***Section* 3.5 – Least Squares Analysis**

The use to ***best*** fit data, we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial.

**Fitting a Curve to Data**

The common problem is to obtain a mathematical relationship between 2 variables *x* and *y* by ***fitting*** a curve to points in the *xy*-plane.

Some possibility of fitting the data

|  |  |  |
| --- | --- | --- |
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**Least Squares Fit of a Straight Line**

Recall that a system of equations  is called inconsistent if it does not have a solution. Suppose we want to fit a straight line  to the determined points 

If the data points were collinear, the line would pass through all *n* points and the unknown coefficients *m* and *b* would satisfy the equations



The problem is to find *m* and *b* that minimize the errors is some sense.

**Least Square Problem**

Given a linear system  of *m* equations in *n* unknowns, find a vector  that minimizes  with respect to the Euclidean inner product on . We call such as  a least squares solution of the system, we call  the least squares error vectors, and we call  the least squares error.



The term “***least square solution***” results from the fact the minimizing 

***Example***

Find the sums of squares of the errors of (2, 4), (4, 8), (6, 6)

***Solution***





The least squares problem for this example to find the values *m* and *b* for which is a minimum.

***Theorem***

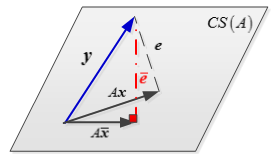
If *A* is an  matrix, the equation  has a solution if and only if  is in the column space of *A*.



 is a vector that is in the column space of *A*. For this *A* the column space is a plane is 

 is a vector, not in the column space of *A* (otherwise  has an exact solution)

 is the error vector, the difference between  and 



The length  is a *minimum* exactly when 

**Best Approximation** ***Theorem***

If  is a finite dimensional subspace of an inner product space, and if  is a vector in ***V***, then  is the best approximation to ***y*** from  is the sense that



For every vector  in  that is different from 

***Theorem***

For every linear system , the associated normal system



Is consistent, and all solutions are least squares solutions of 

If the columns of *A* are linearly independent, then  is invertible so has a unique solution .

This solution is often expressed theoretically as





***Proof***

Let the vector  is a least squares solution to 









***Theorem***

If *A* is an  matrix, then the following are equivalent

1. *A* has linearly independent column vectors.
2.  is invertible.

***Example***

Find the equation of the line that best fits the given points in the least-squares sense.

(40, 482), (45, 467), (50, 452), (55, 432), (60, 421)

***Solution***

Let  be the equation of the line that best fits the given points. Then



Where 

Using the normal equation formula: 











***Or***













Thus, 

***Example***

Given the system equation: 

1. Find the least-squares solution of the linear system 
2. Find the orthogonal projection of  on the column space of *A*
3. Find the ***error*** ***vector*** and the ***error***

***Solution***

1. 











Thus 

1. The orthogonal projection of  on the column space of *A*





1. 



The ***error***: 



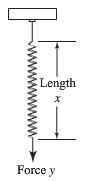
***Exercises Section* 3.5 – Least Squares Analysis**

(**1 – 7**) Find the equation of the line that best fits the given points in the least-squares sense and find the error.

1. 
2. 
3. 
4. 
5. 
6. 
7. 

(**8 – 10**) Find the orthogonal projection of the vector  on the subspace of  spanned by the vectors

1. 
2. 
3. 
4. Find the standard matrix for the orthogonal projection *P* of  on the line passes through the origin and makes an angle *θ* with the positive *x-*axis.
5. Hooke’s law in physics states that the length *x* of a uniform spring is a linear function of the force *y* applied to it. If we express the relationship as , then the coefficient *m* is called the spring constant.



Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., *x* = 6.1 when ). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.

1. Prove: If *A* has a linearly independent column vectors, and if  is orthogonal to the column space of *A*, then the least squares solution of  is.
2. Let *A* be an  matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of  onto the row space of *A*.
3. Let *W* be the line with parametric equations 
4. Find a basis for *W*.
5. Find the standard matrix for the orthogonal projection on *W*.
6. Use the matrix in part (*b*) to find the orthogonal projection of a point  on *W*.
7. Find the distance between the point  and the line *W*.
8. In , consider the line *l* given by the equations 

And the line *m* given by the equations 

Let *P* be the point on *l*, and let *Q* be a point on *m*.

Find the values of *t* and *s* that minimize the distance between the lines by minimizing the squared distance 

1. Determine whether the statement is true or false,
2. If *A* is an  matrix, then  is a square matrix.
3. If  is invertible, then *A* is invertible.
4. If *A* is invertible, then  is invertible.
5. If  is a consistent linear system, then  is also consistent.
6. If  is an inconsistent linear system, then  is also inconsistent.
7. Every linear system has a least squares solution.
8. Every linear system has a unique least squares solution.
9. If *A* is an  matrix with linearly independent columns and  is in , then  has a unique least squares solution.
10. A certain experiment produces the data .

Find the function that it will fit these data in the form of 

1. According to Kepler’s first law, a comet should have an ellipse, parabolic, or hyperbolic orbit (with gravitational attractions from the planets ignored). In suitable polar coordinates, the position  of a comet satisfies an equation of the form



Where  is a constant and *e* is the eccentricity of the orbit, with  for an ellipse,  for a parabolic, and  for a hyperbola.

Suppose observations of a newly discovered comet provide the data below.



Determine the type of orbit, and predict where the orbit will be when ?

1. To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from  to 

The position (in *feet*) were:



1. Find the least square cubic curve  for these data.
2. Estimate the velocity of the plane when , using the result from part (*a*).