***Solution Section* 3.2 – Angle and Orthogonality in Inner Product Spaces**

***Exercise***

Which of the following form orthonormal sets?

1. (1, 0), (0, 2) in 
2.  in 
3.  in 
4.  in 
5.  in 
6.  in 

***Solution***

1. 



They are ***orthonormal*** sets

1. 





They are ***orthonormal*** sets

1. 





They are ***not orthonormal*** sets

1. 







They are ***not orthonormal*** sets

1. 







They are ***not orthonormal*** sets

1. 



They are ***orthonormal*** sets

***Exercise***

Find the cosine of the angle between  and .

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 

















1. 















1. 













1. 













1. 













1. 















1. 













1. 















***Exercise***

Find the cosine of the angle between ***A*** and ***B***.

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 





















1. 



















1. 









1. 









***Exercise***

Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

|  |  |
| --- | --- |
|  |  |

***Solution***

1. 



Therefore, the given vectors are orthogonal.

1. 



Therefore, the given vectors are orthogonal.

1. 



Therefore, the given vectors are ***not*** orthogonal.

1. 



Therefore, the given vectors are ***not*** orthogonal.

1. 

















The vectors  are ***not*** orthogonal with respect to the Euclidean

***Exercise***

Do there exist scalars *k* and *l* such that the vectors  are mutually orthogonal with respect to the Euclidean inner product?

***Solution***



















Thus, there are no scalars such that the vectors are mutually orthogonal.

***Exercise***

Let  have the Euclidean inner product. For which values of *k* are  and  orthogonal?

1. 
2. 

***Solution***

1. 



 and  are orthogonal for 

1. 



 and  are orthogonal for 

***Exercise***

Let *V* be an inner product space. Show that if  and  are orthogonal unit vectors in *V*, then 

***Solution***







 since  and  are orthogonal unit vectors





Thus 

***Exercise***

Let ***S*** be a subspace of . Explain what  means and why it is true.

***Solution***

 is the orthogonal complement of , , which is itself the orthogonal complement of ***S***, so  means that ***S*** is the orthogonal of its orthogonal complement.

We need to show that ***S*** is contained in  and, conversely, that  is contained in ***S*** to be true.

1. Suppose  and . Then  by definition of .

Thus, ***S*** is certainly contained is  (which consists of all vectors in which are orthogonal to ).

1. Suppose  (means is orthogonal to all vectors in ); then we need to show that .

Let assume  be a basis for ***S*** and let  be a basis for . If , then  is linearly independent set. Since each vector ifs that set is orthogonal to all of , the set  is linearly independent.

Since there are  vectors in this set, this means that .

On the other hand, If *A* is the matrix whose *ith* row is , then the row space of *A* is ***S*** and the nullspace of *A* is .

Since ***S*** is *p*-dimensional, the rank of *A* is *p*, meaning that the dimension of  is . Therefore,

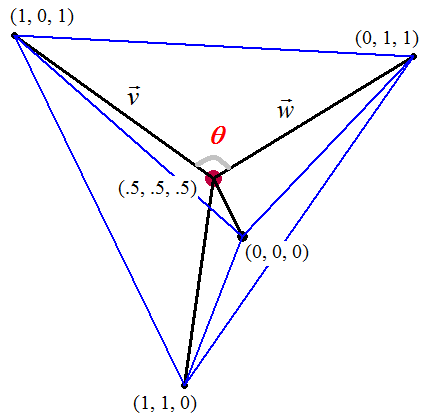


Which contradict the fact that . From this, we see that, if , it must be the case that .

***Exercise***

The methane molecule  is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at , ,  and  − (***note*** that all six edges have length , so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center  to the vertices?

***Solution***



Let  be the vector of the segment (1, 0, 1) and 





Let be the vector of the segment (0, 1, 1) and 





We have:











***Exercise***

Determine if the given vectors are orthogonal.



***Solution***

































The given vectors are ***orthogonal.***

***Exercise***

Which of the following sets of vectors are orthogonal with respect to the Euclidean inner

1. 
2. 

***Solution***

1. 











Therefore, the given vectors are ***not*** orthogonal.

1. 











Therefore, the given vectors are orthogonal.

***Exercise***

Consider vectors 

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  | 1. Cosine between |

***Solution***

1. 





1. 



1. 



1.   

***Exercise***

Consider vectors 

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  | 1. Cosine *θ* between |

***Solution***

1. 





1. 



1. 



1.  

 are orthogonal vectors.

***Exercise***

Consider vectors 

|  |  |  |
| --- | --- | --- |
|  |  | 1. Cosine *α* between 2. Cosine *β* between 3. Cosine *θ* between |

***Solution***

1. 





1. 





1. 





1. 



1. 



1. 



1.  
2.  
3.  
4. 







***Exercise***

Consider polynomial 

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  | 1. Cosine between |

***Solution***

1. 











1. 















1. 











1.  



***Exercise***

Consider polynomial 

|  |  |  |
| --- | --- | --- |
|  |  | 1. Cosine *α* between 2. Cosine *𝛽* between 3. Cosine *𝜃* between |

***Solution***

1.  









1.  









1.  









1.  











1.  













1.  













1.  



1.  



1.  



***Exercise***

Suppose  in a complex inner product space *V*. Find:

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

***Solution***

1. 





1. 





1. 









1. 





***Exercise***

Find the Fourier coefficient *c* and the projection  of  along  in 

***Solution***



















***Exercise***

Suppose . Find the projection of  onto *W* or find  that minimizes , where *W* is the subspace of  spanned by:

1. 
2. 

***Solution***

1. 





Therefore,  are orthogonal.





























1. 





Therefore,  are *not* orthogonal.

Applying Gram-Schmidt algorithm

























***Exercise***

Suppose  is an orthogonal set of vectors. Prove that (*Pythagoras*)



***Solution***







***Exercise***

Suppose *A* is an orthogonal matrix. Show that  for any 

***Solution***

*A* is an orthogonal matrix 

And 









 ***√***

***Exercise***

Suppose *A* is an orthogonal matrix. Show that  for every 

***Solution***

*A* is an orthogonal matrix

 and 











 ***√***

***Exercise***

Let *V* be an inner product space over  or . Show that



If and only if



***Solution***

Suppose that . For 





















***Exercise***

Let *V* be an inner product vector space over .

1. If  are three vectors in *V* with pairwise product negative, that is,



Show that  are linearly independent.

1. Is it possible for three vectors on the *xy*−plane to have pairwise negative products?
2. Does part (a) remain valid when the word “negative: is replaced with positive?
3. Suppose  are three−unit vectors in the *xy*−plane. What are the maximum and minimum values that



Can attain? And when?

***Solution***

1. Suppose that  are linearly dependent.

Then, assume that  are unit vectors and that



Then















Therefore,  are linearly independent.

1. To have all three vectors on the *xy*−plane which is in 2 dimensional.

Therefore, it is ***impossible*** for three to have pairwise negative products.

1. No
2. Given:  are three−unit vectors in the *xy*−plane and











Since 





Since the 3 vectors are unit vectors in the *xy*−plane and which it will divide the plane into a three equal angles 









Therefore, the minimum 

The maximum: 