***Solution*** ***Section* 2.9 – Rank and the Fundamental Matrix Spaces**

***Exercise***

Verify that 



***Solution***



























***Exercise***

Find the rank and nullity of the matrix; then verify that the values obtained satisfy 



***Solution***













***Exercise***

Find the rank and nullity of the matrix; then verify that the values obtained satisfy 



***Solution***















***Exercise***

Find the rank and nullity of the matrix; then verify that the values obtained satisfy 



***Solution***















***Exercise***

Find the rank and nullity of the matrix; then verify that the values obtained satisfy 



***Solution***

















Number of columns = 5



***Exercise***

If *A* is an  matrix, what is the largest possible value for its rank and the smallest possible value of the nullity of *A*.

***Solution***

The largest possible value for the rank of an  matrix:

*  (when every column of the *rref* (A) contains a leading 1)
*  (when every row of the *rref* (A) contains a leading 1)

The smallest possible value for the nullity of an  matrix:

*  (when every column of the *rref* (A) contains a leading 1)
*  (when every row of the *rref* (A) contains a leading 1)

***Exercise***

Discuss how the rank of *A* varies with *t*.

*a*)  *b*) 

***Solution***

1. 



Solve for *t*: 

Therefore,  for 

If *t* = 1, 







If *t* = −2, 











1. 



Solve for *t*: 

Therefore,  for 

If *t* = 1, 











If , 















***Exercise***

Are there values of *r* and *s* for which



Has rank 1? Has rank 2? If so, find those values.

***Solution***

Since the third column will always have a nonzero entry, the *rank* will never be 1. (row 1 and row 4 never have a nonzero entry).

If *r* = 2 and *s* = 1, that implies to





***Exercise***

Find the row reduced form ***R*** and the rank *r* of ***A*** (those depend on *c*).

Which are the pivot columns of ***A***? Which variables are free? What are the special solutions and the nullspace matrix ***N*** (always depending on *c*)?



***Solution***

1. ,

, the pivot columns are 1 and 3, the second variable  is free.

The special solution: which yields to

,

, the pivot column is column 1, the second and third variables  are free.

The special solution goes into 

1. ,

, the pivot column is the first column, the second variable  is free.

The special solution: which yields to



, the matrix has no pivot column, and both variables are free.

The special solution is: 

***Exercise***

Find the row reduced form *R* and the rank *r* of *A* (those depend on *c*).

Which are the pivot columns of *A*? Which variables are free? What are the special solutions and the nullspace matrix *N* (always depending on *c*)?



***Solution***







1. If *c*  = 1, then



This has only one pivot (first column) and 3 free variables .

The nullspace matrix: 

1. If *c*  ≠ 1, then







There are two pivots  and 2 free variables 

The nullspace matrix: 



1. If *c* = 1







This has a single pivot in the second column and one free variable with the nullspace matrix 

1. If *c* = 2









This has a single pivot in the first column with the nullspace matrix 

1. Otherwise 









The result is the identity matrix with 2 pivots, which has (2 – 2) 0 null space.

***Exercise***

If *A* has a rank *r*, then it has an *r* by *r* sub-matrix *S* that is invertible. Remove  rows and columns to find an invertible sub-matrix *S* inside each *A* (you could keep the pivot rows and pivot columns of *A*).



***Solution***

If a matrix ***A*** has rank *r*, then the

**(*dimension of the column space*) *=* (*dimension of the row space*) *= r***

For the invertible sub-matrix S, we need to find *r* linearly independent rows and *r* linearly independent columns.

For matrix ***A***:







The 1st and 3rd columns are linearly independent, and the 1st and 2nd rows are also linearly independent.

Rank (***A***) = 2.

The sub matrices are: 

For matrix ***B***:





Rank (***B***) = 1.

The submatrix is: 

For matrix ***C***:



Rank (***C***) = 2.

The submatrix is by disregarding (deleting) 1st column and 2nd row: 

***Exercise***

Suppose that column 3 of 4 x 6 matrix is all zero. Then must be a \_\_\_\_\_\_ variable. Give one special solution for this matrix.

***Solution***

Themust be a ***free variable***.

A special solution for this variable can be taken to be.



***Exercise***

Fill in the missing numbers to make *A* rank1, rank 2, rank 3. (your solution should be 3 matrices)



***Solution***



If rank (*A*) = 1, then we need the 1st and 3rd to be multiple of the 2nd row to get zero in these rows.









If rank (*A*) = 2, then we need the 1st ***or*** 3rd to be multiple of the 2nd row to get zero row.







If rank (*A*) = 3 (full rank), then the appropriate to start using 0’s or 1’s to fill the blank.















Hence, it has *rank* 3.

***Exercise***

Fill out these matrices so that they have rank 1:



***Solution***

Rank = 1 means that all the rows are multiples of each other.













***Exercise***

Suppose *A* and *B* are *n* by *n* matrices, and *AB* = *I*. Prove from  that the . So, *A* is invertible and *B* must be its two-sided inverse. Therefore *BA* = *I* (which is not so obvious!).

***Solution***

Since *A* is *n* by *n* 



***Exercise***

Every *m* by *n* matrix of rank *r* reduces to (*m* by *r*) times (*r* by *n*):

*A* = (pivot columns of *A*) (first *r* rows of *R*) 

Write the 3 by 4 matrix  as the product of the 3 by 2 from the pivot columns and the 2 by 4 matrix from *R*.

***Solution***









The pivots columns are the 1st and 2nd column.

*A* = (pivot columns of *A*) (first *r* rows of *R*) 





***Exercise***

Suppose *R* is *m* by *n* matrix of rank *r,* with pivot columns first: 

1. What are the shapes of those 4 blocks?
2. Find the right-inverse *B* with *RB* = *I* if *r* = *m*.
3. Find the right-inverse *C* with *CR* = *I* if *r* = *n*.
4. What is the reduced row echelon form of  (with shapes)?
5. What is the reduced row echelon form of  (with shapes)?

Prove that  has the same nullspace as R. Then show that  always has the same nullspace as *A* (a value fact).

1. Suppose you allow elementary column operations on ***A*** as well as elementary row operations (which get to ***R***). What is the “row-and-column reduced form” for an *m* by *n* matrix of rank ***r***?

***Solution***

1. 
2. 











1. 





1. 



1. 



, the inner is not equal but to make work, we can use the *F* transpose.















So, that  for any matrix ***A***. So, 

1. After getting to *R* we can use the column operations to get rid of *F*.



***Exercise***

True or False (check addition or give a counterexample)

1. The symmetric matrices in *M*  from a subspace.
2. The skew-symmetric matrices in *M*  from a subspace.
3. The un-symmetric matrices in *M*  from a subspace.
4. Invertible matrices
5. Singular matrices

***Solution***

1. True:  and  lead to 
2. True:  and  lead to 
3. False: 
4. False:  and  are invertible matrices but  is not invertible.

∴ The zero matrix is not invertible but any linear subspace should contain the zero matrix. So, invertible matrices do not form a linear subspace.

1. False:  and  are singular matrices

But  is not singular.

***Exercise***

Let 

1. Reduce *A* to row-reduced echelon from.
2. What is the rank of *A*?
3. What are the pivots?
4. What are the free variables?
5. Find the special solutions. What is the nullspace ?
6. Exhibit an *r* x *r* submatrix of *A* which is invertible, where . (An *r* x *r* submatrix of *A* is obtained by keeping *r* rows and *r* columns of *A*)

***Solution***

1. 





1. Rank(***A***) = 3
2. The pivots are 
3. The free variables are 
4. 





Let 





1. Set  

The special solution: 

1. Set  

The special solution: 

The nullspace is the set 

1. The pivot rows and columns must be included in a submatrix. To do that, just take the rows and columns of ***A*** containing pivots, which are columns 1, 3, 5 and rows 1, 2, 3. That will give us a 3 by 3 submatrix. Therefore, this submatrix of ***A*** will be invertible.



***Exercise***

Let 

1. Reduce ***A*** to (ordinary) echelon from.
2. What the pivots?
3. What are the free variables?
4. Reduce ***A*** to row-reduced echelon form.
5. Find the special solutions. What is the nullspace ?
6. What is the rank of ***A***?
7. Give the complete solution to 

***Solution***

1. 







1. The pivots are −1, 5, and −5 (Columns 1, 2, 4)
2. The free variables are 3rd and 5th 
3. 





1. Let 





1. Set 



The special solution: 

1. Set 



The special solution: 

The nullspace is the set 

1. Rank(***A***) = 3
2. 

The complete solution = (the particular solution) + (special solution)





***Exercise***

Let 

1. Reduce *A* to row-reduced echelon form.
2. What is the rank of *A*?
3. What the pivots variables?
4. What are the free variables?
5. Find the special solutions.
6. What is the nullspace ?

***Solution***

1. 







1. Rank(***A***) = 3
2. The pivots variables are: 
3. The free variables are: 
4. Let 



Set 

The special solution: 

Set  ;

The special solution: 

1. The nullspace is the set 



***Exercise***

Let 

1. Reduce *A* to row-reduced echelon form.
2. What is the rank of *A*?
3. What the pivots?
4. What are the free variables?
5. Find the special solutions.
6. What is the nullspace ?

***Solution***

1. 









1. Rank(***A***) = 3
2. The pivots variables are: 
3. The free variables are: 
4. Let 



Set 

The special solution: 

Set  ;

The special solution: 

1. The nullspace is the set 



***Exercise***

The 3 by 3 matrix *A* has rank 2.



1. Reduce  to , so that  becomes triangular system .
2. Find the condition on  for  to have a solution
3. Describe the column space of *A*. Which plane in ?
4. Describe the nullspace of *A*. Which special solutions in ?
5. Find a particular solution to  and then complete solution.

***Solution***

1. 





1. The last equation  shows the solvability condition.
2. (***i***) The column space is the plane containing all combinations of the pivot columns: 1st (1, 2, 3) and 3rd (3, 8, 7).

(***ii***) The column space contains all vectors with . That makes  solvable, so ***b*** is in the column space. All columns of *A* pass this test . This is the equation for the plane in (***i***).

1. The special solutions have free variables:





Let 





Let 





The nullspace *N*(*A*) in contains all



1. One particular solution has free variables = zero.







Let 





The complete solution to is



***Exercise***

Find the special solutions and describe the complete solution to  for

= 3 by 4 zero matrix  

Which are the pivot columns? Which are the free variables? What is the *R* (Reduced Row Echelon matrix) in each case?

***Solution***

has 4 solutions. They are the columns  of the identity matrix (4 by 4).

The Nullspace is of .

The complete solution: in .

There are no pivot columns; all variables are free; the reduced *R* is the same zero matrix as.





The vector solution: , The first column of  is its pivot column, and  is the free variable.









All variables are free. There are three special solutions to 



The complete solution:





***Exercise***

Create a 3 by 4 matrix whose special solutions to  are :



You could create the matrix *A* in row reduced form *R*. Then describe all possible matrices *A* with the required Nullspace all combinations of .

***Solution***

We can write the solution:











The entries 3, 2, 6 are the negatives of −3, −2, −6 in the special solutions.

Every 3 by 4 matrix has at least one special solution. These *A*’s have two.

***Exercise***

The plane  is parallel to the plane . One particular point on this plane is . All points on the plane have the form (fill the first components)



***Solution***









***Exercise***

Construct a matrix whose column space contains (1, 1, 5) and (0, 3, 1) and whose Nullspace contains .

***Solution***









***Exercise***

Construct a matrix whose column space contains (1, 1, 0) and (0, 1, 1) and whose Nullspace contains (1, 0, 1) and (0, 0, 1).

***Solution***

It is impossible. Matrix ***A*** must be 3 by 3.

Since the nullspace is supposed to contain two independent vectors, ***A*** can have at most  pivots.

Since the column space supposes to contain two independent vectors. *A* must has at least 2 pivots.

These conditions can’t both be met.

***Exercise***

Construct a matrix whose column space contains (1, 1, 1) and whose Nullspace contains .

***Solution***

The matrix needs to be 3 by 4 matrix.











***Exercise***

How is the Nullspace related to the spaces N(*A*) and N(*B*), if ?

***Solution***



If and only if 



***Exercise***

Why does no 3 by 3 matrix have a nullspace that equals its column space?

***Solution***

If nullspace = column space, then *n – r* = *r* (there are *r* pivots).

For *n* = 3 ⇒ 3 = 2***r*** is impossible.

***Exercise***

If *AB* = 0 then the column space *B* is contained in the \_\_\_\_\_\_\_ of *A*. Give an example of *A* and *B*.

***Solution***

If *AB* = 0 then the column space *B* is contained in the ***nullspace*** of *A*.

Example: 

***Exercise***

True or false (with reason if true or example to show it is false)

1. A square matrix has no free variables.
2. An invertible matrix has no free variables.
3. An *m* by *n* matrix has no more than***n*** pivot variables.
4. An *m* by *n* matrix has no more than***m*** pivot variables.

***Solution***

1. False. Any matrix with fewer than full number of pivots will. 
2. True. Since it is invertible, we will get the full number of pivots. The nullspace has dimension, so we have 0 free variables.
3. True, the number of pivot variables is the dimension of the nullspace, which is at most the number of columns. The nullspace dimension + column space dimension = number of columns.
4. True, in reduced echelon matrix the pivot columns are all 0 except for a single 1, and there are only up to *m* vectors of this type.

***Exercise***

Suppose an *m* by *n* matrix has *r* pivots. The number of special solutions is \_\_\_\_\_\_.

The Nullspace contains only *x* = 0 when *r* = \_\_\_\_\_\_\_.

The column space is all of  when *r* = \_\_\_\_\_\_.

***Solution***

Suppose an *m* by *n* matrix has *r* pivots. The number of special solutions is \_ ***n – r***\_.

The Nullspace contains only *x* = 0 when *r* = \_ ***n*** \_.

The column space is all of  when *r* = \_***m*** \_.

***Exercise***

Find the complete solution in the form  to these full rank system:

*a*)  *b*) 

***Solution***

***a***) 

The equivalent matrix is given by: 

The complete solution in the form 

 is the homogeneous solution to 

Size of ***A*** is *m* = 1 and *n* = 3, rank(***A***) = *r* = 1







Set 

The special solution: 

Set 

The special solution: 

The nullspace is the set 





Set  that implies to the particular solution: 

The complete solution in the form 

Note: that the null space of A is spanned by the two linearly independent vectors 

***b***) 

The equivalent matrix is given by:

 and 









The pivots are ; The free variable is 

Rank *r* = 2, *n* = 2, *m* = 3.

The nullspace has dimension *m – r* = 1.





If 

The special solution: 

The nullspace is the set 

Set  that implies





Then the particular solution: 

The complete solution in the form:



***Exercise***

Find the complete solution in the form  to the system: 

***Solution***









The pivots are ; The free variables are 





1. Set 

The special solution: 

1. Set 

The special solution: 

The special solution: 





Then the particular solution:



The complete solution in the form:



***Exercise***

If ***A*** is 3 x 7 matrix, its largest possible rank is \_\_\_\_\_\_\_\_. In this case, there is a pivot in every \_\_\_\_\_\_\_\_ of ***U*** and ***R***, the solution to  \_\_\_\_\_\_\_\_\_ (always exists or is unique), and the column space of ***A*** is \_\_\_\_\_\_\_\_\_. Construct an example of such a matrix ***A***.

***Solution***

If ***A*** is 3 x 7 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***row*** of ***U*** and *R*, the solution to  ***always exists***, and the column space of ***A*** is.



, that implies that you have 3 pivots (1 each row)





***Exercise***

If ***A*** is 6 x 3 matrix, its largest possible rank is \_\_\_\_\_\_\_\_. In this case, there is a pivot in every \_\_\_\_\_\_\_\_ of ***U*** and *R*, the solution to  \_\_\_\_\_\_\_\_\_ (always exists or is unique), and the nullspace of ***A*** is \_\_\_\_\_\_\_\_\_. Construct an example of such a matrix ***A***.

***Solution***

If ***A*** is 6 x 3 matrix, its largest possible rank is **3**. In this case, there is a pivot in every ***column*** of ***U*** and *R*, the solution to  ***is unique***, and the column space of ***A*** is.



***Exercise***

Find the rank of  and  for 

***Solution***

































 for any matrix, ***A***.

***Exercise***

Explain why these are all false:

1. The complete solution is any linear combination of .
2. A system  has at most one particular solution.
3. The solution  with all free variables zero is the shortest solution (minimum length ). Find a 2 by 2 counterexample.
4. If *A* is invertible there is no solution  in the null space.

***Solution***

1. The coefficient of  must be one.
2. If  is the nullspace of ***A*** and  is one particular solution, then  is also a particular solution.
3. If ***A*** is a 2 by 2 matrix of rank 1, then the solution to  form a line parallel to the line that the nullspace. The line  gives such an example.



Then 





while the particular solutions having some coordinate equal to zero are (1, 0) and (0, 1) and they both have 

1. There is always 

***Exercise***

Write down all known relation between *r* and *m* and *n* if  has

1. No solution for some .
2. Infinitely many solutions for every .
3. Exactly one solution for some , no solution for another .
4. Exactly one solution for every .

***Solution***

1. The system has less than full row rank: .
2. The system has full row rank and less than full column rank: .
3. The system has full column rank and less than full row rank: .
4. The system has full row and column rank (it is invertible): .

***Exercise***

Find a basis for its row space, find a basis for its column space, and determine its rank

 

***Solution***

1. ***Row Space***: every row

***Column Space***: , , , 

***Rank*** = 4

1. 







***Row Space***: , 

***Column Space***: , 

***Rank*** = 2

***Exercise***

Find a basis for the row space, find a basis for the null space, find , find , and verify 



***Solution***









Row Space: , 

Column Space: , 







***Exercise***

Determine if  lies in the column space of the given matrix. If it does, express as linear combination of the column.



***Solution***









 does not lie in the column space

***Exercise***

Find the transition matrix from *B* to *C* and find 

1. 
2. 



***Solution***

1. 











1. 













***Exercise***

Does *A* and  have the same number of pivots.

***Solution***

True

The number of pivots of *A* is its column rank, *r*.

We know that the column rank of *A* equals the row rank of *A*, which is the column rank of .

Hence,  must have the same number of pivots as *A*.

***Exercise***

Let  

1. What is the rank of *A*?
2. What is the dimension of *A*?
3. What are the pivots variables?
4. What are the free variables?
5. Find the special (homogeneous) solutions.
6. What is the nullspace ?
7. Find the particular solution to 
8. Give the complete solution.

***Solution***

1. Rank(***A***) = 2
2. Dimension of *A* = 2
3. The pivots variables are: 
4. The free variables are: 
5. Let 



Set 

The special solution: 

Set  ;

The special solution: 

1. The nullspace is the set 



1. 
2. 

***Exercise***

Let  

1. What is the rank of *A*?
2. What is the dimension of *A*?
3. What are the pivots variables?
4. What are the free variables?
5. Find the special (homogeneous) solutions.
6. What is the nullspace ?
7. Find the particular solution to 
8. Give the complete solution.

***Solution***

1. Rank(***A***) = 3
2. Dimension of *A* = 1
3. The pivots variables are: 
4. The free variables are: 
5. Let 



The special solution: 

1. 
2. 
3. 

***Exercise***

Let  

1. What is the rank of *A*?
2. What is the dimension of *A*?
3. What are the pivots variables?
4. What are the free variables?
5. Find the special (homogeneous) solutions.
6. What is the nullspace ?
7. Find the particular solution to 
8. Give the complete solution.

***Solution***





 

1. Rank(***A***) = 2
2. Dimension of *A* = 2
3. The pivots variables are: 
4. The free variables are: 
5. Let 

Set 

The special solution: 

Set  ;

The special solution: 

1. The nullspace is the set 



1. 
2. 

***Exercise***

Let  

1. What is the rank of *A*?
2. What is the dimension of *A*?
3. What are the pivots variables?
4. What are the free variables?
5. Find the special (homogeneous) solutions.
6. What is the nullspace ?
7. Find the particular solution to 
8. Give the complete solution.

***Solution***

1. Rank(***A***) = 3
2. Dimension of *A* = 1
3. The pivots variables are: 
4. The free variables are: 
5. Let 



Set 

The special solution: 

1. 
2. 
3. 

***Exercise***

Let  

1. What is the rank of *A*?
2. What is the dimension of *A*?
3. What are the pivots variables?
4. What are the free variables?
5. Find the special (homogeneous) solutions.
6. What is the nullspace ?
7. Find the particular solution to 
8. Give the complete solution.

***Solution***

1. Rank(***A***) = 3
2. Dimension of *A* = 1
3. The pivots variables are: 
4. The free variables are: 
5. Let 



Set 

The special solution: 

1. 
2. 
3. 

***Exercise***

Let  

1. What is the rank of *A*?
2. What is the dimension of *A*?
3. What are the pivots variables?
4. What are the free variables?
5. Find the special (homogeneous) solutions.
6. What is the nullspace ?
7. Find the particular solution to 
8. Give the complete solution.

***Solution***

1. Rank(***A***) = 3
2. Dimension of *A* = 2
3. The pivots variables are: 
4. The free variables are: 
5. Let 



Set 

The special solution: 

Set 

The special solution: 

1. 
2. 
3. 

***Exercise***

Find a basis for each of the four subspaces associated with the given matrix



***Solution***







Rank (***A***) = 1

Dimension of *A* = 1

1. Basis for ***row space***: 

The pivots variables are: 

1. Basis of the **column spaces**:

The free variable is: 

Set 

1. Basis of the **Nullspace**:









1. Basis of the **Left** **Nullspace**: 

***Exercise***

Find a basis for each of the four subspaces associated with the given matrix



***Solution***





Rank (***A***) = 2

Dimension of *A* = 2

1. Basis for ***row space***: 

The pivots variables are: 

1. Basis of the **column spaces**:

The free variable is: 



Set 

The special solution: 

Set 

The special solution: 

1. Basis of the **Nullspace**: 









Let 

1. Basis of the **Left** **Nullspace**: 

***Exercise***

Find a basis for each of the four subspaces associated with the given matrix



***Solution***







Rank (***A***) = 2

Dimension of *A* = 2

1. Basis for ***row space***: 

The pivots variables are: 

1. Basis of the **column spaces**:

The free variable is: 



Set 

The special solution: 

Set 

The special solution: 

1. Basis of the **Nullspace**: 













1. Basis of the **Left** **Nullspace**: 

***Exercise***

Find a basis for each of the four subspaces associated with the given matrix



***Solution***







Rank (***A***) = 2

Dimension of *A* = 2

1. Basis for ***row space***: 

The pivots variables are: 

1. Basis of the **column spaces**:

The free variable is: 



Set 

The special solution: 

Set 

The special solution: 

1. Basis of the **Nullspace**: 











Let 

1. Basis of the **Left** **Nullspace**: 

***Exercise***

Find a basis for each of the four subspaces associated with the given matrix



***Solution***









Rank (***A***) = 2

Dimension of *A* = 2

1. Basis for ***row space***: 

The pivots variables are: 

1. Basis of the **column spaces**:

The free variable is: 



Set 

The special solution: 

Set 

The special solution: 

1. Basis of the **Nullspace**: 











Let 

1. Basis of the **Left** **Nullspace**: 

***Exercise***

Find a basis for each of the four subspaces associated with the given matrix



***Solution***









Rank (***A***) = 2

Dimension of *A* = 2

1. Basis for ***row space***: 

The pivots variables are: 

1. Basis of the **column spaces**:

The free variable is: 



Set . The special solution: 

1. Basis of the **Nullspace**: 















Let 

1. Basis of the **Left** **Nullspace**: 