***Section* 2.7 – Coordinates, Basis and Dimension**

**Coordinate Systems in Linear Algebra**

In ***analytic geometry***, we use rectangular coordinate systems to create a point either in 2-space or 3-space

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| --- | --- |
| ***Coordinates of P in a rectangular coordinate system in* 2*-space*** | ***Coordinates of P in a rectangular coordinate system in* 3*-space*** |

|  |  |
| --- | --- |
| ***Coordinates of P in a nonrectangular coordinate system in* 2*-space*** | ***Coordinates of P in a nonrectangular coordinate system in* 3*-space*** |

In ***linear algebra*** coordinate systems are commonly specified using vectors rather than coordinate axes.



***Basis***

***Definition***

If *V* is any vector space and  is a finite set of vectors in *V*, then *S* is called a ***basis*** for *V* if the following two conditions hold:

1. *S* is linearly independent.
2. *S* spans *V*.

***Example***

The columns of  produce the “standard basis” for .

***Solution***

The basis vectors:  are independent. They span .

***Example***

The columns of any invertible *n* by *n* matrix give a basis for .



***Example***

The standard unit vectors  form a basis in .

***Solution***

1.  it follows that . That implies they are linearly independent.
2. Every vector  in  can be expressed as  which is linear combination of  . Thus, the standard vector span 

Thus, they form a basis for  that we call the ***standard basis*** for .

***Example***

Show that the vectors  form a basis in 

***Solution***

1. We need to show that the vectors are linearly independent.





Which yields the homogeneous linear system





 has a trivial solution.

The vectors  are linearly independent.

1. Every vector can be expressed as  which is linear combination. Thus, the standard vector span 

That proves that the vectors  form a basis in 

* The vectors  are a basis for  exactly when they are the ***columns*** of an ***n* by *n* *invertible matrix***. Thus  has infinitely many different bases.
* ***The pivots columns of A are a basis for its column space***. The pivot rows of *A* are a basis for its row space. So are the pivot rows of its echelon form *R*.

***Example***

Find bases for the column and row spaces of a rank two matrices: 

***Solution***

Columns 1 and 3 are the pivot columns. They are a basis for the column space. It is a subspace of .

Column 2 and 4 are a basis for the same column space.

***Coordinates Relative to a Basis***

***Theorem* − Uniqueness of Basis Representation**

If  is a basis for a vector space *V*, then every vector in *V* can be expressed in the form  in exactly one way.

***Proof***

Suppose that some vector can be written as



Also 

Subtracting the second from the first equation



Since the right side of this equation is a linear combination of vectors in *S*, the linear independence of *S* implies that



That implies 

Thus, the two expressions for are the same.

***Definition***

If  is a basis for a vector space *V*, and 

is the expression for a vector in terms of the basis ***S***, then the scalars  are called coordinates of  relative to the basis *S*. The vector  in  constructed from these coordinates is called ***coordinate vector of v relative to S***; it is denoted by



**A *one-to-one correspondence***



***Example***

1. Given the vectors  form a basis for . Find the coordinate vector of  relative to the basis .
2. Find the coordinate vector of in  whose coordinate relative to *S* is .

***Solution***

1. To find  we must first express  as a linear combination of the vectors in *S*;





Which gives: 

Solving this system, we obtain .

Therefore 

1. 





***Dimension***

If  and  are both bases for the same vector space, then *m* = *n*.

***Note***

***V*** may have many different bases, but they all must have the same number of elements.

***Proof***

Let **S** and **W** be bases of **V** can be written as a linear combination of vectors in **S**.



But since **V** is a basis,  (to be linearly independent, otherwise to be linearly dependent with at least 1 of t)







∴  linear independent.

Now all bases of **V** have some number of elements, we can define the dimension (is # of vectors in a basis)

***Definition***

The dimension of a finite-dimensional vector space *V* is denoted by dim(*V*) and is defined to be the number of vectors in a basis for *V*. in addition, the zero-vector space is defined to have dimension zero.

1. Dim(**V**) = # elements in basis. If **V** is finite.
2. If , then Dim(**V**) = 0, even though there is no basis.
3. Dim(**V**) may be infinite.

*  **The standard basis has *n* vectors**.
*  **The standard basis has *n +* 1 vectors**.
*  **The standard basis has *mn* vectors**.

**Bases for Matrix Spaces *and Function Spaces***

Independence, basis, and dimension are not all restricted to column vectors.

* The dimension of the whole *n* by *n* space is 
* The dimension of the subspace of upper triangular matrices is 
* The dimension of the subspace of diagonal matrices is 
* The dimension of the subspace of symmetric matrices is 

***Function Spaces***

The equations:

 is solved by any linear function 

 is solved by any combination 

 is solved by any combination 

***Example***

Find a basis for and the dimension of the solution space of the homogeneous system



***Solution***















The solution 



The solution space has dimension 2.

**Plus/Minus Theorem**

***Theorem***

Let *S* be a nonempty set of vector space *V*.

1. If *S* is a linearly independent set, and if is a vector in *V* that is outside of span(S), the set  that results by inserting into *S* is still linearly independent.
2. If is a vector in *S* that is expressible as a linear combination of other vectors in *S*, and if  denotes the set obtained by removing from *S*, then *S* and  span the same space; that is,



|  |  |  |
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| Untitled.png | | |
| The vector outside the plane can be adjoined to the other two without affecting their linear independence | Any of the vectors can be removed, and the remaining two still span the plane | Either of the collinear vectors can be removed, and the remaining two will still span the plane |

***Theorem***

If *W* is a subspace of a finite-dimensional vector space *V*, then

1. *W* is finite-dimensional
2. 
3.  if and only if 

***Exercises*** ***Section* 2.7 – Coordinates, Basis and Dimension**

1. Suppose  is a basis for  and the *n* by *n* matrix *A* is invertible. Show that  is also a basis for .
2. Consider the matrix 
3. Which vectors  will make the columns of ***A*** linearly dependent?
4. Which vectors  will make the columns of ***A*** a basis for ?
5. For , compute a basis for the four subspaces.
6. Find a basis for  in .

Find a basis for the intersection of that plane with *xy* plane. Then find a basis for all vectors perpendicular to the plane.

1. **U** comes from ***A*** by subtracting row 1 from row 3:



Find the bases for the two column spaces. Find the bases for the two row spaces. Find bases for the two nullspaces.

1. Write a 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives , and check entries to prove is zero.) The five permutation matrices are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.
2. Choose three independent columns of . Then choose a different three independent columns. Explain whether either of these choices forms a basis for .
3. Which of the following sets of vectors are bases for ?

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1. Which of the following sets of vectors are bases for ?

|  |  |
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1. Let *V* be the space spanned by 
2. Show that  is not a basis for *V*.
3. Find a basis for *V*.
4. Find the coordinate vector of  relative to the basis  for 

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|  |  |

1. Find the coordinate vector of relative to the basis 
2. 
3. 
4. Show that  is a basis for , and express *A* as a linear combination of the basis vectors
5. 
6. 
7. 
8. Consider the eight vectors



1. List all of the one-element. Linearly dependent sets formed from these.
2. What are the two-element, linearly dependent sets?
3. Find a three-element set spanning a subspace of dimension three, and dimension of two? One? Zero?
4. Which four-element sets are linearly dependent? Explain why.

(**14 – 18**) Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space

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1. If  for the shift matrix *S*. Show that ***A*** must have this special form:





“The subspace of matrices that commute with the shift *S* has dimension \_\_\_\_\_\_.”

1. Find bases for the following subspaces of 
2. All vectors of the form (*a, b, c*, 0)
3. All vectors of the form (*a, b, c*, *d*), where *d* = *a + b* and *c = a – b*.
4. All vectors of the form (*a, b, c*, *d*), where *a = b* = *c = d*.
5. Find a basis for the null space of *A*.
6. 
7. 
8. 
9. Find a basis for the subspace of spanned by the given vectors
10. 
11. 
12. Determine whether the given vectors form a basis for the given vector space
13. 
14. 
15. 
16. Find a basis for, and the dimension of, the null space of the given matrix 
17. Let  be the set of all real numbers and let  be the set of all positive real numbers. Show that  is a vector space over under the addition



And the scalar multiplication



Find the dimension of the vector space. Is  also a vector space if the scalar multiplication is instead defined as

