***Solution Section* 2.6 – Linear Independence**

***Exercise***

State the following statements as true or false

1. If *S* is a linearly dependent set, then each vector in *S* is a linear combination of other vectors in *S*.
2. Any set containing the zero vector is linearly dependent.
3. The empty set is linearly dependent.
4. Subsets of linearly dependent sets are linearly dependent.
5. Subsets of linearly independent sets are linearly independent.
6. If  and  are linearly independent, the null the scalars  are zero

***Solution***

1. False
2. True
3. False
4. False
5. True
6. True

***Exercise***

Given three independent vectors . Take combinations of those vectors to produce . Write the combinations in a matrix form as 



which is 

What is the test on a matrix **V** to see if its columns are linearly independent?

If  show that  are linearly independent.

If  show that  are linearly *dependent*.

***Solution***

The nullspace of **V** must contain only the *zero* vector. Then  is the only combination of the columns that gives **V** = zero vector.







If , then the matrix *M* is invertible. So if *x* is any nonzero vector we know that *Mx* is nonzero. Since ***w***’s are given as independent and  is nonzero. Since , this says that *x* is not in the nullspace of **V**. therefore;  are independent.

If , that implies



, which means that  are linearly *dependent*.

The other way, the vector  is in that nullspace, and . Then certainly  which is the same as . So, the  are dependent.

***Exercise***

Find the largest possible number of independent vectors among



***Solution***

Since , there are at most three

independent vectors among these: furthermore, applying row reduction to the matrix  gives three pivots, showing that are independent.

***Exercise***

Show that are independent but  are dependent:



Solve either . The *v*’s go in the columns of ***A***.

***Solution***



This matrix has 3 pivots with rank of 3 equals to rows that implies the  are independent.



That shows that  are dependent.

***Exercise***

Decide the dependence or independence of

1. The vectors (1, 3, 2), (2, 1, 3), and (3, 2, 1).
2. The vectors , , and .

***Solution***

1. These are *linearly independent*.

 only if 

1. These are *linearly dependent*:



***Exercise***

Find two independent vectors on the plane  in . Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

***Solution***

This plane is the nullspace of the matrix 



The pivot is 1st column, and the rest are 3 variables.

If  . The vector is 

If  . The vector is 

If  . The vector is 

The 3 vectors  are linearly independent.

We can’t find 4 independent vectors because the nullspace only has dimension 3 (have 3 variables).

***Exercise***

Determine whether the vectors are linearly dependent or linearly independent in 

|  |  |
| --- | --- |
| 1. (4, −1, 2), (−4, 10, 2) 2. (8, −1, 3), (4, 0, 1) | 1. (−3, 0, 4), (5, −1, 2), (1, 1, 3) 2. (−2, 0, 1), (3, 2, 5), (6, −1, 1), (7, 0, −2) |

***Solution***

1. The vector equation *a* (4, −1, 2) + *b* (−4, 10, 2) = (0, 0 ,0)











Therefore, the system has only the trivial solution *a* = *b* = 0.

We conclude that the given set of vectors is linearly independent.

1. *A* (8, −1, 3) + *b* (4, 0, 1) = (0, 0 ,0)











Therefore, the system has only one trivial solution *a* = *b* = 0.

We conclude that the given set of vectors is linearly independent

1. The vector equation:

*a* (−3, 0, 4) + *b* (5, −1, 2) + *c*(1, 1, 3) = (0, 0 ,0)















Therefore, the system has only the trivial solution *a* = *b* = *c* = 0.

We conclude that the given set of vectors is linearly independent.

1. The vector equation:

*a* (−2, 0, 1) + *b* (3, 2, 5) + *c* (6, −1, 1) + *d* (7, 0, −2) = (0, 0 ,0)











Therefore, the system has nontrivial solutions 

We conclude that the given set of vectors is linearly dependent.

***Exercise***

Determine whether the vectors are linearly dependent or linearly independent in 

1. 
2. 
3. 
4. 
5. 
6. 
7. 

***Solution***

1. 

The system has only the trivial solution and the vectors are *linearly independent*.

1. 















The system has only the trivial solution and the vectors are *linearly independent*.

1. 

The system has only the trivial solution and the vectors are *linearly independent*.

1. 















Therefore, the system has only one trivial solution *a* = *b* = *c* = *d* = 0.

The given set of vectors is *linearly independent*

1. 



The system has only the trivial solution and the vectors are *linearly independent*.

1. 

















∴ The set is *linearly independent*.

1. 











∴ The set is *linearly independent*.

***Exercise***

*a* ) Show that the three vectors  form a linearly dependent set in .

*b*) Express each vector in part (*a*) as a linear combination of the other two.

***Solution***

1. The vector equation:











The solution: 

Since the system has nontrivial solutions, the given set of vectors is *linearly dependent*.

1. Since  and if we let *t* = 1, then 



***Exercise***

For which real values of λ do the following vectors form a linearly dependent set in 



***Solution***







For , the determinant is zero and the vectors form a *linearly dependent* set.

***Exercise***

Show that if  is a linearly independent set of vectors, then so is every nonempty subset of S.

***Solution***

Let  be a nonempty subset of *S*.

If this set is linearly dependent, then there would be a nonzero solution  to . This can be expanded to a nonzero solution of  by taking all other coefficients as 0. This contradicts the linear independence of *S*, so the subset must be *linearly independent*.

***Exercise***

Show that if  is a linearly dependent set of vectors in a vector space *V*, and if  are vectors in *V* that are not in *S*, then  is also linearly dependent.

***Solution***

If *S* is linearly dependent, then there is a nonzero solution  to .

Thus  is a nonzero solution to 

So, the set  is *linearly dependent*.

***Exercise***

Show that  is linearly independent and  does not lie in span , then  is a linearly independent.

***Solution***

If  are linearly dependent, there exist a nonzero solution to  with  (since  and  are linearly independent).

 which contradicts that  is not in span . Thus  is a linearly independent.

***Exercise***

By using the appropriate identities, where required, determine  are linearly dependent.

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Solution***

1. From the identity 





Therefore, the set is *linearly dependent*.

1. 





Therefore, the set is *linearly independent*.

1. 







Therefore, the set is *linearly independent*.

1. 







Therefore, the set is linearly dependent.

1. By using the double angle:

 are *linearly dependent*.

***Exercise***

 are linearly independent in  because neither function is a scalar multiple of the other. Confirm the linear independence using Wroński’s test.

***Solution***

The Wronskian: 







 are *linearly independent*

***Exercise***

Show  are linearly independent in 

***Solution***

 









 are *linearly independent*

***Exercise***

Use the Wronskian to show that  span a three-dimensional subspace of 

***Solution***

The Wronskian: 









Since  for all real *x* values, the vectors are *linearly independent.*

***Exercise***

Show by inspection that the vectors are linearly dependent.



***Solution***















***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***















The system has only he trivial solution *a = b = c =* 0.



The system has only the trivial solution and the vectors are *linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***



The system has only the trivial solution and the vectors are *linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***













The vectors are *linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

 in 

***Solution***













It is *linearly dependent*.





***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

 in 

***Solution***





∴ *Linearly dependent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)

 in 

***Solution***







∴ *Linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***







∴ *Linearly dependent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***



∴ *Linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***





∴ *Linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***







∴ *Linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***







∴ *Linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***



∴ *Linearly independent*









***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***





∴ *Linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***





∴ *Linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***











∴ *Linearly dependent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***









∴ *Linearly independent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***









If 



∴ *Linearly dependent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***



∴ *Linearly dependent*

***Exercise***

Determine if the given vectors are linearly dependent or independent, (any method)



***Solution***









If 



∴ *Linearly dependent*

***Exercise***

Suppose that the vectors  are linearly dependent. Are the vectors , , and  also linearly dependent?

(***Hint***: Assume that , and see what the  can be.)

***Solution***

***Given***:  are linearly dependent, then there are scalar  such that.

Assume that 







If  and since  are linearly dependent, therefore,  are *linearly dependent*.

***Exercise***

Show that the set  is a linearly independent subset of .

***Solution***





 ∴ *Linearly* *Independent*.



 

Since the only solution to this system is the trivial one. *F* is Linearly Independent subset of 

***Exercise***

Suppose that *A* is linearly dependent set of vectors and *B* is any set containing *A*. Show that *B* must be linearly dependent.

***Solution***

If *A* is linearly dependent, then there are vectors  in *A* and ,  with all not  and 

If *B* any set that contains *A*, then this same relation holds in *B* set.

*B* is also dependent.

***Exercise***

Show that  is a linearly independent, subset of . Does it span 

***Solution***









∴ *Linearly* *Independent*.



If 

Since all the polynomials are in  and there is no other way that we can write them as linear combinations of .

The set can’t possible span 

***Exercise***

Show that the set  is linearly dependent on .

***Solution***







∴ The set is linearly dependent on 



If 







***Exercise***

Show that if  are linearly independent and  are linearly dependent, then  can be uniquely expressed as a linear combination of 

***Solution***

Since,  are linearly independent, then

 when all .

Let assume that:





If , then  and  doesn’t exist.

If , and  are linearly dependent, then





If 



Then  since  are linearly independent and contradict that  are linearly dependent.

Therefore,  can be uniquely expressed as a linear combination of in the form

***Exercise***

Show that if  are linearly dependent with  if and only if there exists an integer , such that  is a linear combination of 

***Solution***

Since,  are linearly dependent, then if



then there exists an 

If we let  where , then 

