1. Prove there is no $x, y \in \mathbb{Z}$ satisfying $x^2 - 5y^2 = 12$.

Assume there exist $x, y \in \mathbb{Z}$ such that $x^2 - 5y^2 = 12$ then,

$x^2 - 5y^2 \equiv x^2 \mod 5$
$12 \equiv 2 \mod 5$

$\rightarrow$ there exist $x \in \mathbb{Z}$ such that $x^2 \equiv 2 \mod 5$.

$x \equiv 0$, then $x^2 \equiv 0 \not\equiv 2 \mod 5$.
$x \equiv 1$, then $x^2 \equiv 1 \not\equiv 2 \mod 5$.
$x \equiv 2$, then $x^2 \equiv 4 \not\equiv 2 \mod 5$.
$x \equiv 3$, then $x^2 \equiv 4 \not\equiv 2 \mod 5$.
$x \equiv 4$, then $x^2 \equiv 1 \not\equiv 2 \mod 5$.

There not exist $x \in \mathbb{Z}$ such that $x^2 \equiv 2 \mod 5$. which gives a contradiction.
$\rightarrow$ Assumption is false.

Therefore, there not exist $x, y \in \mathbb{Z}$ such that $x^2 - 5y^2 = 12$.

2. Find all rational solution for $x^2 - y^2 = 1$.

We can find a trivial solution: $(x, y) = (-1, 0)$.

Let $m \in \mathbb{Q}$ and define a line equation by $y = m(x+1)$, then $x^2 - m^2(x+1)^2 = 1$.

If $m^2 \neq 1$, by Vieta's formula, two solution $\alpha + \beta = r \in \mathbb{Q}$, where one of the solution is -1.
So the other solution is also a rational number. (say $\alpha = -1$ then $\beta = r + 1$.)

If $m^2 = 1$, there is no solution.

We showed that if $m \neq \pm 1$ is a rational number, we can generate a solution for the problem.
Now we need to show that this is the only solution for the problem.

Assume $(x, y) = (a, b)$ is a rational solution of $x^2 - y^2 = 1$.
if $a \neq -1$ then $(x, y) = (-1, 0)$ is a trivial solution, so $y = \dfrac{b}{a+1}(x+1)$ is satisfied.
$a + 1$ is not zero and rational, b is rational so $\dfrac{b}{a+1}$ is rational. We can generate this solution by putting $m = \dfrac{b}{a+1}$.
if $a = -1$ then $b = 0$.

Now we can generate all solution for the problem.

if $m \neq \pm 1$, from $x^2 - m^2(x+1)^2 = 1$ we have $x = \dfrac{1+m^2}{1-m^2}$, $y = \dfrac{2m}{1-m^2}$.

We also have $x = -1,\ y = 0$ which cannot be generated from above.

The only and all solution is in form of $(x, y) = \left(\dfrac{1+m^2}{1-m^2}, \dfrac{2m}{1-m^2}\right)$ where $m \in \mathbb{Q} \setminus \{-1, 1\}$ or $(x, y) = (-1, 0)$

3. Find another rational solution of $y^2 = x^3 + 8$ given 2 solutions $(1, -3)$, $\left(\frac{-7}{4}, \frac{13}{8}\right)$.

$y^2 = x^3 + 8 \to (-y)^2 = x^3 + 8$.
We can easily find another solution by negating y.

$(1, 3)$, $\left(\frac{-7}{4}, \frac{-13}{8}\right)$ is also a solution, but probably not the intended solution.

If we draw a line between two points, we have $y = \dfrac{-37x - 29}{22}$.

$\dfrac{(37x + 29)^2}{484} = x^3 + 8$ is a cubic equation, so we have $\alpha + \beta + \gamma = \dfrac{1369}{484}$.

Answer $x = \dfrac{433}{121}, y = \dfrac{-9765}{1331}$.

4. Find all integer solution for $\gcd(252, 198) = 252x + 198y$.

Perform Euclidean Algorithm

$252 = 1 \times 198 + 54$
$198 = 3 \times 54 + 36$
$54 = 1 \times 36 + 18$
$36 = 2 \times 18 + 0$
$\gcd(252, 198) = 18.$

$54 = 1 \times 252 - 1 \times 198$
$36 = 1 \times 198 - 3 \times (1 \times 252 - 1 \times 198) = -3 \times 252 + 4 \times 198$
$18 = 1 \times (1 \times 252 - 1 \times 198) - 1 \times (-3 \times 252 + 4 \times 198) = 4 \times 252 - 5 \times 198$

We found one solution $x = 4$, $y = -5$.

If (x, y) is a solution, $(x + 11k, y - 14k)$ is also a solution. (There is no other solution.)

Answer $x = 4 - 11k$, $y = 14k - 5$, $k \in \mathbb{Z}$.

5. Find all solution for congruence equation $9x \equiv 12 \mod 15$.

$\gcd(9, 15) = 3 \mid 12$

$3x \equiv 4 \pmod 5$

$x \equiv 3 \pmod 5$

Answer $x \equiv 3 \text{ or } 8 \text{ or } 13 \pmod{15}$.