***Section* 3.2 – Angle and Orthogonality in Inner Product Spaces**

***Cosine Formula***

If  and  are nonzero vectors that implies







***Example***

Let  have the Euclidean inner product. Find the cosine angle *θ*  between the vectors  and .

***Solution***















 





***Theorem* − Cauchy-Schwarz Inequality**

If  and  are vectors in a real inner product space *V*, then



***Proof***

If either  or  is equal to zero, then both sides equal to zero

Inequality holds.

Suppose that  and if  any vector



Let , then:







 Let 







 Since 







The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:





***Theorem***

If ,  and  are vectors in a real inner product space *V*, and if *k* is any scalar, then

1.  (***Triangle inequality for vectors***)
2.  (***Triangle inequality for distances***)

***Proof* (*a*)**

















***Definition***

Two vectors  and  in an inner product space are called orthogonal if 

***Example***

The vectors  and  are orthogonal with respect to the Euclidean inner product on , since





They are not orthogonal with the respect to the weighted Euclidean inner product , since





***Example***

 are orthogonal, since





***Definition***

If *W* is a subspace of an inner product space *V*, then the set of all vectors are orthogonal to every vector in *W* is called the ***orthogonal complement*** of *W* and is denoted by the symbol 

***Theorem***

If *W* is a subspace of an inner product space *V*, then:

1.  is a subspace of *V*.
2. 

***Proof***

1. Let set  contains at least the zero vector, since  for every vector  in *W*. We need to show that  is closed under addition and scalar multiplication.

Suppose that  and  are vectors in , so every vector  in *W* we have  and 





 ***Closed under addition***





 ***Closed under scalar multiplication***

Which proves that and  are in 

1. If  is any vector in both *W* and , then  is orthogonal to itself; that is, . It follows from the positivity axiom for inner products that 

***Theorem***

If *W* is a subspace of a finite-dimensional inner product space *V*, then the orthogonal complement of is *W*; that is



***Example***

Let *W* be the subspace of  spanned by the vectors



Find a basis for the orthogonal complement of *W*.

***Solution***

The Space *W* is the same as the row space of the matrix











The solution







***Definition***

A collection of vectors in  (or inner space) is called orthogonal if any 2 are perpendicular.



***Theorem***

If  are nonzero orthogonal vectors, then they are linearly independent.

***Definition***

A vector  is called normal if 

A collection of vectors  is called orthonormal if they are orthogonal and each .

An orthonormal basis is a basis made up of orthonormal vectors.

***Example***

***Q*** rotates every vector in the plane through the angle *θ*.



 





The dot product , the columns are orthogonal.

They are unit vectors because . Those columns give an orthonormal basis for the plane .

We have:  (This type is called ***rotation***)

***Exercises Section* 3.2 – Angle and Orthogonality in Inner Product Spaces**

1. Which of the following form orthonormal sets?
2. (1, 0), (0, 2) in 
3.  in 
4.  in 
5.  in 
6.  in 
7.  in 
8. Find the cosine of the angle between  and .

|  |  |
| --- | --- |
|  |  |

1. Find the cosine of the angle between ***A*** and ***B***.

|  |  |
| --- | --- |
|  |  |

1. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

|  |  |
| --- | --- |
|  |  |

1. Do there exist scalars *k* and *l* such that the vectors  are mutually orthogonal with respect to the Euclidean inner product?
2. Let  have the Euclidean inner product. For which values of *k* are  and  orthogonal?

|  |  |
| --- | --- |
|  |  |

1. Let *V* be an inner product space. Show that if  and  are orthogonal unit vectors in *V*, then 
2. Let ***S*** be a subspace of . Explain what  means and why it is true.
3. The methane molecule  is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at , ,  and  − (***note*** that all six edges have length , so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center  to the vertices?
4. Determine if the given vectors are orthogonal.



1. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner
2. 
3. 
4. Consider vectors 

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  | 1. Cosine between |

1. Consider vectors 

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  | 1. Cosine *θ* between |

1. Consider vectors 

|  |  |  |
| --- | --- | --- |
|  |  | 1. Cosine *α* between 2. Cosine *β* between 3. Cosine *θ* between |

1. Consider polynomial 

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  | 1. Cosine between |

1. Consider polynomial 

|  |  |  |
| --- | --- | --- |
|  |  | 1. Cosine *α* between 2. Cosine *𝛽* between 3. Cosine *𝜃* between |

1. Suppose  in a complex inner product space *V*. Find:

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |

1. Find the Fourier coefficient *c* and the projection  of  along  in 
2. Suppose . Find the projection of  onto *W* or find  that minimizes , where *W* is the subspace of  spanned by:
3. 
4. 
5. Suppose  is an orthogonal set of vectors. Prove that (*Pythagoras*)



1. Suppose *A* is an orthogonal matrix. Show that  for any 
2. Suppose *A* is an orthogonal matrix. Show that  for every 
3. Let *V* be an inner product space over  or . Show that



If and only if



1. Let *V* be an inner product vector space over .
2. If  are three vectors in *V* with pairwise product negative, that is,



Show that  are linearly independent.

1. Is it possible for three vectors on the *xy*−plane to have pairwise negative products?
2. Does part (a) remain valid when the word “negative: is replaced with positive?
3. Suppose  are three unit vectors in the *xy*−plane. What are the maximum and minimum values that



Can attain? And when?