***Solution Section* 1.4 – Inverse Matrices - Finding **

***Exercise***

Apply Gauss-Jordan method to find the inverse of this triangular “Pascal matrix”

***Triangular Pascal matrix*** 

***Solution***











* The inverse matrix  looks like *A*, except odd-numbered diagonals are multiplied by -1.

***Exercise***

If *A* is invertible and , prove that 

***Solution***

 ***Multiply by  both sides***.

 ***Multiplication is associative***

 





***Exercise***

If , find two matrices  such that 

***Solution***

Let 











***Exercise***

If *A* has ***row*** 1 + ***row*** 2 = ***row*** 3, show that *A* is not invertible

1. Explain why  can’t have a solution.
2. Which right sides  might allow a solution to
3. What happens to ***row*** 3 in elimination?

***Solution***

1. Let be the row vectors of *A* and *x* is a solution to .

Then .

Since 

Means 

Implies  a contradiction

1. If 

Since 





1. In the elimination matrix, the third row will be zero.

***Exercise***

True or false (with a counterexample if false and a reason if true):

1. A 4 by 4 matrix with a row of zeros is not invertible.
2. A matrix with 1’s down the main diagonal is invertible.
3. If *A* is invertible then  is invertible.
4. If *A* is invertible then  is invertible.

***Solution***

1. True, because it can have at most 3 pivots.
2. False, if the matrix:  and only has 2 pivots, thus is not invertible.
3. True, If *A* is invertible then necessarily is invertible.
4. True,  where *x* is nonzero matrix.



Since *A* is invertible, this can only be true if x was zero to begin with. Thus  must also be invertible.

***Exercise***

Do there exist 2 by 2 matrices *A* and *B* with real entries such that , where *I* is the identity matrix?

***Solution***

Let 



















Therefore,  for any 2 by 2 matrices.

***Exercise***

If *B* is the inverse of , show that  is the inverse of *A*.

***Solution***

Since *B* is the inverse of  that implies: 

Show that  is the inverse of *A*











Therefore,  is the inverse of *A*.

***Exercise***

Find and check the inverses (assuming they exist) of these block matrices.



***Solution***

































***Exercise***

For which three numbers *c* is this matrix not invertible, and why not?



***Solution***

,  (zero column 2 / row 2)

,  (equal rows)

,  (equal columns)

***Exercise***

Find  and  (if they exist) by elimination.



***Solution***

























 ***doesn’t*** exist, and if we add the columns in *B*, the result is zero.

***Exercise***

Find  using the Gauss-Jordan method, which has a remarkable inverse.



***Solution***











***Exercise***

Find the inverse, if exists of 

***Solution***





***Exercise***

Find the inverse, if exists of 

***Solution***







***Exercise***

Find the inverse, if exists of 

***Solution***







***Exercise***

Find the inverse, if exists, of 

***Solution***







***Exercise***

Find the inverse, if exists, of 

***Solution***





∴ Inverse ***doesn’t exist***

***Exercise***

Find the inverse of *A* = 

***Solution***

 

 

 

 





***Exercise***

Find the inverse of *A* = 

***Solution***







***Exercise***

Find the inverse of *A* = 

***Solution***







***Exercise***

Find the inverse of *A* = 

***Solution***







***Exercise***

Find the inverse of 

***Solution***







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***Solution***





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***Solution***





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***Solution***





***Exercise***

Find the inverse of 

***Solution***



∴ Inverse ***doesn’t exist***

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***Solution***



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***Exercise***

Find the inverse of 

***Solution***





***Exercise***

Find the inverse of 

***Solution***



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***Solution***



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***Exercise***

Find the inverse of 

***Solution***





***Exercise***

Find the inverse of 

***Solution***



∴ Inverse ***doesn’t exist***

***Exercise***

Find the inverse of 

***Solution***



∴ Inverse ***doesn’t exist***

***Exercise***

Find 

***Solution***

 





 





***Exercise***

Find, where 

***Solution***

 



 

 

 





***Exercise***

Find, where 

***Solution***

 

 

 

 

  





***Exercise***

Find, where 

***Solution***

  

  

  

  

  



∴ The inverse matrix ***doesn’t exist***

***Exercise***

Find the inverse, if exists, of 

***Solution***

 



  



  





***Exercise***

Find the inverse, if exists, of 

***Solution***

 



  









***Exercise***

Find the inverse, if exists, of 

***Solution***















***Exercise***

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∴ Inverse ***does not exist***

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***Exercise***

Find the inverse, if exists of 

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***Exercise***

Find the inverse, if exists of 

***Solution***

















***Exercise***

Find the inverse, if exists of 

***Solution***











***Exercise***

Find the inverse, if exists of 

***Solution***



Since this matrix is ***singular***, row 3 all zeros.

***Exercise***

Find the inverse, if exists, of 

***Solution***









∴ Inverse ***does not exist***

***Exercise***

Find the inverse, if exists, of 

***Solution***



















***Exercise***

Find the inverse, if exists, of 

***Solution***













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***Exercise***

Show that *A* is not invertible for any values of the entries



***Solution***

Since the matrix *A* had zero’s on its diagonals, therefore *A* is not invertible.

***Exercise***

Prove that if *A* is an invertible matrix and *B* is row equivalent to *A*, then *B* is also invertible.

***Solution***

Since *B* is row equivalent to *A*, there exist some elementary matrices  such that . Because  and *A* are invertible, then *B* is also invertible.

***Exercise***

Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying 

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 





1. 





The inverse matrix doesn’t exist

***Exercise***

Show that the inverse of  is 

***Solution***





 







***Exercise***

If the product  is invertible (and *A* & *B* are square matrices), find a formula for  that involves  and *B*.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

***Solution***

Since  is invertible, the 











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***Exercise***

Prove that if *A* is an  matrix, there is an invertible matrix *C* such that  is in reduced row-echelon form.

***Solution***

The reduced row-echelon form of *A* can be written in the form  . where  are elementary matrices.

Let , then *C* is invertible since  are invertible.

Hence, there exists such a matrix *C*.

***Exercise***

Prove that 2  matrices *A* and *B* are row equivalent if and only if there exists a nonsingular matrix *P* such that 

***Solution***

Suppose that , then there exist elementary matrices  such that .

Let  ⇒ by the theorem, *P* is nonsingular.

Suppose that , for some nonsingular matrix *P*. By the theorem, *P* is row equivalent to . That is, .

Thus,  and this implies that *A* is row equivalent to *B*.

***Exercise***

Let *A* and *B* be 2  matrices. Suppose *A* is row equivalent to *B*. Prove that *A* is nonsingular if and only if *B* is nonsingular.

***Solution***

Suppose that *A* is row equivalent to *B*. Then, there exists a nonsingular matrix *P* such that .

If *A* is nonsingular then *B* is nonsingular.

Conversely, if *B* is nonsingular then  is nonsingular.

***Exercise***

Show that if *A* and *B* are two  invertible matrices then *A* is row equivalent to *B*.

***Solution***

Since *A* is invertible, then *A* is a row equivalent to . That is, there exist elementary matrices  such that .

Similarly, there exist elementary matrices  such that .

Hence, 





That is, *A* row equivalent to *B*.

***Exercise***

Prove that a square matrix *A* is nonsingular if and only if *A* is a product of elementary matrices.

***Solution***

Suppose that *A* is nonsingular. Then *A* is row equivalent to . That is, there exist elementary matrices  such that .

But each  is an elementary matrix.

Conversely, suppose that , then 

That is, *A* is nonsingular.

***Exercise***

Show that if  (that is, if they are row equivalent), then  for some matrix *E* which is a product of elementary matrices.

***Solution***

If , there is some sequence of elementary row operations which, when performed on *A*, produce *B*.

Further, multiplying on the left by the corresponding elementary matrix is the same as performing that row operation. So we have









Thus, if , then we have 

***Exercise***

Show that if  for some matrix *E* which is a product of elementary matrices, then  for every  matrix *C*.

***Solution***

Let , where each  is an elementary matrix.







 since 



Therefore; 

***Exercise***

Let  be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system  also has only trivial solution.

***Solution***

Since *A* is a square matrix, thus *A* has only the trivial solution. That implies that *A* is invertible.

But  is also invertible so  has only trivial solution.

***Exercise***

Let  be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible  matrix. Show that  has just trivial solution if and only if  has just trivial solution.

***Solution***

*A* is a square  matrix. If  has just a trivial solution, then *A* is invertible. Since *Q* is an invertible  matrix, then  is also invertible.

Thus,  has trivial solution.

On the other hand, if  has trivial solution, then  is also invertible.

Since *Q* is invertible, then  is also invertible.

Thus,  is invertible, i.e  has just trivial solution, equivalent  has just trivial solution if and only if  has just trivial solution.

***Exercise***

Let  be any consistent system of linear equations, and let  be a fixed solution. Show that every solution to the system can be written in the form  where  is a solution to . Show also that every matrix of this form is a solution.

***Solution***

Since  is a solution to , we have .

Adding  to , then





As adding an equation to the original equation does not affect the solution.

If we let  be a fixed solution, then every solution to  is .

Besides,







So, every matrix (vector) in the form  is a solution to 

***Exercise***

If *A* and *B* are  matrices satisfying . Prove that .

***Solution***

Since , then *A*, *B*, *AB* are nonsingular.











 *√*

***Exercise***

Let  . Verify that , then find  in term of *A*.

***Solution***















Since 







***Exercise***

Consider , thus if *B* is the inverse of *A*, then  becomes . On the other hand *B* is a product of elementary matrices since it is invertible. This indicates that the inverse of *A* can be obtained by applying elementary row operations to  to get .

Find the inverses of

|  |  |
| --- | --- |
|  |  |

***Solution***

1.  

 





1. First, we have move row 4 to row 1, for the calculation

 

 

 

 





Since we move Row 4 to Row 1, we must move Column 1 to Column 4 to get the inverse matrix.



***Exercise***

Let , *A* and *C* are invertible. Find

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 

















1. 







***Exercise***

Suppose that *A, B*, and *A* – *B* are invertible  matrices. Show that



***Solution***

*A, B*, and *A* – *B* are invertible Then

**



Let:



Then, we need to prove that



























 ***√***

Therefore; 

***Exercise***

Suppose *P* is invertible and . Solve for *B* in terms of *A*.

***Solution***

Since *P* is invertible, then 



 

 



***Exercise***

Suppose **,where *A* and *B* are  matrices and *C* is invertible. Show that  .

***Solution***

Since *C* is invertible, then 

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