***Section* 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices**

***Transpose***

***Definition***

The transpose of a matrix *A* is defined as the matrix that is obtained by interchanging the corresponding rows and columns in *A*. Then the transpose of *A*, denoted by  or .

*The columns of* *are the rows of A*.

When *A* is an *m* by *n* matrix, the transpose is *n* by *m*:



The matrix ***flips over*** the main diagonal. The entry in row *i*, column *j* of  comes from row *j*, column *i* of the original *A*.



**Properties of Transpose**

1. 
2. 
3. 
4. 
5. 

*The transpose of a product of any number of matrices is the product of the transposes in the reverse order.*

***Theorem***

If *A* is an invertible matrix, then  is also invertible and



***Proof***













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***Trace***

***Definition***

If *A* is a square matrix, then the trace of *A*, denoted by **tr(*A*)**, is defined to the sum of the entries on the main diagonal of *A*. The trace of *A* is undefined if *A* is not a square matrix.

***Example***





***Diagonal***

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. A general  diagonal matrix can be written as



A diagonal matrix is invertible iff all of its diagonal entries are nonzero; the



Powers of diagonal matrices are:



***Triangular Matrices***

A square matrix in which all the entries above the main diagonal are zero is called ***lower diagonal triangular***.

A square matrix in which all the entries below the main diagonal are zero is called ***upper diagonal triangular***.

A matrix that is either upper triangular or lower triangular is called ***triangular***.

 

***lower diagonal triangular upper diagonal triangular***

***Theorem***

* The transpose of a lower triangular matrix is upper triangular, and the transpose of a upper triangular matrix is lower triangular.
* The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.
* A triangular matrix is invertible iff its diagonal entries are all nonzero.
* The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

***Example***

, 

***Solution***

  

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

***The factors are triangular matrices.***

***The factorization that comes from elimination is*** ***.***

***Symmetric Matrices***

***Definition***

A square matrix *A* is said to be ***symmetric*** if . That means a square matrix must satisfies 

***Example***





* The ***inverse*** of a symmetric matrix is also ***symmetric***.

***Example***

Given , show that the inverse is symmetric too?

***Solution***



***Theorem***

If *A* and *B* are symmetric matrices with the same size, and if *k* is any scalar, then:

1.  is symmetric
2. *A* + *B* and *A* – *B* are symmetric.
3. *kA* is symmetric

* If *A* is an invertible symmetric matrix, then  is *symmetric*.

***Proof***

Assume that *A* is symmetric and invertible then 



Which proves that  is *symmetric*

* Multiplying *M* by  gives a symmetric matrix.

***Proof***

The entry  of , it is the dot product of ***row*** *i* of  (column *i* of *M*) with column *j* of *M*. The  entry is the same dot product, column *j* with column *i*. so  is symmetric.

The matrix  is also symmetric and  is a different matrix from .

* If *A* is an invertible symmetric matrix, then and  are also invertible.
* Matrix *A*is symmetric across its main diagonal. So is 
* Matrix *A* is tridiagonal (only three nonzero diagonals). But  is a full matrix with no zeros. (another reason we don’t compute )

***Example***

Given  and . Find  and 

***Solution***









***Symmetric in* LDU**

When elimination is applied to a symmetric matrix,  is an advantage.





* If  can be factored into *LDU* with no row exchanges, then . The ***symmetric*** ***factorization*** ***of a symmetric matrix is*** 

***Exercises Section* 1.5 – Transpose, Diagonal, Triangular, and Symmetric Matrices**

1. Solve  to find***c***. Then solve  to find ***x***. What was *A*?



1. Find *L* and *U* for the symmetric matrix



Find four conditions on *a, b, c, d* to get  with four pivots

1. Determine whether the given matrix is invertible



1. Find  by inspection

  

1. Decide whether the given matrix is symmetric

  

1. Find all values of the unknown constant(s) in order for *A* to be symmetric



1. Find a diagonal matrix *A* that satisfies the given condition 
2. Let *A* be an  symmetric matrix
3. Show that  is symmetric
4. Show that  is symmetric
5. Prove if , then *A* is symmetric and 
6. A square matrix *A* is called ***skew-symmetric*** if . Prove
7. If *A* is an invertible skew-symmetric matrix, then  is skew-symmetric.
8. If *A* and *B* are skew-symmetric matrices, then so are  for any scalar *k*.
9. Every square matrix *A* can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.



1. Suppose *R* is rectangular (*m* by *n*) and *A* is symmetric (*m* by *m*)
2. Transpose  to show its symmetric
3. Show why  has no negative numbers on its diagonal.
4. If *L* is a lower-triangular matrix, then  is \_\_\_\_\_\_\_Triangular
5. True or False
6. The block matrix  is automatically symmetric
7. If *A* and *B* are symmetric then their product is symmetric
8. If *A* is not symmetric then  is not symmetric
9. When *A, B, C* are symmetric, the transpose of *ABC* is *CBA*.
10. The transpose of a diagonal matrix is a diagonal.
11. The transpose of an upper triangular matrix is an upper triangular matrix.
12. The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.
13. All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.
14. All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.
15. The inverse of an invertible lower triangular matrix is an upper triangular matrix.
16. A diagonal matrix is invertible if and only if all of its diagonal entries are positive.
17. The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.
18. A matrix that is both symmetric and upper triangular must be a diagonal matrix.
19. If *A* and *B* are  matrices such that  is symmetric, then *A* and *B* are symmetric.
20. If *A* and *B* are  matrices such that  is upper triangular, then *A* and *B* are upper triangular.
21. If  is a symmetric matrix, then *A* is a symmetric matrix.
22. If  is a symmetric matrix for some , then *A* is a symmetric matrix.
23. Find 2 by 2 symmetric matrices  with these properties
24.  is not invertible
25. *A* is invertible but cannot be factored into *LU* (row exchanges needed)
26. *A* can be factored into  but not into  (because of negative *D*)
27. A group of matrices includes *AB* and  if it includes *A* and *B* . “Products and inverses stay in the group.” Which of these sets are groups?

Lower triangular matrices *L* with 1’s on the diagonal, symmetric matrices *S*, positive matrices *M*, diagonal invertible matrices *D*, permutation matrices *P*, matrices with . ***Invent two more matrix groups***.

1. Write  as the product *EH* of an elementary row operation matrix *E* and a symmetric matrix *H*.
2. When is the product of two symmetric matrices symmetric? Explain your answer.
3. Express  in terms of  and 
4. Find the transpose of the given matrix: 
5. Show that if *A* is symmetric and invertible, then  is also symmetric.
6. Prove that 
7. For the given matrix, compute , , , and , then compare and 



1. Show that a  lower triangular matrix is invertible if and only if  and in this case the inverse is also lower triangular.
2. Let *A* be any  diagonal matrix. Give a necessary and sufficient condition on the diagonal entries so that *A* has an inverse. Compute the inverse of any such matrix.