***Solution*** ***Section* 2.8 – Row and Column Spaces**

***Exercise***

List the row vectors and column vectors of the matrix



***Solution***

Row vectors:



Column vectors:



***Exercise***

Express the product  as a linear combination of the column vectors of *A*. 

***Solution***



***Exercise***

Express the product  as a linear combination of the column vectors of *A*. 

***Solution***



***Exercise***

Express the product  as a linear combination of the column vectors of *A*. 

***Solution***



***Exercise***

Determine whether  is in the column space of *A*, and if so, express as a linear combination of the column vectors of *A*.



***Solution***











***Exercise***

Determine whether  is in the column space of *A*, and if so, express as a linear combination of the column vectors of *A*.



***Solution***

















The system  is consistent and  is in the column space of *A*.

***Exercise***

Determine whether  is in the column space of *A*, and if so, express as a linear combination of the column vectors of *A*.



***Solution***











The system is inconsistent and  is not in the column space of *A*.

***Exercise***

Determine whether  is in the column space of *A*, and if so, express as a linear combination of the column vectors of *A*.



***Solution***













The system  is consistent and  is in the column space of *A*

***Exercise***

Suppose that  is a solution of a nonhomogeneous linear system  and that the solution set of the homogeneous system  is given by the formulas



1. Find a vector form of the general solution of 
2. Find a vector form of the general solution of 

***Solution***

1. 



1. Special Solution: 





***Exercise***

Find the vector form of the general solution of the given linear system ; then use that result to find the vector form of the general solution of .



***Solution***





The solution of  is

 ***or***



The general form of the solution  is 

***Exercise***

Find the vector form of the general solution of the given linear system ; then use that result to find the vector form of the general solution of .



***Solution***









The solution of  is

 or



The general form of the solution of  is



***Exercise***

Find the vector form of the general solution of the given linear system ; then use that result to find the vector form of the general solution of .



***Solution***









The solution of  is



The general form of the solution of  is



***Exercise***

Find the vector form of the general solution of the given linear system ; then use that result to find the vector form of the general solution of .



***Solution***





Let 

The solution of  is



The general form of the solution of  is



***Exercise***

Given the vectors  and 

1. Are they linearly independent?
2. Are they a basis for any space?
3. What space **V** do they span?
4. What is the dimension of that space?
5. What matrices ***A*** have **V** as their column space?
6. Which matrices have **V** as their nullspace?
7. Describe all vectors  that complete a basis for .

***Solution***

1.  are independent – the only combination to give ****  is .
2. Yes, they are a basis for whatever space **V** they span.
3. That space **V** contains all vectors . It is the *xy* plane in .
4. The dimension of **V** is 2 since the basis contains 2 vectors.
5. This **V** is the column space of any 3 by *n* matrix ***A*** of rank 2, if every column is a combination of . In particular ***A*** could just have columns .
6. This **V** is the nullspace of any *m* by 3 matrix ***B*** of rank 1, if every row is a multiple of . In particular, take . Then .
7. Any third vector  will complete a basis for  provided .

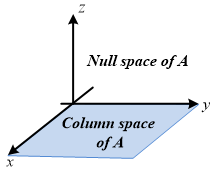
***Exercise***

*a*) Let 

Show that relative to an *xyz*-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane.

*b*) Find a 3 x 3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.

***Solution***

1. 



The general form of the solution of  is,



Therefore, the null space of *A* is the *z*-axis, and the column space is the span of  which is all linear combinations of *y* and *x* (*xy*-plane)

1. 

***Exercise***

If we add an extra column  to a matrix *A*, then the column space gets larger unless \_\_\_\_\_. Give an example where the column space gets larger and an example where it doesn’t. Why is  solvable exactly when the column space doesn’t get larger – it is the same for *A* and ?

***Solution***

If we add an extra column  to a matrix ***A***, then the column space gets larger unless ***it contains***  that is a linear combination of the columns of ***A***.

Let ; then the column space gets larger if  and it doesn’t if .

The equation  is solvable exactly when  is a (nontrivial) linear combination of the column of ***A***.

The equation  is solvable exactly when  lies in the column space, when the column space doesn’t get larger.

***Exercise***

For which right sides (find a condition on ) are these solvable. (Use the column space  and the equation )

1. 
2. 

***Solution***

1. The column space consists of the vectors for

 is 

They are scalar multiples of 

1. By substituting with new variable *z*, then the column space consists of the vectors 

They are linear combinations of , 

***Exercise***

Show that the matrices *A* and  (with extra columns) have the same column space. But find a square matrix with  smaller than . Important point: An *n* by *n* matrix has  exactly when *A* is an \_\_\_\_\_\_ matrix.

***Solution***

Each column of ***AB*** is a combination of the columns of ***A*** (the combining coefficients are the entries in the corresponding column of B). So, any combination of the columns of  is a combination of the columns of ***A*** alone. Thus, ***A*** and  have the same column space.

Let ; then , so .

 is the line through .

Any *n* by *n* matrix has  exactly when ***A*** is an ***invertible*** matrix, because  is solvable for any given ***b*** when ***A*** is invertible.

***Exercise***

The column of *AB* are combinations of the columns of *A*. This means: The column space of *AB* is contained in (possibly equal to) to the column space of *A*. Give an example where the column spaces *A* and *AB* are not equal.

***Solution***

The column space of ***AB*** is contained in (possibly equal to) to the column space of ***A***.

 is a case when  has a smaller column space than ***A***.

***Exercise***

Find a square matrix *A* where  (the column space of  is smaller than .

***Solution***

For example, ; then .

Thus  is generated by vector , which is of one dimensional, but  is a zero space.

Hence,  is strictly smaller than .

***Exercise***

Suppose  and  have the same (complete) solutions for every . Is true that ?

***Solution***

Yes, if , let  be any vector of the correct size, and set . Then ***y*** is a solution to  and it is also a solution to ;



***Exercise***

Apply Gauss-Jordan elimination to  and . Reach  and :

Solve  to find  (its free variable is ).

Solve  to find  (its free variable is ).

***Solution***







The free variable is , since it is the only one. We have to let 



The special solution is 









The free variable is that implies to 



The particular solution is 

***Exercise***

Which of the following subsets of are actually subspaces?

1. The plane of vectors  with 
2. The plane of vectors with .
3. The vectors with .
4. All linear combinations of  and .
5. All vectors that satisfies 
6. All vectors with .

***Solution***

1. This is subspace

* For  with and  with  the sum

 is in the same set as 

* For an element  with ,  and , thus it is in the same set.

1. This is not a subspace. For example, for  and  is not in the set.
2. This is not a subspace. For example, for  and  are in the set, but their sum  is not in the set.
3. This is subspace, by definition of linear combination.

* For 2 vectors  and  the sum





is still the linear combination of *v* and *w*.

* For an element  ,  is still the linear combination of  and , thus it is the same set

1. This is subspace, these are the vectors orthogonal to 

* For  with 

and  with 

The sum  is in the same set as 

* For an element  with ,  and , thus it is in the same set.

1. This is not a subspace. For example, for  and  is not in the set.

***Exercise***

We are given three different vectors . Construct a matrix so that the equations  and  are solvable, but is not solvable.

1. How can you decide if this possible?
2. How could you construct *A*?

***Solution***

The equations  and  will be solvable.

 (solvable?)

Ifis not solvable, we have the desired matrix *A*.

Ifis solvable, then it is not possible to construct *A*.

When the column space contains  and , it will have to contain their linear combinations.

So would necessarily be in that column space and would necessarily be solvable.

***Exercise***

For which vectors  do these systems have a solution?

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 



Solution for every *b*.

1. 



Solvable only if 

1. 









Solvable only if 

***Exercise***

Find a basis for the null space of *A*. 

***Solution***













Let 



The general form of the solution of  is



Therefore, the vectors  and  form a basis for the null space of *A*.

***Exercise***

Is it true that is  then the row space of *A* equals the column space.

***Solution***

False

Counterexample, let 

We have , but the row space of *A* contains multiple of  while the column space of A contains multiples of .

***Exercise***

If the row space equals the column space the 

***Solution***

False,

Counterexample, let  .

Here, the row space and column space are both equal to all of  (since *A* is invertible).

But 

***Exercise***

If , then the row space of *A* equals the column space.

***Solution***

True,

The row space of *A* equals to the column space of , which for this particular *A* equals the column space of .

Since *A* and  have the same fundamental subsequences. We conclude that the row space of *A* equals the column space of *A*.

***Exercise***

Does the matrices *A* and  share the same 4 subspaces?

***Solution***

True.

The nullspaces are identical because 

The column spaces are identical because any vector  that can be expressed as  for some  can also be expressed as 

***Exercise***

Is *A* and *B* share the same 4 subspaces then *A* is multiple of *B*.

***Solution***

False

Any invertible  matrix will have  as its column space and row space and zero vector as its (left and right) nullspace.

However, it is easy to produce 2 invertible  matrices that are not multiples of each other, as example



***Exercise***

Suppose  have the same (complete) solutions for every  . Is it true that 

***Solution***

If  for all vectors  of the correct size.

Then, it is true that 

***Exercise***

*A* and  have the same left nullspace?

***Solution***

False,

Counterexample, take any a  matrix, such as  .

The left nullspace of *A* contains vectors in  while the left nullspace of , which is the right nullspace of *A*, contains vectors in .

So, they can’t be the same.