry, the *rank* will never be 1. (row 1 and row 4 never have a nonzero entry).

If *r* = 2 and *s* = 1, that implies to



***Exercise***

Find the row reduced form ***R*** and the rank *r* of ***A*** (those depend on *c*).

Which are the pivot columns of ***A***? Which variables are free? What are the special solutions and the nullspace matrix ***N*** (always depending on *c*)?



***Solution***

1. ,

, the pivot columns are 1 and 3, the second variable  is free.

The special solution: which yields to

,

, the pivot column is column 1, the second and third variables  are free.

The special solution goes into 

1. ,

, the pivot column is the first column, the second variable  is free.

The special solution: which yields to



, the matrix has no pivot column, and both variable are free.

The special solution is: 

***Exercise***

Find the row reduced form *R* and the rank *r* of *A* (those depend on *c*).

Which are the pivot columns of *A*? Which variables are free? What are the special solutions and the nullspace matrix *N* (always depending on *c*)?



***Solution***



1. If *c*  = 1, then



This has only one pivot (first column) and 3 free variables .

The nullspace matrix: 

1. If *c*  ≠ 1, then



There are two pivots  and 2 free variables 

The nullspace matrix: 



1. If *c* = 1 ⇒ 



This has a single pivot in the second column and one free variable with the nullspace matrix 

1. If *c* = 2 ⇒ 



This has a single pivot in the first column with the nullspace matrix 

1. Otherwise  ⇒ 



The result is the identity matrix with 2 pivots, which has (2 – 2) 0 null space.

***Exercise***

If *A* has a rank *r*, then it has an *r* by *r* sub-matrix *S* that is invertible. Remove  rows and columns to find an invertible sub-matrix *S* inside each *A* (you could keep the pivot rows and pivot columns of *A*).



***Solution***

If a matrix ***A*** has rank *r*, then the

**(*dimension of the column space*) *=* (*dimension of the row space*) *= r***

For the invertible sub-matrix S, we need to find *r* linearly independent rows and *r* linearly independent columns.

For matrix ***A***:



The 1st and 3rd columns are linearly independent, and the 1st and 2nd rows are also linearly independent.

Rank (***A***) = 2.

The sub matrices are: 

For matrix ***B***:



Rank (***B***) = 1.

The submatrix is: 

For matrix ***C***:



Rank (***C***) = 2.

The submatrix is by disregarding (deleting) 1st column and 2nd row: 

***Exercise***

Suppose that column 3 of 4 x 6 matrix is all zero. Then must be a \_\_\_\_\_\_ variable. Give one special solution for this matrix.

***Solution***

Themust be a ***free variable***.

A special solution for this variable can be taken to be.



***Exercise***

Fill in the missing numbers to make *A* rank1, rank 2, rank 3.(your solution should be 3 matrices)



***Solution***



If rank (*A*) = 1, then we need the 1st and 3rd to be multiple of the 2nd row to get zero in these rows.







If rank (*A*) = 2, then we need the 1st ***or*** 3rd to be multiple of the 2nd row to get zero row.





If rank (*A*) = 3 (full rank), then the appropriate to start using 0’s or 1’s to fill the blank.









Hence, it has rank 3.

***Exercise***

Fill out these matrices so that they have rank 1:



***Solution***

Rank = 1 means that all the rows are multiples of each other.











***Exercise***

Suppose *A* and *B* are *n* by *n* matrices, and *AB* = *I*. Prove from  that the . So *A* is invertible and *B* must be its two-sided inverse. Therefore *BA* = *I* (which is not so obvious!).

***Solution***

Since *A* is *n* by *n* 



***Exercise***

Every *m* by *n* matrix of rank *r* reduces to (*m* by *r*) times (*r* by *n*):

*A* = (pivot columns of *A*) (first *r* rows of *R*) 

Write the 3 by 4 matrix  as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from *R*.

***Solution***







The pivots columns are the 1st and 2nd column.



***Exercise***

Suppose *R* is *m* by *n* matrix of rank *r,* with pivot columns first: 

1. What are the shapes of those 4 blocks?
2. Find the right-inverse *B* with *RB* = *I* if *r* = *m*.
3. Find the right-inverse *C* with *CR* = *I* if *r* = *n*.
4. What is the reduced row echelon form of  (with shapes)?
5. What is the reduced row echelon form of  (with shapes)?

Prove that  has the same nullspace as R. Then show that  always has the same nullspace as *A* (a value fact).

1. Suppose you allow elementary column operations on ***A*** as well as elementary row operations (which get to ***R***). What is the “row-and-column reduced form” for an *m* by *n* matrix of rank ***r***?

***Solution***

1. 
2. 









1. 





1.  
2. 

, the inner is not equal but to make work, we can use the *F* transpose.









So that  for any matrix ***A***. So, 

1. After getting to *R* we can use the column operations to get rid of *F*.



***Exercise***

Let 

1. Reduce *A* to row-reduced echelon from.
2. What is the rank of *A*?
3. What are the pivots?
4. What are the free variables?
5. Find the special solutions. What is the nullspace ?
6. Exhibit an *r* x *r* submatrix of *A* which is invertible, where . (An *r* x *r* submatrix of *A* is obtained by keeping *r* rows and *r* columns of *A*)

***Solution***

1. 



1. The pivots are 
2. The free variables are 
3. 
4. Let 

1. Set  

The special solution: 

1. Set  

The special solution: 

The nullspace is the set 

1. Rank(***A***) = 3
2. The pivot rows and columns must be included in a submatrix. To do that, just take the rows and columns of ***A*** containing pivots, which are columns 1, 3, 5 and rows 1, 2, 3. That will give us a 3 by 3 submatrix. Therefore, this submatrix of ***A*** will be invertible.



***Exercise***

Let 

1. Reduce ***A*** to (ordinary) echelon from.
2. What the pivots?
3. What are the free variables?
4. Reduce ***A*** to row-reduced echelon form.
5. Find the special solutions. What is the nullspace ?
6. What is the rank of ***A***?
7. Give the complete solution to 

***Solution***

1. 



1. The pivots are -1, 5, and -5 (Columns 1, 2, 4)
2. The free variables are 3rd and 5th 
3. 

1. Let 

1. Set  

The special solution: 

1. Set  

The special solution: 

The nullspace is the set 

1. Rank(***A***) = 3
2. 

The complete solution = (the particular solution) + (special solution)





***Exercise***

The 3 by 3 matrix *A* has rank 2.



1. Reduce  to , so that  becomes triangular system .
2. Find the condition on  for  to have a solution
3. Describe the column space of *A*. Which plane in ?
4. Describe the nullspace of *A*. Which special solutions in ?
5. Find a particular solution to  and then complete solution.

***Solution***

1. 



1. The last equation  shows the solvability condition.
2. (***i***) The column space is the plane containing all combinations of the pivot columns: 1st (1, 2, 3) and 3rd (3, 8, 7).

(***ii***) The column space contains all vectors with . That makes  solvable, so ***b*** is in the column space. All columns of *A* pass this test . This is the equation for the plane in (***i***).

1. The special solutions have free variables:







The nullspace *N*(*A*) in contains all 

1. One particular solution has free variables = zero.







The complete solution to is

***Exercise***

Find the special solutions and describe the complete solution to  for

= 3 by 4 zero matrix  

Which are the pivot columns? Which are the free variables? What is the *R* (Reduced Row Echelon matrix) in each case?

***Solution***

has 4 solutions. They are the columns  of the identity matrix (4 by 4).

The Nullspace is of .

The complete solution: in .

There are no pivot columns; all variables are free, the reduced *R* is the same zero matrix as.





The vector solution: , The first column of  is its pivot column, and  is the free variable.





All variables are free. There are three special solutions to 



The complete solution:  .

***Exercise***

Create a 3 by 4 matrix whose special solutions to  are :



You could create the matrix A in row reduced form R. Then describe all possible matrices A with the required Nullspace all combinations of .

***Solution***

We can write the solution:









The entries 3, 2, 6 are the negatives of -3, -2, -6 in the special solutions.

Every 3 by 4 matrix has at least one special solution. These *A*’s have two.

***Exercise***

The plane  is parallel to the plane . One particular point on this plane is . All points on the plane have the form (fill the first components)



***Solution***





***Exercise***

Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains .

***Solution***









***Exercise***

Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

***Solution***

It is impossible. Matrix ***A*** must be 3 by 3. Since the nullspace is supposed to contain two independent vectors, ***A*** can have at most  pivots. Since the column space supposes to contain two independent vectors. A must have at least 2 pivots. These conditions can’t both be met.

***Exercise***

Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains .

***Solution***

The matrix needs to be 3 by 4 matrix.







***Exercise***

How is the Nullspace N(*C*) related to the spaces N(*A*) and N(*B*), if ?

***Solution***



If and only if *Ax* =0 and *Bx* = 0.



***Exercise***

Why does no 3 by 3 matrix have a nullspace that equals its column space?

***Solution***

If nullspace = column space then *n – r* = *r* (there are *r* pivots). For *n* = 3 ⇒ 3 = 2***r*** is impossible.

***Exercise***

If *AB* = 0 then the column space *B* is contained in the \_\_\_\_\_\_\_ of *A*. Give an example of *A* and *B*.

***Solution***

If *AB* = 0 then the column space *B* is contained in the ***nullspace*** of *A*.

Example: 

***Exercise***

True or false (with reason if true or example to show it is false)

1. A square matrix has no free variables.
2. An invertible matrix has no free variables.
3. An *m* by *n* matrix has no more than***n*** pivot variables.
4. An *m* by *n* matrix has no more than***m*** pivot variables.

***Solution***

1. False. Any matrix with fewer than full number of pivots will. 
2. True. Since it is invertible, we will get the full number of pivots. The nullspace has dimension, so we have 0 free variables.
3. True, the number of pivot variables is the dimension of the nullspace, which is at most the number of columns. The nullspace dimension + column space dimension = number of columns.
4. True, in reduced echelon matrix the pivot columns are all 0 except for a single 1, and there are only up to *m* vectors of this type.

***Exercise***

Suppose an *m* by *n* matrix has *r* pivots. The number of special solutions is \_\_\_\_\_\_.

The Nullspace contains only *x* = 0 when *r* = \_\_\_\_\_\_\_.

The column space is all of  when *r* = \_\_\_\_\_\_.

***Solution***

Suppose an *m* by *n* matrix has *r* pivots. The number of special solutions is \_ ***n – r***\_.

The Nullspace contains only *x* = 0 when *r* = \_ ***n*** \_.

The column space is all of  when *r* = \_***m*** \_.

***Exercise***

Find the complete solution in the form  to these full rank system:

*a*)  *b*) 

***Solution***

***a***) 

The equivalent matrix is given by: 

The complete solution in the form 

 is the homogeneous solution to 

Size of ***A*** is *m* = 1 and *n* = 3, rank(***A***) = *r* = 1



Set  The special solution: 

Set  The special solution: 

The nullspace is the set 

Set  that implies to the particular solution: 

The complete solution in the form 

Note: that the null space of A is spanned by the two linearly independent vectors 

***b***) 

The equivalent matrix is given by:  and 





The pivots are ; The free variable is 

Rank *r* = 2, *n* = 2, *m* = 3. The nullspace has dimension *m – r* = 1.



If  The special solution: 

The nullspace is the set 

Set  that implies 

Then the particular solution: 

The complete solution in the form 

***Exercise***

Find the complete solution in the form  to the system:



***Solution***





The pivots are ; The free variables are 



1. Set  The special solution: 
2. Set  The special solution: 

The special solution: 



Then the particular solution: 

The complete solution in the form 

***Exercise***

If ***A*** is 3 x 7 matrix, its largest possible rank is \_\_\_\_\_\_\_\_. In this case, there is a pivot in every \_\_\_\_\_\_\_\_ of ***U*** and ***R***, the solution to  \_\_\_\_\_\_\_\_\_ (always exists or is unique), and the column space of ***A*** is \_\_\_\_\_\_\_\_\_. Construct an example of such a matrix ***A***.

***Solution***

If ***A*** is 3 x 7 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***row*** of ***U*** and *R*, the solution to  ***always exists***, and the column space of ***A*** is.



, that implies that you have 3 pivots (1 each row)



***Exercise***

If ***A*** is 6 x 3 matrix, its largest possible rank is \_\_\_\_\_\_\_\_. In this case, there is a pivot in every \_\_\_\_\_\_\_\_ of ***U*** and *R*, the solution to  \_\_\_\_\_\_\_\_\_ (always exists or is unique), and the nullspace of ***A*** is \_\_\_\_\_\_\_\_\_. Construct an example of such a matrix ***A***.

***Solution***

If ***A*** is 6 x 3 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***column*** of ***U*** and *R*, the solution to  ***is unique***, and the column space of ***A*** is.



***Exercise***

Find the rank of  and  for 

***Solution***













 for any matrix ***A***.

***Exercise***

Explain why these are all false:

1. The complete solution is any linear combination of .
2. A system  has at most one particular solution.
3. The solution  with all free variables zero is the shortest solution (minimum length ). Find a 2 by 2 counterexample.
4. If *A* is invertible there is no solution  in the null space.

***Solution***

1. The coefficient of  must be one.
2. If  is the nullspace of ***A*** and  is one particular solution, then  is also a particular solution.
3. If ***A*** is a 2 by 2 matrix of rank 1, then the solution to  form a line parallel to the line that the nullspace. The line  gives such an example.



Then  while the particular solutions having some coordinate equal to zero are (1, 0) and (0, 1) and they both have 

1. There is always 

***Exercise***

Write down all known relation between *r* and *m* and *n* if  has

1. No solution for some ***b***.
2. Infinitely many solutions for every ***b***.
3. Exactly one solution for some ***b***, no solution for other ***b***.
4. Exactly one solution for every ***b***.

***Solution***

1. The system has less than full row rank: .
2. The system has full row rank and less than full column rank: .
3. The system has full column rank and less than full row rank: .
4. The system has full row and column rank (it is invertible): .

***Solution Section* 2.8 – Matrix Transformations from  to **

***Exercise***

Find the standard matrix for the transformation defined by the equations

1. 
2. 
3. 

***Solution***

1.  The standard matrix is 
2.  The standard matrix is 
3.  The standard matrix is

***Exercise***

Find the standard matrix for the operator *T* defined by the formula

1. 
2. 
3. 

***Solution***

1. 



The standard matrix is 

1. 



The standard matrix is 

1. 

The standard matrix is 

***Exercise***

Find the standard matrix for the transformation *T* defined by the formula

1. 
2. 
3. 

***Solution***

1. 

The matrix is 

1. The matrix is 
2. The matrix is 