***Solution Section* 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices**

***Exercise***

Solve  to find***c***. Then solve  to find ***x***. What was *A*?



***Solution***





 





 





***Exercise***

Find *L* and *U* for the symmetric matrix



Find four conditions on *a, b, c, d* to get  with four pivots

***Solution***





***Exercise***

Determine whether the given matrix is invertible



***Solution***

The matrix is a diagonal matrix with nonzero entries on the diagonal, so it is invertible.



***Exercise***

Find  by inspection 

***Solution***













***Exercise***

Find  by inspection 

***Solution***













***Exercise***

Find  by inspection 

***Solution***







***Exercise***

Decide whether the given matrix is symmetric 

***Solution***

Not *symmetric*, since 

***Exercise***

Decide whether the given matrix is symmetric 

***Solution***

*Symmetric*

***Exercise***

Decide whether the given matrix is symmetric 

***Solution***

Not *symmetric*, since 

***Exercise***

Find all values of the unknown constant(s) in order for *A* to be symmetric



***Solution***

 

***Exercise***

Find a diagonal matrix *A* that satisfies the given condition 

***Solution***











***Exercise***

Let *A* be an  symmetric matrix

1. Show that  is symmetric
2. Show that  is symmetric

***Solution***

1. The property of the transpose states that 





 ***A is symmetric***



∴  is symmetric

1. 

 ***A and I are symmetric***



***∴***  is ***Symmetric***

***Exercise***

Prove if , then *A* is symmetric and 

***Solution***

If , then









So *A* is symmetric.

Since 

 



***Exercise***

A square matrix *A* is called ***skew-symmetric*** if . Prove

1. If *A* is an invertible skew-symmetric matrix, then  is skew-symmetric.
2. If *A* and *B* are skew-symmetric matrices, then so are  for any scalar *k*.
3. Every square matrix *A* can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.



***Solution***

1. 

 ***skew-symmetric***



∴  is also skew-symmetric

1. Let *A* and *B* are skew-symmetric matrices























1. We need to prove from the hint that is symmetric and  is skew-symmetric





*Thus*  ***is symmetric***







*Thus*  ***is skew-symmetric***

***Exercise***

Suppose *R* is rectangular (*m* by *n*) and *A* is symmetric (*m* by *m*)

1. Transpose  to show its symmetric
2. Show why  has no negative numbers on its diagonal.

***Solution***

1. 







1. 

= ***Product of the diagonal entry by itself.***

= length squared of column j.

***Exercise***

If *L* is a lower-triangular matrix, then  is \_\_\_\_\_\_\_Triangular

***Solution***

 is ***upper*** triangular.

 is a lower-triangular because *L* is.

The transpose carries the lower-triangular matrices to the upper-triangular (and vice versa).

***Exercise***

True or False

1. The block matrix  is automatically symmetric
2. If *A* and *B* are symmetric then their product is symmetric
3. If *A* is not symmetric then  is not symmetric
4. When *A, B, C* are symmetric, the transpose of *ABC* is *CBA*.
5. The transpose of a diagonal matrix is a diagonal.
6. The transpose of an upper triangular matrix is an upper triangular matrix.
7. The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.
8. All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.
9. All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.
10. The inverse of an invertible lower triangular matrix is an upper triangular matrix.
11. A diagonal matrix is invertible if and only if all of its diagonal entries are positive.
12. The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.
13. A matrix that is both symmetric and upper triangular must be a diagonal matrix.
14. If *A* and *B* are  matrices such that  is symmetric, then *A* and *B* are symmetric.
15. If *A* and *B* are  matrices such that  is upper triangular, then *A* and *B* are upper triangular.
16. If  is a symmetric matrix, then *A* is a symmetric matrix.
17. If  is a symmetric matrix for some , then *A* is a symmetric matrix.

***Solution***

1. ***False***: 
2. ***False*** 
3. ***True*** by definition.
4. ***True***  Since 
5. ***True*** Since a diagonal matrix must be square and have zeros off the main diagonal, its transpose is also diagonal.
6. ***False*** The transpose of an upper triangular matrix is lower triangular.
7. ***False*** 
8. ***True*** The entries above the main diagonal determine the entries below the main diagonal in a symmetric matrix.
9. ***True*** in an upper triangular matrix, the series below the main diagonal are all zeros.
10. ***False*** The inverse of an invertible lower triangular matrix is lower triangular.
11. ***False*** The diagonal entries may be negative, as long as they are nonzero.
12. ***True*** Adding a diagonal matrix to a lower triangular matrix will not create nonzero entries above the main diagonal.
13. ***True*** Since the entries below the main diagonal must be zero, so also must be the entries above the main diagonal.
14. ***False *** which is symmetric
15. ***False *** which is upper triangular.
16. ***False*** 
17. ***True***  then





 since  then 

Therefore, *A* is a symmetric matrix

***Exercise***

Find 2 by 2 symmetric matrices  with these properties

1.  is not invertible
2. *A* is invertible but cannot be factored into *LU* (row exchanges needed)
3. *A* can be factored into  but not into  (because of negative *D*)

***Solution***

1. 
2.  only need a *zero* in the diagonal.
3. 









***Exercise***

A group of matrices includes *AB* and  if it includes *A* and *B* . “Products and inverses stay in the group.” Which of these sets are groups?

Lower triangular matrices *L* with 1’s on the diagonal, symmetric matrices *S*, positive matrices *M*, diagonal invertible matrices *D*, permutation matrices *P*, matrices with . ***Invent two more matrix groups***.

***Solution***

The lower triangular matrices *L* with 1’s on the diagonal form a group.

Clearly the product of two is a third. The Gauss-Jordan method shows that the inverse of one is another.

The symmetric matrices don’t form a group. An example of the 2 symmetric matrices *A* and *B* whose product is not symmetric



The positive matrices do not form a group.

, the inverse is not symmetric.

The diagonal invertible matrices form a group.

The permutation matrices form a group.

The matrices with  form a group. If *A* and *B* are two matrices, then so are *AB* and , as





There are many more matrix groups. For example, given two, the block matrices  form a third as *A* ranges over the first group and *B* ranges over the second.

Another example is the set of all products *cP* where *c* is a nonzero scalar and *P* is a permutation matrix of given size.

***Exercise***

Write  as the product *EH* of an elementary row operation matrix *E* and a symmetric matrix *H*.

***Solution***







An elementary row operation matrix has the form 

The inverse is: 





Since matrix *H* is symmetric, therefore:







***Exercise***

When is the product of two symmetric matrices symmetric? Explain your answer.

***Solution***

 is symmetric *iff *



 ***A and B are symmetric***



 is symmetric iff *A* and *B* commute

***Exercise***

Express  in terms of  and 

***Solution***





***Exercise***

Find the transpose of the given matrix: 

***Solution***



***Exercise***

Show that if *A* is symmetric and invertible, then  is also symmetric.

***Solution***

*A* is symmetric and invertible, then 





 is symmetric.

***Exercise***

Prove that 

***Solution***

Let  and 

Then the *ij*-entry of *AB* is:



The reverse order, *ji*-entry of 

Column *j* of *B* becomes row *j* of , and row *i* of *A* becomes column *i* of .

Thus, the *ij*-entry of  is:



Thus 

***Exercise***

For the given matrix, compute , , , and , then compare and 



***Solution***







 





 





***Exercise***

Show that a  lower triangular matrix is invertible if and only if  and in this case the inverse is also lower triangular.

***Solution***

Let *A* to be the lower triangular matrix



 is invertible iff  and then





***Exercise***

Let *A* be any  diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that *A* has an inverse. Compute the inverse of any such matrix.

***Solution***

Let 



So,  exists when both entries on the main diagonal are nonzero.