***Section* 1.3 – The Algebra of Matrices**

***Matrices***



This is called Matrix (*Matrices*)

Each number in the array is an ***element*** or ***entry***

The matrix is said to be of order *m x n*

*m*: numbers of rows,

*n*: number of columns

When *m = n*, then matrix is said to be ***square***.

*Given the system equations*



Write into an ***augmented matrix*** form



The Matrix:  is called the ***coefficient matrix*** of the system.

The matrix *A* above has 3 rows and 3 columns, therefore the order of the matrix *A* is (3 *x* 3)



***Equality of Matrices***

**Definition of Equality of Matrices**

Two matrices ***A*** and ***B*** are equal if and only if they have the same order (size) *m* *x* *n* and if each pair corresponding elements is equal

 for *i* = 1, 2, …, *m* ***and*** *j* = 1, 2, …, *n*

***Example***

Find the values of the variables for which each statement is true, if possible.

1. 



1. 

*can’t be true*

1. 



***Addition* and *Subtraction* of Matrices**

***Definition***

If  and  are  matrices, their sum, is the  matrix obtained by adding the corresponding entries; that is



Matrices can be added if their shapes are the same, meaning have the same ***order***.





***Scalar* Multiplication Matrices**

***Definition***

If *k* is a scalar and  is an  matrices, then scalar product ***kA*** is the  matrix obtained by multiplying each entry of *A* by *k*; that is







***Example***





***Definition***

If  are matrices of the same size, and if  are scalars, then expression of the form



Is called a ***linear combination*** of  with *coefficients* .

***Matrix Multiplication***

**Product of Two Matrices**

Let ***A*** be an *m x n* matrix and let ***B*** be an *n x k* matrix. To find the element in the *ith* row and *jth* column of the product matrix ***AB***, multiply each element in the *ith* row of ***A*** by the corresponding element in the *jth* column of ***B***, and then add these products. The product matrix ***AB*** is an *m x k* matrix.

*Matrix* ***A*** *Matrix B*

*m x n n x k*

***Outer***: Order of *AB* is *m x k*

***Inner*** *must be equal*

* To multiply ***AB*** or dot product, if ***A*** has ***n*** columns, ***B*** must have ***n*** rows.
* Squares matrices can be multiplied if and only if (***iff***) they have the same size.
* The entry in row *i* and column *j* of ***AB*** is 

The result: 







 

 

 

 



***Example***

Find: 

***Solution***





***Special Case***

When *A* is a square matrix, then









***Block Multiplication***

If the cuts between columns of ***A***match the cuts between rows of ***B***, then the block multiplication of ***AB*** allowed.



***Important special case***





**Matrix Form of the Equations**

The coefficient matrix is 

The equivalent matrix equation is in the form :



Multiplication by ***rows*** 

Multiplication by ***columns*** 



***Identity Matrix***

The identity matrix is given by the form:  

***Properties of Matrix***

**Addition and Scalar Multiplication**

 *Commutative Property of Addition*

 *Associative Property of Addition*

 *Associative Property of Scalar Multiplication*

 *Distributive Property*

 *Distributive Property*

 *Distributive Property*

 *Distributive Property*

 *Additive Identity Property*

 *Additive Inverse Property*



***Multiplication***

 *Commutative “****law****” is usually broken*

 *Associative Property of Multiplication* (***Parentheses not needed***)

 *Distributive Property*

 *Distributive Property*

 *Distributive Property*

 *Distributive Property*

**\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\***

Consider the three vectors:

  

The linear combinations in three-dimensional space are 

***Combination*** 

Combine the three vectors  into on matrix *A*.



Multiplies the matrix *A* by a vector *x*, where  are the component of a vector *x*.



We can rewrite the form, matrix *A* times the vector *x*, as the combination 



Write the matrix in the form 



Where the *x* is the input and *b* is the output.

**Cyclic Difference**

The linear combinations of three vectors  lead to a cyclic difference matrix *C* and is given by:

  



The matrix *C* is not triangular. It is not easy to find the solution to , because either we are going to have ***infinitely many solution*** or ***no solution***.

Let looks at these problems geometrically.



***Exercises Section* 1.3 – The Algebra of Matrices**

1. For the matrices:  and , when does 

(**2 − 8**) Find values for the variables so that the matrices are equal.

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. Find a combination  that gives the zero vector:



Those vectors are independent or dependent?

The vectors lie in a \_\_\_\_\_\_.

The matrix *W* with those columns is not invertible.

1. The very last words say that the 5 by 5 centered difference matrix is not invertible, Write down the 5 equations . Find a combination of left sides that gives zero. What combination of must be zero?
2. A direct graph starts with *n* nodes. There are possible edges, each edge leaves one of the *n* nodes and enters one of the *n* nodes (possibly itself). The *n* by *n* adjacency matrix has  when edge leaves node *i* and enter node *j*; if no edge then . Here are directed graphs and their adjacency matrices:

 

 

The *i*, *j* entry of is .

Why does that sum count the two-step paths from *i* to any node to *j*?

The *i*, *j* entry of  counts *k*-steps paths:



List all 3-step paths between each pair of nodes and compare with . When  has ***no zeros***, that number *k* is the diameter of the graph – the number of edges needed to connect the most pair of nodes. What is the diameter of the second graph?

1. *A* is 3 by 5, *B* is 5 by 3, *C* is 5 by 1, and *D* is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

*a*) *AB* *b*) *BA* *c*) *ABD* *d*) *DBA*

*e*) *ABC* *f*) *ABCD* *g*) *A*(*B + C*)

1. What rows or columns or matrices do you multiply to find.
2. The third column of *AB*?
3. The second column of *AB*?
4. The first row of *AB*?
5. The second row of *AB*?
6. The entry in row 3, column 4 of *AB*?
7. The entry in row 2, column 3 of *AB*?
8. Add *AB* to *AC* and compare with :



1. True or False
2. If  is defined then *A* is necessarily square.
3. If  and  are defined then *A* and *B* are squares.
4. If  and  are defined then and  are squares.
5. If , then 
6. *a*) Find a nonzero matrix *A* such that 

*b*) Find a matrix that has  but 

1. Suppose you solve  for three special right sides *b*:



If the three solutions  are the columns of a matrix *X*, what is *A* times *X*?

1. Show that  is different from , when



Write down the correct rule for 

(19 – 22) Find the product of the 2 matrices by rows or by columns:

|  |  |
| --- | --- |
|  |  |

1. Given   Find 

(**24 – 37**) Find  and , if possible

|  |  |
| --- | --- |
|  |  |

1. Consider the matrices

    

Compute the following (where possible):

*a*)  *b*)  *c*)  *d*)  *e*)  *f*) 

1. Consider the matrices

   

Compute the following (where possible):

*a*)  *b*)  *c*)  *d*)  *e*)  *f*) 

*g*)  *h*)  *i*)  *j*)  *k*) 

1. Let , show that 
2. Let , show that 
3. Let . Prove that  if 
4. Let . Prove that  if 
5. The following system of recurrence relations holds for all 



Solve the system for  and  in terms of  and 

1. If , prove that 
2. If , use the fact  and mathematical induction, to prove that



1. A sequence of numbers  satisfies the recurrence relation  for , where *a* and *b* are constants. Prove that



Where  and hence express  in terms of .

If  and , use the previous question to find a formula for  in terms  and 