***Section* 4.4 – Eigenvalues and Eigenvectors**

In many problems in science and mathematics, linear equations  come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of  (is changing with time) has nonzero solutions. (***All matrices are square***)

***Definition***

Suppose *A* is an *n* x *n* matrix and



The values of  are called eigenvalues of the matrix ***A*** and the nonzero vectors  in  are called the eigenvectors corresponding to that eigenvalue .

 is the eigenvalue associated with or corresponding to the eigenvector .

* One of the meanings of the word “***eigen***” in German is “***proper***”; eigenvalues are also called ***proper values, characteristic values,*** or ***latent roots***.

***Example***

The vector  is an eigenvector of  corresponding to the eigenvalue λ = 3 since









Eigenvalues and eigenvectors have a useful geometric interpretation in  and .

**The equation for the *eigenvalues***

Let’s rewrite the equation .

 : are the eigenvalues and not a vector





The matrix  times the eigenvectors  is the zero vector.

The eigenvectors make up the nullspace of .

***Definition***

The number λ is an eigenvalue of ***A*** if and only if  is singular:



This equation  is called ***characteristic equation*** of ***A***; the scalars satisfying this equation are the eigenvalues of ***A***. when expanding the determinant  is a polynomial in *λ* of degree *n*, called the ***characteristic polynomial*** of ***A***.

***Example***

Find the eigenvalues of the matrix 

***Solution***









The characteristic equation of ***A*** is:



The eigenvalues of ***A*** are 

***Theorem***

If ***A*** is an *n* x *n* triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of ***A*** are the entries on the main diagonal of ***A***.

***Example***

Find the eigenvalues of the lower triangular matrix



***Solution***

The eigenvalues are: 

***Theorem***

If ***A*** is an *n* x *n* matrix, the following are equivalent.

1. λ is an eigenvalue of ***A***.
2. The system of equations  has nontrivial solutions.
3. There is a nonzero vector  in  such that .
4. λ is a real solution of the characteristic equation 

***Eigenvectors***

To find the eigenvector , for each eigenvalue λ solve 

From the eigenvalues, the eigenvectors, in the form , of the system can be determined by letting:

 and

***Example***

Find the eigenvalues and the eigenvectors of the matrix 

***Solution***











The eigenvalues of ***A*** are: 

For , we have:









If *y* = −1 ⇒ *x* = 2

Therefore, the eigenvector 

Or 

For :







Therefore, the eigenvector 

**Power of a Matrix**

***Theorem***

If *k* is a positive integer, λ is an eigenvalue of a matrix *A*, and  is a corresponding eigenvector, then  is an eigenvalue of  and  is a corresponding eigenvector.

***Example***

Find the eigenvalues of  for 

***Solution***





The eigenvalues of *A*: 

The eigenvalues of  are:



***Theorem***

A square matrix *A* is invertible *iff*  is not an eigenvalue of *A*.

***Summary***

To solve the eigenvalue problem for an *n* by *n* matrix:

1. Compute the determinant of . With λ subtracted along the diagonal, this determinant starts with  or . It is a polynomial in λ of degree *n*.
2. Find the roots of this polynomial, by solving . The *n* roots are the *n* eigenvalues of *A*. They make  singular.
3. For each eigenvalue λ, solve  ***to find an eigenvector*** ***x***.

**Imaginary Eigenvalues**

***Example***

Find the eigenvalues and the eigenvectors of the matrix 

***Solution***









The eigenvalues are: 

For : 





Therefore, the eigenvector 

:





Therefore, the eigenvector 

***Example***

Find the eigenvalues and the eigenvectors of the matrix 

***Solution***







The eigenvalues are: 

The matrix ***A*** is a 90° rotation which has no real eigenvalues or eigenvectors.

No vector  stays in the same direction as  (except the zero vector which is useless).

If we add the eigenvalues together the result is zero which is the trace of ***A***.

: 





Therefore, the eigenvector 

: 





Therefore, the eigenvector 

***Exercises*** ***Section* 4.4 – Eigenvalues and Eigenvectors**

1. Find the eigenvalues and eigenvectors of :



Check the trace  and the determinant  for *A* and also .

1. Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues



1. For which real numbers c does this matrix A have



1. Two real eigenvalues and eigenvectors.
2. A repeated eigenvalue with only one eigenvector
3. Two complex eigenvalues and eigenvectors.
4. Find the eigenvalues of ***A***, ***B***, ***AB***, and ***BA***:



1. The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***A*** times eigenvalues of ***B***.
2. The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***BA***.
3. When  show that (1, 1) is an eigenvector and find both eigenvalues of



1. The eigenvalues of *A* equal to the eigenvalues of . This is because  equals . That is true because \_\_\_\_\_. Show by an example that the eigenvectors of *A* and  are not the same.
2. Let . Compute the eigenvalues and eigenvectors of *A*.
3. Let 
4. What is the characteristic polynomial for *A* (i.e. compute ?
5. Verify that 1 is an eigenvalue of *A*. What is a corresponding eigenvector?
6. What are the other eigenvalues of *A*?

(**9 – 58**) For the following matrices:

1. Find the characteristic equation.
2. Find the eigenvalues.
3. Find the eigenvectors.

|  |  |  |
| --- | --- | --- |
|  |  |  |

|  |  |  |
| --- | --- | --- |
|  | 4. . |  |

1. Find the eigenvalues of  for 
2. Given: . Compute 
3. Find the eigenvalues of the matrices



1. Given the matrix 
2. Find the characteristic polynomial.
3. Find the eigenvalues
4. Find the bases for its eigenspaces
5. Graph the eigenspaces
6. Verify directly that , for all associated eigenvectors and eigenvalues.
7. Given the matrix 
8. Find the characteristic polynomial.
9. Find the eigenvalues
10. Find the bases for its eigenspaces
11. Graph the eigenspaces
12. Verify directly that , for all associated eigenvectors and eigenvalues.
13. Explain why a  matrix can have at most two distinct eigenvalues. Explain why an  matrix can have at most *n* distinct eigenvalues.
14. Construct an example of a  matrix with only one distinct eigenvalue.
15. Let  be an eigenvalue of an invertible matrix *A*. Show that  is an eigenvalue of .
16. Show that if  is the zero matrix, then the only eigenvalue of *A* is 0.
17. Show that  is an eigenvalue of *A* if and only if  is an eigenvalue of .
18. For , find one eigenvalue, without calculation. Justify your answer.
19. For , find one eigenvalue, and two linearly independent eigenvectors, without calculation. Justify your answer.
20. Consider an matrix *A* with the property that the row sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of *A*.
21. Consider an matrix *A* with the property that the column sums all equal the same number ***s***. Show that ***s*** is an eigenvalue of *A*.
22. Let *A* be the matrix of the linear transformation *T* on 

*T*: reflects points across some line through the origin.

Without writing *A*, find an eigenvalue of *A* and describe the eigenspace.

1. Let *A* be the matrix of the linear transformation *T* on 

*T*: reflects points about some line through the origin.

Without writing *A*, find an eigenvalue of *A* and describe the eigenspace.

1. Show that if  is an eigenvector of the matrix product  and , then  is an eigenvector of 
2. Explain and demonstrate that the eigenspace of a matrix *A* corresponding to some eigenvalue  is a subspace.
3. If  is an eigenvalue of the matrix *A*, prove that  is an eigenvalue of .