***Solution Section* 4.1 – Matrix Transformations from  to** 

***Exercise***

Find the standard matrix for the transformation defined by the equations

1. 
2. 
3. 

***Solution***

1.  The standard matrix is 
2.  The standard matrix is 
3.  The standard matrix is

***Exercise***

Find the standard matrix for the operator *T* defined by the formula

1. 
2. 
3. 

***Solution***

1. 



The standard matrix is 

1. 



The standard matrix is 

1. 

The standard matrix is 

***Exercise***

Find the standard matrix for the transformation *T* defined by the formula

1. 
2. 
3. 
4. 

***Solution***

1.  The matrix is 
2. The matrix is 
3. The matrix is 
4. 



***Solution Section* 4.2 – General Linear Transformations**

***Exercise***

The matrix  gives a shearing transformation .

What happens to (1, 0) and (2, 0) on the *x*-axis.

What happens to the points on the vertical lines *x* = 0 and *x* = *a*?.

***Solution***

The points (1, 0) and (2, 0) on the *x*-axis transform by *T* to (1, 3) and (2, 6). The horizontal *x*-axis transforms to the straight line with slope 3 (going through (0, 0) of course). The points on the *y*-axis are not moved because *T*(0, *y*) – (0, *y*). The *y* -axis is the line of eigenvectors of *T* with λ = 1.

The vertical line *x* = *a* is moved up by 3*a*, since 3*a* is added to the *y* component. This is ***shearing***. Vertical lines slide higher as you go from left to right.

***Exercise***

A nonlinear transformation *T* is invertible if every ***b*** in the output space comes from exactly one x in the input space. *T*(***x***) = ***b*** always has exactly one solution. Which of these transformation (on real numbers ***x*** is invertible and what is ? None are linear, not even . When you solve *T*(***x***) = ***b***, you are inverting T: 

***Solution***

is not invertible because  and  has no solution.

is not invertible because  has no solution.

 is invertible. The solutions to 

 is invertible. The solutions to 

 is invertible. The solutions to 

***Exercise***

If *S* and *T* are linear transformations, is *S* (*T* (***v***)) linear or quadratic?

1. If *S* (***v***) = ***v*** and *T* (***v***) = ***v***, then *S* (*T* (***v***)) = ***v*** or ?
2.  and  combine into



***Solution***

1.  **since *T* (*v*) = *v***
2. 



It is quadratic.

***Exercise***

Find the range and kernel (like the column space and nullspace) of *T*:

1. 
2. 
3. 
4. 

***Solution***

1. Range is the line *y* = 0, Kernel is the line *x = y* in the *xy* plane.
2. Range is the *xy* plane, Kernel is the complementary line in .
3. Range is the point (0, 0), Kernel is plane
4. Range is the line *x = y* in the *xy* plane, Kernel is the line *x* = 0.

***Exercise***

*M* is any 2 by 2 matrix and . The transformation *T* is defined by . What rules of matrix multiplication show that *T* is linear?

***Solution***

The distribution law and the association law for multiplication give the linearity







***Exercise***

Which of these transformations satisfy  and which satisfy ?

1. 
2. 
3. 
4. = largest component of ***v***.

***Solution***

1. This is scaling the vector into a normal vector. This it is impossible that we get additivity, because the sums of normal vectors don’t have to be normal. For example *T*(0, 1) and *T*(1, 0) for instance. However, true to its name this does have the scaling property. For ***c*** value, this value will be canceled from ***v*** and .
2. This satisfies both. One immediate way to see that it is matrix multiplication by [1, 1, 1], which is a linear operation and thus satisfies both properties.
3. This satisfies both. This a matrix multiplication by 
4. Doesn’t satisfy additivity [(0, 1) and (1, 0) still work]. Scaling doesn’t work either, if we scale by -1 we now pick out the negative of the smallest component, which doesn’t have to be related in any way to the largest component.

***Exercise***

Consider the basis for , where  and let  be the linear transformation for which



Find a formula for , and then use that formula to compute 

***Solution***





















***Exercise***

Consider the basis for , where  and let  be the linear transformation for which



Find a formula for , and then use that formula to compute 

***Solution***































***Exercise***

Let  be vectors in a vector space *V*, and let  be the linear transformation for which



Find 

***Solution***









***Exercise***

Let  be the linear operation given by the formula 

Which of the following vectors are in 



***Solution***

1. 



This is a consistent system, therefore (1, −4) is in 

1. 



This is an inconsistent system, therefore (5, 0) is not in 

1. 



This is a consistent system, therefore (−3, 12) is in 

***Exercise***

Let  be the linear operation given by the formula 

Which of the following vectors are in 



***Solution***

1. ; therefore (5, 10) is in 
2. ; therefore (3, 2) is not in 
3. ; therefore (1, 1) is not in 

***Exercise***

Let  be the linear operation given by the formula



Which of the following vectors are in  

***Solution***

1. 





This is a consistent system, therefore (0, 0, 6) is in 

1. 





This is a consistent system, therefore (1, 3, 0) is in 

1. 





This is a consistent system, therefore (2, 4, 1) is in 

***Exercise***

Let  be the linear operation given by the formula



Which of the following vectors are in  

***Solution***

1. 

Therefore, (3, −8, 2, 0) is in 

1. 

Therefore, (0, 0, 0, 1) is ***not*** in 

1. 

Therefore, (0, −4, 1, 0) is ***not*** in 

***Exercise***

Determine if the given function *T* is a linear transformation



***Solution***

Let  and 













Function *T* is NOT a linear transformation

***Exercise***

Determine if the given function *T* is a linear transformation



***Solution***

Let  and 







 ***√***











 ***√***

Since  and , then function *T* is a linear transformation.

***Exercise***

Determine if the given function *T* is a linear transformation where *A* is fixed  matrix



***Solution***













Function *T* is a linear transformation

***Exercise***

Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*; if *T* is linear, find the *A* such . 

***Solution***

Let 





















Since  and , then function *T* is a linear transformation.

***Domain***: 





***Exercise***

Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*. 

***Solution***

Let 













The function *T* is ***not*** a linear transformation.

Domain: 

***Exercise***

Determine if the given function *T* is a linear transformation. Also give the domain and range of *T*; if *T* is linear, find the *A* such .



***Solution***

Let 





















Since  and , then function *T* is a linear transformation.

Domain: 





***Exercise***

Show that the function  given the formula  is linear transformation

***Solution***

Let 



















Since  and , then function *T* is a linear transformation.

***Solution Section* 4.3 – LU−Decompositions**

***Exercise***

What matrix *E* puts *A* into triangular form ? Multiply by  to factor A into :



***Solution***

















***Exercise***

Solve  to find***c***. Then solve  to find ***x***. What was *A*?



***Solution***













***Exercise***

Find *L* and *U* for the symmetric matrix



Find four conditions on *a, b, c, d* to get  with four pivots

***Solution***



***Exercise***

For which *c* is  impossible – with three pivots?



***Solution***











*LU* will be impossible for 

***Exercise***

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system



***Solution***

|  |  |
| --- | --- |
|  |  |







The solution: 

***Exercise***

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system



***Solution***

|  |  |
| --- | --- |
|  |  |







The solution: 

***Exercise***

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system



***Solution***

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |





Solution: 

***Exercise***

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system



***Solution***

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |







Solution: 

***Exercise***

Find an *LU*-decomposition of the coefficient matrix, and then use to solve the system



***Solution***

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

***For lower triangular***: 







Solution: 

***Solution*** ***Section* 4.4 – Introduction to Eigenvalues**

***Exercise***

Find the eigenvalues and eigenvectors of :



Check the trace  and the determinant  for *A* and also .

***Solution***

***For*** ***A***:







The eigenvalues of ***A*** are .

The trace of a square matrix A is the sum of the elements on the main diagonal: 2 + 2 agrees with 1+ 3. The det(***A***) = 3 agrees with the product .

The eigenvectors for ***A*** are:

:





Therefore the eigenvector 

:



Therefore the eigenvector 

***For*** :



The eigenvalues of are . ***Or*** 





:



Therefore the eigenvector 

:



Therefore the eigenvector 

***For*** :





The eigenvalues of are .

:





Therefore the eigenvector 

:





Therefore the eigenvector 

***For*** :





The eigenvalues of are .

:





Therefore the eigenvector 

:



Therefore the eigenvector 

The eigenvalues

The eigenvalues 

The eigenvalues 

***Exercise***

Show directly that the given vectors are eigenvectors of the given matrix. What are the corresponding eigenvalues



***Solution***



 is an eigenvectors corresponding to the eigenvalue 7.



 is an eigenvectors corresponding to the eigenvalue 0.









The eigenvalues are: 

***Exercise***

For which real numbers c does this matrix A have



1. Two real eigenvalues and eigenvectors.
2. A repeated eigenvalue with only one eigenvector
3. Two complex eigenvalues and eigenvectors.

***Solution***











1. Two real eigenvalues and eigenvectors, when 
2. A repeated eigenvalue with only one eigenvector, when 
3. Two complex eigenvalues and eigenvectors, when 

***Exercise***

Find the eigenvalues of ***A***, ***B***, ***AB***, and ***BA***:



1. The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***A*** times eigenvalues of ***B***.
2. The eigenvalues of ***AB*** (are equal to) (are not equal to) eigenvalues of ***BA***.

***Solution***

Since ***A*** is a lower triangular, then 

Since ***B*** is a upper triangular, then 

1. The eigenvalues of ***AB*** are not equal to eigenvalues of ***A*** times eigenvalues of ***B***.
2. The eigenvalues of ***AB*** are equal to the eigenvalues of ***BA***.

***Exercise***

When  show that (1, 1) is an eigenvector and find both eigenvalues of



***Solution***



If 









The eigenvalues for :







The eigenvector: 

***Exercise***

The eigenvalues of *A* equal to the eigenvalues of . This is because  equals . That is true because \_\_\_\_\_. Show by an example that the eigenvectors of *A* and  are not the same.

***Solution***



Therefore, A and have the same eigenvalues.

Let consider the matrix: 



The eigenvalues are: 

For 









***Exercise***

Let . Compute the eigenvalues and eigenvectors of *A*.

***Solution***







The eigenvalues of ***A*** are:

For 





The eigenvector is: 

For 



The eigenvector is: 

***Exercise***

Let 

1. What is the characteristic polynomial for *A* (i.e. compute ?
2. Verify that 1 is an eigenvalue of *A*. What is a corresponding eigenvector?
3. What are the other eigenvalues of *A*?

***Solution***

1. 









1. 







1 is an eigenvalue of *A*.







The eigenvector for  is 

1. 

***Exercise***

For the matrix:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

1. Find the characteristic equation
2. Find the eigenvalues
3. Find the eigenvectors

***Solution***

1. 





The characteristic equation: 

1. 

The eigenvalues are 

1. 



Therefore the eigenvector 





Therefore the eigenvector 

1. For the matrix: 
2. 





⇒ The characteristic equation: 

1. 

⇒ The eigenvalues are 

1. 



Therefore the eigenvector 

1. For the matrix: 
2. 



⇒ The characteristic equation: 

1. 

The eigenvalues are 

1. For 



Therefore the eigenvector 

For 



Therefore the eigenvector 

1. For the matrix 
2. 





The characteristic equation: 

1. 
2. For  , we have: 





Therefore the eigenvector 

For  , we have: 





Therefore the eigenvector 

1. For the matrix: 
2.  



 ⇒ The characteristic equation: 

1. 
2. 

Therefore the eigenvector 

Therefore the eigenvector 

Therefore the eigenvector 

1. For the matrix: 
2. 









⇒ The characteristic equation: 

1. 
2. 



Therefore the eigenvector 



Therefore the eigenvector 





Therefore the eigenvector 

1. For the matrix: 
2. 









⇒ The characteristic equation: 

1. 



1. 



Therefore the eigenvector 

For 



Therefore the eigenvector 

For 



Therefore the eigenvector 

1. For the matrix: 
2. 







⇒ The characteristic equation: 

1. 
2. 



Therefore the eigenvector 





Therefore the eigenvector 





Therefore the eigenvector 

 Therefore the eigenvector 

1. For the matrix: 
2. 











⇒ The characteristic equation: 

1. 
2. 



Therefore the eigenvector 

 Therefore the eigenvector 







Therefore the eigenvector 





Therefore the eigenvector 

1. For the matrix 
2. 







The characteristic equation: 

1. 
2. For  , we have: 



If we let ; therefore the eigenvector 

For  , we have: 





If we let ; therefore the eigenvector 

For  , we have: 





If we let ; therefore the eigenvector 

***Exercise***

Find the eigenvalues of  for 

***Solution***

The eigenvalues are: 

The eigenvalues of  are: 

***Exercise***

Find the eigenvalues of the matrices



***Solution***

The eigenvalues for:



The eigenvalues are: 

The eigenvalues for: 

The eigenvalues for:  

The eigenvalues for:

The eigenvalues are: 

***Exercise***

Given the matrix 

1. Find the characteristic polynomial.
2. Find the eigenvalues
3. Find the bases for its eigenspaces
4. Graph the eigenspaces
5. Verify directly that , for all associated eigenvectors and eigenvalues.

***Solution***

1. 





The characteristic polynomial is 

1. 
2. For  , we have: 

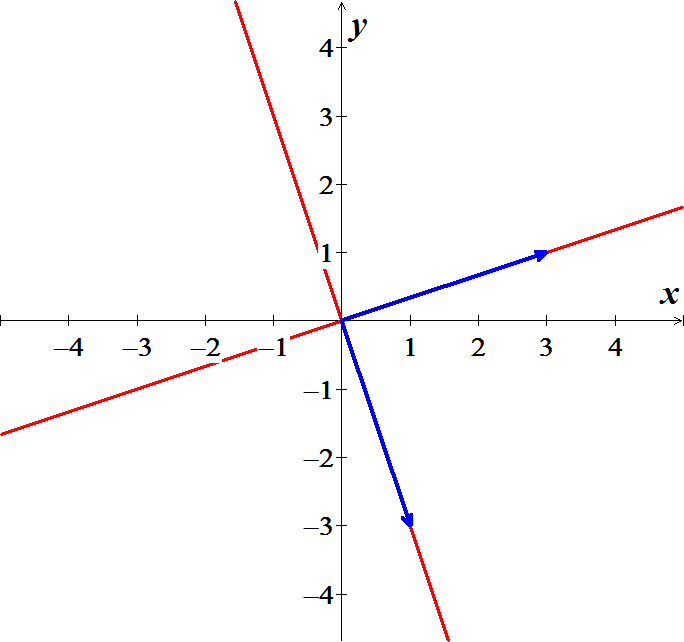
 

Therefore the eigenvector 

For  , we have: 

Therefore the eigenvector 



1. 

***√***



***√***

***Exercise***

Given the matrix 

1. Find the characteristic polynomial.
2. Find the eigenvalues
3. Find the bases for its eigenspaces
4. Graph the eigenspaces
5. Verify directly that , for all associated eigenvectors and eigenvalues.

***Solution***

1. 







The characteristic polynomial is 

1. 
2. For  , we have: 

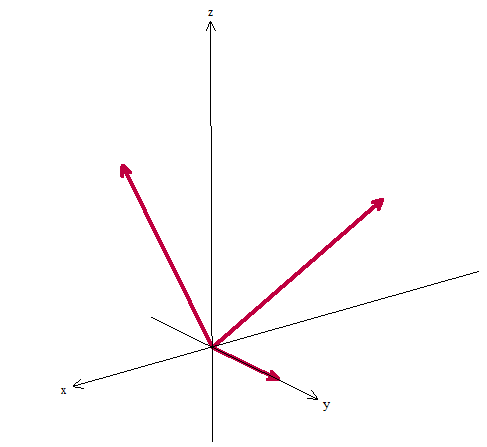


Therefore the eigenvector 

For  , we have: 



Therefore the eigenvector 



1. 

***√***



***√***



***√***

***Exercise***

Given: . Compute 

***Solution***





The eigenvalues are: 

For  , we have: 



The eigenvector 

For  , we have: 



If we let ;

The eigenvector 

For  , we have: 



The eigenvector 











***Solution Section* 4.5 – Diagonalization**

***Exercise***

The Lucas numbers are like Fibonacci numbers except they start with . Following the rule . The next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number .

***Solution***

Let , the rule  becomes . 



The characteristic equation is  and the solutions are







The linear combination: 



The solution 



***Exercise***

Find all eigenvector matrices *S* that diagonalize *A* (rank 1) to give :



What is ? Which matrices *B* commute with *A* (so that *AB* = *BA*)

***Solution***

Since *A* has rank 1, its nullspace is a two-dimensional plane. Any vector with  solves *A****x*** = 0. So λ = 0 is an eigenvalue with multiplicity 2. There are two independent eigenvectors. The other eigenvalues must be λ = 3 because the trace *A* is 1 + 1 + 1 = 3.



The eigenvalues are .

The eigenvectors for  is:





The eigenvectors for  are any two independent vectors in the plane of 

The possible matrices *S*:

 and 

.

The powers  come:  and 

If *AB* = *BA*, all the column and row of ***B*** must be the same. One possible ***B*** is***A*** itself, since , ***B*** is any linear combination of permutation matrices.

***Exercise***

Determine whether the matrix is diagonalizable

*a*)  *b*) 

*c*)  *d*) 

***Solution***

1. 

The only eigenvalue: , the eigenvectors are:



 and the inverse doesn’t exist. Therefore the matrix *A* is not diagonalizable.

1. 



The only eigenvalue: , the eigenvectors are:

 (***linearly dependent***)

 the inverse doesn’t exist. Therefore the matrix *A* is not diagonalizable.

This space is 1-dimensional, *A* does not have 2 linearly independent eigenvectors.

1. 









The eigenvalues are: 

The eigenvector for  is:





1. Since the matrix is upper triangular with diagonal entries 2, 2, 3, and 3, the eigenvalues are 2 and 3 (each multiplicity of 2)

For 

 has dimension 1.

For 



The matrix is not diagonalizable since it has only 2 distinct eigenvectors. Note that showing that the geometric multiplicity of either eigenvalue is less than its algebraic multiplicity is sufficient to show that the matrix is not diagonalizable,

***Exercise***

Find a matrix *P* that diagonalizes *A*, and compute 

***a***)  ***b***)  ***c*)** 

***Solution***

1. 

The eigenvalues are: 

For 

Therefore the eigenvector: 

For 

Therefore the eigenvector: 

The eigenvectors: 







1. 

The eigenvalues are: 

For 

Therefore the eigenvector: 

For 

Therefore the eigenvector: 

The eigenvectors: 







1. 







The eigenvalues are: 

The eigenvector for  is:





The eigenvectors for  is:









***Exercise***

Determine if the matrices are diagonalizable. If so, find a matrix ***P*** that diagonalizes *A* and determine 

|  |  |  |
| --- | --- | --- |
|  |  |  |

***Solution***

1. 

The eigenvalues are: 



1. 

The eigenvalues are: 



1. 

The eigenvalues are: 



1. 

Since the matrix *A* is an upper triangular, then the eigenvalues are: 

For 



For 











***Exercise***

The 4 by 4 triangular Pascal matrix and its inverse (alternating diagonals) are



Check that  and  have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives , so  is similar to . Show that  with columns of alternating signs is its own inverse.

Since  and  are similar they have the same Jordan form *J*. Find *J* by checking the number of independent eigenvectors of  with λ = 1.

***Solution***

The triangular matrices  and  both have λ = 1, 1, 1, 1 on their main diagonals. Choose *D* with alternating 1 and −1 on its diagonal. *D* equals :



***Check***:

Changing signs in rows 1 and 3 of , and columns 1 and 3, produces the four negative entries in . Multiply row *i* by  and column *j* by , which gives the alternating diagonals.

Then  has columns with alternating signs and equals its own inverse!



 has only one line of eigenvectors  with λ = 1. The rank of  is certainly 3. So its Jordan form *J* has only one block (also with λ = 1):

 and  are somehow similar to Jordan’s 

***Exercise***

These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don’t match and they are not similar:



For any matrix *M* compare *JM* with MK. If they are equal show that *M* is not invertible. Then  is Impossible; *J* is not similar to *K*.

***Solution***

Let , then





If *JM* = *MK* then 

Which in particular means that the second row is either a multiple of the fourth row, or the fourth row is all 0’s. In either of these cases *M* is not invertible.

Suppose that *J* were similar to *K*. Then there would be some invertible matrix *M* such that . But we just showed that in this case *M* is never invertible (contradiction). Thus *J* is not similar to *K*.

***Exercise***

If ***x*** is in the nullspace of *A* show that  is in the nullspace of .

The nullspaces of *A* and  have the same (vectors) (basis) (dimension)

***Solution***





So any vector in  is a linear combination of those in , hence is contained in it. That is, the two vector spaces consists of the same vectors.

***Exercise***

Prove that  is always similar to A (λ′s are the same):

1. For one Jordan block , find  so that .
2. For any *J* with blocks , build  from blocks so that .
3. For any : Show that  is similar to  and so to J and so to *A*.

***Solution***

1. For one Jordan block , then



So *J* is similar to 

1. For any *J* with block , that satisfies 

Let  be the block-diagonal matrix consisting of the  along the diagonal. Then









1. 

So is similar to , which is similar to *J*, which is similar to *A*, Thus any matrix is similar to its transpose.

***Exercise***

Why are these statements all true?

1. If *A* is similar to *B* then  is similar to .
2.  and  can be similar when *A* and *B* are not similar.
3.  is similar to 
4.  is not similar to 
5. If we exchange rows 1 and 2 of *A*, and then exchange columns 1 and 2 the eigenvalues stay the same. In this case *M* =?

***Solution***

1. If *A* is similar to *B* then  for some *M*. Then , so  is similar to .
2. Let , then  so they are similar but *A* is not similar to *B* because nothing but zero matrix.
3. 
4. They are not similar because the first matrix has a plane of eigenvectors for the eigenvalues 3, while the second only has a line.
5. In order to exchange two rows of *A* we multiply on the left by



In order to exchange two columns we multiply on the right by the same *M*. As  the new matrix is similar to the old one, so the eigenvalues stay the same.

***Exercise***

If an *n* x *n* matrix *A* has all eigenvalues λ = 0 prove that  is the zero matrix.

***Solution***

Suppose that the Jordan Block has a size of ***i*** with eigenvalue 0. Then  will have a diagonal of 1’s two diagonals above the main diagonal and zeroes elsewhere.  will have a diagonal of 1’s three diagonals above the main diagonal and zeroes elsewhere. Therefore , since there is no diagonal ***i*** diagonals above the main diagonal. If A has all eigenvalues λ = 0 then A is similar to some matrix with Jordan block  with each  of size  and .

Each Jordan block will have eigenvalue of 0, so that , and thus 

As  is similar to a block-diagonal matrix with blocks  and each of these is 0 we know that .

Another way, if *A* has all eigenvalues 0 this means that the characteristic polynomial of A must be , as this is the only polynomial of degree *n* all of whose roots are 0. Thus  by the Cayley-Hamilton theorem.

***Exercise***

If *A* is similar to , must all the eigenvalues equal to 1 or −1?.

***Solution***

***No***



Thus  is similar to 

***Exercise***

Show that *A* and *B* are not similar matrices

1. 
2. 
3. 

***Solution***

1. 

; therefore *A* and *B* are not similar

1. 

; therefore *A* and *B* are not similar

1. 

; therefore *A* and *B* are not similar

***Exercise***

Prove that two similar matrices have the same determinant. Explain geometrically why this is reasonable.

***Solution***

Suppose that 

Then  











***Geometric Explanation***: The determinant tells us what Factor area changes when using a linear transformation. This “factor” doesn’t care about the particular basis you use.

***Exercise***

Prove that two similar matrices have the same characteristic polynomial and thus the same eigenvalues. Explain geometrically why this is reasonable.

***Solution***

Suppose that 

Then the characteristic polynomial is equal to .













***Geometric Explanation***: At least in terms of the eigenvalues, these values are numbers λ such that there exists a vector  such that the linear transformation *T* satisfies .

***Exercise***

Suppose that *A* is a matrix. Suppose that the linear transformation associated to *A* has two linearly independent eigenvectors. Prove that *A* is similar to a diagonal matrix.

***Solution***

Let *T* be the linear transformation associated with *A*. Consider the basis  of the 2 linearly independent eigenvectors of *A* where  the eigenvalues associated with. Then,



Let *T* be a matrix with respect to the basis , then we obtain the matrix 

This completes the proof because A is similar to this diagonal matrix by definition.

***Exercise***

Prove that if *A* is a  matrix that has two distinct eigenvalues, then *A* is similar to a diagonal matrix.

***Solution***

Suppose *A* has 2 distinct eigenvalues .

Let  be an eigenvector for .

Suppose that  are not linearly independent, thus they are scalar multiples of each other.

So there exists  such that . Then



So that 

But then  which contradicts the initial assumption.

Thus  are linearly independent then 

Let *T* be a matrix with respect to the basis , then we obtain the matrix 

This completes the proof because A is similar to this diagonal matrix by definition.

***Exercise***

Suppose that the characteristic polynomial of a matrix has a double root. Is it true that the matrix has two linearly independent eigenvectors? Consider the example . Is it true that matrices with equal characteristic polynomial are necessarily similar?

***Solution***



The characteristic polynomial:  which has a double root .



Therefore, the eigenvectors are vectors of the form  which can transform to 

Thus matrices whose characteristic polynomials have a double root do not necessarily have 2 linear independent.

Let , then the characteristic polynomial:  which has a double root . But they are not similar. The eigenvector is the **0** vector.

The linear transformation associated to the second matrix send every vector to **0**. Thus the 2 matrices can’t represent the same linear transformation.

Thus, matrices with equal characteristic polynomial are not necessarily similar.

***Exercise***

Show that the given matrix is not diagonalizable. 

***Solution***



Since the determinant is 0, the inverse doesn’t exist; therefore the matrix is not diagonalizable

***Exercise***

Determine if the given matrix is diagonalizable. If, so, find matrices S and  such that the given matrix equals 

1. 
2. 

***Solution***

1. 



For 



Therefore the eigenvector: 

For 



Therefore the eigenvector: 

 and 









1. 









The given matrix is not diagonalizable, since the eigenvalues are not distinct.

***Solution Section* 4.6 – Orthogonal Diagonalization**

***Exercise***

Find a matrix *P* that orthogonally diagonalizes *A*, and determine 

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

***Solution***

1. 





The eigenvalues are: 

For , we have: 



Therefore the eigenvector 

For , we have: 



Therefore the eigenvector 

















1. 





The eigenvalues are: 

For , we have: 



Therefore the eigenvector 

For , we have: 



Therefore the eigenvector 









1. 









The eigenvalues are: 

For , we have: 



Therefore the eigenvector 

For , we have: 



Therefore the eigenvector 

For , we have: 



Therefore the eigenvector 













1. 





The eigenvalues are: 

For , we have: 



Therefore the eigenvector 

For , we have: 



Therefore the eigenvector 













1. 







The eigenvalues are: 

For , we have: 





Therefore the eigenvector 

For , we have: 



Therefore the eigenvector 





























1. 









The eigenvalues are: 

For , we have:





Therefore the eigenvector 

For , we have:





Therefore the eigenvector 















1. For 











Therefore the matrix has eigenvalues 

For  , then 



The eigenvectors are: 

For  , then 



The eigenvectors are:  or 

For  , then 



The eigenvectors are: 





***Exercise***

Find the eigenvalues of *A* and *B* and check the Orthogonality of their first two eigenvectors. Graph these eigenvectors to see the discrete sines and cosines:



The −1, 2, −1 pattern in both matrices is a “second derivative. Then  and  are like . This has eigenvectors  and  that are the bases for Fourier series. The matrices lead to “discrete sines” and “discrete cosines” that are the bases for the discrete Fourier Transform. This DFT is absolutely central to all areas of digital signal processing.

***Solution***

The eigenvalues of *A* are .

Their sum is 6 (the trace of *A*) and their product is 4 (the determinant).

The eigenvector matrix *S* gives the “Discrete Sine Transform”.



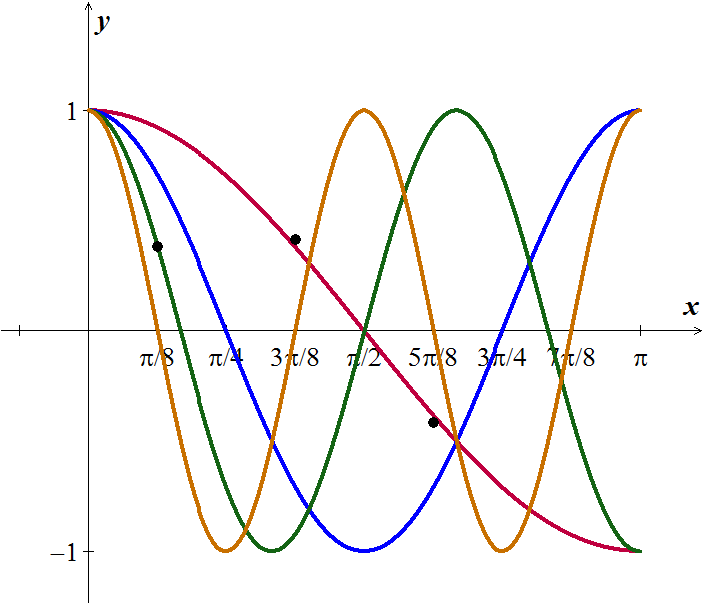






The eigenvalues of *B* are .





***Exercise***

Suppose  and  and λ ≠ 0. Then y is in the nullspace and x is in the column space. They are perpendicular because \_\_\_\_\_\_\_. (why are these subspaces orthogonal?) If the second eigenvalue is a nonzero number β, apply this argument to . The eigenvalue moves to zero and the eigenvectors stay the same – so they are perpendicular.

***Solution***

Suppose that  and , , and λ ≠ 0. Then *x* is in the column space of *A*, and *y* is in the left nullspace of *A* since . But *C*(*A*) and  are orthogonal complements, so *x* and *y* are perpendicular.

If with  then  and . Since  it follows that *x* is in the column space *A- βI*  and *y* is in the nullspace of *A- βI*, and , Therefore we can replace A with  in the argument of previous paragraph and it follows that *x* and *y* are perpendicular.

***Exercise***

True or false. Give a reason or a counterexample.

1. A matrix with real eigenvalues and eigenvectors is symmetric.
2. A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
3. The inverse of a symmetric matrix is symmetric
4. The eigenvector matrix *S* of a symmetric matrix is symmetric.
5. A complex symmetric matrix has real eigenvalues.
6. If *A* is symmetric, then  is symmetric.
7. If *A* is Hermitian, then  is Hermitian.

***Solution***

1. ***False***. Let 

Then 

So *A* has eigenvalues 

The eigenvectors are:  so both the eigenvalues and eigenvectors are real but *A* is not symmetric.

1. ***True***. If the matrix *A* has orthogonal eigenvectors  with eigenvalues , we can define  for all *i*; then  for all *i* and the  are orthonormal. Then we can diagonalize *A* as:  where the *ith* column of S is , and Λ is the diagonal matrix, so  and .



So *A* is symmetric.

1. ***True***. If *A* is symmetric then it can be diagonalized by an orthogonal matrix Q, , and then . Since is still a diagonal matrix, it follows: 
2. ***False***. The eigenvalues of  are:  and the eigenvectors are: . We can diagonalize *A* with eigenvector matrix  which is not symmetric.
3. ***False***. For example *A* = (*i*), the 1 by 1 matrix. The eigenvalue is ***i***, it is not a real number.
4. ***True***. 
5. ***False***. . It is typically not the same as . Taking *A* = (1), the 1 by 1 matrix, would be a enough example because  which is not a real number.

***Exercise***

Find a symmetric matrix  that has a negative eigenvalue.

1. How do you know it must have a negative pivot?
2. How do you know it can’t have two negative eigenvalues?

***Solution***

1. The eigenvalues of that matrix are / so take any . In this case, the determinant is .
2. The signs of the pivots coincide with the signs of the eigenvalues. Alternatively, the product of the pivots is the determinant, which is negative in this case. So, one of the two pivots must be negative.
3. The product of the eigenvalues equals the determinant, which is negative in this case. So, precisely one numbers cannot have a negative product.

***Exercise***

Which of these classes of matrices do *A* and *B* belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?



Which of these factorizations are possible for *A* and *B*: *LU*, *QR*, , ?

***Solution***

Matrix *A* is invertible, orthogonal, a permutation matrix, diagonalizable, and Markov! (Everything but a projection).

Matrix *A* satisfies  , , and also , This means it is invertible, symmetric, and orthogonal. Since it is symmetric, it is diagonalizable (with real eigenvalues!). It is a permutation matrix by just looking at it. It is Markov since the columns add to 1. It is not a projection since .

All of the factorization are possible for it: LU and QR are always possible.  is possible since it is diagonalizable, and  is possible since it is symmetric.

Matrix *B* is a projection, diagonalizable, and Markov. It is not invertible, not orthogonal, and not a permutation.

*B* is a projection since , it is symmetric and thus diagonalizable, and it is Markov since the columns add to 1. It is not invertible since the columns are visibly linearly dependent, it is not orthogonal since the columns are far from orthonormal, and it’s clearly not a permutation.

All the factorizations are possible for it: LU and QR are always possible.  is possible since it is diagonalizable, and  is possible since it is symmetric.

***Exercise***

Prove that *A* is any  matrix, then  has an orthonormal set of *n* eigenvectors

***Solution***

, then  is symmetric, therefore there is an eigenvector  for .

Let 





 Since 

Therefore; 

Then the vectors  are orthogonal



***Example***

Construct a 3 by 3 matrix *A* with no zero entries whose columns are mutually perpendicular. Compute . Why is it a diagonal matrix?

***Solution***

Consider the matrix  to be columns mutual perpendicular

Let assume  





***Exercise***

Assuming that , find a matrix that orthogonally diagonalizes 

***Solution***









Therefore the eigenvalues are: 

Assume that .

For  , then 



The eigenvectors are: 

For  , then 



The eigenvectors are: 

Applying the Gram Schmidt process.









