***Section* 4.5 – Diagonalization**

When  is an eigenvector, multiplication by ***A*** is just multiplication by a single number: .

The matrix ***A*** turns into a diagonal matrix ***A*** when we use the eigenvectors property.

**Diagonalization**

Suppose the *n* by *n* matrix ***A*** has *n* linearly independent eigenvectors . Put them into the column of an ***eigenvector*** ***matrix*** ***P***. Then  is the eigenvalue matrix ***A***:



***Example***

The projection matrix  has 

***Solution***

For 





Therefore, 

For 





Therefore, 

The eigenvectors are:  that are the value of *P*.













***Definition***

A square matrix *A* is called ***diagonalizable*** if there is an invertible matrix *P* such that  is diagonal; the matrix *P* is said to ***diagonalize*** *A*.

***Theorem***

***Independent x from different λ*** - Eigenvectors  that correspond to distinct (all different) eigenvalues are linearly independent. An *n* by *n* matrix that has *n* different eigenvalues (no repeated λ’s) must be diagonalizable.

***Proof***

Suppose 





Multiply  by , that implies to













Since  and λ’s are different , we forced 

Similarly; Multiply  by , that implies to 







Therefore,  must be independent.

***Theorem***

If  are eigenvectors of *A* corresponding to distinct eigenvalues , then  is linearly independent set.

***Theorem***

If an *n* x *n* matrix ***A*** has *n* distinct eigenvalues, then the following are equivalent:

1. ***A*** is diagonalizable
2. ***A*** has *n* linearly independent eigenvectors.

***Example***

Given the Markov matrix 

***Solution***









The eigenvalues are: 

For , we have: 





Therefore, the eigenvector 

For , we have: 





Therefore, the eigenvector 

The eigenvector matrix is given by:















***Eigenvalues of AB and A* + *B***

An eigenvalue of ***A*** times an eigenvalue of ***B*** usually does not give an eigenvalue of ***AB***.



***Commuting matrices share eigenvectors***: Suppose ***A*** and ***B*** can be diagonalized. They share the eigenvector matrix ***P*** if and only if *AB* = *BA*.

**Matrix Powers **









The eigenvector matrix for  is still *S*, and the eigenvalue matrix is . The eigenvectors don’t change, and the eigenvalues are taken to the *kth*  power. When *A* is diagonalized,  is easy.

Here are steps (taken from Fibonacci):

1. Find the eigenvalues of *A* and look for *n* independent eigenvectors.
2. Write as a combination of the eigenvectors.
3. Multiply each eigenvector . Then





***Example***

Compute 

***Solution***





The eigenvalues are: 

For 





For 





The eigenvector matrix is given by:















***Similar Matrices***

***Definition***

If *A* and *B* are square matrices, then we say that ***B is similar to A*** if there exists an invertible matrix *P* such that 

* Similar matrices *B* and  have the same eigenvalues. If  is an eigenvector of *A* then  is an eigenvector of .

***Proof***

Since 

Suppose :





The eigenvalue of *B* is the same λ. The eigenvector is now 

***Example***

The projection  is similar to 

Choose  ; the similar matrix 

Also choose  ; the similar matrix 

These matrices  all have the same eigenvalues 1 and 0.

***Every 2 by 2 matrix with those eigenvalues is similar to A***.

The eigenvectors change with *M*.

***Example***

 is similar to every matrix  except .

These matrices *B* all have zero determinant (like *A*). T

hey all have rank one (like *A*). Their trace is *cd – cd* = 0.

Their eigenvalues are 0 and 0 (like *A*).

Choose  and 

Connections between similar matrices *A* and *B*:

|  |  |
| --- | --- |
| ***Not Changed*** | ***Changed*** |
| Eigenvalues | Eigenvectors |
| Trace and determinant | Nullspace |
| Rank | Column space |
| Number of independent  eigenvectors | Row space  Left nullspace |
| Jordan form | Singular values |

***Example***

Jordan matrix *J* has triple eigenvalues 5, 5, 5. Its only eigenvectors are multiples of (1, 0, 0). Algebraic multiplicity 3, geometric multiplicity 1:

If  has rank 2.

Every similar matrix  has the same triple eigenvalues 5, 5, 5. Also *B* – 5*I* must have the same rank 2. Its nullspace has dimension 3 − 2 = 1. So each similar matrix B also has only one independent eigenvector.

The transpose matrix  has the same eigenvalues 5, 5, 5, and  has the same rank 2. ***Jordan’s theory says that  is similar to J***. The matrix that produces the similarity happens to be the reserve identity *M*:



There is one line of eigenvectors  for *J* and another line  for .

***Fibonacci* Numbers**

Every new Fibonacci number is the sum of the two previous *F*’s.

The ***sequence***  comes from 

***Problem***

Find the Fibonacci number 

We can apply the rule one step at a time, or just use Linear algebra.

Let consider the matrix equation: . Fibonacci rule gave us a two-step rule for scalars.

Let , the rule  becomes .

Every step multiplies by , after 100 steps we reach 









The characteristic equation is  and the solutions are



For 







For 







The eigenvector matrix is given by:



The combination of these eigenvectors that give :













**The *Jordan* Form**

For every *A*, we want to choose *M* so that  is as nearly diagonal as possible. When *A* has a full set of *n* eigenvectors, they go into the columns of *M*. Then *M* = *P*. The matrix is diagonal.

If *A* has ***s*** independent eigenvectors, it is similar to a matrix *J* that has ***s*** Jordan blocks on its diagonal. There is a matrix *M* such that



Each block in *J* has one eigenvalue , one eigenvector, and 1’s above the diagonal:



***A is similar to B if they share the same Jordan form J – not otherwise.***

***Exercises Section* 4.5 – Diagonalization**

1. The Lucas numbers are like Fibonacci numbers except they start with . Following the rule . The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number .
2. Find all eigenvector matrices *S* that diagonalize *A* (rank 1) to give :



What is ? Which matrices *B* commute with *A* (so that *AB* = *BA*)

(**3 – 6**) Determine whether the matrix is diagonalizable

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(**7 – 26**) Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine .

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1. The 4 by 4 triangular Pascal matrix and its inverse (alternating diagonals) are



Check that  and  have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives , so  is similar to . Show that  with columns of alternating signs is its own inverse.

Since  and  are similar they have the same Jordan form *J*. Find *J* by checking the number of independent eigenvectors of  with λ = 1.

1. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don’t match and they are not similar:



For any matrix *M* compare *JM* with MK. If they are equal show that *M* is not invertible. Then  is Impossible; *J* is not similar to *K*.

1. If ***x*** is in the nullspace of *A* show that  is in the nullspace of .

The nullspaces of *A* and  have the same (vectors) (basis) (dimension)

1. Prove that  is always similar to A (λ′s are the same):
2. For one Jordan block , find  so that .
3. For any *J* with blocks , build  from blocks so that .
4. For any : Show that  is similar to  and so to J and so to *A*.
5. Why are these statements all true?
6. If *A* is similar to *B* then  is similar to .
7.  and  can be similar when *A* and *B* are not similar.
8.  is similar to 
9.  is not similar to 
10. If we exchange rows 1 and 2 of *A*, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case *M* =?
11. If an *n* x *n* matrix *A* has all eigenvalues λ = 0 prove that  is the zero matrix.
12. If *A* is similar to , must all the eigenvalues equal to 1 or −1?.

(**34 – 42**) Determine whether the *two matrices* are similar matrices

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1. Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.
2. Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.
3. Suppose that *A* is a matrix. Suppose that the linear transformation associated to *A* has two linearly independent eigenvectors. Prove that *A* is similar to a diagonal matrix.
4. Prove that if *A* is a  matrix that has two distinct eigenvalues, then *A* is similar to a diagonal matrix.
5. Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example . Is it true that matrices with equal characteristic polynomial are necessarily similar?
6. Show that the given matrix is not diagonalizable. 
7. Determine if the given matrix is diagonalizable. If, so, find matrices S and  such that the given matrix equals 

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1. *A* is a  matrix with *two* eigenvalues. One eigenspace is *three*−dimensional, and the other eigenspace is *two*−dimensional. Is *A* diagonalizable? Why?
2. *A* is a  matrix with *two* eigenvalues. Each eigenspace is *one*−dimensional. Is *A* diagonalizable? Why?
3. *A* is a  matrix with *three* eigenvalues. One eigenspace is *one*−dimensional, and one of the other eigenspace is *two*−dimensional. Is it possible that *A* is *not* diagonalizable? Justify your answer?
4. *A* is a  matrix with *three* eigenvalues. One eigenspace is *two*−dimensional, and one of the other eigenspace is *three*−dimensional. Is it possible that *A* is *not* diagonalizable? Justify your answer?
5. Show that if *A* is diagonalizable and invertible, then so is .
6. Show that if *A* has *n* linearly independent eigenvectors, then so does .
7. A factorization  is not unique. Demonstrate this for the matrix  with  , find a matrix  such that .
8. Construct a nonzero  matrix that is invertible but not diagonalizable.
9. Construct a nonzero  matrix that is diagonalizable but not invertible.
10. What are the matrices that are similar to themselves only?
11. For any scalars *a, b*, and *c*, show that



are similar.

Moreover, if, then *A* has two zero eigenvalues.

(**61 – 64**) For positive integer , compute

|  |  |
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1. Let . Show that  is similar to *A* fro every positive integer *k*. It is true more generally for any matrix with all eigenvalues equal to 1.
2. Can a matrix be similar to two different diagonal matrices?
3. Prove that if *A* is diagonalizable, then  is diagonalizable.
4. Prove that if the eigenvalues of a diagonalizable matrix *A* are all , then the matrix is equal to its inverse.
5. Prove that if *A* is diagonalizable with *n* real eigenvalues , then 
6. If *x* is a real number, then we can define  by the series



In similar way, If *X* is a square matrix, then we can define  by the series



Evaluate , where *X* is the indicated square matrix.

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