***Solution*** ***Section* 2.7 – Coordinates, Basis and Dimension**

***Exercise***

Suppose  is a basis for  and the *n* by *n* matrix *A* is invertible. Show that  is also a basis for .

***Solution***

Put the basis vectors  in the columns of an invertible matrix **V**. then  are the columns of ***A*V**. Since ***A*** is invertible, so is ***A*V** and its column give a basis.

Suppose . This is  with . Multiply by  to get . By linear independence of , all . So, the  are independent.

***Exercise***

Consider the matrix 

1. Which vectors  will make the columns of ***A*** linearly dependent?
2. Which vectors  will make the columns of ***A*** a basis for ?
3. For , compute a basis for the four subspaces.

***Solution***

1. All linear combination of 
2. To satisfy *b + d* = 0. For example, 



1. 









The first 2 columns span the column space *C*(***A***).

If  that implies that the nullspace

*N*(***A***): 

Rank(***A***) = 2 and  is a basis for the one-dimensional *N*(***A***).

 

***Exercise***

Find a basis for  in .

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

***Solution***

This plane is the nullspace of the matrix



The special solutions:  give a basis for the nullspace, and for the plane.

The intersection of this plane with the *xy*-plane is a line  and the vector  lies in the *xy*-plane.

The vector  is perpendicular to both vectors : the space vectors perpendicular to a plane  is one-dimensional, it gives a basis.

***Exercise***

**U** comes from ***A*** by subtracting row 1 from row 3:



Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

***Solution***





1. The pivots are in the first two columns, so one possible basis for *C*(***A***) is  and for *C*(***U***) is 
2. Both ***A*** and ***U*** have the same nullspace *N*(***A***) = *N*(***U***),

with basis 

1. Both ***A*** and ***U*** have the same row space



***Exercise***

Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives , and check entries to prove is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.

***Solution***

Assume:





and 











***Exercise***

Choose three independent columns of . Then choose a different three independent columns. Explain whether either of these choices forms a basis for .

***Solution***















Rank(***A***) = 3, the columns space is 3 which form a basis of . The variable is 

If 





*N*(*A*) is spanned by , which gives the relation of the columns.

The special solution  gives a relation . If we take any two columns from the first three columns and the column 4, they will span a three-dimensional space since there will be no relation among them. Hence, they form a basis of .

***Exercise***

Which of the following sets of vectors are bases for ?

1. 
2. 

***Solution***

1. 





Therefore, the vectors  are linearly independent and span  , so they form a basis for .

1. 





Therefore; the vectors  are linearly dependent, so they don’t form a basis for .

***Exercise***

Which of the following sets of vectors are bases for ?

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 

Therefore, the set of vectors are linearly independent.

The set form a basis for.

1. 

Therefore, the set of vectors are linearly independent.

The set form a basis for.

1. 

Therefore, the set of vectors are linearly dependent.

The set don’t form a basis for.

1. 

Therefore; the set of vectors are linearly dependent.

The set don’t form a basis for.

***Exercise***

Let *V* be the space spanned by 

1. Show that  is not a basis for *V*.
2. Find a basis for *V*.

***Solution***

1. 









If 



This shows that  is linearly dependent, therefore it is ***not*** a basis for *V*.

1. For  to hold for all real *x* values, we must have  and .

Therefore, the vectors  are linearly independent.





This proves that the vectors  span *V*.

We can conclude that  can form a basis for *V*.

***Exercise***

Find the coordinate vector of  relative to the basis  for 

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 

We must first express  as a linear combination of the vectors in *S;* 







Therefore, 

1. 

Solve: 

















Therefore, 

1. 

Solve: 







Therefore, 

1. 

Solve: 





 



Therefore, 

1. 

Solve: 





 



Therefore, 

***Exercise***

Find the coordinate vector of  relative to the basis 

1. 
2. 

***Solution***

1. 

Solve: 







Therefore, 

1. 

Solve: 













Therefore, 

***Exercise***

Show that  is a basis for , and express *A* as a linear combination of the basis vectors

1. 
2. 
3. 

***Solution***

1. Matrices  are linearly independent if the equation





Has only the trivial solution.

For these matrices to span , it must be expressed every matrix  as





The 2 equations can be written as linear systems



,

That the homogeneous system has only the trivial solution.

span 

















1. Matrices  are linearly independent if the equation





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1. 









***Exercise***

Consider the eight vectors



1. List all of the one-element. Linearly dependent sets formed from these.
2. What are the two-element, linearly dependent sets?
3. Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
4. Which four-element sets are linearly dependent? Explain why.

***Solution***

1.  zero vector is the only linearly dependent.
2. The set that contains zero vector and any other vector.
3. 2-dimension:



1-dimensional subspace if we allow duplicates (zero vector) 

1. All four-element sets are linearly dependent in three-dimensional space.

***Exercise***

Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 





The solution: 

The solution space has dimension 1 and a basis 

1. 







The solution:





The solution space has dimension 2 and a basis 

1. 



The solution:





The solution space has dimension 2 and a basis 

1. 







The solution: 

The solution space has dimension 1 and a basis 

1. 









No basis and dimension = 0

***Exercise***

If  for the shift matrix *S*. Show that ***A*** must have this special form:





“The subspace of matrices that commute with the shift *S* has dimension \_\_\_\_\_\_.”

***Solution***









The subspace of matrices that commute with the shift *S* has dimension 3, because the matrix has only three variables.

***Exercise***

Find bases for the following subspaces of 

1. All vectors of the form (*a, b, c*, 0)
2. All vectors of the form (*a, b, c*, *d*), where *d* = *a + b* and *c = a – b*.
3. All vectors of the form (*a, b, c*, *d*), where *a = b* = *c = d*.

***Solution***

1. The subspace can be expressed as span is a set of linearly independent vectors. Therefore; *S* forms a basis for the subspace, so its dimension is 3.
2. The subspace contains all vectors , the setis linearly independent vectors. Therefore; *S* forms a basis for the subspace, so its dimension is 2.
3. The subspace contains all vectors , we can express the set as span *S* and it is linearly independent. Therefore, *S* forms a basis for the subspace, so its dimension is 1.

***Exercise***

Find a basis for the null space of *A*.

1. 
2. 
3. 

***Solution***

1. 





The general form of the solution of  is 

Therefore, the vector  forms a basis for the null space of *A*.

1. 









The general form of the solution of  is 

Therefore, the vectors  form a basis for the null space of *A*.

1. 





 

The general form of the solution of is 

Therefore, the vectors  form a basis for the null space of *A*.

***Exercise***

Find a basis for the subspace of spanned by the given vectors

1. 
2. 

***Solution***

1. 













A basis for the subspace is 

1. 















A basis for the subspace is 

***Exercise***

Determine whether the given vectors form a basis for the given vector space

1. 
2. 
3. 

***Solution***

1. 



The given vectors are linearly independent and span , so they form a basis for .

1. 



The given vectors are linearly independent and span , so they form a basis for .

1. 



They form a basis for .

***Exercise***

Find a basis for, and the dimension of, the null space of the given matrix 

***Solution***











The *bases* are: 

*Dimension*: 2

***Exercise***

Let  be the set of all real numbers and let  be the set of all positive real numbers. Show that  is a vector space over  under the addition



And the scalar multiplication



Find the dimension of the vector space. Is  also a vector space if the scalar multiplication is instead defined as



***Solution***

 







Since for , then



Thus  is a basis, therefore the dimension of the vector space is **1**.

 is not a vector space over  with respect to .

Since,



















