***Solution Section* 1.6 – The Properties of Determinants**

***Exercise***

Verify that  when: 

***Solution***

















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***Exercise***

For which value(s) of ***k*** does *A* fail to be invertible? 

***Solution***

For ***A*** to have an invertible the determinant cannot be equal to zero. To ***fail*** det(A) = 0.











***Exercise***

Without directly evaluating, show that 

***Solution***



It is equal to zero, since first row and third row are proportional.





***Exercise***

If the entries in every row of *A* add to zero, solve ***Ax*** = 0 to prove det *A* = 0. If those entries add to one, show that det (*A – I*) = 0. Does this mean det *A = I*?

***Solution***

If ***x*** = (1, 1, … , 1), then ***Ax*** = the sums of the rows of ***A***. Since every row of *A* add to zero, that implies ***Ax*** = 0. Since A has non-zero nullspace, it is not invertible and det *A* = 0. If the entries in every row of *A* sum to one, then the entries in every row of *A –* I sum to zero. A – I has a non-zero nullspace and det (*A –* I) =0. This does not mean that det *A* = I.

***Example***:

 every row of *A* adds up to zero



***Exercise***

Does  in general?

1. True or false if *A* and *B* are square *n* x *n* matrices?
2. True or false if *A* is *m* x *n* and B is *n* x *m* with ?

***Solution***

1. Matrices *A* and *B* are square matrices, then by the property:







Therefore; it is true for any *A* and *B* square matrices.

1. False, example if 















***Exercise***

True or false, with a reason if true or a counterexample if false:

1. The determinant of  is 1 + det ***A***.
2. The determinant of ABC is .
3. The determinant of 4*A* is 
4. The determinant of *AB – BA* is zero. (try an example)
5. If *A* is not invertible then *AB* is not invertible.
6. The determinant of *A – B* equals to det *A* – det *B*.

***Solution***

1. ***False***, if 













1. ***True***, .
2. ***False***, in general  if *A* is *n* x *n*.
3. ***False***, 











1. False, any matrix is invertible, iff its determinant is nonzero. So det *A* = 0 which

. Therefore, *AB* can’t be invertible.

1. 







***Exercise***

Use row operations to show the 3 by 3 “Vandermonde determinant” is



***Solution***









 ***Multiply the main diagonal by* (*b - a*)**



***Exercise***

The inverse of a 2 by 2 matrix seems to have determinant = 1:



What is wrong with this calculation? What is the correct 

***Solution***

The  (*ad – bc*) it is part of the determinant and it is not the solution.







***Exercise***

A *Hessenberg* matrix is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci’s rule . The same rule will continue for all sizes . Which Fibonacci number is ?



***Solution***



The cofactor  for  is the determinant .



The cofactor 



 









The actual number: .

Since  follows Fibonacci’s rule , it must be .

***Exercise***

Evaluate 

***Solution***





***Exercise***

Evaluate 

***Solution***





***Exercise***

Evaluate 

***Solution***







***Exercise***

Evaluate 

***Solution***







***Exercise***

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***Exercise***

Find all the values of λ for which det(***A***) = 0: 

***Solution***





 ***Solve for λ.***



***Exercise***

Find all the values of λ for which det(***A***) = 0: 

***Solution***













***Exercise***

Prove that if a square matrix ***A*** has a column of zeros, then det(***A***) = 0

***Solution***

Consider a 3 by 3 matrix with a zero column, however to find the determinant we can interchange any column of that matrix; therefore:



By definition, the determinant of ***A*** using the cofactor:







***Exercise***

With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:



1. Why is the first statement true? Somehow *B* doesn’t enter.
2. Show by example that equality fails (as shown) when *C* enters.
3. Show by example that the answer  is also wrong.

***Solution***

1. If we don’t pick any 0 entries, then the first two columns are picked from ***A*** and the last two rows are from *D*. We can’t pick any columns or rows from *B*, because there aren’t any left.
2. 



and 

1. Use the example from part (*b*): 



***Exercise***

Show that the value of the following determinant is independent of *θ*.



***Solution***









Therefore, the determinant is independent of *θ*.

***Exercise***

Show that the matrices  commute if and only if 

***Solution***







Iff 



 ***√***

***Exercise***

Show that  for every  matrix A.

***Solution***

Let 

















***Exercise***

What is the maximum number of zeros that a  matrix can have without a zero determinant? Explain your reasoning.

***Solution***

The maximum number of zeros that a  matrix can have without a zero determinant is 12 zeros.

If the main diagonal has nonzero entries and the rest are zero, then the determinant of the matrix is equal to the product of the main diagonal entries.

***Exercise***

Evaluate *,* , and . Then verify that 



***Solution***

























 ***√***

***Exercise***

Show that  is not invertible for any values of *α, β, γ*

***Solution***















Therefore, this matrix in not invertible.

***Exercise***

The determinant of a  matrix  is .

Assuming no rows swaps are required, perform elimination on A and show explicitly that  is the product of the pivots.

***Solution***











***Exercise***

If *A* is a  matrix and let . What is ?

***Solution***





Multiplying a single row by 3 multiplies the determinant by 3.

Multiplying the whole  matrix by 3 multiplies all 7 rows by 3 .

∴ 



***Exercise***

Explain without computations why the following determinant is equal to zero



***Solution***

The determinant is equal to zero because there are too many zeros (as block ).

***Or***



Since row 5 is has zero entries, therefore the determinant is zero.

***Exercise***

Let *A* be an  real matrix.

1. Show that if  and *n* is odd, then .
2. Show that if , then *n* must be even.
3. Does part (*b*) remain true for complex matrices?

***Solution***

1. Given:  and *n* is odd





 Since *n* is odd



 only when 

1. 









If *n* is odd, then 

If *n* is even, then 

1. It can’t be true because 

And *A* is real matrix, the determinant has to be a real number.

***Exercise***

Let *A* and *C* be  and  matrices, respectively.

1. Show that 
2. Evaluate
3. 
4. 
5. 
6. 
7. Find a formula for 

***Solution***

1. If we let matrices *B* be  an **0** be , so the determinant of the matrix size will be , then









If we let matrices *B* be  an **0** be  , so the determinant of the matrix size will be , then









1. ***i*** - 







***ii*** - 





***iii*** - 







***iv*** - 

 





From that we can see that the signs are: 



1. 





***Exercise***

Let  and let



1. Show that, if  ,



1. Show that, if  ,



Where  means  with factor  missing.

1. Use part (*b*) to evaluate



***Solution***

1. ; with 

Using the mathematical Induction to prove the equality.

For :



















For , the proof is true.

Assume that is true for 







We need to prove it is also true for  









 ***√***

 is also true.

∴ by the mathematical induction, the proof is completed.

1. If 

Where  means  with factor  missing.

 

  missing











 



















1. 













***Exercise***

Let *A, B, C, D* 

1. Show that when *A* is invertible: 
2. Show that when : 
3. Can *B* and *C* on the right-hand side of the identity be switched?
4. Does part (*b*) remain true if the condition is dropped?

***Solution***

1. Since A in invertible, then  exists and 













1. When 



 







1. To switch *B* and *C* it is not necessary that 

Let 



























No, *B* and *C* on the right-hand side of the identity cannot be switched since 

1. No, since from previous part (*c*) *D* doesn’t commute necessarily.