***Solution Section* 2.5 – Subspaces, Span and Null Space**

***Exercise***

Suppose *S* and *T* are two subspaces of a vector space ***V***.

1. The sum  contains all sums  of a vector *s* in *S* and a vector *t* in *T*. Show that  satisfies the requirements (addition and scalar multiplication) for a vector space.
2. If *S* and *T* are lines in , what is the difference between  and ? That union contains all vectors from *S* and *T* or both. Explain this statement: The span of  is .

***Solution***

1. Let  be vectors in *S*, Let  be vectors in *T*, and let *c* be a scalar. Then

 and



Thus  is closed under addition and scalar multiplication, it satisfies the two requirements for a vector space.

1. If *S* and *T* are distinct lines, then *S* and *T* is a plane, whereas  is not even closed under addition. The span of  is the set of all combinations of vectors in this union. In particular, it contains all sums  of a vector ***s*** in *S* and a vector ***t*** in *T*, and these sums form .  contains both *S* and *T*; so, it contains .  is a vector space.
2. So, it contains all combinations of vectors in itself; in particular, it contains the span of . Thus, the span of  is .

***Exercise***

Determine which of the following are subspaces of ?

1. All vectors of the form (*a*, 0, 0)
2. All vectors of the form (*a*, 1, 1)
3. All vectors of the form (*a*, *b*, *c*), where *b* = *a* + *c*
4. All vectors of the form (*a*, *b*, *c*), where *b* = *a* + *c +* 1
5. All vectors of the form (*a*, *b*, 0)

***Solution***

1. 



This is a subspace of 

1.  which is not in the set.

Therefore, this is not a subspace of 

1. 













This is a subspace of 

1.  so  is not in the set.

Therefore, this is not a subspace of 

1. 



This is a subspace of 

***Exercise***

Determine which of the following are subspaces of ?

1. All sequences  in  of the form  = (*v*, 0, *v*, 0, …)
2. All sequences  in  of the form  = (*v*, 1, *v*, 1, …)
3. All sequences  in  of the form 

***Solution***

1. Let 



 





 



This is a ***subspace*** of

1. Let 







 is not in the set

Since *k* ≠ 1, then is ***not*** a ***subspace*** of

1. Let 





 







 

This is a ***subspace*** of

***Exercise***

Which of the following are linear combinations of  = (0, −2, 2) and  = (1, 3, −1)?

1. (2, 2, 2) *b)* (3, 1, 5) *c)* (0, 4, 5) *d)* (0, 0 ,0)

***Solution***



1. 











 is a linear combination of  and .

1. 











 is a linear combination of  and .

1. 













(0, 4, 5) is not a linear combination of  and .

1. 











 is a linear combination of  and .

***Exercise***

Which of the following are linear combinations of  = (2, 1, 4),  = (1, −1, 3) and  = (3, 2, 5)?

1. (−9, −7, −15)
2. (6, 11, 6)
3. (0, 0 ,0)

***Solution***



1. 









Therefore, 

1. 









Therefore, 

1. 









Therefore, 

***Exercise***

Determine whether the given vectors span 

1. 
2. 
3. 

***Solution***

1. 

The system is consistent for all values so the given vectors span .

1. 

The system is not consistent for all values so the given vectors do not span .

1. 







The system has a solution only if . But since this is a restriction that the given vectors don’t span on all of . So the given vectors do not span .

***Exercise***

Which of the following are linear combinations of 

1.  *b)*  *c)* 

***Solution***



1. 













 is a linear combinations of *A*, *B*, and *C*.

1. 













 is a linear combinations of *A*, *B*, and *C*.

1. 











 is a linear combination of *A*, *B*, and *C*.

***Exercise***

Suppose that . Which of the following vectors are in span 

1. (2, 3, −7, 3) *b)* (0, 0, 0, 0) *c)* (1, 1, 1, 1) *d)* (−4, 6, −13, 4)

***Solution***

In order to be span , there must exists scalars *a, b, c* that 





1. (2, 3, −7, 3)













This system is consistent, it has only solution is *a* = 2, *b* = −1, *c* = −1 

Therefore, (2, 3, −7, 3) is in span 

1. The vector (0, 0 , 0, 0) is obviously in span 

Since 

1. For the vector *b* = (1, 1, 1, 1)















This system is inconsistent, therefore (1, 1, 1, 1) is *not* in span 

1. For the vector *b* = (−4, 6, −13, 4)













This system is consistent, it has only solution is *a* = 3, *b* = −3, *c* = 1 

Therefore, (−4, 6, −13, 4) is in span 

***Exercise***

Let  and . Which of the following lie in the space spanned by  and 

1.  *b)*  *c)*  *d)* 

***Solution***

1. , therefore  is in span 
2. In order for  to be in span , there must exist scalars *a* and *b* such that



When 

Therefore  is *not* in span 

1. In order for  to be in span , there must exist scalars *a* and *b* such that



When 

Therefore  is *not* in span 

1. In order for 0 to be in span , there must exist scalars *a* and *b* such that



Therefore  is in span 

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let , and









 



*S* is closed under addition

1. 











*S* is closed under scalar multiplication

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let , and



















*S* is not closed under addition

1. 











*S* is closed under scalar multiplication

1. Since *S* is *not* closed under addition, then *S* is ***not*** a subspace of.

***Exercise***

Let, Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let , and

















*S* is not closed under addition

1. 











*S* is closed under scalar multiplication

1. Since *S* is *not* closed under addition, then *S* is ***not*** a subspace of.

***Exercise***

Let, Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let , and











*S* is closed under addition

1. 









*S* is closed under scalar multiplication

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of.

***Exercise***

Let, Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let , and













*S* is not closed under addition

1. 









*S* is not closed under scalar multiplication

1. Since *S* is not closed under addition and not closed scalar multiplication, then *S* is ***not*** a subspace of.

***Exercise***

 where *V* is a vector space and *S* is subset of *V*

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of *V*?

***Solution***

1. Let 





Yes, *S* is closed under addition

1. 



Yes, *S* is closed under scalar multiplication

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

 where *V* is a vector space and *S* is subset of *V*

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of *V*?

***Solution***

1. Let 

 



Yes, *S* is closed under addition

1. 

*S* is ***not*** closed under scalar multiplication since 

1. Since *S* is closed under addition but it is not closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

 where *V* is a vector space and *S* is subset of *V*

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of *V*?

***Solution***

1. Let 





*S* is *not* closed under addition

1. 



 



*S* is closed under scalar multiplication

1. Since *S* is not closed under addition and closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 







 





*S* is closed under addition

1. 



 





*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let  



 



*S* is *not* closed under addition

1. 









*S* is *not* closed under scalar multiplication.

1. Since *S* is not closed under addition and not closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 









 Let 

 



*S* is closed under addition

1. 



 Let 

 



*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let 







 Let 

 



*S* is closed under addition

1. 



 Let 

 



*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let 







 Let 

 



*S* is closed under addition

1. 



 Let 

 



*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let 







 Let 

 



*S* is *not* closed under addition

1. 



 



*S* is *not* closed under scalar multiplication.

1. Since *S* is not closed under addition and not closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***



1. Let 







*S* is *not* closed under addition

1. 









*S* is closed under scalar multiplication.

1. Since *S* is *not* closed under addition and is closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 





 Let 

 Then, 



*S* is closed under addition

1. 





 

 

*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 







Since 

Then, 

*S* is closed under addition

1. 





*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 



 If we let 





*S* is ***not*** closed under addition

1. 

 If we let 

 

*S* is ***not*** closed under scalar multiplication.

1. Since *S* is *not* closed under addition and is *not* closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 



 





*S* is closed under addition

1. 

 If we let 







*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Let  and , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 



 If we let 



*S* is ***not*** closed under addition

1. 

 If we let 

 

*S* is ***not*** closed under scalar multiplication.

1. Since *S* is *not* closed under addition and is *not* closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let  and , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 



 If we let 



*S* is ***not*** closed under addition

1. 

 If we let 

 

*S* is ***not*** closed under scalar multiplication.

1. Since *S* is *not* closed under addition and is *not* closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let  and , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 







*S* is ***not*** closed under addition

1. 





Since, 



*S* is closed under scalar multiplication.

1. Since *S* is *not* closed under addition and is closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

 where *V* is a vector space and *S* is subset of *V*

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of *V*?

***Solution***

1. Let assume:  are invertible

But  is not invertible.

*S* is *not* closed under addition

1. *S* is *not* closed under scalar multiplication if 
2. Since *S* is *not* closed under addition and is *not* closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let  and , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 



 



*S* is closed under addition

1. 

 



*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and is closed scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 







Has no 3rd degree polynomial.

*S* is *not* closed under addition

1. 





It is 3rd degree polynomial.

*S* is closed under scalar multiplication.

1. Since *S* is *not* closed under addition and is closed scalar multiplication, then *S* is ***not*** a subspace of *V*.

***Exercise***

Let , Determine:

1. Is *S* closed under addition?
2. Is *S* closed under scalar multiplication?
3. Is *S* a subspace of ?

***Solution***

1. Let 









*S* is closed under addition

1. 







*S* is closed under scalar multiplication.

1. Since *S* is closed under addition and is closed scalar multiplication, then *S* is a subspace of *V*.

***Exercise***

Given: 

1. Find 
2. For which *n* is  a subspace of 
3. Sketch  in  or 

***Solution***

1. 





1. 



***Exercise***

Determine which of the following are subspaces of 

1. All 2 × 2 matrices with integer entries
2. All matrices  where 

***Solution***

1. Let  and 

where  are integers.



where  are integers too.

Then, it is closed under addition.





It is not closed under multiplication if the scalar is a real number.

Therefore; it is ***not*** a subspace of 

1. Let  

and  







Then, it is closed under addition.





It is closed under multiplication

Therefore; it is a subspace of 

***Exercise***

Let . Is *V* a vector space?

***Solution***









∴ *V* is *not* a vector space

***Exercise***

Let . Define addition and scalar multiplication as follows:



Is *V* a vector space?

***Solution***

Let 









∴ *V* is *not* a vector space

***Exercise***

Construct a matrix whose column space contains , , and whose nullspace contains  and 

***Solution***

It is *not* possible.

Since a matrix (*A*) must be .

Since the nullspace contains 2 independent vectors, then *A* can have at most  pivot.

But the column space contains 2 independent vectors, *A* must have at least 2 pivots.

These 2 conditions can’t both be met.

***Exercise***

How is the nullspace  related to the spaces  and , is  ?

***Solution***





*Iff* 

***Exercise***

True or False (check addition or give a counterexample)

1. If *V* is a vector space and *W* is a subset of *V* that is a vector space, then *W* is a subspace of *V*.
2. The empty set is a subspace of every vector space.
3. If *V* is a vector space other than the zero vector space, then *V* contains a subspace *W* such that .
4. The intersection of any two subsets of *V* is a subspace of *V*.
5. Let *W* be the *xy-*plane in ; that is, . Then 

***Solution***

1. False

*W* is a subset of *V*, but not necessary that the scalar of a vector in *W* is in *V*.

Therefore, *W* is *not* a subspace of *V*

1. False

Since not every subspace has an empty space, example 

1. True

If *V* is a vector space in  and *W* is a vector space in . Then *V* contains a subspace *W* and 

1. False
2. False

***Exercise***

Let  be a homogeneous system of *n* linear equations in *n* unknowns that has only the trivial solution. Show that of *k* is any positive integer, then the system  also has only trivial solution.

***Solution***

Since *A* is a square matrix, thus *A* has only the trivial solution that implies to *A* is invertible.

But  is also invertible so  has only trivial solution.

***Exercise***

Let  be a homogeneous system of *n* linear equations in *n* unknowns and let *Q* be an invertible  matrix. Show that of  has just trivial solution if and only if  has just trivial solution.

***Solution***

Since *A* is a square matrix . If  has just trivial solution, then *A* is invertible. Since *Q* is an invertible  matrix that implies *QA* is also invertible. Thus, has trivial solution.

On the other hand, if  has trivial solution then *QA* is invertible.

Since *Q* is invertible that implies  is also invertible.

Thus,  is invertible i.e.  has just trivial solution.

 has just trivial solution *iff*  has just trivial solution.

***Exercise***

Let  be a consistent system of linear equations and let  be a fixed solution. Show that every solution to the system can be written in the form  where  is a solution to . Show also that every matrix of this form is a solution.

***Solution***

Since  is a solution to , we have .

The sum of  and 





As adding an equation to the original equation does not affect the solution.

If we let  be a fixed solution, then every solution to  is 

Besides that







So, every matrix (vector) in the form is a solution to .