***Lecture Three***

***Section* 3.1 – Inner Products**

***Definition***

An ***inner product*** on a real vector space *V* is a function that associates a real number  with each pair of vectors in *V* in such a way that the following axioms are satisfies for all vectors ***u***, ***v***, and ***w*** in *V* and all scalars *k*.

1.  ***Symmetry axiom***
2.  ***Additivity axiom***
3.  ***Homogeneity axiom***
4.  and  iff  ***Positivity axiom***

A real vector space with an inner product is called a ***real inner product space***.



This is called the ***Euclidean inner product*** (or the ***standard*** ***inner product***)

***Definition***

If *V* is a real inner product space, then the norm (or length) of a vector *v* in *V* is denoted by  and is defined by



And the ***distance*** between two vectors is denoted by  and is defined by



A vector of norm 1 is called a ***unit vector***.

***Theorem***

If ***u*** and ***v*** are vectors in a real inner product space *V*, and if *k* is a scalar, then:

1.  with equality *iff* 
2. 
3. 
4.  with equality *iff*  

Although the Euclidean inner product is the most important inner product on , there are various applications in which is desirable to modify it by weighing each term differently. More precisely, if

 are positive real numbers, which we will call weighs, and if  and are vectors in , then it can be shown that the formula



Defines an inner product on  that we call the ***weighted Euclidean inner product*** with weights 

***Example***

Let  and  be vectors in , verify that the weighted Euclidean inner product  satisfies the four inner product axioms.

***Solution***

*Axiom* 1: 

*Axiom* 2: 









*Axiom* 3: 

**



*Axiom* 3: 

**



***Exercises Section* 3.1 – Inner Products**

1. Let  be the Euclidean inner product on , and let , , , and . Compute the following.
2. 
3. 
4. 
5. 
6. 
7. 
8. Let  be the Euclidean inner product on , and let , , and . Compute the following for the weighted Euclidean inner product  .
9. 
10. 
11. 
12. 
13. 
14. 
15. Let  be the Euclidean inner product on , and let , , , and . Verify the following.

|  |  |
| --- | --- |
|  |  |

1. Let  be the Euclidean inner product on , and let , , , and . Verify the following for the weighted Euclidean inner product

|  |  |
| --- | --- |
|  |  |

1. Let  and . Show that the following are inner product on  by verifying that the inner product axioms hold. 
2. Show that the following identity holds for the vectors in any inner product space



***Section* 3.2 – Angle and Orthogonality in Inner Product Spaces**

***Cosine Formula***

If  and  are nonzero vectors that implies 



***Example***

Let  have the Euclidean inner product. Find the cosine angle θ between the vectors  and .

***Solution***















***Theorem* − Cauchy-Schwarz Inequality**

If  and  are vectors in a real inner product space *V*, then



The following two alternative forms of the Cauchy-Schwarz inequality are useful to know:





***Theorem***

If ,  and  are vectors in a real inner product space *V*, and if *k* is any scalar, then

1.  (***Triangle inequality for vectors***)
2.  (***Triangle inequality for distances***)

***Proof* (*a*)**

















***Definition***

Two vectors ***u*** and ***v*** in an inner product space are called orthogonal if 

***Example***

The vectors  and  are orthogonal with respect to the Euclidean inner product on , since



They are not orthogonal with the respect to the weighted Euclidean inner product , since



***Example***

 are orthogonal, since



***Definition***

If *W* is a subspace of an inner product space *V*, then the set of all vectors are orthogonal to every vector in *W* is called the ***orthogonal complement*** of *W* and is denoted by the symbol 

***Theorem***

If *W* is a subspace of an inner product space *V*, then:

1.  is a subspace of *V*.
2. 

***Proof***

1. Let set  contains at least the zero vector, since  for every vector ***w*** in *W*. We need to show that  is closed under addition and scalar multiplication.

Suppose that ***u*** and ***v*** are vectors in , so every vector ***w*** in *W* we have  and 

 ***Closed under addition***

 ***Closed under scalar multiplication***

Which proves that and *k****u*** are in 

1. If ***v*** is any vector in both *W* and , then ***v*** is orthogonal to itself; that is, . It follows from the positivity axiom for inner products that 

***Theorem***

If *W* is a subspace of a finite-dimensional inner product space *V*, then the orthogonal complement of is *W*; that is



***Example***

Let *W* be the subspace of  spanned by the vectors



Find a basis for the orthogonal complement of W.

***Solution***

The Space W is the same as the row space of the matrix



The solution







***Definition***

A collection of vectors in  (or inner space) is called orthogonal if any 2 are perpendicular.



***Theorem***

If  are nonzero orthogonal vectors, then they are linearly independent.

***Definition***

A vector ***v*** is called normal if 

A collection of vectors  is called orthonormal if they are orthogonal and each .

An orthonormal basis is a basis made up of orthonormal vectors.

***Example***

***Q*** rotates every vector in the plane through the angle θ.



The dot product , the columns are orthogonal.

They are unit vectors because . Those columns give an orthonormal basis for the plane .

We have:  (This type is called ***rotation***)

***Exercises Section* 3.2 – Angle and Orthogonality in Inner Product Spaces**

1. Which of the following form orthonormal sets?
2. (1, 0), (0, 2) in 
3.  in 
4.  in 
5.  in 
6.  in 
7.  in 
8. Find the cosine of the angle between ***u*** and ***v***.

|  |  |
| --- | --- |
|  |  |

1. Find the cosine of the angle between ***A*** and ***B***.

|  |  |
| --- | --- |
|  |  |

1. Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

|  |  |
| --- | --- |
|  |  |

1. Do there exist scalars *k* and *l* such that the vectors are mutually orthogonal with respect to the Euclidean inner product?
2. Let  have the Euclidean inner product. For which values of *k* are ***u*** and ***v*** orthogonal?

|  |  |
| --- | --- |
|  |  |

1. Let *V* be an inner product space. Show that if ***u*** and ***v*** are orthogonal unit vectors in *V*, then 
2. Let **S** be a subspace of . Explain what  means and why it is true.
3. The methane molecule  is arranged as if the carbon atom were at the center of a regular tetrahedron with four hydrogen atoms at the vertices. If vertices are placed at , ,  and  − (***note*** that all six edges have length , so the tetrahedron is regular). What is the cosine of the angle between the rays going from the center  to the vertices?
4. Determine if the given vectors are orthogonal.



1. Which of the following sets of vectors are orthogonal with respect to the Euclidean inner
2. 
3. 

***Section* 3.3 – Gram-Schmidt Process**

***Definition***

A set of two or more vectors in a real inner product space is said to be ***orthogonal*** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be ***orthonormal***.

***Theorem***

1. If  is an orthogonal basis for an inner product space *V*, and if ***u*** is any vector in *V*, then



1. If  is an orthonormal basis for an inner product space *V*, and if ***u*** is any vector in *V*, then



***Proof***

1. Since  is a basis for V, every vector ***u*** in *V* can be expressed in the form



Let show that 





Since *S* is an orthogonal set, all of the inner products in the last equality are zero except the *ith*, so we have



***The Gram-Schmidt Process***

To convert a basis  into an orthogonal basis , perform the following computations:

***Step* 1**: 

***Step* 2**: 

***Step* 3**: 

***Step* 4**: 

To convert the orthogonal basis into an orthonormal basis , normalize the orthogonal basis vectors. 

***Example***

Assume that the vector space  has the Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vectors



Into the orthogonal basis , and then normalize the orthogonal basis vectors to obtain an orthonormal basis 

***Solution***





































***Gram-Schmidt* Process (*Orthonormal*)**

Suppose  linearly independent in , construct *n* ***orthonormal***  that span the same space: span  = span 

***Step* 1**: Since  are linearly independent (≠ 0), so  (to create a normal vector)

Let , then  since is orthonormal and span  = span 



***Step* 2**: 







***Step* 3**: 



|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |

***Example***

Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of 

***Solution***

***Step*** **1**: 





***Step*** **2**: 

















***Step*** **3**: 



















***QR−Decomposition***

***Problem***

If *A* is an  matrix with linearly independent column vectors, and if *Q* is the matrix that results by applying the Gram-Schmidt process to the column vectors of *A*, what relationship, if any, exists between *A* and *Q*?

To solve this problem, suppose that the column vectors of *A* are  and the orthonormal column vectors of *Q* are .





The equation  is a factorization of *A* into the product of a matrix *Q* with orthonormal column vectors and an invertible upper triangular matrix *R*. We call it the ***QR-decomposition of A***.

***Theorem***

If *A* is an  matrix with linearly independent column vectors, then *A* can be factored as



Where *Q* is an  matrix with orthonormal column vectors, and *R* is an  invertible upper triangular matrix.

***Example***

Find the *QR*-decomposition of



***Solution***

The column vectors of are



From the previous example









***Exercises Section* 3.3 – Gram-Schmidt Process**

1. Use the Gram-Schmidt process to find an orthonormal basis for the subspaces of .
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. Find the ***QR***-decomposition of

|  |  |  |
| --- | --- | --- |
|  |  |  |

1. Verify that the Cauchy-Schwarz inequality holds for the given vectors using the Euclidean inner product



***Section* 3.4 – Orthogonal Matrices**

***Definition***

A square matrix *A* is said to be orthogonal if its transpose is the same as its inverse, that is, if



or, equivalently, if



***Example***

The matrix 

***Solution***



***Example***

The matrix 

***Solution***



***Theorem***

The following are equivalent for  matrix *A*.

1. *A* is orthogonal.
2. The row vectors of *A* form an orthonormal set in  with the Euclidean inner product.
3. The column vectors of *A* form an orthonormal set in  with the Euclidean inner product.

***Theorem***

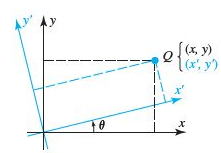
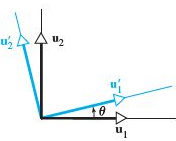
1. The inverse of an orthogonal matrix is orthogonal
2. A product of orthogonal matrices is orthogonal
3. If *A* is orthogonal, then  or 

***Theorem***

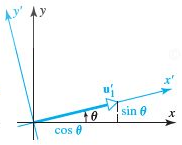
If *A* is an  matrix, then the following are equivalent

1. *A* is orthogonal.
2.  for all ***x*** in .
3.  for all ***x*** and ***y*** in .

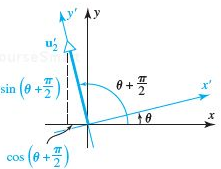
Let  and  be the unit vectors along the *x*- and *y*-axes and unit vectors  and  along the *x′*- and *y*′-axes.

The new coordinates  and the old coordinates  of a point Q will be related by







These are sometimes called the ***rotation equations***.

***Example***

Use the form  to find the new coordinates of the point  if the coordinate axes of a rectangular coordinate system are rotated through an angle of .

***Solution***







The new coordinates of *Q* are 

***Exercises Section* 3.4 – Orthogonal Matrices**

1. Show that the matrix is orthogonal

|  |  |
| --- | --- |
|  |  |

1. Determine if the matrix is orthogonal. For those that is orthogonal find the inverse.

|  |  |  |
| --- | --- | --- |
|  |  |  |

1. Prove that if *A* is orthogonal, then  is orthogonal.
2. Find a last column so that the resulting matrix is orthogonal



1. Determine if the given matrix is orthogonal. If it is, find its inverse



***Section* 3.5 – Least Squares Analysis**

The use to ***best*** fit data, we will use results about orthogonal projections in inner product spaces to obtain a technique for fitting a line or other polynomial.

**Fitting a Curve to Data**

The common problem is to obtain a mathematical relationship between 2 variables *x* and *y* by ***fitting*** a curve to points in the *xy*-plane.

Some possibility of fitting the data

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

**Least Squares Fit of a Straight Line**

Recall that a system of equations  is called inconsistent if it does not have a solution. Suppose we want to fit a straight line  to the determined points 

If the data points were collinear, the line would pass through all *n* points and the unknown coefficients *m* and *b* would satisfy the equations



The problem is to find *m* and *b* that minimize the errors is some sense.

**Least Square Problem**

Given a linear system  of *m* equations in *n* unknowns, find a vector ***x*** that minimizes  with respect to the Euclidean inner product on . We call such as ***x*** a least squares solution of the system, we call  the least squares error vectors, and we call  the least squares error.



The term “***least square solution***” results from the fact the minimizing 

***Example***

Find the sums of squares of the errors of (2, 4), (4, 8), (6, 6)

***Solution***





The least squares problem for this example to find the values *m* and *b* for which is a minimum.

***Theorem***

If *A* is an  matrix, the equation  has a solution if and only if *y* is in the column space of *A*.



 is a vector that is in the column space of A. For this A the column space is a plane is 

 is a vector, not in the column space of A (otherwise  has an exact solution)

 is the error vector, the difference between  and 



The length  is a minimum exactly when 

**Best Approximation** ***Theorem***

If  is a finite dimensional subspace of an inner product space, and if ***y*** is a vector in ***V***, then  is the best approximation to ***y*** from  is the sense that



For every vector ***w*** in  that is different from 

***Theorem***

For every linear system , the associated normal system



Is consistent, and all solutions are least squares solutions of 

If the columns of *A* are linearly independent, then  is invertible so has a unique solution . This solution is often expressed theoretically as





***Proof***

Let the vector  is a least squares solution to 









***Theorem***

If *A* is an  matrix, then the following are equivalent

1. *A* has linearly independent column vectors.
2.  is invertible.

***Example***

Find the equation of the line that best fits the given points in the least-squares sense.

(40, 482), (45, 467), (50, 452), (55, 432), (60, 421)

***Solution***

Let  be the equation of the line that best fits the given points. Then



Where 

Using the normal equation formula: 





|  |  |
| --- | --- |
|  | ***Or*** |

Thus 

***Example***

Given the system equation: 

1. Find the least-squares solution of the linear system 
2. Find the orthogonal projection of  on the column space of *A*
3. Find the error vector and the error

***Solution***

1. 









Thus 

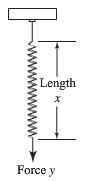
1. The orthogonal projection of  on the column space of *A*



1. 

The error: 

***Exercises Section* 3.5 – Least Squares Analysis**

1. Find the equation of the line that best fits the given points in the least-squares sense.
2. 
3. 
4. 
5. 
6. Find the orthogonal projection of the vector  on the subspace of  spanned by the vectors
7. 
8. 
9. 
10. Find the standard matrix for the orthogonal projection *P* of  on the line passes through the origin and makes an angle *θ* with the positive *x-*axis.
11. Hooke’s law in physics states that the length *x* of a uniform spring is a linear function of the force *y* applied to it. If we express the relationship as , then the coefficient *m* is called the spring constant. Suppose a particular unstretched spring has a measured length of 6.1 *inches*.(i.e., *x* = 6.1 when *y* = 0). Forces of 2 pounds, 4 pounds, and 6 pounds are then applied to the spring, and the corresponding lengths are found to be 7.6 inches, 8.7 inches, and 10.4 inches. Find the spring constant.
12. Prove: If *A* has a linearly independent column vectors, and if  is orthogonal to the column space of *A*, then the least squares solution of  is.
13. Let *A* be an  matrix with linearly independent row vectors. Find a standard matrix for the orthogonal projection of  onto the row space of *A*.
14. Let *W* be the line with parametric equations 
15. Find a basis for *W*.
16. Find the standard matrix for the orthogonal projection on *W*.
17. Use the matrix in part (*b*) to find the orthogonal projection of a point  on *W*.
18. Find the distance between the point  and the line *W*.
19. In , consider the line *l* given by the equations 

And the line *m* given by the equations 

Let *P* be the point on *l*, and let *Q* be a point on *m*. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance 

1. Determine whether the statement is true or false,
2. If *A* is an  matrix, then  is a square matrix.
3. If  is invertible, then *A* is invertible.
4. If *A* is invertible, then  is invertible.
5. If  is a consistent linear system, then  is also consistent.
6. If  is an inconsistent linear system, then  is also inconsistent.
7. Every linear system has a least squares solution.
8. Every linear system has a unique least squares solution.
9. If *A* is an  matrix with linearly independent columns and ***b*** is in , then  has a unique least squares solution.