***Section* 3.3 – Gram-Schmidt Process**

***Definition***

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

***Theorem***

1. If  is an orthogonal basis for an inner product space *V*, and if is any vector in *V*, then



1. If  is an orthonormal basis for an inner product space *V*, and if  is any vector in *V*, then



***Proof***

1. Since  is a basis for *V*, every vector  in *V* can be expressed in the form



Let show that 





Since *S* is an orthogonal set, all of the inner products in the last equality are zero except the *ith*, so we have





***The Gram-Schmidt Process***

To convert a basis  into an orthogonal basis , perform the following computations:

***Step* 1**: 

***Step* 2**: 

***Step* 3**: 

***Step* 4**: 

To convert the orthogonal basis into an orthonormal basis , normalize the orthogonal basis vectors. 

***Example***

Assume that the vector space  has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors



Into the orthogonal basis , and then normalize the ***orthogonal*** basis vectors to obtain an orthonormal basis 

***Solution***















































***Gram-Schmidt* Process (*Orthonormal*)**

Suppose  linearly independent in , construct *n* ***orthonormal***  that span the same space: span  = span 

***Step* 1**: Since  are linearly independent (≠ 0), so  (to create a normal vector)

Let , then  since is orthonormal and span 



***Step* 2**: 







***Step* 3**: 



|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |

***Example***

Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of



***Solution***

***Step*** **1**:  





***Step*** **2**: 























***Step*** **3**: 



























The ***orthonormal*** basis:



***QR−Decomposition***

***Problem***

If *A* is an  matrix with linearly independent column vectors, and if *Q* is the matrix that results by applying the Gram-Schmidt process to the column vectors of *A*, what relationship, if any, exists between *A* and *Q*?

To solve this problem, suppose that the column vectors of *A* are  and the orthonormal column vectors of *Q* are .





The equation  is a factorization of *A* into the product of a matrix *Q* with orthonormal column vectors and an invertible upper triangular matrix *R*. We call it the ***QR-decomposition of A***.

***Theorem***

If *A* is an  matrix with linearly independent column vectors, then *A* can be factored as



Where *Q* is an  matrix with orthonormal column vectors, and *R* is an  invertible upper triangular matrix.

***Example***

Find the *QR*-decomposition of



***Solution***

The column vectors of are



From the previous example

  









***Calculus***: Applying the Gram-Schmidt Process

We can apply the Gram-Schmidt orthogonalization procedure to generate some polynomials that are orthonormal on the interval  with inner product



***Example***

Apply the Gram-Schmidt orthonormalization process to the basis  in  using the inner product

***Solution***



Let 







































































The ***orthonormal*** basis is 

***Exercises Section* 3.3 – Gram-Schmidt Process**

(**1 – 14**) Use the Gram-Schmidt process to find an ***orthonormal*** basis for the subspaces of .

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. 
14. 

(**15 – 26**) Use the Gram-Schmidt process to find an ***orthogonal*** basis for the subspaces of .

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 
11. 
12. 
13. Find the ***QR***-decomposition of

|  |  |  |
| --- | --- | --- |
|  |  |  |

1. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product



(**29 – 33**) Apply the Gram-Schmidt ***orthonormalization*** process in  spanned by the functions, using the inner product

1. 
2. 
3. 
4. 
5. 
6. For , define the inner product over  as



1. If  is a unit vector in ?
2. Find an orthonormal basis for the subspace spanned by .
3. Complete the basis in part (*b*) to an orthonormal basis for  with respect to the inner product.
4. Is



Also, an inner product for 

1. Find a pair of vectors  and  such that



1. Is the basis found in part (*c*) are orthonormal basis for  with respect to the inner product in part (*d*)?