***Lecture Four***

***Section* 4.1 – Matrix Transformations from  to **

***Definition***

If *V* and *W* are vector spaces, and if *f* is a function with domain *V* and codomain *W*, then we say that *f* is a transformation from *V* to *W* or that f maps *V* to *W*, which we denote by writing



In the special case where *V* = *W*, the transformation is also called an operator on *V*.

**Matrix Transformation**



Which we can write in matrix formation





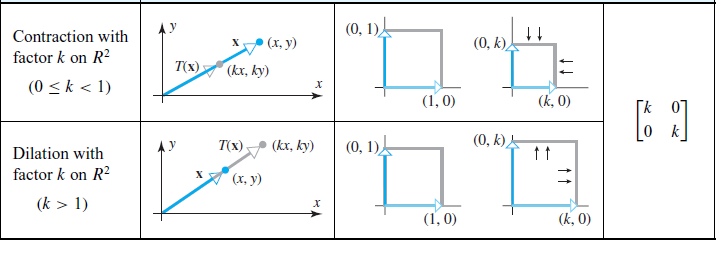
Although we could view this as a linear system, we will view it instead as a transformation that maps the column vector ***x*** in  into the column vector w in  by multiplying ***x*** on the left by *A*. We call this a ***matrix transformation*** (or ***matrix operator*** if *m = n*) and we denote it by

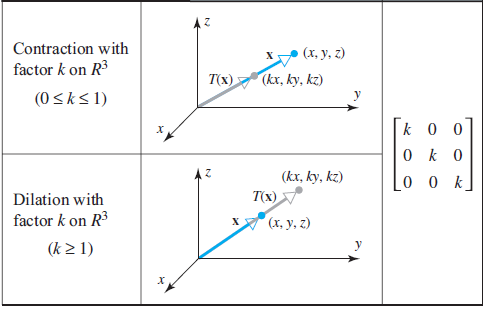


|  |  |  |  |
| --- | --- | --- | --- |
|  | |  | |
| ***T*** *maps vectors to vectors* | | ***T*** *maps points to points* | |
| Reflection about the *y*-axis |  | |  | |  |
| Reflection about the *x*-axis |  | |  | |  |
| Reflection about the line  y = *x* |  | |  | |  |
| Reflection about the  *xy*-plane |  | |  | |  |
| Reflection about the  *xy*-plane |  | |  | |  |
| Reflection about the  *yz*-plane |  | |  | |  |
| Orthogonal projection on the *x*-axis |  | |  | |  |
| Orthogonal projection on the *y*-axis |  | |  | |  |

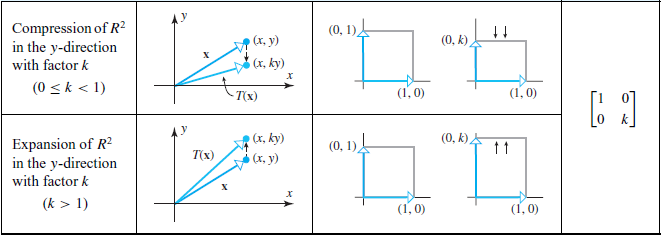
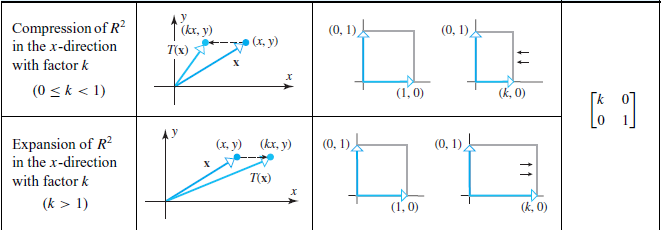
|  |  |  |  |
| --- | --- | --- | --- |
| Orthogonal projection on the *xy*-Plane |  |  |  |
| Orthogonal projection on the *xz*-Plane |  |  |  |
| Orthogonal projection on the *yz*-Plane |  |  |  |
| ***Rotation Operators*** | | | |
| Rotation through an angle *θ* |  |  |  |
| Counterclockwise rotation about the positive *x*-axis through an angle *θ* |  |  |  |
| Counterclockwise rotation about the positive *y*-axis through an angle *θ* |  |  |  |
| Counterclockwise rotation about the positive *z*-axis through an angle *θ* |  |  |  |

***Dilations and Contractions***

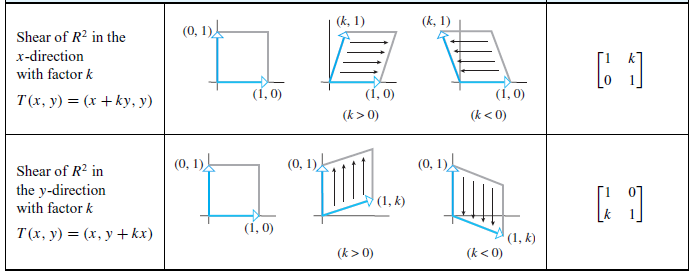




***Expansion or Compression***



***Shear***



**Orthogonal Projections on Lines through the Origin**





***Example***

Find the orthogonal projection of the vector ***x*** = (1, 5) on the line through the origin that makes an angle of  with the *x*-axis

***Solution***











***Four Fundamental Subspaces***

1. The ***row space*** is , a subspace of .
2. The ***column space*** is , a subspace of .
3. The ***nullspace*** is , a subspace of .
4. The ***left nullspace*** is , a subspace of .

***The Four Subspaces for R***

Consider the matrix 3 by 5:



1. Rows 1 and 2 are a basis. The row space contains combination of all 3 rows.

The ***row space*** of ***R*** has dimension 2 (= ***rank***).

***The dimension of the row space is r***. The nonzero rows of ***R*** form a basis.

1. The ***column space*** of ***R*** has dimension *r* = 2.

The pivot columns 1 and 4 form a basis. They are independent because they start with the *r* by *r* identity matrix.

There are 3 special solutions:



***The dimension of the column space is r***. The pivot columns form a basis.

1. The ***nullspace*** has dimension *n* – *r* = 5 – 2 = 3 (free variables). Here are free (no pivots in those columns). They yield the three special solutions to . Set a free variable to 1, and solve for .



 has the complete solution: 

***The nullspace has dimension*** *n* – *r*. The special solutions form a basis.

1. The ***nullspace*** of  has dimension *m* – *r* = 3 – 2 = 1

The equation : 

The nullspace of  contains all vectors  and it is the line of the basis vector .

***The left nullspace has dimension m – r***. The solutions are 

* In  the row space and nullspace have dimensions *r* and *n* – *r* (adding to *n*)
* In  the column space and left nullspace have dimensions *r* and *m* – *r* (total *m*)



***The Four Subspaces for A***

***The subspace dimensions for A are the same as for R***.

These matrices are connected by an invertible matrix *E*. 

1. ***A*** *has the same row space as* ***R***. Same dimension *r* and same basis

Every row of ***A*** is a combination of the rows of ***R***. Also every row of ***R*** is a combination of the rows of ***A***.

1. *The column space of* ***A*** *has dimension* *r*. The number of independent columns equals the number of independent rows.
2. ***A*** *has the same nullspace as* ***R***. Dimension *n* – *r* and same basis.

**(*dimension of column space*) *+* (*dimension of nullspace*) *= dimension of ***

1. *The left nullspace* ***A*** (the nullspace of) *has dimension m – r*.

**Fundamental Theorem of Linear Algebra**, (Part 1)

The column space and row space both have dimension *r*.

The nullspaces have dimensions *n – r* and *m – r*.

***Example***

Consider 

***A*** has *m* = 1, *n* = 3, and rank: *r* = 1.

The row space is a line in .

The nullspace is the plane . This plane has dimension 2 (which is 3 – 1).

The columns of this is 1 by 3 matrix are in . The column space is all of .

The left nullspace contains only the zero vector.

The only solution to , the only combination of the row that gives the zero row. Thus , the zero space with dimension 0 (*m – r*). In  the dimensions (1 + 0) = 1.

***Example***

Consider 

***A*** has *m* = 2, *n* = 3, and rank: *r* = 1.

The row space is a line in .

The nullspace is the plane . This plane has dimension 3 (1 + 2).

The columns are multiples of the first column (1, 1).

The left nullspace contains more than one zero vector. The solution to  has the solution .

The column space and nullspace are perpendicular lines in . Their dimensions are 1 and 1 = 2.

Column space = line through 

Left nullspace=line through 

***Exercises***  ***Section* 4.1 – Matrix Transformations from  to **

1. Find the standard matrix for the transformation defined by the equations
2. 
3. 
4. 
5. Find the standard matrix for the operator *T* defined by the formula
6. 
7. 
8. 
9. Find the standard matrix for the transformation *T* defined by the formula
10. 
11. 
12. 
13. 

***Section* 4.2 – General Linear Transformations**

***Definition***

A transformation *T* assigns an output  to each input vector *v*. The transformation is ***linear*** if it meets these requirements for all *v* and *w*:



We can combine both into one: 

***Theorem***

If  is a linear transformation, then:

1. 
2.  for all ***u*** and ***v*** in *V*.

***Example***

If *V* is a vector space and *k* is any scalar, then the mapping  given by  is a linear operator on *V*, for if *c* is any scalar and if ***u*** and ***v*** are any vectors in *V*, then





If 0 < *k* < 1, then *T* is called ***contraction*** of *V* with factor *k*, and if *k* > 1, then *T* is called ***dilation*** of *V* with factor *k*

|  |  |
| --- | --- |
|  |  |
| ***Dilation of V*** | ***Contraction of V*** |

***Example***

Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*; if *T* is linear, find the *A* such . 

***Solution***

Let 





















Since  and , then function *T* is a linear transformation.

***Domain***: 





***Example*** − the Zero Transformations

Let *V* and *W* be any vector spaces. The mapping  such that  for every ***v*** in *V* is a linear transformation called the zero transformation. To see that *T* is linear, observe that:



Therefore; 

***Example***

Choose a fixed vector ***a*** = (1, 3, 4), and let  be the dot product ***a.v***:

***Solution***

Let 



His is linear. The inputs *v* come from three dimensional space, so . The output just numbers, so the output space is . We are multiplying by the row matrix *A* = [1, 3, 4].

Then 

***Example***

Show that the length  is not linear.

***Solution***

 There are not equal because the sides of a triangle satisfy an inequality 

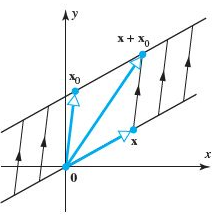
 Not - because the length 

***Example***

If  is a fixed nonzero vector in , then the transformation



It has a geometric effect of translating each point ***x*** in a direction parallel to  through a distance of . S This cannot be a linear transformation since 



***Theorem***

Let  be the linear transformation, where *V* is finite dimensional. If  is a basis for *V*, then the image of any vector ***v*** in *V* can be expressed as



Where  are the coefficients required to express ***v*** as a linear combination of the vectors in *S*.

***Example***

Consider the basis for , where 

Let  be the linear transformation for which



Find a formula for , and then use that formula to compute 

***Solution***

















***Example***

*T* is the transformation that rotates every vector by 30°, the domain is the *xy*-plane (where the input vector ***v*** is). The range is also the *xy*-plane (where the rotated *T*(***v***) is). Is the rotation linear?

***Solution***

Yes it is. We can rotate two vectors and add the results. The sum of rotation *T*(***v***) + *T*(***w***) is the same as the rotation *T*(***v + w***) of the sum.

The whole plane is turning together, in this linear transformation.

***Definition***

If  is a linear transformation, then the set of vectors in *V* that *T* maps into **0** is called ***kernel*** of *T* and is denoted by ***ker***(*T*). The set of all vectors in *W* that are images under *T* of at least one vector in *V* is called the ***range*** of *T* and is denoted by ***R***(*T*).

***Note***:

Transformations have a language of their own. Where there is no matrix, we can’t talk about a column space. But the idea can be rescued and used. The column space consisted of all ouputs *A****v***. The nullspace consisted of all inputs for which *A****v*** – 0. Translate those into “range” and “kernel”

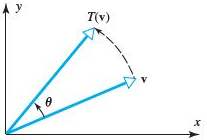
**Range** of *T* = set of all outputs *T*(***v***): corresponds to column space

**Kernel** of *T* = set of all outputs for which *T*(***v***) = 0: corresponds to nullspace

***Example***

Project every 3-dimensional vector down onto the *xy* plane.

The range is that plane, which contains every *T*(***v***). The kernel is the ***z*** axis (which projects down to zero). This projection is linear.

***Example*** − ***Kernel and Range of a Rotation***

Let  be the linear operator that rotates each vector in the *xy−*plane through the angle *θ*. Since every vector in the *xy−*plane can be obtained by rotating some vector through the angle *θ*, it follows that . Moreover, the only vector that rotates into **0** is **0**, so 

***Theorem***

If  is a linear transformation, then:

1. The *kernel* of *T* is a subspace of *V*
2. The *range* of *T* is a subspace of *W*

***Theorem***

If  is a linear transformation from an *n-*dimensional vector space *V* to a vector space *W*, then



***Example***

Project every 3-dimensional vector down onto horizontal plane ***z*** = 1.

The vector ***v*** = (*x, y, z*) is transformed to *T*(***v***) = (*x, y*, 1). This transformation is not linear, it doesn’t even transform ***v*** = 0 into *T*(***v***) = 0.

Multiply every 3-dimensional vector by a 3 by 3 matrix *A*. This is definitely a linear transformation

 which does equal 

***Example***

Suppose *A* is an invertible matrix. The kernel of *T* is the zero vector; the range ***W*** equals the domain ***V***. Another linear transformation is multiplication by . This is the inverse transformation , which brings every vector *T*(***v***) back to ***v***:

 matches the matrix multiplication 

***Are all linear transformation produced by matrices?***

Each *m* by *n* matrix does produce a linear transformation from  to . When a linear *T* is described as a “rotation” or “projection” or “...” is there always a matrix hiding behind *T*?

The answer is yes. This is an approach to linear algebra that doesn’t start with matrices. The next section shows that we still end up with matrices.

**Matrices for General Linear Transformations**

Suppose that *V* is an *n*-dimensional vector space, *W* is an *m*-dimensional vector space, and that  is a linear transformation. Suppose further that *B* is a basis for *V*, that  is a basis for *W*, and that for each ***x*** in *V*, the coordinate matrices for ***x*** and  are  and , respectively



By using matrix multiplication, we can execute the linear transformation and the following indirect procedure:

1. Compute the coordinate vector 
2. Multiply  on the left by *A* to produce 
3. Reconstruct  from its coordinate vector 



***Example***

Let  be the linear transformation defined by 

Find the matrix for *T* with respect to the standard bases



Where 

***Solution***







The matrix for *T* with respect to *B* and  is



***Example***

Let  be the linear transformation defined by  describe in the following figure to perform the computation







***Solution***

***Step*** 1: The coordinates matrix for  relative to the basis  is



***Step*** 2: Multiply  by the matrix  found in previous example, we obtain



***Step*** 3: Reconstructing  from  we obtain



***Exercises Section* 4.2 – General Linear Transformations**

1. The matrix  gives a shearing transformation .

What happens to (1, 0) and (2, 0) on the *x*-axis.

What happens to the points on the vertical lines *x* = 0 and *x* = *a*?.

1. A nonlinear transformation *T* is invertible if every ***b*** in the output space comes from exactly one x in the input space. *T*(***x***) = ***b*** always has exactly one solution. Which of these transformation (on real numbers ***x*** is invertible and what is ? None are linear, not even . When you solve *T*(***x***) = ***b***, you are inverting T:



1. If *S* and *T* are linear transformations, is *S* (*T* (***v***)) linear or quadratic?
2. If *S* (***v***) = ***v*** and *T* (***v***) = ***v***, then *S* (*T* (***v***)) = ***v*** or ?
3.  and  combine into



1. Find the range and kernel (like the column space and nullspace) of *T*:
2. 
3. 
4. 
5. 
6. *M* is any 2 by 2 matrix and . The transformation *T* is defined by . What rules of matrix multiplication show that *T* is linear?
7. Which of these transformations satisfy  and which satisfy ?
8. 
9. 
10. 
11. = largest component of ***v***.
12. Consider the basis for , where  and let  be the linear transformation for which



Find a formula for , and then use that formula to compute 

1. Consider the basis for , where  and let  be the linear transformation for which



Find a formula for , and then use that formula to compute 

1. let  be vectors in a vector space *V*, and let  be the linear transformation for which . Find 
2. Let  be the linear operation given by the formula 

Which of the following vectors are in 



1. Let  be the linear operation given by the formula 

Which of the following vectors are in 



1. Let  be the linear operation given by the formula



Which of the following vectors are in 



1. Let  be the linear operation given by the formula



Which of the following vectors are in 



1. Determine if the given function *T* is a linear transformation



1. Determine if the given function *T* is a linear transformation



1. Determine if the given function *T* is a linear transformation where *A* is fixed  matrix



1. Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*; if *T* is linear, find the *A* such . 
2. Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*. 
3. Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*; if *T* is linear, find the *A* such .



1. Show that the function  given the formula  is linear transformation

***Section* 4.3 – LU-Decompositions**

The goal is to describe Gaussian elimination in the most useful way by looking at them closely, which are factorizations of a matrix.

***The factors are triangular matrices.***

***The factorization that comes from elimination is .***

**The Method of *LU*−Decomposition**

***Step*** 1: Rewrite the system  as 

***Step*** 2: Define a new  matrix ***y*** by 

***Step*** 3: Use  to rewrite  as  and solve this system for ***y***.

***Step*** 4: Substitute ***y*** in  and solve for ***x***.



***Example***

Given 2 by 2 matrix 

To make ***row*** 2 ***column*** 1 is ***zero*** then we need to subtract 3 times *row* 2 from *row* 2



That step is  in the forward direction such that:



The return step from *U* to *A* is 

Back from *U* to *A*: 

Therefore; 

***Example***

What matrix *L* and *U* puts *A* into triangular form  where



***Solution***





***The lower triangular L has all 1’s on its diagonal. The multipliers  are below the diagonal of L***





* 

*The inverses go in opposite order*.

*  This is ***elimination without row exchanges***. The *upper triangular* ***U*** has the pivots on its diagonal. The *lower triangular* ***L*** has all 1’s on its diagonal. ***The multipliers  are below the diagonal of L***.

***One* Square System = *Two* Triangular Systems**

***Factor:*** into *L* and *U*, by forward elimination on *A*.

***Solve***: forward on ***b*** using *L*, then back substitution using *U*.

Solve  and then solve 

***Example***

Forward elimination on  ends at 



***Solution***

The multiplier was 4. 

The lower triangular system: 



The upper triangular system: 



To solve 1000 equations on a PC

* Elimination on *A* requires about  multiplications and  subtractions.
* Each right side needs  multiplications and  subtractions.

***Exercises Section* 4.3 – LU-Decompositions**

1. What matrix *E* puts *A* into triangular form ? Multiply by  to factor A into :



1. Solve  to find***c***. Then solve  to find ***x***. What was *A*?



1. Find *L* and *U* for the symmetric matrix



Find four conditions on *a, b, c, d* to get  with four pivots

1. For which *c* is  impossible – with three pivots?



1. Find an *LU-*decomposition of the coefficient matrix, and then use to solve the system

|  |  |
| --- | --- |
|  |  |

***Section* 4.4 – Eigenvalues and Eigenvectors**

In many problems in science and mathematics, linear equations ***Ax*** = ***b*** come from steady state problems. Eigenvalues have their greatest importance in dynamic problems. The solution of  (is changing with time) has nonzero solutions. (***All matrices are square***)

***Definition***

Suppose *A* is an *n* x *n* matrix and



The values of  are called eigenvalues of the matrix ***A*** and the nonzero vectors ***x*** in  are called the eigenvectors corresponding to that eigenvalue .

* One of the meanings of the word “***eigen***” in German is “***proper***”; eigenvalues are also called ***proper values, characteristic values,*** or ***latent roots***.

***Example***

The vector  is an eigenvector of  corresponding to the eigenvalue λ = 3 since



Eigenvalues and eigenvectors have a useful geometric interpretation in  and .

**The equation for the *eigenvalues***

Let’s rewrite the equation .

 : are the eigenvalues and not a vector





The matrix  times the eigenvectors ***x*** is the zero vector. The eigenvectors makes up the nullspace of .

***Definition***

The number λ is an eigenvalue of ***A*** if and only if  is singular:



This is called ***characteristic equation*** of ***A***; the scalars satisfying this equation are the eigenvalues of ***A***. when expanding the determinant  is a polynomial in *λ* called the ***characteristic polynomial*** of ***A***.

***Example***

Find the eigenvalues of the matrix 

***Solution***









The characteristic equation of ***A*** is:

; these are the eigenvalues of ***A***.

***Theorem***

If ***A*** is an *n* x *n* triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of ***A*** are the entries on the main diagonal of ***A***.

***Example***

Find the eigenvalues of the lower triangular matrix



***Solution***

The eigenvalues are: 

***Theorem***

If ***A*** is an *n* x *n* matrix, the following are equivalent.

1. λ is an eigenvalue of ***A***.
2. The system of equations  has nontrivial solutions.
3. There is a nonzero vector ***x*** in  such that .
4. λ is a real solution of the characteristic equation 

***Eigenvectors***

To find the eigenvector ***x***, for each eigenvalue λ solve 

From the eigenvalues, the eigenvectors, in the form , of the system can be determined by letting:

 and

***Example***

Find the eigenvalues and the eigenvectors of the matrix 

***Solution***











The eigenvalues of ***A*** are: 

For , we have:







If *y* = −1 ⇒ *x* = 2, therefore the eigenvector 

Or 

For , we have:







If *x* = 1 ⇒  *y* = 2, therefore the eigenvector 

**Power of a Matrix**

***Theorem***

If *k* is a positive integer, λ is an eigenvalue of a matrix *A*, and ***x*** is a corresponding eigenvector, then  is an eigenvalue of  and ***x*** is a corresponding eigenvector.

***Example***

Find the eigenvalues of  for 

***Solution***



The eigenvalues of *A*: 

The eigenvalues of  are: 

***Theorem***

A square matrix *A* is invertible *iff*  is not an eigenvalue of *A*.

***Summary***

To solve the eigenvalue problem for an *n* by *n* matrix:

1. Compute the determinant of . With λ subtracted along the diagonal, this determinant starts with  or . It is a polynomial in λ of degree *n*.
2. Find the roots of this polynomial, by solving . The *n* roots are the *n* eigenvalues of A. They make  singular.
3. For each eigenvalue λ, solve  ***to find an eigenvector*** ***x***.

**Imaginary Eigenvalues**

***Example***

Find the eigenvalues and the eigenvectors of the matrix 

***Solution***











The solutions are: .

:



Therefore the eigenvector 

:



Therefore the eigenvector 

***Example***

Find the eigenvalues and the eigenvectors of the matrix 

***Solution***





The matrix ***A*** is a 90° rotation which has no real eigenvalues or eigenvectors.

No vector ***Ax*** stays in the same direction as ***x*** (except the zero vector which is useless).

If we add the eigenvalues together the result is zero which is the trace of ***A***.

:



Therefore the eigenvector 

:



Therefore the eigenvector 

***Exercises*** ***Section* 4.4 – Eigenvalues and Eigenvectors**

1. Find the eigenvalues and eigenvectors of :



Check the trace  and the determinant  for *A* and also .

1. Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues



1. For which real numbers c does this matrix A have



1. Two real eigenvalues and eigenvectors.
2. A repeated eigenvalue with only one eigenvector
3. Two complex eigenvalues and eigenvectors.
4. Find the eigenvalues of ***A***, ***B***, ***AB***, and ***BA***:



1. The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***A*** times eigenvalues of ***B***.
2. The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***BA***.
3. When  show that (1, 1) is an eigenvector and find both eigenvalues of



1. The eigenvalues of *A* equal to the eigenvalues of . This is because  equals . That is true because \_\_\_\_\_. Show by an example that the eigenvectors of *A* and  are not the same.
2. Let . Compute the eigenvalues and eigenvectors of *A*.
3. Let 
4. What is the characteristic polynomial for *A* (i.e. compute ?
5. Verify that 1 is an eigenvalue of *A*. What is a corresponding eigenvector?
6. What are the other eigenvalues of *A*?
7. For the following matrices:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

1. Find the characteristic equation
2. Find the eigenvalues
3. Find the eigenvectors
4. Find the eigenvalues of  for 
5. Find the eigenvalues of the matrices



1. Given the matrix 
2. Find the characteristic polynomial.
3. Find the eigenvalues
4. Find the bases for its eigenspaces
5. Graph the eigenspaces
6. Verify directly that , for all associated eigenvectors and eigenvalues.
7. Given the matrix 
8. Find the characteristic polynomial.
9. Find the eigenvalues
10. Find the bases for its eigenspaces
11. Graph the eigenspaces
12. Verify directly that , for all associated eigenvectors and eigenvalues.
13. Given: . Compute 

***Section* 4.5 – Diagonalization**

When ***x*** is an eigenvector, multiplication by ***A*** is just multiplication by a single number: .

The matrix ***A*** turns into a diagonal matrix ***A*** when we use the eigenvectors property.

**Diagonalization**

Suppose the *n* by *n* matrix ***A*** has *n* linearly independent eigenvectors . Put them into the column of an ***eigenvector*** ***matrix*** ***P***. Then  is the eigenvalue matrix ***A***:



***Example***

The projection matrix  has 

The eigenvectors are:  that are the value of *P*. 





***Definition***

A square matrix *A* is called ***diagonalizable*** if there is an invertible matrix *P* such that  is diagonal; the matrix *P* is said to ***diagonalize*** *A*.

***Theorem***

***Independent x from different λ*** - Eigenvectors  that correspond to distinct (all different) eigenvalues are linearly independent. An *n* by *n* matrix that has *n* different eigenvalues (no repeated λ’s) must be diagonalizable.

***Proof***

Suppose 





Multiply  by , that implies to













Since  and λ’s are different , we forced 

Similarly; Multiply  by , that implies to 







Therefore,  must be independent.

***Theorem***

If  are eigenvectors of *A* corresponding to distinct eigenvalues , then  is linearly independent set.

***Theorem***

If an *n* x *n* matrix ***A*** has *n* distinct eigenvalues, then the following are equivalent:

1. ***A*** is diagonalizable
2. ***A*** has *n* linearly independent eigenvectors.

***Example***

Given the Markov matrix 

***Solution***







The eigenvalues are: 

For , we have: 



If , therefore the eigenvector 

For , we have: 



If , therefore the eigenvector 





***Eigenvalues of AB and A* + *B***

An eigenvalue of ***A*** times an eigenvalue of ***B*** usually does not give an eigenvalue of ***AB***.



***Commuting matrices share eigenvectors***: Suppose ***A*** and ***B*** can be diagonalized. They share the eigenvector matrix ***P*** if and only if *AB* = *BA*.

**Matrix Powers **





The eigenvector matrix for  is still *S*, and the eigenvalue matrix is . The eigenvectors don’t change, and the eigenvalues are taken to the *kth*  power. When *A* is diagonalized,  is easy.

Here are steps (taken from Fibonacci):

1. Find the eigenvalues of *A* and look for *n* independent eigenvectors.
2. Write as a combination of the eigenvectors.
3. Multiply each eigenvector . Then



***Example***

Compute 

***Solution***

The matrix *A* has 



***Similar Matrices***

***Definition***

If *A* and *B* are square matrices, then we say that ***B is similar to A*** if there exists an invertible matrix *P* such that 

* Similar matrices *B* and  have the same eigenvalues. If *x* is an eigenvector of *A* then  is an eigenvector of .

***Proof***

Since 

Suppose :





The eigenvalue of *B* is the same λ. The eigenvector is now 

***Example***

The projection  is similar to 

Choose  ; the similar matrix 

Also choose  ; the similar matrix 

These matrices  all have the same eigenvalues 1 and 0. ***Every 2 by 2 matrix with those eigenvalues is similar to A***. The eigenvectors change with *M*.

***Example***

 is similar to every matrix  except .

These matrices *B* all have zero determinant (like *A*). They all have rank one (like *A*). Their trace is *cd – cd* = 0. Their eigenvalues are 0 and 0 (like *A*).

Choose  and 

Connections between similar matrices *A* and *B*:

|  |  |
| --- | --- |
| ***Not Changed*** | ***Changed*** |
| Eigenvalues | Eigenvectors |
| Trace and determinant | Nullspace |
| Rank | Column space |
| Number of independent  eigenvectors | Row space  Left nullspace |
| Jordan form | Singular values |

***Example***

Jordan matrix *J* has triple eigenvalues 5, 5, 5. Its only eigenvectors are multiples of (1, 0, 0). Algebraic multiplicity 3, geometric multiplicity 1:

If  has rank 2.

Every similar matrix  has the same triple eigenvalues 5, 5, 5. Also *B* – 5*I* must have the same rank 2. Its nullspace has dimension 3 − 2 = 1. So each similar matrix B also has only one independent eigenvector.

The transpose matrix  has the same eigenvalues 5, 5, 5, and  has the same rank 2. ***Jordan’s theory says that  is similar to J***. The matrix that produces the similarity happens to be the reserve identity *M*:



There is one line of eigenvectors  for *J* and another line  for .

***Fibonacci* Numbers**

Every new Fibonacci number is the sum of the two previous *F*’s.

The ***sequence***  comes from 

***Problem***

Find the Fibonacci number 

We can apply the rule one step at a time, or just use Linear algebra.

Let consider the matrix equation: . Fibonacci rule gave us a two-step rule for scalars.

Let , the rule  becomes .

Every step multiplies by , after 100 steps we reach 





The characteristic equation is  and the solutions are



The eigenvectors are:







The combination of these eigenvectors that give :











**The *Jordan* Form**

For every *A*, we want to choose *M* so that  is as nearly diagonal as possible. When *A* has a full set of *n* eigenvectors, they go into the columns of *M*. Then *M* = *P*. The matrix is diagonal.

If *A* has ***s*** independent eigenvectors, it is similar to a matrix *J* that has ***s*** Jordan blocks on its diagonal. There is a matrix *M* such that



Each block in *J* has one eigenvalue , one eigenvector, and 1’s above the diagonal:



***A is similar to B if they share the same Jordan form J – not otherwise.***

***Exercises Section* 4.5 – Diagonalization**

1. The Lucas numbers are like Fibonacci numbers except they start with . Following the rule . The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number .
2. Find all eigenvector matrices *S* that diagonalize *A* (rank 1) to give :



What is ? Which matrices *B* commute with *A* (so that *AB* = *BA*)

1. Determine whether the matrix is diagonalizable

*a*)  *b*) 

*c*)  *d*) 

1. Find a matrix ***P*** that diagonalizes *A*, and compute 

*a*)  *b*) 

*c*) 

1. Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine .

|  |  |  |
| --- | --- | --- |
|  |  |  |

1. The 4 by 4 triangular Pascal matrix and its inverse (alternating diagonals) are



Check that  and  have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives , so  is similar to . Show that  with columns of alternating signs is its own inverse.

Since  and  are similar they have the same Jordan form *J*. Find *J* by checking the number of independent eigenvectors of  with λ = 1.

1. If ***x*** is in the nullspace of *A* show that  is in the nullspace of .

The nullspaces of *A* and  have the same (vectors) (basis) (dimension)

1. These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don’t match and they are not similar:



For any matrix *M* compare *JM* with MK. If they are equal show that *M* is not invertible. Then  is Impossible; *J* is not similar to *K*.

1. Prove that  is always similar to A (λ′s are the same):
2. For one Jordan block , find  so that .
3. For any *J* with blocks , build  from blocks so that .
4. For any : Show that  is similar to  and so to J and so to *A*.
5. Why are these statements all true?
6. If *A* is similar to *B* then  is similar to .
7.  and  can be similar when *A* and *B* are not similar.
8.  is similar to 
9.  is not similar to 
10. If we exchange rows 1 and 2 of *A*, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case *M* =?
11. If an *n* x *n* matrix *A* has all eigenvalues λ = 0 prove that  is the zero matrix.
12. If *A* is similar to , must all the eigenvalues equal to 1 or −1?.
13. Show that *A* and *B* are not similar matrices
14. 
15. 
16. 
17. Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.
18. Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.
19. Suppose that *A* is a matrix. Suppose that the linear transformation associated to *A* has two linearly independent eigenvectors. Prove that *A* is similar to a diagonal matrix.
20. Prove that if *A* is a  matrix that has two distinct eigenvalues, then *A* is similar to a diagonal matrix.
21. Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example . Is it true that matrices with equal characteristic polynomial are necessarily similar?
22. Show that the given matrix is not diagonalizable. 
23. Determine if the given matrix is diagonalizable. If, so, find matrices S and  such that the given matrix equals 
24. 
25. 

***Section* 4.6 – Orthogonal Diagonalization**

***Definition***

A square matrix ***A*** is called orthogonally diagonalizable if there is an orthogonal matrix *P* such that  is diagonal; the matrix *P* is said to orthogonally diagonalize *A*.



We say that *A* is orthogonally diagonalizable and that *P* orthogonally diagonalizes *A*.

***Theorem***

If ***A*** is an *n* x *n* matrix, then the following are equivalent.

1. ***A*** is orthogonally diagonalizable
2. ***A*** has an orthonormal set of *n* eigenvectors.
3. ***A*** is symmetric.

***Theorem***

If ***A*** is symmetric matrix, then:

1. The eigenvalues of ***A*** are all real numbers.
2. Eigenvectors from different eigenspaces are orthogonal.

***Example***

Find an orthogonal matrix *P* that diagonalizes



***Solution***











The eigenvalues are: 

For , we have: 



If , therefore the eigenvector 

For , we have: 



If , therefore the eigenvector 

For , we have: 





If , therefore the eigenvector 





























**Spectral Decomposition**

The spectral decomposition of *A* is:



***Example***

The matrix 

***Solution***





The eigenvalues are: 

The corresponding eigenvectors are: 



















The spectral decomposition about the image of the vector 











***Example***

Consider a 2 by 2 symmetric matrix 

***Solution***

The eigenvalues are:





The eigenvectors are:























Therefore, these eigenvectors are perpendicular.

***Theorem***

***Orthogonal Eigenvectors:*** Eigenvectors of a real symmetric matrix (when they correspond to different λ′s) are always perpendicular.

***Proof***

Suppose  ,  and .

The dot products of the first equation with *y* and the second with *x*:





Since , this proves that .

The eigenvector *x*  is perpendicular to the eigenvector *y* 

***Example***

Find the λ′s and *v*′s for this symmetric matrix with trace zero:

***Solution***









The eigenvalues are: 

The eigenvectors are:







Thus, the eigenvectors are perpendicular.

The unit vector of the eigenvectors by dividing by their length 

The are the columns of *Q*.





* Every symmetric matrix *A* has a complete set of orthogonal eigenvectors:



**Complex Eigenvalues of Real Matrices**

For real matrices, complex λ′s and *x*′s come in “conjugate pairs”



***Example***

Given 

***Solution***

The eigenvalues of *A*:















The eigenvalues are conjugate to each other.

The eigenvectors:









The vector 







This fact holds for the eigenvalues of every orthogonal matrix.

***Theorem*** − ***Equivalent Statements***

If *A* is an  matrix, then the following statements are equivalent.

1. *A* is invertible
2. A***x*** = **0** has only the trivial solution
3. The reduced row echelon form of *A* is 
4. A is expressible as a product of elementary matrices
5. A***x*** = ***b*** is consistent for every  matrix ***b***
6. A***x*** = ***b*** has exactly one solution for every  matrix ***b***
7. 
8. The column vectors of *A* are linearly independent
9. The row vectors of *A* are linearly independent
10. The column vectors of *A* span 
11. The row vectors of *A* span 
12. The column vectors of *A* form a basis for 
13. The row vectors of *A* form a basis for 
14. *A* has a rank *n*.
15. *A* has nullity 0.
16. The orthogonal complement of the null space of *A* is 
17. The orthogonal complement of the row space of *A* is 
18. The range of  is 
19.  is one-to-one.
20.  is not an eigenvalue of A.
21.  is invertible,

***Exercises Section* 4.6 – Orthogonal Diagonalization**

1. Find a matrix *P* that orthogonally diagonalizes *A*, and determine 

|  |  |  |
| --- | --- | --- |
|  |  |  |

1. Find the eigenvalues of *A* and *B* and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:



The −1, 2, −1 pattern in both matrices is a “second derivative. Then  and  are like . This has eigenvectors  and  that are the bases for Fourier series. The matrices lead to “discrete sines” and “discrete cosines” that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

1. Suppose  and  and λ ≠ 0. Then y is in the nullspace and x is in the column space. They are perpendicular because \_\_\_\_\_\_\_. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to . The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.
2. True or false. Give a reason or a counterexample.
3. A matrix with real eigenvalues and eigenvectors is symmetric.
4. A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
5. The inverse of a symmetric matrix is symmetric
6. The eigenvector matrix *S* of a symmetric matrix is symmetric.
7. A complex symmetric matrix has real eigenvalues.
8. If *A* is symmetric, then  is symmetric.
9. If *A* is Hermitian, then  is Hermitian.
10. Find a symmetric matrix  that has a negative eigenvalue.
11. How do you know it must have a negative pivot?
12. How do you know it can’t have two negative eigenvalues?
13. Which of these classes of matrices do *A* and *B* belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?



Which of these factorizations are possible for *A* and *B*: LU, QR, , ?

1. Construct a 3 by 3 matrix *A* with no zero entries whose columns are mutually perpendicular. Compute . Why is it a diagonal matrix?
2. Assuming that , find a matrix that orthogonally diagonalizes 