***Section* 1.4 – Inverse Matrices - Finding** 

***Definition***

The matrix *A* is invertible if there exists a matrix  such that:



 and *A* has to be a ***square matrix***.

***Not all matrices have inverses***.

1. The inverse exists *iff* elimination produces *n* pivots (row exchanges allow).
2. The matrix *A* cannot have two different inverses.
3. If *A* is invertible, the one and only one solution to  is 



 ***Multiply both side by A-1***

 ***Associate property***

 ***Multiplicative inverse property***

 ***Identity property***

1. Suppose there is a ***nonzero*** vector *x* such that . Then *A* cannot have an inverse
2. A 2 by 2 matrix is invertible iff  is not zero.

 ⇒  ***Only for 2 by 2 matrices***

If is the determinant, then doesn’t exist

**The Inverse of a Product** 

***Theorem***

If an  matrix has an inverse, that inverse is unique.

***Proof***

Suppose that *A* has an inverse  and *B* is a matrix such that 











Therefore, the inverse is unique

***Theorem***

If *A* and *B* are invertible then so is  Theinverse of a product is 

***Proof***









***Reverse Order***





***Theorem***

If *A* is invertible and *n* is a nonnegative integer, then:

1.  is *invertible* and 
2.  is *invertible* and 
3.  is *invertible* for any nonzero scalar *k*, and 

***Proof***

















**Finding**  **using Gauss-Jordan Elimination**



Find 

  





  

 

* Matrix ***A*** is ***symmetric*** across its main diagonal. So is 
* Matrix ***A*** is ***tridiagonal*** (only three nonzero diagonals). But  is a full matrix with no zeros. (another reason we don’t compute )

**Singular *versus* Invertible**

 exists when *A* has a full set of *n* pivots. (Row exchanges allowed)

* With *n* pivots, elimination solves all the equations . The columns  go into . Then  is at least a ***right-inverse***.
* Elimination is really a sequence of multiplications.

***Conclusion***

* If *A* doesn’t have *n* pivots, elimination will lead to a ***zero row***.
* Elimination steps are taken by an invertible *M*. So a row of *MA* is zero.
* If  then . The zero row of *MA*, times *B*, gives a zero row of *M*.
* The invertible matrix *M* can’t have a zero row! A must have *n* pivots if .

***Elementary Matrices***

***Definition***

Let ***e*** be an elementary row operation. Then the  ***elementary matrix*** *E* associated with ***e*** is the matrix obtained by applying ***e*** to the identity matrix. Thus



***Example***

1. 
2. 
3. 
4. 

***Theorem***

Let *e* be an elementary operation and let *E* be the corresponding elementary matrix . Then for every  matrix *A*



That is, an elementary row operation can be performed on *A* by multiplying *A* on the left by the corresponding elementary matrix.

***Example***

Let    





This result can be obtained from *A* by multiplying the first row by 2.





This result can be obtained from *A* by interchanging rows 2 and 3.





This result can be obtained from *A* by adding 3 times row 1 to row 3.

**Uniqueness of Echelon Form**

Two matrices *A* and *B* are row-equivalent if and only if they have the same reduced echelon form.

***Proof***

If *A* and *B* have the same reduced echelon form *E*, then *A* is row-equivalent to *E* and *E* is row-equivalent to *B*. It follows that *A* is row-equivalent to *B*.

Now Suppose *A* and *B* are row-equivalent. Let  be a reduced echelon form of *A* and  be a reduced echelon form of *B*. Then  and  are row equivalent.

Suppose . Since  and  are row equivalent, for some matrix *C*. This means  and . But then .

***Example***

Show that the two matrices are row equivalent



***Solution***







***Definition***

A relationship ~ (equivalent) between elements of a set is called an equivalence relation if

* *A* ~ *A* is always true,
* *A* ~ *B* always implies *B* ~ *A*,
* *A* ~ *B* and *B* ~ *C* always implies *A* ~ *C*.

***Exercises Section* 1.4 – Inverse Matrices - Finding** 

1. Apply Gauss-Jordan method to find the inverse of this triangular “Pascal matrix”

***Triangular Pascal matrix*** 

1. If *A* is invertible and , prove that 
2. If , find two matrices  such that 
3. If *A* has ***row*** 1 + ***row*** 2 = ***row*** 3, show that *A* is not invertible
4. Explain why  can’t have a solution.
5. Which right sides  might allow a solution to
6. What happens to ***row*** 3 in elimination?
7. True or false (with a counterexample if false and a reason if true):
8. A 4 by 4 matrix with a row of zeros is not invertible.
9. A matrix with 1’s down the main diagonal is invertible.
10. If *A* is invertible then  is invertible.
11. If *A* is invertible then  is invertible.
12. Do there exist 2 by 2 matrices *A* and *B* with real entries such that , where *I* is the identity matrix?
13. If *B* is the inverse of , show that  is the inverse of *A*.
14. Find and check the inverses (assuming they exist) of these block matrices.



1. For which three numbers *c* is this matrix not invertible, and why not?



1. Find  and  (if they exist) by elimination.



1. Find  using the Gauss-Jordan method, which has a remarkable inverse.



Find the inverse, if exists, of

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| 6. *A* = 7. *A* = 8. *A* = 9. *A* = |  | |  | |
|  | |  | |

1. Show that *A* is not invertible for any values of the entries



1. Prove that if *A* is an invertible matrix and *B* is row equivalent to *A*, then *B* is also invertible.
2. Determine if the given matrix has an inverse, and find the inverse if it exists. Check your answer by multiplying 

|  |  |
| --- | --- |
|  |  |

1. Show that the inverse of  is 
2. If the product  is invertible (and *A* & *B* are square matrices), find a formula for  that involves  and *B*.

Hence, it is not possible to multiply a non-invertible matrix by another matric and obtain an invertible matrix as a result.

1. Prove that if *A* is an  matrix, there is an invertible matrix *C* such that  is in reduced row-echelon form.
2. Prove that 2  matrices *A* and *B* are row equivalent if and only if there exists a nonsingular matrix *P* such that 
3. Let *A* and *B* be 2  matrices. Suppose *A* is row equivalent to *B*. Prove that *A* is nonsingular if and only if *B* is nonsingular.
4. Show that if *A* and *B* are two  invertible matrices then *A* is row equivalent to *B*.
5. Prove that a square matrix *A* is nonsingular if and only if *A* is a product of elementary matrices.
6. Show that if  (that is, if they are row equivalent), then  for some matrix *E* which is a product of elementary matrices.
7. Show that if  for some matrix *E* which is a product of elementary matrices, then  for every  matrix *C*.
8. Let  be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system  also has only trivial solution.
9. Let  be a homogeneous system of *n* linear equations in *n* unknowns, and let *Q* be an invertible  matrix. Show that  has just trivial solution if and only if  has just trivial solution.
10. Let  be any consistent system of linear equations, and let  be a fixed solution. Show that every solution to the system can be written in the form  where  is a solution to . Show also that every matrix of this form is a solution.
11. If *A* and *B* are  matrices satisfying . Prove that .
12. Let  . Verify that , then find  in term of *A*.
13. Consider , thus if *B* is the inverse of *A*, then  becomes . On the other hand *B* is a product of elementary matrices since it is invertible. This indicates that the inverse of *A* can be obtained by applying elementary row operations to  to get .

Find the inverses of

|  |  |
| --- | --- |
|  |  |

1. Let , *A* and *C* are invertible. Find

|  |  |
| --- | --- |
|  |  |

1. Suppose that *A, B*, and *A* – *B* are invertible  matrices. Show that



1. Suppose *P* is invertible and . Solve for *B* in terms of *A*.
2. Suppose **,where *A* and *B* are  matrices and *C* is invertible. Show that  .