Low Reynolds number gravitational settling of a sphere through a fluid-fluid interface: Modelling using a boundary integral method

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Abstract

1 Introduction

2 Fundamentals of Stokes Flow

We present here a background to the fundamentals of Stokes flow, covering the equations of motion and non-dimensionalisation, different types of boundary condition, Greens functions and the integral representation of Stokes flow. Throughout this document we will

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be making use of the Einstein summation convention and tensor notation (Riley et al., 2006).

2.1 Equations of Motion

The starting point for all fluid dynamical problems are the continuity (equation 1) and Navier Stokes (equation 2) equations (Batchelor, 1967). Defining the fluid density ρ, the dynamic viscosity η, the fluid velocity field u_i and the pressure field P these are expressed as

$$\frac{\partial \rho(\boldsymbol{x},t)}{\partial t} + \partial_i[\rho(\boldsymbol{x},t)u_i(\boldsymbol{x},t)] = 0, \tag{1}$$

and

$$\rho(\boldsymbol{x},t)\left(\frac{\partial u_i(\boldsymbol{x},t)}{\partial t} + [u_j(\boldsymbol{x},t)\partial_j]u_i(\boldsymbol{x},t)\right) = -\partial_i P(\boldsymbol{x},t) - \rho(\boldsymbol{x},t)g + \eta\left(\partial_j\partial_j u_i(\boldsymbol{x},t) + \frac{\partial_i(\partial_j u_j(\boldsymbol{x},t))}{3}\right), \tag{2}$$

Forming a coupled set of non-linear, partial differential equations for the velocity and pressure fields as functions of space $\boldsymbol{x}$ and time t, these represent mass and momentum conservation respectively and must be satisfied by all fluid phases within the system. For most practical applications, the fluids are assumed to be incompressible (have constant density) and so the continuity equation reduces to the incompressibility relation;

$$\partial_i u_i(\boldsymbol{x},t) = 0. \tag{3}$$

This can be combined with equation 2 to form the incompressible Navier Stokes equation;

$$\rho\left(\frac{\partial u_i(\boldsymbol{x},t)}{\partial t} + [u_j(\boldsymbol{x},t)\partial_j]u_i(\boldsymbol{x},t)\right) = -\partial_i P(\boldsymbol{x},t) - \rho g + \eta\partial_j\partial_j u_i(\boldsymbol{x},t). \tag{4}$$

The equations of motion can be expressed in an alternative form by defining the stress tensor $T_{ij}(\boldsymbol{x},t)$ (Batchelor, 1967; Manga, 1994) and dynamic pressure $P_{\mathrm{d}}(\boldsymbol{x},t)$:

$$T_{ij}(\boldsymbol{x},t) = -P_{\mathrm{d}}(\boldsymbol{x},t)\delta_{ij} + \eta(\partial_i u_j(\boldsymbol{x},t) + \partial_j u_i(\boldsymbol{x},t)), \tag{5}$$

$$P_{\mathrm{d}}(\boldsymbol{x},t) = P(\boldsymbol{x},t) - \rho g_i x_i. \tag{6}$$

This definition of the stress tensor removes the gravitational body force from the equations of motion, meaning that it only appears in the boundary conditions. The Navier Stokes equation then becomes

$$\rho\left(\frac{\partial u_i(\boldsymbol{x},t)}{\partial t} + [u_j(\boldsymbol{x},t)\partial_j]u_i(\boldsymbol{x},t)\right) = \partial_j T_{ij}(\boldsymbol{x},t). \tag{7}$$

When working in fluid dynamics, it is usual to non-dimensionalise the equations of motion and boundary conditions (White, 1999). This can be achieved by scaling the quantities involved by parameters specific to the problem. For example, consider a problem with typical scales of length L_{c} and velocity U_{c}. This allows us to define dimensionless variables (denoted by a $'$)

$$x_i = L_{\mathrm{c}} x_i', \tag{8}$$

$$u_i(\boldsymbol{x},t) = U_{\mathrm{c}} u_i'(\boldsymbol{x}',t'), \tag{9}$$

and

$$t = \frac{L_{\mathrm{c}} t'}{U_{\mathrm{c}}} \tag{10}$$

In the case of highly viscous flows the relevant scaling for the dynamic pressure uses a characteristic viscosity η_{c} and is given byLee and Leal (1982)

$$P_{\mathrm{d}}(\boldsymbol{x},t) = \frac{\eta_{\mathrm{c}} U_{\mathrm{c}} P'_{\mathrm{d}}(\boldsymbol{x'},t')}{L_{\mathrm{c}}}. \tag{11}$$

This choice of pressure scaling means that upon substitution of equations 8 to 11 into equation 5 the stress tensor can also be non-dimensionalised,

$$T_{ij}(\boldsymbol{x},t) = \frac{\eta_{\mathrm{c}} U_{\mathrm{c}} T'_{ij}(\boldsymbol{x'},t')}{L_{\mathrm{c}}} \quad \text{where} \quad T'_{ij}(\boldsymbol{x'},t') = p'_{\mathrm{d}}(\boldsymbol{x'},t')\delta_{ij} + \Lambda(\partial'_i u'_j(\boldsymbol{x'},t') + \partial'_j u'_i(\boldsymbol{x'},t')), \tag{12}$$

where $\Lambda = \eta/\eta_{\mathrm{c}}$. Hence, the dimensionless continuity and Navier Stokes equations are

$$\partial'_i u'_i(\boldsymbol{x'},t') = 0, \tag{13}$$

and

$$\mathrm{Re}\left(\frac{\partial u'_i(\boldsymbol{x'},t')}{\partial t'} + (u'_j(\boldsymbol{x'},t')\partial'_j)u'_i(\boldsymbol{x'},t')\right) = \partial'_j T'_{ij}(\boldsymbol{x'},t'), \tag{14}$$

where the Reynolds number is defined as

$$\mathrm{Re} = \frac{\rho L_{\mathrm{c}} U_{\mathrm{c}}}{\eta_{\mathrm{c}}} \tag{15}$$

As we are considering the case of low Reynolds number ($\mathrm{Re} \ll 1$), we can neglect the inertial terms on the right hand side and the equation reduces to the Stokes equation (Batchelor, 1967; Kim and Karrila, 2005)

$$\partial'_i T'_{ij}(\boldsymbol{x'},t') = 0. \tag{16}$$

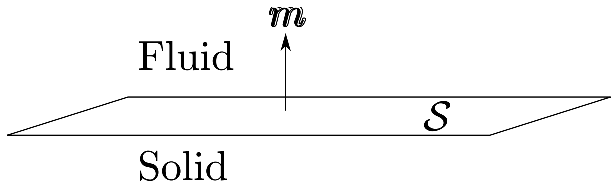

Figure 1: Fluid-solid boundary $\mathcal{S}$ with normal vector $\boldsymbol{m}$ directed into the fluid phase.

2.2 Boundary Conditions

In order to complete the formulation of any fluid dynamics problem, it is necessary to state the boundary conditions alongside the equations of motion (Riley et al., 2006). For fluids of infinite (or semi-infinite) extent in some dimension, these include the value of the flow velocity at infinity. For bounded flows, the conditions are imposed at the boundaries of the fluid domain, and their exact nature depends on the phase of the material bounding it. At a boundary, two types of boundary condition can exist: a kinematic boundary condition of the velocity field and a dynamic boundary condition on the stress field. Kinematic boundary conditions are an expression of mass conservation and dynamic boundary conditions are a balance of forces, an expression of Newton's third law.

2.2.1 Fluid-Solid Boundary

At low Reynolds number for a fluid-solid boundary defined the surface $\mathcal{S}$ (see figure 1), the kinematic boundary condition is one of no-slip; the fluid velocity at the boundary is the same as that of the solid $U'_{\mathrm{s},i}$. This is easily expressed in dimensionless form as

$$u'_i(\boldsymbol{x'}) = U'_{\mathrm{s},i}, \quad \text{when } \boldsymbol{x} \in \mathcal{S}. \tag{17}$$

There also needs to be a dynamic boundary condition applied at the interface. If the solid

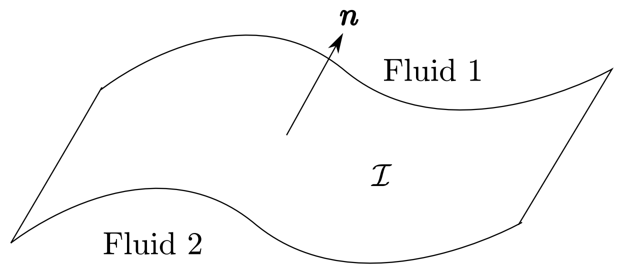

Figure 2: Fluid-fluid boundary $\mathcal{I}$ with normal vector $\boldsymbol{n}$.

exerts a force F_i onto the fluid then the condition states

$$\int_{\mathcal{S}} m_i(\boldsymbol{x})T_{ij}(\boldsymbol{x})\mathrm{d}\mathcal{S} = F_j, \tag{18}$$

where $m_i(\boldsymbol{x'})$ is the normal vector to $\mathcal{S}$ directed into the fluid. Using the non-dimensionalisation scheme presented above this becomes

$$\eta_{\mathrm{c}}U_{\mathrm{c}}L_{\mathrm{c}}\int_{\mathcal{S}} f_i(\boldsymbol{x'})\mathrm{d}\mathcal{S}' = F_i, \tag{19}$$

where we have defined $f_i(\boldsymbol{x'})$ as the dimensionless traction vector defined on the surface $\mathcal{S}$, $m_i(\boldsymbol{x'})T'_{ij}(\boldsymbol{x'})$.

2.2.2 Fluid-Fluid Boundary

For a boundary $\mathcal{I}$ between two fluids labelled 1 and 2 (figure 2), the kinematic boundary condition states that the velocity of the two fluids must be continuous across the interface (Kim and Karrila, 2005). Defining the velocity is fluid l as u_l this can be expressed in dimensionless form as

$$u'_{1,i}(\boldsymbol{x'}) = u'_{2,i}(\boldsymbol{x'}), \quad \text{when } \boldsymbol{x'} \in \mathcal{I}. \tag{20}$$

The dynamic boundary condition is an expression of the balance between the stress discontinuity across the interface and the interfacial tension (IFT) σ (Batchelor, 1967). With out definition of the stress tensor this is given as (Manga, 1994):

$$n_i(\boldsymbol{x})[T_{1,ij}(\boldsymbol{x})-\rho_1 g_k x_k \delta_{ij}]-n_i(\boldsymbol{x})[T_{2,ij}(\boldsymbol{x})-\rho_2 g_k x_k \delta_{ij}] = \sigma(\boldsymbol{x})n_j(\boldsymbol{x})[\partial_{\mathrm{s},i} n_i(\boldsymbol{x})]-\partial_{\mathrm{s},j}\sigma(\boldsymbol{x}), \quad \text{when } \boldsymbol{x} \in \mathcal{I}. \tag{21}$$

The operator $\partial_{\mathrm{s},i}$ is defined as the tangential gradient operator within the surface $\mathcal{I}$:

$$\partial_{\mathrm{s},i} = (\delta_{ij} - \partial_i \partial_j)\partial_j. \tag{22}$$

When this takes the normal vector as its argument it can be shown that (Brackbill et al., 1992)

$$\partial_{\mathrm{s},i} n_i = \partial_i n_i. \tag{23}$$

The presence of spatial gradients in the interfacial tension can lead to so-called Marangoni effects (Thomson, 1855; Gibbs, 1878). However, for our purposes we will assume that the interfacial tension is uniform across the interface $\mathcal{I}$ and so the last term on the right hand side vanishes;

$$n_i(\boldsymbol{x})[T_{1,ij}(\boldsymbol{x}) - \rho_1 g_k x_k \delta_{ij}] - n_i[T_{2,ij}(\boldsymbol{x}) - \rho_2 g_k x_k \delta_{ij}] = \sigma(\boldsymbol{x})n_i(\boldsymbol{x})\partial_i n_j(\boldsymbol{x}), \quad \text{when } \boldsymbol{x} \in \mathcal{I}. \tag{24}$$

Like the equations of motion, this can be non-dimensionalised using equations 8 to 12:

$$\mathrm{Ca}\ n_i(\boldsymbol{x'})[T'_{1,ij}(\boldsymbol{x'}) - T'_{2,ij}(\boldsymbol{x'})] + \mathrm{Bo}(\hat{z}_i x'_i)n_j(\boldsymbol{x'}) = n_j(\boldsymbol{x'})\partial'_i n_i(\boldsymbol{x'}). \tag{25}$$

The capillary number Ca and Bond number Bo are dimensionless numbers defined as:

$$\mathrm{Ca} = \frac{\eta_{\mathrm{c}} U_{\mathrm{c}}}{\sigma} \tag{26}$$

$$\mathrm{Bo} = \frac{(\rho_2 - \rho_1) g L_\mathrm{c}^2}{\sigma} \tag{27}$$

2.3 Greens functions

In order to derive the integral representation of the Stokes equations, it is necessary to make use of the Greens functions (Riley et al., 2006) for Stokes flow, $\hat{u}_i(\boldsymbol{x'} - \boldsymbol{y'})$ and $\hat{T}_{ij}(\boldsymbol{x'} - \boldsymbol{y'})$, defined such that (Kim and Karrila, 2005)

$$\partial_i' \hat{u}_i(\boldsymbol{x'} - \boldsymbol{y'}) = 0, \tag{28}$$

and

$$\partial_i' \hat{T}_{ij}(\boldsymbol{x'} - \boldsymbol{y'}) + \mathcal{F}_j \delta(\boldsymbol{x'} - \boldsymbol{y'}) = 0, \tag{29}$$

where $\mathcal{F}_i$ is a arbitrary constant vector. Equations 28 and 29 can be solved following Ladyzhenskaya (1963), using Fourier transforms (appendix A) to show that

$$\hat{u}_j(\boldsymbol{\xi}) = \frac{\mathcal{F}_i J_{ij}(\boldsymbol{\xi})}{\Lambda}, \tag{30}$$

and

$$\hat{T}_{ij}(\boldsymbol{\xi}) = K_{ijk}(\boldsymbol{\xi}) \mathcal{F}_k, \tag{31}$$

where $\boldsymbol{\xi} = \boldsymbol{x'} - \boldsymbol{y'}$ and

$$J_{ij}(\boldsymbol{\xi}) = \frac{1}{8\pi\xi} \left(\delta_{ij} + \frac{\xi_i \xi_j}{\xi^2} \right), \tag{32}$$

and

$$K_{ijk}(\boldsymbol{\xi}) = \frac{-3\xi_i\xi_j\xi_k}{4\pi\xi^5}. \tag{33}$$

We have defined $\xi = \xi_i\xi_i$.

2.4 Integral Representation of Stokes Equations

We now substitute the Greens functions and unknown velocity and stress field solutions into the Lorentz Reciprocal Theorem (equation 115 in appendix D) and simplfify using equations 43 and 29 to find

$$\int_{\mathcal{V}} u'_k(\boldsymbol{x'})\delta(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x'}^3 = \frac{1}{\Lambda}\int_{\mathcal{S}} J_{ik}(\boldsymbol{\xi})T'_{ij}(\boldsymbol{x'})n_j(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^2 - \int_{\mathcal{S}} u'_i(\boldsymbol{x'})K_{ijk}(\boldsymbol{\xi})n_j(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^2. \tag{34}$$

Finally make the transformation $\boldsymbol{x'} \leftrightarrow \boldsymbol{y'}$ and use the symmetry properties of the kernels (equations 85 and 86 in appendix A) and the delta function (equation 105 in appendix B) to obtain the general form of the integral representation of the Stokes equations;

$$\int_{\mathcal{V}} u'_k(\boldsymbol{y'})\delta(\boldsymbol{\xi})\mathrm{d}\boldsymbol{y'}^3 = \frac{1}{\Lambda}\int_{\mathcal{S}} J_{ik}(\boldsymbol{\xi})T'_{ij}(\boldsymbol{y'})n_j(\boldsymbol{y'})\mathrm{d}\boldsymbol{y'}^2 + \int_{\mathcal{S}} u'_i(\boldsymbol{y'})K_{ijk}(\boldsymbol{\xi})n_j(\boldsymbol{y'})\mathrm{d}\boldsymbol{y'}^2. \tag{35}$$

Using the definition of the delta function (equation 103 in appendix B) this means

$$\frac{1}{\Lambda}\int_{\mathcal{S}} J_{ik}(\boldsymbol{\xi})T'_{ij}(\boldsymbol{y'})n_j(\boldsymbol{y'})\mathrm{d}\boldsymbol{y'}^2 + \int_{\mathcal{S}} u'_i(\boldsymbol{y'})K_{ijk}(\boldsymbol{\xi})n_j(\boldsymbol{y'})\mathrm{d}\boldsymbol{y'}^2 = \begin{cases} u'_k(\boldsymbol{x'}) & \boldsymbol{x'} \in \mathcal{V} \\ \frac{u'_k(\boldsymbol{x'})}{2} & \boldsymbol{x'} \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases}. \tag{36}$$

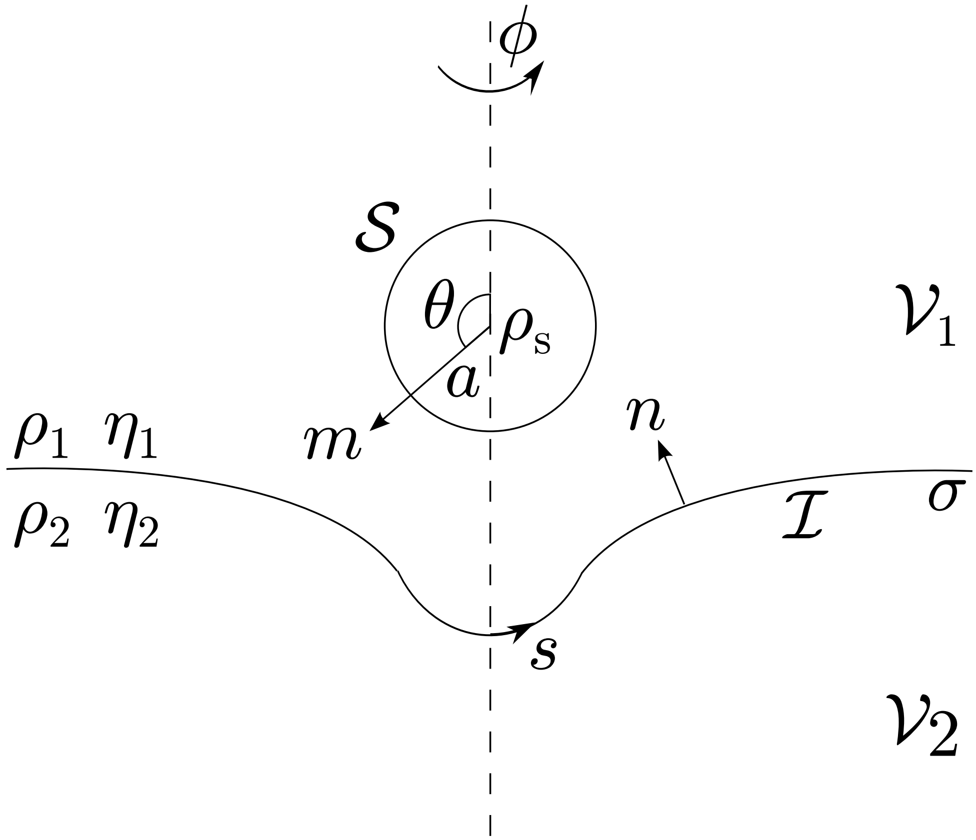

Figure 3: Diagrammatic representation of the system. A sphere falls under gravity, at low Reynolds number, towards an initially horizontal interface between two density stratified, immiscible semi-infinite fluids. See table **??** for definition of symbols.

3 Theoretical Development

3.1 Problem Statement

The system is formulated as in figure 3. The physical parameters motivate the choice of scaling variables. The characteristic lengthscale is chosen to be the sphere radius a, characteristic viscosity that of the upper fluid η_1 and characteristic velocity to be the Stokes velocity (Reynolds, 1886),

$$U_\mathrm{c} = \frac{2(\rho_\mathrm{s} - \rho_1) g a^2}{9\eta_1}. \tag{37}$$

This means the capillary and Bond numbers can be expressed as:

$$\mathrm{Ca} = \frac{(\rho_\mathrm{s} - \rho_1)ga^2}{\sigma}, \tag{38}$$

$$\mathrm{Bo} = \frac{(\rho_2 - \rho_1)ga^2}{\sigma}. \tag{39}$$

The dimensionless stress tensor for each fluid can be written as

$$T'_{\alpha,ij}(\boldsymbol{x'}) = -P'_{\mathrm{d},\alpha}(\boldsymbol{x'})\delta_{ij} + \Lambda_\alpha[\partial'_i u'_{\alpha,j}(\boldsymbol{x'}) - \partial'_j u'_{\alpha,i}(\boldsymbol{x'})]. \tag{40}$$

The parameter Λ_α is defined as

$$\Lambda_\alpha = \frac{\eta_\alpha}{\eta_1} = \begin{cases} 1, & \alpha = 1 \\ \frac{\eta_2}{\eta_1} = \lambda, & \alpha = 2 \end{cases}. \tag{41}$$

Note λ is the viscosity ratio of the two fluids. It is straightforward to apply the general equations of motion and boundary conditions to the problem. The equations of motion, expressed in the Einstein summation convention (Riley et al., 2006) which will be used from now on, appear as

$$\partial'_\mathrm{i} u'_{\alpha,i}(\boldsymbol{x'}) = 0, \tag{42}$$

and

$$\partial'_\mathrm{i} T'_{\alpha,ij}(\boldsymbol{x'}) = 0. \tag{43}$$

Here, $\alpha = 1, 2$ and denotes the fluid, i, j denote components of tensoral quantities.

The first boundary condition that we impose is that the undisturbed fluid is quiescent;

$$u'_{\alpha,i}(\boldsymbol{x'}) \to 0 \text{ as } |\boldsymbol{x'}| \to \infty. \tag{44}$$

The kinematic boundary condition on the fluid interface (equation 20) can be expressed as

$$u'_{1,i}(\boldsymbol{x'}) = u'_{2,i}(\boldsymbol{x'}), \quad \boldsymbol{x'} \in S_{\text{int}}. \tag{45}$$

The dynamic boundary condition is also imposed at the interface;

$$\text{Ca } n_i[T'_{1,ij}(\boldsymbol{x'}) - T'_{2,ij}(\boldsymbol{x'})] + \text{Bo}\hat{z}_i x'_i n_j = n_j \partial'_i n_i. \tag{46}$$

The kinematic boundary condition on the sphere surface is one of no-slip meaning the fluid velocity at the surface has to equal the sphere velocity ;

$$u'_{1,i}(\boldsymbol{x'}) = u'_{\text{s},i}, \quad \boldsymbol{x'} \in \mathcal{S}. \tag{47}$$

The final boundary condition is the dynamic boundary condition on the sphere. The force on the fluid due to the sphere originates from the balance between gravity and buoyancy;

$$F_i = \frac{4\pi a^3(\rho_\text{s} - \rho_1)g\hat{z}_i}{3}. \tag{48}$$

Substituting this into equation 19 and using equation 37 we obtain

$$\int_{\mathcal{S}} n_i T'_{1,ij}(\boldsymbol{x'})\text{d}\mathcal{S}' = 6\pi\hat{z}_j \tag{49}$$

The dimensionless numbers that describe the system are the set $\{\lambda, \text{Ca}, \text{Bo}\}$. However, an equivalent set can be generated by defining the dimensionless density ratio D;

$$D = \frac{\mathrm{Ca}}{\mathrm{Bo}} = \frac{\rho_\mathrm{s} - \rho_1}{\rho_2 - \rho_1}. \tag{50}$$

Therefore, we can also describe the system with the set $\{\lambda, D, \mathrm{Bo}\}$. This allows us to re-express equation 46 as

$$D\mathrm{Bo}\; n_i[T'_{1,ij}(\boldsymbol{x'}) - T'_{2,ij}(\boldsymbol{x'})] = n_j(\partial'_i n_i - \mathrm{Bo}\hat{z}_i x'_i) \tag{51}$$

To summarise, the problem is completely described by equations 42 to 45, and equations 47, 49 and 51.

3.2 Integral Representation

This equation needs to be considered for each fluid separately. For fluid 1 ($\alpha = 1$), $\mathcal{S}_1 = \mathcal{S} + \mathcal{I}$, $\boldsymbol{n}_1(\boldsymbol{y'}) = \boldsymbol{m}(\boldsymbol{y'})$ for $\boldsymbol{y'} \in \mathcal{S}$ and $\boldsymbol{n}_1(\boldsymbol{y'}) = \boldsymbol{n}(\boldsymbol{y'})$ for $\boldsymbol{y'} \in \mathcal{I}$. Noting that for $\boldsymbol{y'} \in \mathcal{S}$, $u'_{1,i}(\boldsymbol{y'}) = u_{\mathrm{s},i}$ and that $\partial'_j K_{ijk}(\boldsymbol{\xi}) = -\delta_{ik}\delta(\boldsymbol{\xi})$ (as must follow from equations 29 and 88) the boundary integral equation for fluid 1 can be written as

$$\begin{aligned}\fint_{\mathcal{S}} J_{ik}(\boldsymbol{\xi})T'_{1,ij}(\boldsymbol{y'})m_j(\boldsymbol{y'})\mathrm{d}^2\boldsymbol{y'} + \fint_{\mathcal{I}} J_{ik}(\boldsymbol{\xi})T'_{1,ij}(\boldsymbol{y'})n_j(\boldsymbol{y'})\mathrm{d}^2\boldsymbol{y'} +& \\ \fint_{\mathcal{I}} u'_{1,i}(\boldsymbol{y'})K_{ijk}(\boldsymbol{\xi})n_j(\boldsymbol{y})\mathrm{d}^2\boldsymbol{y'} = \begin{cases} \frac{u'_{1,k}(\boldsymbol{x'})}{2} & \boldsymbol{x'} \in \mathcal{I} \\ u'_{s,k} & \boldsymbol{x'} \in \mathcal{S} \end{cases} &.\end{aligned} \tag{52}$$

For fluid 2, $\mathcal{S}_2 = \mathcal{I}$ and $\boldsymbol{n}_2(\boldsymbol{y'}) = -\boldsymbol{n}(\boldsymbol{x'})$ for $\boldsymbol{x'} \in \mathcal{I}$. Using equation 45 the boundary integral equation for fluid 2 can be written as

$$-\fint_{\mathcal{I}} J_{ik}(\boldsymbol{\xi})T'_{1,ij}(\boldsymbol{y'})n_j(\boldsymbol{y'})\mathrm{d}^2\boldsymbol{y'} - \lambda\fint_{\mathcal{I}} u'_{1,i}(\boldsymbol{y'})K_{ijk}(\boldsymbol{\xi})n_j(\boldsymbol{y})\mathrm{d}^2\boldsymbol{y'} = \frac{\lambda u'_{1,k}(\boldsymbol{x'})}{2} \quad \boldsymbol{x'} \in \mathcal{I}. \tag{53}$$

Equations 52 and 53 can be added together and combined with equation 51 to obtain

$$\begin{aligned}\fint_{\mathcal{S}} J_{ik}(\boldsymbol{\xi}) f_{\mathrm{s},i}(\boldsymbol{y'})\mathrm{d}^2\boldsymbol{y'} + \frac{9}{2D\mathrm{Bo}}\fint_{\mathcal{I}} J_{ik}(\boldsymbol{\xi}) n_i(\boldsymbol{y'})[\partial'_j n_j(\boldsymbol{y'}) - \hat{z}_j y'_j \mathrm{Bo}]\mathrm{d}^2\boldsymbol{y'}+ \\ (1-\lambda)\fint_{\mathcal{I}} u'_{1,i}(\boldsymbol{y'}) K_{ijk}(\boldsymbol{\xi}) n_j(\boldsymbol{y'})\mathrm{d}^2\boldsymbol{y'} = \begin{cases} \frac{(1+\lambda)u'_{1,k}(\boldsymbol{x'})}{2} & \boldsymbol{x'} \in \mathcal{I} \\ u'_{s,k} & \boldsymbol{x'} \in \mathcal{S} \end{cases}.\end{aligned} \tag{54}$$

This together with equation 49 completely describes the system in an integral representation.

3.3 Axisymmetric Simplification

We can exploit the axial symmetry of the system to chose the point $\boldsymbol{x'}$ such that it lies in the plane defined by $\phi = 0$. Hence in Cartesian coordinates $\boldsymbol{x'} = (x_r, 0, x_z)$. This also means we can write $\boldsymbol{y'} = (y_r \cos\phi, y_r \sin\phi, y_z)$. On the surface of the sphere $y_{r(z)} = y_{r(z)}(\theta)$ and on the interface $y_{r(z)} = y_{r(z)}(s)$. Additionally $\boldsymbol{f}_{\mathrm{s}} = [f_{\mathrm{s},r}(\theta)\cos\phi, f_{\mathrm{s},r}(\theta)\sin\phi, f_{\mathrm{s},z}(\theta)]$ and $\boldsymbol{n} = [n_r(s)\cos\phi, n_r(s)\sin\phi, n_z(s)]$. Since the system is axisymmetric, it is useful to extract the azimuthal integration from the surface integrals in equations 54 and 49. To achieve this, the Cartesian components of each equation is considered separately. For equation 54, it can be shown that both the left and right hand sides of the 2-component are identically zero. For equation 49 this is true for the 1- and 2-components. To show this, expand J_{ij} and K_{ijk} in terms of the components of $\boldsymbol{x'}$ and $\boldsymbol{y'}$. This leaves three integral equations which can be expressed as

$$\begin{aligned}\int_{\theta=0}^{\pi} B_{\alpha\beta}(\boldsymbol{x'},\theta) b_\beta(\theta)\sin\theta\mathrm{d}\theta + \int_{s=0}^{\infty}\left(A_{\alpha\beta}(\boldsymbol{x'},s)y_r(s) - \frac{(1+\lambda)\delta_{\alpha\beta}\delta(s-s_0)}{2}\right)a_\beta(s)\mathrm{d}s \\ = -\int_{s=0}^{\infty} C_\alpha(\boldsymbol{x'},s)y_r(s)\mathrm{d}s, \quad \text{when } \boldsymbol{x'} \in \mathcal{I},\end{aligned} \tag{55}$$

$$\int_{\theta=0}^{\pi} B_{\alpha\beta}(\boldsymbol{x'},\theta)b_\beta(\theta)\sin\theta \mathrm{d}\theta + \int_{s=0}^{\infty} A_{\alpha\beta}(\boldsymbol{x'},s)a_\beta(s)y_r(s)\mathrm{d}s - d_\alpha = \int_{s=0}^{\infty} C_\alpha(\boldsymbol{x'},s)y_r(s)\mathrm{d}s, \quad \text{when } \boldsymbol{x'} \in \mathcal{S}, \tag{56}$$

and

$$\int_{\theta=0}^{\pi} b_2(\theta)\sin\theta\mathrm{d}\theta = 3, \tag{57}$$

where the quantities $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{d}$ are defined as:

$$\boldsymbol{A} = (1-\lambda)\int_{\phi=0}^{2\pi} \begin{pmatrix} \begin{matrix} n_r(K_{111}\cos^2\phi + K_{221}\sin^2\phi + 2K_{121}\sin\phi\cos\phi) \\ + n_z(K_{131}\cos\phi + K_{231}\sin\phi) \end{matrix} & \begin{matrix} n_r(K_{131}\cos\phi + K_{231}\sin\phi) \\ + n_z K_{331} \end{matrix} \\ \begin{matrix} n_r(K_{113}\cos^2\phi + K_{223}\sin^2\phi + 2K_{123}\sin\phi\cos\phi) \\ + n_z(K_{133}\cos\phi + K_{233}\sin\phi) \end{matrix} & \begin{matrix} n_r(K_{133}\cos\phi + K_{233}\sin\phi) \\ + n_z K_{333} \end{matrix} \end{pmatrix} \mathrm{d}\phi, \tag{58}$$

$$\boldsymbol{B} = \int_{\phi=0}^{2\pi} \begin{pmatrix} J_{11}\cos\phi + J_{21}\sin\phi & J_{31} \\ J_{13}\cos\phi + J_{23}\sin\phi & J_{33} \end{pmatrix} \mathrm{d}\phi, \tag{59}$$

$$\boldsymbol{C} = \frac{9(\partial'_j n_j - \mathrm{Bo} y_z)}{2D\mathrm{Bo}} \int_{\phi=0}^{2\pi} \begin{pmatrix} n_r(J_{11}\cos\phi + J_{21}\sin\phi) + n_z J_{31} \\ n_r(J_{13}\cos\phi + J_{23}\sin\phi) + n_z J_{23} \end{pmatrix} \mathrm{d}\phi, \tag{60}$$

$$\boldsymbol{a} = \begin{pmatrix} u'_{1,r}(s) \\ u'_{1,z}(s) \end{pmatrix}, \tag{61}$$

$$\boldsymbol{b} = \begin{pmatrix} f_{\mathrm{s},r}(s) \\ f_{\mathrm{s},z}(s) \end{pmatrix}, \tag{62}$$

and

$$\boldsymbol{d} = \begin{pmatrix} 0 \\ u'_{\mathrm{s}} \end{pmatrix} \tag{63}$$

For brevity the function arguments have been dropped from the kernels and the normal vectors but $n_i = n_i[\boldsymbol{y'}(s,\phi)]$ and in equation 58, $K_{ijk} = K_{ijk}[\boldsymbol{x'} - \boldsymbol{y'}(s,\phi)]$, in equation 59, $J_{ij} = J_{ij}[\boldsymbol{x'} - \boldsymbol{y'}(\theta,\phi)]$ and in equation 60, $J_{ij} = J_{ij}[\boldsymbol{x'} - \boldsymbol{y'}(s,\phi)]$.

The aziumthal integrals inside the definitions of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ can be expressed as sums of complete elliptic integrals of the first and second kind (Lee and Leal, 1982; Geller et al., 1985; Graziani, 1989; Pozrikids, 1992; Manga, 1994; Roumeliotis, 2000). Previous authors have used different notations and formulations to present these expressions and we will use a formulation similar to Graziani (1989). First, we define the quantities α and β:

4 Numerical Method

A Greens Functions for Stokes Flow

We present here a derivation of equations 30 and 88 following Ladyzhenskaya (1963). First, the Greens function for dynamic pressure $\hat{P}_\alpha(\boldsymbol{\xi})$ is defined such that

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = -\hat{P}_\alpha(\boldsymbol{\xi}) + \Lambda_\alpha[\partial'_i \hat{u}_{\alpha,j}(\boldsymbol{\xi}) + \partial'_j \hat{u}_{\alpha,i}(\boldsymbol{\xi})]. \tag{64}$$

Substituting this into equation 29 and using equation 28 yields

$$-\partial'_j \hat{P}_\alpha(\boldsymbol{\xi}) + \Lambda_\alpha \partial'_i \partial'_i \hat{u}_{\alpha,j}(\boldsymbol{\xi}) + \mathcal{F}_j \delta(\boldsymbol{\xi}) = 0. \tag{65}$$

We also define two further quantities $\bar{P}_{\alpha,i}$ and $\bar{u}_{\alpha,ij}$ such that

$$\hat{P}_\alpha(\boldsymbol{\xi}) = \mathcal{F}_i \bar{P}_{\alpha,i}(\boldsymbol{\xi}), \tag{66}$$

and

$$\hat{u}_{\alpha,j}(\boldsymbol{\xi}) = \mathcal{F}_i \bar{u}_{\alpha,ij}(\boldsymbol{\xi}). \tag{67}$$

Substitution of these expressions into equations 65 and 28 and rearranging results in

$$-\partial'_j \bar{P}_{\alpha,i}(\boldsymbol{\xi}) + \Lambda_\alpha \partial'_k \partial'_k \bar{u}_{\alpha,ij}(\boldsymbol{\xi}) + \delta_{ij}\delta(\boldsymbol{\xi}) = 0, \tag{68}$$

and

$$\partial'_i \bar{u}_{\alpha,ij}(\boldsymbol{\xi}) = 0. \tag{69}$$

To derive functional forms for the Greens functions it is necessary to express equations 68 and 69 in Fourier representation. To do this we need to define the Fourier transformed variables $\tilde{P}_{\alpha,i}$ and $\tilde{u}_{\alpha,ij}$ Riley et al. (2006):

$$\bar{P}_{\alpha,i}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{P}_{\alpha,i}(\boldsymbol{k}) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}, \tag{70}$$

and

$$\bar{u}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{u}_{\alpha,ij}(\boldsymbol{k}) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}. \tag{71}$$

Using these definitions, and equation 106 in appendix B, in equations 68 and 69 gives the Fourier representations of the Stokes equations and the continuity equation. Following some manipulation these can be written as

$$-ik_j\tilde{P}_{\alpha,i}(\boldsymbol{k})-\Lambda_\alpha k^2\tilde{u}_{\alpha,ij}(\boldsymbol{k})+\frac{\delta_{ij}}{(2\pi)^{3/2}}=0, \tag{72}$$

and

$$k_i\tilde{u}_{\alpha,ij}(\boldsymbol{k})=0, \tag{73}$$

where $k=|\boldsymbol{k}|$. By multiplying equation 72 by k_j, substituting in equation 73 and rearranging, it is then possible to obtain the Fourier representation of the Greens function for pressure;

$$\tilde{P}_{\alpha,i}(\boldsymbol{k})=\frac{-ik_i}{(2\pi)^{3/2}k^2}. \tag{74}$$

A final substitution of this into equation 70 gives the Greens function for pressure;

$$\bar{P}_{\alpha,i}(\boldsymbol{\xi})=\frac{-i}{(2\pi)^3}\int\frac{k_i\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}^3\boldsymbol{k}}{k^2}. \tag{75}$$

This integral is evaluated in appendix A.1 and it is shown that

$$\bar{P}_{\alpha,i}(\boldsymbol{\xi})=-\frac{1}{4\pi}\partial_i'\left(\frac{1}{\xi}\right)=\frac{\xi_i}{4\pi\xi^3}\quad,\quad\xi=\xi_i\xi_i. \tag{76}$$

We also need to find an equivalent expression for $\bar{u}_{\alpha,ij}$. To do so, substitute equation 74 into equation 72 and rearrange;

$$\tilde{u}_{\alpha,ij}(\boldsymbol{k})=\frac{k^2\delta_{ij}-k_ik_j}{(2\pi)^{3/2}k^4\Lambda_\alpha}. \tag{77}$$

Combining this with equation 71 results in an expression for the Greens function for velocity;

$$\bar{u}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^3\Lambda_\alpha}\left(\delta_{ij}\int\frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}^3\boldsymbol{k}}{k^2} - \int\frac{k_ik_j\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}^3\boldsymbol{k}}{k^4}\right). \tag{78}$$

These integrals are evaluated in appendix A.2 (equations 96 and 102) and following some manipulation we find

$$\bar{u}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{1}{8\pi\Lambda_\alpha\xi}\left(\delta_{ij} + \frac{\xi_i\xi_j}{\xi^2}\right). \tag{79}$$

We can now substitute equations 76 and 79 into 66 and 67 to obtain

$$\hat{P}_\alpha(\boldsymbol{\xi}) = \frac{\mathcal{F}_i\xi_i}{4\pi\xi^3}, \tag{80}$$

and

$$\hat{u}_{\alpha,j}(\boldsymbol{\xi}) = \frac{\mathcal{F}_i}{8\pi\Lambda_\alpha\xi}\left(\delta_{ij} + \frac{\xi_i\xi_j}{\xi^2}\right). \tag{81}$$

Substitution of equations 80 and 81 into equation 64 results in

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{-3\mathcal{F}_k\xi_i\xi_j\xi_k}{4\pi\xi^5} \tag{82}$$

The kernels $\boldsymbol{J}$ and $\boldsymbol{K}$ are defined as

$$J_{ij} = \frac{1}{8\pi\xi}\left(\delta_{ij} + \frac{\xi_i\xi_j}{\xi^2}\right) \tag{83}$$

$$K_{ijk} = \frac{-3\xi_i\xi_j\xi_k}{4\pi\xi^5} \tag{84}$$

Note that under the interchange $\boldsymbol{\xi} \to -\boldsymbol{\xi}$ the kernels are symmetric and antisymmetric respectively

$$J_{ki}(-\boldsymbol{\xi}) = J_{ki}(\boldsymbol{\xi}) \tag{85}$$

$$K_{jik}(-\boldsymbol{\xi}) = -K_{jik}(\boldsymbol{\xi}) \tag{86}$$

This means we can express the velocity and stress Greens functions as

$$\hat{u}_{\alpha,j}(\boldsymbol{\xi}) = \frac{\mathcal{F}_i J_{ij}(\boldsymbol{\xi})}{\Lambda_\alpha} \tag{87}$$

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = K_{ijk}(\boldsymbol{\xi})\mathcal{F}_k \tag{88}$$

A.1 Integral for Greens Function for Pressure

To evaluate the integral in equation 75 first define a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$. The consider

$$\int_{\mathcal{V}} \nabla^2 \left(\frac{1}{\xi}\right) \mathrm{d}^3\boldsymbol{\xi} = \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \left(\frac{1}{\xi}\right) \mathrm{d}^3\boldsymbol{\xi}. \tag{89}$$

Using the divergence theorem (appendix C) this can be written as

$$\int_{\mathcal{V}} \nabla^2 \left(\frac{1}{\xi}\right) \mathrm{d}^3\boldsymbol{r} = \int_{\mathcal{S}} \boldsymbol{\nabla} \left(\frac{1}{\xi}\right) \cdot \mathrm{d}\mathcal{S}. \tag{90}$$

This can be evaluated for the case that $\mathcal{V}$ is a sphere of radius R;

$$\int_{\mathcal{V}} \nabla^2 \left(\frac{1}{\xi}\right) \mathrm{d}^3\boldsymbol{\xi} = -4\pi = -4\pi \int_{\mathcal{V}} \delta(\boldsymbol{\xi})\mathrm{d}^3\boldsymbol{\xi}, \tag{91}$$

where the final step follows from comparison with equation 104. Using equation 106 leads to

$$\nabla^2 \left(\frac{1}{\xi}\right) = \frac{-4\pi}{(2\pi)^3} \int \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}. \tag{92}$$

Inspection of this then suggests

$$\partial_i \left(\frac{1}{\xi}\right) = \frac{4i\pi}{(2\pi)^3} \int \frac{k_i \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2}. \tag{93}$$

Hence

$$\frac{-i}{(2\pi)^3} \int \frac{k_i \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2} = -\frac{1}{4\pi} \partial_i \left(\frac{1}{\xi}\right). \tag{94}$$

A.2 Integrals for the Greens Function for Velocity

Equation 78 contains two integrals that need to be evaluated. By inspecting equation 92 it can be seen

$$\frac{1}{\xi} = \frac{4\pi}{(2\pi)^3} \int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2}. \tag{95}$$

Hence

$$\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2} = \frac{(2\pi)^3}{4\pi\xi}. \tag{96}$$

The second integral requires a bit more work. Firstly, express it in a different form;

$$\int \frac{k_i k_j \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^4} = \partial_i \partial_j \left(\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^4} \right). \tag{97}$$

To evalaute this, first consider $\nabla^4\xi = \nabla^2(\nabla^2\xi)$. Expanding ∇^2 in spherical polar coordinates centred on $\xi = 0$ shows

$$\nabla^4\xi = 2\nabla^2\left(\frac{1}{\xi}\right). \tag{98}$$

Combining this with equation 92 we obtain

$$\nabla^4\xi = \frac{-8\pi}{(2\pi)^3}\int \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}^3\boldsymbol{k}. \tag{99}$$

Insepection of this yields

$$\xi = \frac{-8\pi}{(2\pi)^3}\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}^3\boldsymbol{k}}{k^4}. \tag{100}$$

Rearrangement of this produces an expression for the integral on the right hand side of equation 97;

$$\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}^3\boldsymbol{k}}{k^4} = -\frac{(2\pi)^3\xi}{8\pi}. \tag{101}$$

Hence

$$\int \frac{k_i k_j \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}^3\boldsymbol{k}}{k^4} = \frac{(2\pi)^3\partial_i'\partial_j'\xi}{8\pi}. \tag{102}$$

B Dirac Delta Function

In a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$, the Dirac delta function $\delta(\boldsymbol{x}-\boldsymbol{y})$ is defined as (Riley et al., 2006)

$$\int_{\mathcal{V}} f(\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}^3\boldsymbol{y} = \begin{cases} f(\boldsymbol{x}) & \boldsymbol{x} \in \mathcal{V} \\ \frac{f(\boldsymbol{x})}{2} & \boldsymbol{x} \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases} . \tag{103}$$

The result for $\boldsymbol{x} \in \mathcal{S}$ is only valied for the case that the surface is Lyapunov smooth (a local tangent plane exists everywhere) (REFERENCE FOR LYAPUNOV SMOOTH SURFACE). Equation 103 means that

$$\int_{\mathcal{V}} \delta(\boldsymbol{x}-\boldsymbol{y})\mathrm{d}^3\boldsymbol{y} = 1 \quad \boldsymbol{x} \in \mathcal{V}. \tag{104}$$

A key property of the delta function is that it is symmetric under a change of sign of the argument

$$\delta(-\boldsymbol{x}) = \delta(\boldsymbol{x}) \tag{105}$$

It also needs to be noted that the Dirac delta function can be expressed as (Riley et al., 2006)

$$\delta(\boldsymbol{\xi}) = \frac{1}{(2\pi)^3}\int \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}}\mathrm{d}^3\boldsymbol{k} \tag{106}$$

C Divergence Theorem

The divergence theorem states that for a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$, then for a continuous and differentiable vector field $\boldsymbol{a}$ Riley et al. (2006)

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{a} \mathrm{d}\mathcal{V} = \oint_{\mathcal{S}} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} \tag{107}$$

D Lorentz Reciprocal Theorem

Here we present a proof of the Lorentz reciprocal theorem following Kim and Karrila (2005). The dimensionless strain field in fluid alpha is defined as

$$e'_{\alpha,ij}(\boldsymbol{\xi}) = \frac{\partial'_i u'_{\alpha,j}(\boldsymbol{\xi}) + \partial'_j u'_{\alpha,i}(\boldsymbol{\xi})}{2}, \tag{108}$$

and the corresponding Greens function as

$$\hat{e}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{\partial'_i \hat{u}_{\alpha,j}(\boldsymbol{\xi}) + \partial'_j \hat{u}_{\alpha,i}(\boldsymbol{\xi})}{2}. \tag{109}$$

This allows the Greens function for the stress tensor to be expressed as

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = \hat{P}_{\alpha}(\boldsymbol{\xi}) + 2\hat{e}_{\alpha.ij}(\boldsymbol{\xi}). \tag{110}$$

Now consider the double contraction of the dimensionless stress tensor and strain Green function tensor;

$$T'_{\alpha,ij}(\boldsymbol{x})\hat{e}_{\alpha,ij}(\boldsymbol{\xi}) = -P'_{\mathrm{d},\alpha}(\boldsymbol{x})\hat{e}_{\alpha,ii}(\boldsymbol{\xi}) + 2e'_{\alpha,ij}(\boldsymbol{x})\hat{e}_{\alpha,ij}(\boldsymbol{\xi}) = 2e'_{\alpha,ij}(\boldsymbol{x})\hat{e}_{\alpha,ij}(\boldsymbol{\xi}). \tag{111}$$

The last step here follows by using equations 109 and 28 to show that the first term on the right hand side is equal to zero. In an analagous fashion it can be shown that

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi})e'_{\alpha,ij}(\boldsymbol{x}) = 2\hat{e}_{\alpha,ij}(\boldsymbol{\xi})e'_{\alpha,ij}(\boldsymbol{x}). \tag{112}$$

Since the right hand sides of equations 111 and 112 are identical this means

$$T'_{\alpha,ij}(\boldsymbol{x})\hat{e}_{\alpha,ij}(\boldsymbol{\xi}) = \hat{T}_{\alpha,ij}(\boldsymbol{\xi})e'_{\alpha,ij}(\boldsymbol{x}). \tag{113}$$

Expanding the strain tensors in terms of the velocity gradients and integrating over a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$ with respect to $\boldsymbol{x}$ leads to

$$\fint_{\mathcal{V}} \partial'_j[T'_{ij}(\boldsymbol{x'})\hat{u}_i(\boldsymbol{\xi})]\mathrm{d}\boldsymbol{x'}^3 - \fint_{\mathcal{V}}[\partial'_j T'_{ij}(\boldsymbol{x'})]\hat{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x'}^3 = \fint_{\mathcal{V}} \partial'_j[\hat{T}_{ij}(\boldsymbol{\xi})u'_i(\boldsymbol{x'})]\mathrm{d}\boldsymbol{x'}^3 - \fint_{\mathcal{V}}[\partial'_j\hat{T}_{ij}(\boldsymbol{\xi})]u'_i(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^3. \tag{114}$$

Here the integration has been defined in the sense of the Cauchy Principle Value (CPV) (Riley et al., 2006) to allow for the singular point in the Greens functions at $\boldsymbol{\xi} = \mathbf{0}$. Defining $\boldsymbol{n}$ as the normal to the surface $\mathcal{S}$, the divergence theorem (appendix C) can then to applied to the first term on both sides;

$$\fint_{\mathcal{S}} n_j(\boldsymbol{x'})T'_{ij}(\boldsymbol{x'})\hat{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x'}^2 - \fint_{\mathcal{V}}[\partial'_j T'_{ij}(\boldsymbol{x'})]\hat{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x'}^3 = \fint_{\mathcal{S}} n_j(\boldsymbol{x'})\hat{T}_{ij}(\boldsymbol{\xi})u'_i(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^2 - \fint_{\mathcal{V}}[\partial'_j\hat{T}_{ij}(\boldsymbol{\xi})]u'_i(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^3. \tag{115}$$

This is the Lorentz reciprocal theorem with the additional caveat that the integrals have been defined as CPV integrals so that the singular nature of the Greens functions can be dealt with.

E Elliptic Integrals

The complete elliptic integrals of the first and second kind are defined as (Abramowitz and Stegun, 1972)

$$K(k^2) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1 - k^2 \sin^2\theta)^{1/2}}, \quad 0 \leq k^2 < 1, \tag{116}$$

and

$$E(k^2) = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{1/2} \mathrm{d}\theta, \quad 0 \leq k^2 < 1. \tag{117}$$

k^2 is defined as the modulus of the integral. Polynomial approximations can be found to evaluate the integrals (Roumeliotis, 2000) and we use the following expressions from Abramowitz and Stegun (1972):

$$K(k^2) = \sum_{i=0}^{4} a_i (1 - k^2)^i + \ln\left(\frac{1}{1 - k^2}\right) \sum_{i=0}^{4} b_i (1 - k^2)^i, \tag{118}$$

$$E(k^2) = 1 + \sum_{i=1}^{4} a'_i (1 - k^2)^i + \ln\left(\frac{1}{1 - k^2}\right) \sum_{i=1}^{4} b'_i (1 - k^2)^i \tag{119}$$

The values of the coefficients in the expansion are in table 1.

F Components of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$

Here we present expressions for the components of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ in terms of complete elliptic integrals of the first and second kind (appendix E). The expressions for $\boldsymbol{A}$ anf $\boldsymbol{B}$ are from Graziani (1989). As far as the authors are aware equivalent expressions for $\boldsymbol{C}$ have never been published before although, they were undoubtedly used in the models of

Table 1: The coefficients for equations 118 and 119.

a_0	1.38629436112	b_0	0.5
a_1	0.09666344259	b_1	0.12498593597
a_2	0.03590092383	b_2	0.06880248576
a_3	0.03742563713	b_3	0.03328355346
a_4	0.01451196212	b_4	0.00441787012
a'_1	0.44325141463	b'_1	0.24998368310
a'_2	0.06260601220	b'_2	0.09200180037
a'_3	0.04757383546	b'_3	0.04069697526
a'_4	0.01736506451	b'_4	0.00526449639

Lee and Leal (1982); Geller et al. (1985) and Manga and Stone (1995). The quantities α and β are defined as (Manga, 1994)

$$\alpha^2 = x_r^2 + y_r^2 + (x_z - y_z)^2, \tag{120}$$

and

$$\beta^2 = 2x_r y_r. \tag{121}$$

K and E are complete elliptic integrals of the first and second kind respectively and they all take $k^2 = 2\beta^2/(\alpha^2 + \beta^2)$ as their modulus.

The components of $\boldsymbol{A}$ are:

$$A_{11} = (a_1 n_r + a_2 n_z)K + (a_3 n_r + a_4 n_z)E, \tag{122}$$

$$A_{12} = (a_2 n_r + a_6 n_z)K + (a_4 n_r + a_8 n_z)E, \tag{123}$$

$$A_{21} = (a_9 n_r + a_{10} n_z)K + (a_{11} n_r + a_{12} n_z)E, \tag{124}$$

and

$$A_{22} = (a_{10} n_r + a_{14} n_z)K + (a_{12} n_r + a_{16} n_z)E. \tag{125}$$

The coefficients a_i are given as

$$a_1 = \frac{(1-\lambda)[x_r\alpha_2(4\alpha^4 - 18x_r^2y_r^2) - x_r(2y_r^2 + x_r^2)(2\alpha^4 - 3\beta^4) - y_r\alpha^2\beta^2(y_r^2 + 2x_r^2) + x_r y_r^2\beta^4]}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)\beta^4}, \tag{126}$$

$$a_2 = \frac{(1-\lambda)(x_z - y_z)[2\alpha^4 - 2\beta^4 - \alpha^2(x_z - y_z)^2]}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)\beta^2}, \tag{127}$$

$$a_3 = \frac{1-\lambda}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)^2\beta^4}\left(\frac{x_r(-8\alpha^8 + 15\alpha^4\beta^4 - 3\beta^8)}{2}\right.$$
$$\left. -2x_r\alpha^2(2y_r^2 + x_r^2)(-\alpha^4 + 3\beta^4) + y_r\beta^2(y_r^2 + 2x_r^2)(\alpha^4 + 3\beta^4) - 4x_r y_r^2\alpha^2\beta^4\right), \tag{128}$$

$$a_4 = \frac{-(1-\lambda)(x_z - y_z)}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)^2\beta^2}\left(\alpha^4(\alpha^4 - 5\beta^4) + [\alpha^2 - (x_z - y_z)^2](\alpha^4 + 3\beta^4)\right), \tag{129}$$

$$a_6 = \frac{(1-\lambda)(x_z - y_z)^2(2x_r^2 - \alpha^2)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)x_r}, \tag{130}$$

$$a_8 = \frac{(1-\lambda)(x_z - y_z)^2(\alpha^4 + 3\beta^4 - 8x_r^2\alpha^2)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)^2 x_r}, \tag{131}$$

$$a_9 = \frac{(1-\lambda)(x_z - y_z)(-2\alpha^4 + 3\beta^4 - 4y_r^2\alpha^2 + 4y_r^4)}{4\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)y_r}, \tag{132}$$

$$a_{10} = \frac{(1-\lambda)(x_z - y_z)^2(\alpha^2 - 2y_r^2)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)y_r}, \tag{133}$$

$$a_{11} = \frac{(1-\lambda)(x_z - y_z)(\alpha^6 - 3\alpha^2\beta^4 + 2y_r^2\alpha^4 + 6y_r^2\beta^4 - 8y_r^4\alpha^2)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)^2 y_r^2}, \tag{134}$$

$$a_{12} = \frac{(1-\lambda)(x_z - y_z)^2(8y_r^2\alpha^2 - \alpha^4 - 3\beta^4)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)y_r}, \tag{135}$$

$$a_{14} = \frac{(1-\lambda)(x_z - y_z)^3}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)}, \tag{136}$$

and

$$a_{16} = \frac{-4(1-\lambda)(x_z - y_z)^3\alpha^2}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2+\beta^2)^2}. \tag{137}$$

The components of $\boldsymbol{B}$ are:

$$B_{11} = \frac{1}{2\pi\beta^2(\alpha^2+\beta^2)^{1/2}}\left[[\alpha^2 + (x_z - y_z)^2]K - \left(\alpha^2 + \beta^2 + \frac{\alpha^2(x_z-y_z)^2}{\alpha^2-\beta^2}\right)E\right], \tag{138}$$

$$B_{12} = \frac{x_z - y_z}{4\pi x_r(\alpha^2+\beta^2)^{1/2}}\left(\frac{(2x_r^2-\alpha^2)E}{\alpha^2-\beta^2} + K\right), \tag{139}$$

$$B_{21} = \frac{x_z - y_z}{4\pi y_r(\alpha^2+\beta^2)^{1/2}}\left(\frac{(\alpha^2-2y_r^2)E}{\alpha^2-\beta^2} - K\right), \tag{140}$$

and

$$B_{22} = \frac{1}{2\pi(\alpha^2+\beta^2)^{1/2}}\left(K + \frac{(x_z-y_z)^2E}{\alpha^2-\beta^2}\right). \tag{141}$$

The components of $\boldsymbol{C}$ are:

$$C_1 = \frac{9(\partial'_j n_j - y_z \mathrm{Bo})}{4\pi D\mathrm{Bo}(\alpha^2+\beta^2)^{1/2}} \Bigg[\Bigg([\alpha^2 + (x_z - y_z)^2] n_r + y_r(x_z - y_z) \Bigg) K$$
$$+ \frac{E}{\alpha^2 - \beta^2} \Bigg(n_r[\beta^4 - \alpha^2(\alpha^2 + (x_z - y_z)^2)] + n_z(x_z - y_z)(x_r\beta^2 - y_r\alpha^2) \Bigg) \Bigg], \tag{142}$$

and

$$C_2 = \frac{9(\partial'_j n_j - y_z \mathrm{Bo})}{4\pi D\mathrm{Bo}(\alpha^2+\beta^2)^{1/2}} \Bigg([\beta^2 n_z - x_r(x_z - y_z) n_r] K$$
$$+ \frac{[n_r(x_r\alpha^2 - y_r\beta^2) + (x_z - y_z)\beta^2 n_z](x_z - y_z)E}{\alpha^2 - \beta^2} \Bigg). \tag{143}$$

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