Low Reynolds number gravitational settling of a sphere through a fluid-fluid interface: Modelling using a boundary integral method

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Abstract

1 Introduction

2 Theoretical Background

We present a theoretical construction of the boundary integral equations. Symbols used are defined in table 1.

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Table 1: Definition of symbols.

Symbol	Definition
a	Sphere radius
e	Exponential constant
$\hat{e}_{\alpha,ij}$	Greens function for strain in fluid α
$f_{\mathrm{s},i}(\boldsymbol{x}) = m_j(\boldsymbol{x})T_{1,ij}(\boldsymbol{x})$	Traction on surface of sphere
$f_{\alpha,i}(\boldsymbol{x}) = n_j(\boldsymbol{x})T_{\alpha,ij}(\boldsymbol{x})$	Traction of fluid α on interface
$\mathcal{F}_i$	Arbitrary constant vector
$\boldsymbol{F}$	Force vector
$\boldsymbol{g} = (-9.81\mathrm{m\ s}^{-2})\hat{\boldsymbol{z}}$	Acceleration due to gravity
$\mathcal{I}$	Surface of interface
$i = \sqrt{-1}$	The unit imaginary number
J_{ij}	Components of tensor kernel for velocity Greens function
K_{ijk}	Components of tensor kernel for stress Greens function
$\boldsymbol{k}$	Dummy variable in Fourier space
$k = \|\boldsymbol{k}\|$	Magnitude of $\boldsymbol{k}$
m	Outward normal to sphere surface
n	Normal to interface (points into fluid 1)
$P_\alpha(\boldsymbol{x})$	Pressure field of fluid α
$P_{\mathrm{d},\alpha}(\boldsymbol{x})$	Dynamic pressure of fluid α
$\hat{P}_\alpha$	Greens function for dynamic pressure of fluid α
$\bar{P}_{\alpha,i}(\boldsymbol{\xi})$	Components of the vector of solutions for $\hat{P}_\alpha$ in equation 68
$\tilde{P}_{\alpha,i}(\boldsymbol{k})$	Fourier transform of $\bar{P}_{\alpha,i}(\boldsymbol{\xi})$
s	Arc length along interface measured from axis
$\mathcal{S}$	Surface of sphere
$T_{\alpha,ij}(\boldsymbol{x})$	Stress tensor field of fluid α
$\hat{T}_{\alpha,ij}(\boldsymbol{x'} - \boldsymbol{y'})$	Stress Greens function for fluid α
t	Time
$\boldsymbol{u}_\alpha(\boldsymbol{x})$	Velocity field of fluid α
$\boldsymbol{u}_{\mathrm{s}} = u_{\mathrm{s}}\hat{\boldsymbol{z}}$	Velocity of sphere

$\hat{u}_{\alpha,i}(\boldsymbol{x'}-\boldsymbol{y'})$	Velocity Greens function for fluid α
$\bar{u}_{\alpha,i,j}(\boldsymbol{\xi})$	Components of the tensor of solutions for $\hat{u}_{\alpha,i}$ in equation 68
$\tilde{u}_{\alpha,ij}(\boldsymbol{k})$	Fourier transform of $\bar{u}_{\alpha,i,j}(\boldsymbol{\xi})$
$\mathcal{V}_\alpha$	Volume of fluid α
$\boldsymbol{x}$	Position vector
$\boldsymbol{y}$	Position vector
$\hat{\boldsymbol{z}}$	Unit vector in the upward vertical direction
$\alpha = 1, 2$	Fluid label
$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$	Kronecker delta
$\delta(\boldsymbol{x'}-\boldsymbol{y'}) = \begin{cases} 1 & \text{if } \boldsymbol{x} = \boldsymbol{y} \\ 0 & \text{if } \boldsymbol{x} \neq \boldsymbol{y} \end{cases}$	Dirac delta function
η_α	Viscosity of fluid α
θ	Polar angle with respect to sphere centre
ρ_α	Density of fluid α
ρ_s	Sphere density
σ	Interfacial Tension
ϕ	Azimuhtal angle with respect to axis of motion
$\boldsymbol{\xi} = \boldsymbol{x'} - \boldsymbol{y'}$	Dimensionless separation vector
$\xi = \lvert\boldsymbol{\xi}\rvert$	Magnitude of $\boldsymbol{\xi}$

2.1 Equations of Motion

The starting point for all fluid dynamical problems are the continuity (equation 1) and Navier Stokes (equation 2) equations (Batchelor, 1967):

$$\frac{\partial \rho_\alpha(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{\nabla} \cdot [\rho_\alpha(\boldsymbol{x}, t)\boldsymbol{u}_\alpha(\boldsymbol{x}, t)] = 0, \tag{1}$$

$$\begin{aligned}&\rho_\alpha(\boldsymbol{x},t)\left(\tfrac{\partial \boldsymbol{u}_\alpha(\boldsymbol{x},t)}{\partial t}+(\boldsymbol{u}_\alpha(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\boldsymbol{u}_\alpha(\boldsymbol{x},t)\right)=\\&-\boldsymbol{\nabla}P_\alpha(\boldsymbol{x},t)-\rho_\alpha(\boldsymbol{x},t)g+\eta_\alpha\left[\nabla^2\boldsymbol{u}_\alpha(\boldsymbol{x},t)+\tfrac{\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha(\boldsymbol{x},t))}{3}\right].\end{aligned} \tag{2}$$

Forming a coupled set of non-linear, partial differential equations for the velocity and pressure fields, these represent mass and momentum conservation respectively and must be satisfied by all fluid phases within the system. For most practical applications, the fluids are assumed to be incompressible (have constant density) and so the continuity equation reduces to the incompressibility relation

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha(\boldsymbol{x},t)=0. \tag{3}$$

This can be combined with equation 2 to form the incompressible Navier Stokes equation

$$\rho_\alpha\left(\frac{\partial \boldsymbol{u}_\alpha(\boldsymbol{x},t)}{\partial t}+(\boldsymbol{u}_\alpha(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\boldsymbol{u}_\alpha(\boldsymbol{x},t)\right)=-\boldsymbol{\nabla}P_\alpha(\boldsymbol{x},t)-\rho_\alpha g+\eta_\alpha\nabla^2\boldsymbol{u}_\alpha(\boldsymbol{x},t). \tag{4}$$

The equations of motion can be expressed in an alternative form by defining the stress tensor $\boldsymbol{T}_\alpha(\boldsymbol{x},t)$ (Manga, 1994) and dynamic pressure $P_{\mathrm{d},\alpha}(\boldsymbol{x},t)$:

$$\boldsymbol{T}_\alpha(\boldsymbol{x},t)=-P_{\mathrm{d},\alpha}(\boldsymbol{x},t)\boldsymbol{I}+\eta_\alpha[\boldsymbol{\nabla}\boldsymbol{u}_\alpha(\boldsymbol{x},t)+(\boldsymbol{\nabla}\boldsymbol{u}_\alpha(\boldsymbol{x},t))^T], \tag{5}$$

$$P_{\mathrm{d},\alpha}(\boldsymbol{x},t)=P_\alpha(\boldsymbol{x},t)-\rho_\alpha\boldsymbol{g}\cdot\boldsymbol{x}. \tag{6}$$

This definition of the stress tensor removes the gravitational body force from the equations of motion, meaning that it only appears in the boundary conditions. The Navier Stokes equation then becomes

$$\rho_\alpha\left(\frac{\partial \boldsymbol{u}_\alpha(\boldsymbol{x},t)}{\partial t}+(\boldsymbol{u}_\alpha(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\boldsymbol{u}_\alpha(\boldsymbol{x},t)\right)=\boldsymbol{\nabla}\cdot\boldsymbol{T}_\alpha(\boldsymbol{x},t). \tag{7}$$

When working in fluid dynamics, it is usual to non-dimensionalise the equations of motion and boundary conditions (White, 1999). This can be achieved by scaling the quantities involved by parameters specific to the problem. For example, consider a problem with typical scales of length L_{c} and velocity U_{c}. This allows us to define dimensionless variables (denoted by a $'$)

$$\boldsymbol{x} = L_{\mathrm{c}}\boldsymbol{x'}, \tag{8}$$

$$\boldsymbol{u}_{\alpha}(\boldsymbol{x}, t) = U_{\mathrm{c}}\boldsymbol{u'}_{\alpha}(\boldsymbol{x'}, t'), \tag{9}$$

and

$$t = \frac{L_{\mathrm{c}}t'}{U_{\mathrm{c}}} \tag{10}$$

In the case of highly viscous flows, the pressure is dominated by viscosity and so the relevant scaling for the dynamic pressure uses a characteristic viscosity η_{c} and is given byLee and Leal (1982)

$$P_{\mathrm{d},\alpha}(\boldsymbol{x}, t) = \frac{\eta_{\mathrm{c}}U_{\mathrm{c}}P'_{\mathrm{d},\alpha}(\boldsymbol{x'}, t')}{L_{\mathrm{c}}}. \tag{11}$$

This choice of pressure scaling means that upon substitution of equations 8 to 11 into equation 5 the stress tensor can also be non-dimensionalised,

$$\boldsymbol{T}_{\alpha}(\boldsymbol{x}, t) = \frac{\eta_{\mathrm{c}}U_{\mathrm{c}}\boldsymbol{T'}_{\alpha}(\boldsymbol{x'}, t')}{L_{\mathrm{c}}} \quad \text{where} \quad \boldsymbol{T'}_{\alpha}(\boldsymbol{x'}, t') = p'_{\mathrm{d},\alpha}(\boldsymbol{x'}, t')\boldsymbol{I} + \frac{\eta_{\alpha}[\boldsymbol{\nabla}'\boldsymbol{u'}_{\alpha}(\boldsymbol{x'}, t') + (\boldsymbol{\nabla}'\boldsymbol{u'}_{\alpha}(\boldsymbol{x'}, t'))^{T}]}{\eta_{c}}. \tag{12}$$

In this case, the continuity and Navier Stokes equations become

$$\boldsymbol{\nabla}' \cdot \boldsymbol{u'}_\alpha(\boldsymbol{x'}, t') = 0 \tag{13}$$

and

$$\mathrm{Re}\left(\frac{\partial \boldsymbol{u'}_\alpha(\boldsymbol{x'}, t')}{\partial t'} + (\boldsymbol{u'}_\alpha(\boldsymbol{x'}, t') \cdot \boldsymbol{\nabla}')\boldsymbol{u'}_\alpha(\boldsymbol{x'}, t')\right) = \boldsymbol{\nabla}' \cdot \boldsymbol{T'}_\alpha(\boldsymbol{x'}, t'), \tag{14}$$

where the Reynolds number is defined as

$$\mathrm{Re}_\alpha = \frac{\rho_\alpha L_\mathrm{c} U_\mathrm{c}}{\eta_\mathrm{c}} \tag{15}$$

As we are considering the case of low Reynolds number ($\mathrm{Re}_\alpha \ll 1$), we can neglect the inertial terms on the right hand side and the equation reduces to the Stokes equation Kim and Karrila (2005)

$$\boldsymbol{\nabla}' \cdot \boldsymbol{T'}_\alpha(\boldsymbol{x'}, t') = \mathbf{0}. \tag{16}$$

2.2 Boundary Conditions

In order to complete the formulation of any fluid dynamics problem, it is necessary to state the boundary conditions alongside the equations of motion (Riley et al., 2006). For fluids of infinite (or semi-infinite) extent in some dimension, these include the value of the flow velocity at infinity. For bounded flows, the conditions are imposed at the boundaries of the fluid domain, and their exact nature depends on the phase of the material bounding it.

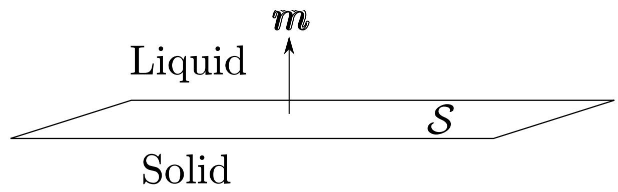

Figure 1: Definition of $\boldsymbol{m}$ and $\mathcal{S}$ for a fluid-solid boundary.

2.2.1 Fluid-Solid Boundary

At low Reynolds number for a fluid-solid boundary given by the surface $\mathcal{S}$ (see figure 1), the kinematic boundary condition (that on the velocity) states that both the normal and tangential components (with respect to the boundary) of velocity are the same as that of the solid at the boundary. This is easily expressed in dimensionless form as

$$\boldsymbol{u'}_\alpha(\boldsymbol{x'}) = \boldsymbol{U}'_{\mathrm{s}} \quad \boldsymbol{x} \in \mathcal{S}. \tag{17}$$

There also needs to be a dynamic boundary condition applied at the interface. If the solid exerts a force $\boldsymbol{F}$ onto the fluid then the condition states

$$\int \boldsymbol{m} \cdot \boldsymbol{T}_\alpha \mathrm{d}\mathcal{S} = \boldsymbol{F}. \tag{18}$$

Using the non-dimensionalisation scheme presented above this becomes

$$\eta_{\mathrm{c}} U_{\mathrm{c}} L_{\mathrm{c}} \int \boldsymbol{m} \cdot \boldsymbol{T'}_\alpha \mathrm{d}\mathcal{S}' = \boldsymbol{F}. \tag{19}$$

2.2.2 Fluid-Fluid Boundary

In the case of a fluid-fluid boundary (see figure 2), both a kinematic and dynamic boundary condition need to be satisfied. The kinematic boundary condition states that the velocity of the two fluids must be continuous across the interface (Manga, 1994). Expressed in

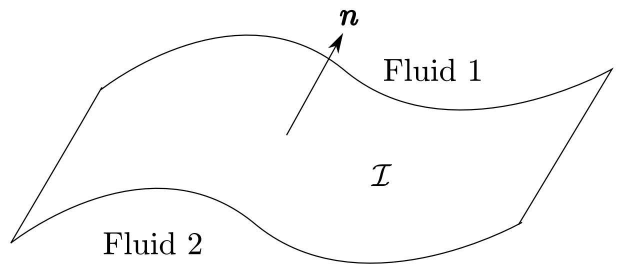

Figure 2: Definition of $\boldsymbol{n}$ and $\mathcal{I}$ for a fluid-fluid boundary.

dimensionless form this looks like

$$\boldsymbol{u'}_1(\boldsymbol{x'}) = \boldsymbol{u'}_2(\boldsymbol{x'}), \quad \boldsymbol{x'} \in \mathcal{I}. \tag{20}$$

The dynamic boundary condition is an expression of the balance between the stress discontinuity across the interface and the interfacial tension (IFT)(Manga, 1994):

$$\boldsymbol{n}\cdot[\boldsymbol{T}_1(\boldsymbol{x})-\rho_1(\boldsymbol{g}\cdot\boldsymbol{x})\boldsymbol{I}]-\boldsymbol{n}\cdot[\boldsymbol{T}_2(\boldsymbol{x})-\rho_2(\boldsymbol{g}\cdot\boldsymbol{x})\boldsymbol{I}] = \sigma(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{n})-\boldsymbol{\nabla}_\mathrm{s}\sigma(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{I}. \tag{21}$$

The operator $\boldsymbol{\nabla}_\mathrm{s}$ is defined as the tangential gradient operator within the surface $\mathcal{I}$:

$$\boldsymbol{\nabla}_\mathrm{s} = (\boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}) \cdot \boldsymbol{\nabla}. \tag{22}$$

When this takes the normal vector as its argument it can be shown that (Brackbill et al., 1992)

$$\boldsymbol{\nabla}_\mathrm{s} \cdot \boldsymbol{n} = \boldsymbol{\nabla} \cdot \boldsymbol{n}. \tag{23}$$

The presence of gradients in the interfacial tension can lead to so-called Marangoni effects (Kim and Karrila, 2005). However, for present purposes we will assume that the interfacial tension is uniform across the interface S and so the last term on the right hand side equals zero;

$$\boldsymbol{n} \cdot [\boldsymbol{T}_1(\boldsymbol{x}) - \rho_1(\boldsymbol{g} \cdot \boldsymbol{x})\boldsymbol{I}] - \boldsymbol{n} \cdot [\boldsymbol{T}_2(\boldsymbol{x}) - \rho_2(\boldsymbol{g} \cdot \boldsymbol{x})\boldsymbol{I}] = \sigma(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{\nabla} \cdot \boldsymbol{n}), \quad \boldsymbol{x} \in \mathcal{S}. \tag{24}$$

Like the equations of motion, this can be non-dimensionalised using equations 8 to 12:

$$\mathrm{Ca}\ \boldsymbol{n} \cdot (\boldsymbol{T'}_1 - \boldsymbol{T'}_2) + \mathrm{Bo}(\hat{\boldsymbol{z}} \cdot \boldsymbol{x'})\boldsymbol{n} = (\boldsymbol{\nabla'} \cdot \boldsymbol{n})\boldsymbol{n}. \tag{25}$$

The capillary number Ca and Bond number Bo are dimensionless numbers defined as:

$$\mathrm{Ca} = \frac{\eta_\mathrm{c} U_\mathrm{c}}{\sigma} \tag{26}$$

$$\mathrm{Bo} = \frac{(\rho_2 - \rho_1) g L_\mathrm{c}^2}{\sigma} \tag{27}$$

The $\nabla \cdot \mathbf{n}$ factor in equation 21 can be expressed in a more physically meaningful manner by noting that it is related to the mean curvature, K, of the interface (Hobson et al., 2011).

$$2K = -\nabla \cdot \mathbf{n} \tag{28}$$

2.3 Problem Statement

The system is formulated as in figure 3. The physical parameters motivate the choice of scaling variables. The characteristic lengthscale is chosen to be the sphere radius a, characteristic viscosity that of the upper fluid η_1 and characteristic velocity to be the

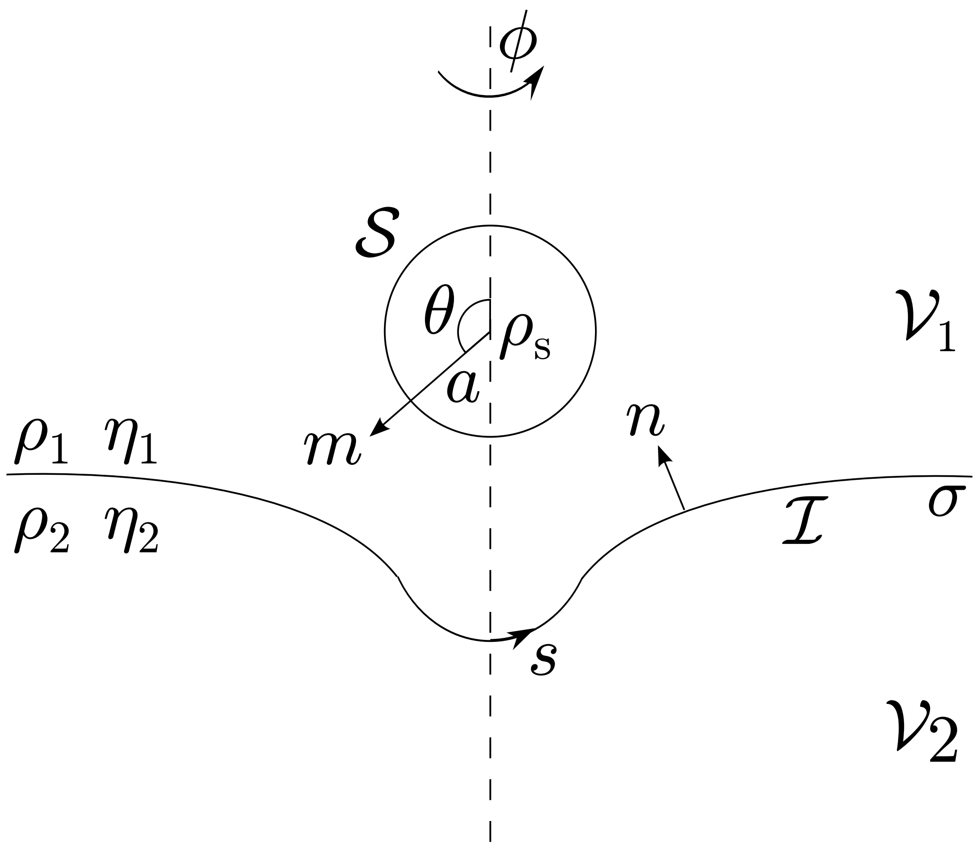

Figure 3: Diagrammatic representation of the system. A sphere falls under gravity, at low Reynolds number, towards an initially horizontal interface between two density stratified, immiscible semi-infinite fluids. See table 1 for definition of symbols.

Stokes velocity (Reynolds, 1886),

$$U_{\mathrm{c}} = \frac{2(\rho_{\mathrm{s}} - \rho_1) g a^2}{9\eta_1}. \tag{29}$$

This means the capillary and Bond numbers can be expressed as:

$$\mathrm{Ca} = \frac{(\rho_{\mathrm{s}} - \rho_1) g a^2}{\sigma}, \tag{30}$$

$$\mathrm{Bo} = \frac{(\rho_2 - \rho_1) g a^2}{\sigma}. \tag{31}$$

The dimensionless stress tensor for each fluid can be written as

$$T'_{\alpha,ij}(\boldsymbol{x'}) = -P'_{\mathrm{d},\alpha}(\boldsymbol{x'})\delta_{ij} + \Lambda_\alpha[\partial'_i u'_{\alpha,j}(\boldsymbol{x'}) - \partial'_j u'_{\alpha,i}(\boldsymbol{x'})]. \tag{32}$$

The parameter Λ_α is defined as

$$\Lambda_\alpha = \frac{\eta_\alpha}{\eta_1} = \begin{cases} 1, & \alpha = 1 \\ \frac{\eta_2}{\eta_1} = \lambda, & \alpha = 2 \end{cases} . \tag{33}$$

Note λ is the viscosity ratio of the two fluids. It is straightforward to apply the general equations of motion and boundary conditions to the problem. The equations of motion, expressed in the Einstein summation convention (Riley et al., 2006) which will be used from now on, appear as

$$\partial'_{\mathrm{i}} u'_{\alpha,i}(\boldsymbol{x'}) = 0, \tag{34}$$

and

$$\partial'_{\mathrm{i}} T'_{\alpha,ij}(\boldsymbol{x'}) = 0. \tag{35}$$

Here, $\alpha = 1, 2$ and denotes the fluid, i, j denote components of tensoral quantities.

The first boundary condition that we impose is that the undisturbed fluid is quiescent;

$$u'_{\alpha,i}(\boldsymbol{x'}) \to 0 \text{ as } |\boldsymbol{x'}| \to \infty. \tag{36}$$

The kinematic boundary condition on the fluid interface (equation 20) can be expressed as

$$u'_{1,i}(\boldsymbol{x'}) = u'_{2,i}(\boldsymbol{x'}), \quad \boldsymbol{x'} \in S_{\mathrm{int}}. \tag{37}$$

The dynamic boundary condition is also imposed at the interface;

$$\text{Ca}\ n_i[T'_{1,ij}(\boldsymbol{x'}) - T'_{2,ij}(\boldsymbol{x'})] + \text{Bo}\hat{z}_i x'_i n_j = n_j \partial'_i n_i. \tag{38}$$

The kinematic boundary condition on the sphere surface is one of no-slip meaning the fluid velocity at the surface has to equal the sphere velocity ;

$$u'_{1,i}(\boldsymbol{x'}) = u'_{\text{s},i}, \quad \boldsymbol{x'} \in \mathcal{S}. \tag{39}$$

The final boundary condition is the dynamic boundary condition on the sphere. The force on the fluid due to the sphere originates from the balance between gravity and buoyancy;

$$F_i = \frac{4\pi a^3(\rho_\text{s} - \rho_1) g \hat{z}_i}{3}. \tag{40}$$

Substituting this into equation 19 and using equation 29 we obtain

$$\int_\mathcal{S} n_i T'_{1,ij}(\boldsymbol{x'}) \mathrm{d}\mathcal{S}' = 6\pi \hat{z}_j \tag{41}$$

The dimensionless numbers that describe the system are the set $\{\lambda, \text{Ca}, \text{Bo}\}$. However, an equivalent set can be generated by defining the dimensionless density ratio D;

$$D = \frac{\text{Ca}}{\text{Bo}} = \frac{\rho_\text{s} - \rho_1}{\rho_2 - \rho_1}. \tag{42}$$

Therefore, we can also describe the system with the set $\{\lambda, D, \text{Bo}\}$. This allows us to re-express equation 38 as

$$D\text{Bo}\ n_i[T'_{1,ij}(\boldsymbol{x'}) - T'_{2,ij}(\boldsymbol{x'})] = n_j(\partial'_i n_i - \text{Bo}\hat{z}_i x'_i) \tag{43}$$

To summarise, the problem is completely described by equations 34 to 37, and equations 39, 41 and 43.

2.4 Greens functions

In order to derive the boundary integral equations, it is necessary to make use of the Greens functions (Riley et al., 2006) for Stokes flow, $\hat{u}_{\alpha,i}(\boldsymbol{x'}-\boldsymbol{y'})$ and $\hat{T}_{\alpha,ij}(\boldsymbol{x'}-\boldsymbol{y'})$, defined such that

$$\partial_i' \hat{u}_{\alpha,i}(\boldsymbol{x'}-\boldsymbol{y'}) = 0 \tag{44}$$

and

$$\partial_i' \hat{T}_{\alpha,ij}(\boldsymbol{x'}-\boldsymbol{y'}) + \mathcal{F}_j \delta(\boldsymbol{x'}-\boldsymbol{y'}) = 0, \tag{45}$$

where $\boldsymbol{\mathcal{F}}$ is a arbitrary constant vector. Equations 44 and 45 can be solved following Ladyzhenskaya (1963) using Fourier transforms (appendix A) to show that

$$\hat{u}_{\alpha,j}(\boldsymbol{\xi}) = \frac{\mathcal{F}_i J_{ij}(\boldsymbol{\xi})}{\Lambda_\alpha}, \tag{46}$$

and

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = K_{ijk}(\boldsymbol{\xi})\mathcal{F}_k, \tag{47}$$

where $\boldsymbol{\xi} = \boldsymbol{x'} - \boldsymbol{y'}$ and

$$J_{ij}(\boldsymbol{\xi}) = \frac{1}{8\pi|\boldsymbol{\xi}|}\left(\delta_{ij} + \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2}\right), \tag{48}$$

and

$$K_{ijk}(\boldsymbol{\xi}) = \frac{-3\xi_i\xi_j\xi_k}{4\pi|\boldsymbol{\xi}|^5} \tag{49}$$

2.5 Integral Representation of Stokes Equations

We now substitute these expressions for the Greens functions into the Lorentz Reciprocal Theorem (equation 118 in appendix D) and simplfify using equations 35 and 45 to find

$$\fint_{\mathcal{V}_\alpha} u'_{\alpha,k}(\boldsymbol{x'})\delta(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x'}^3 = \frac{1}{\Lambda_\alpha}\fint_{\mathcal{S}_\alpha} J_{ik}(\boldsymbol{\xi})T'_{\alpha,ij}(\boldsymbol{x'})n_{\alpha,j}(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^2 - \fint_{\mathcal{S}_\alpha} u'_{\alpha,i}(\boldsymbol{x'})K_{ijk}(\boldsymbol{\xi})n_{\alpha,j}(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^2. \tag{50}$$

Finally make the transformation $\boldsymbol{x'} \leftrightarrow \boldsymbol{y'}$ and use the symmetry properties of the kernels (equations 88 and 89 in appendix A) and the delta function (equation 108 in appendix B) to obtain the general form of the integral representation of the Stokes equations;

$$\fint_{\mathcal{V}_\alpha} u'_{\alpha,k}(\boldsymbol{y'})\delta(\boldsymbol{\xi})\mathrm{d}\boldsymbol{y'}^3 = \frac{1}{\Lambda_\alpha}\fint_{\mathcal{S}_\alpha} J_{ik}(\boldsymbol{\xi})T'_{\alpha,ij}(\boldsymbol{y'})n_{\alpha,j}(\boldsymbol{y'})\mathrm{d}\boldsymbol{y'}^2 + \fint_{\mathcal{S}_\alpha} u'_{\alpha,i}(\boldsymbol{y'})K_{ijk}(\boldsymbol{\xi})n_{\alpha,j}(\boldsymbol{y'})\mathrm{d}\boldsymbol{y'}^2. \tag{51}$$

Using the definition of the delta function (equation 106 in appendix B) this means

$$\frac{1}{\Lambda_\alpha}\fint_{\mathcal{S}_\alpha} J_{ik}(\boldsymbol{\xi})T'_{\alpha,ij}(\boldsymbol{y'})n_{\alpha,j}(\boldsymbol{y'})\mathrm{d}\boldsymbol{y'}^2 + \fint_{\mathcal{S}_\alpha} u'_{\alpha,i}(\boldsymbol{y'})K_{ijk}(\boldsymbol{\xi})n_{\alpha,j}(\boldsymbol{y'})\mathrm{d}\boldsymbol{y'}^2 = \begin{cases} u'_{\alpha,k}(\boldsymbol{x'}) & \boldsymbol{x'} \in \mathcal{V}_\alpha \\ \frac{u'_{\alpha,k}(\boldsymbol{x'})}{2} & \boldsymbol{x'} \in \mathcal{S}_\alpha \\ 0 & \text{otherwise} \end{cases}. \tag{52}$$

This equation needs to be considered for each fluid separately. For fluid 1 ($\alpha = 1$), $\mathcal{S}_1 = \mathcal{S} + \mathcal{I}$, $\boldsymbol{n}_1(\boldsymbol{y'}) = \boldsymbol{m}(\boldsymbol{y'})$ for $\boldsymbol{y'} \in \mathcal{S}$ and $\boldsymbol{n}_1(\boldsymbol{y'}) = \boldsymbol{n}(\boldsymbol{y'})$ for $\boldsymbol{y'} \in \mathcal{I}$. Noting that for

$\boldsymbol{y'} \in \mathcal{S}$, $u'_{1,i}(\boldsymbol{y'}) = u_{\mathrm{s},i}$ and that $\partial'_j K_{ijk}(\boldsymbol{\xi}) = -\delta_{ik}\delta(\boldsymbol{\xi})$ (as must follow from equations 45 and 91) the boundary integral equation for fluid 1 can be written as

$$\begin{aligned} &\fint_{\mathcal{S}} J_{ik}(\boldsymbol{\xi}) T'_{1,ij}(\boldsymbol{y'}) m_j(\boldsymbol{y'}) \mathrm{d}^2\boldsymbol{y'} + \fint_{\mathcal{I}} J_{ik}(\boldsymbol{\xi}) T'_{1,ij}(\boldsymbol{y'}) n_j(\boldsymbol{y'}) \mathrm{d}^2\boldsymbol{y'} + \\ &\fint_{\mathcal{I}} u'_{1,i}(\boldsymbol{y'}) K_{ijk}(\boldsymbol{\xi}) n_j(\boldsymbol{y}) \mathrm{d}^2\boldsymbol{y'} = \begin{cases} \frac{u'_{1,k}(\boldsymbol{x'})}{2} & \boldsymbol{x'} \in \mathcal{I} \\ u'_{s,k} & \boldsymbol{x'} \in \mathcal{S} \end{cases} . \end{aligned} \tag{53}$$

For fluid 2, $\mathcal{S}_2 = \mathcal{I}$ and $\boldsymbol{n}_2(\boldsymbol{y'}) = -\boldsymbol{n}(\boldsymbol{x'})$ for $\boldsymbol{x'} \in \mathcal{I}$. Using equation 37 the boundary integral equation for fluid 2 can be written as

$$-\fint_{\mathcal{I}} J_{ik}(\boldsymbol{\xi}) T'_{1,ij}(\boldsymbol{y'}) n_j(\boldsymbol{y'}) \mathrm{d}^2\boldsymbol{y'} - \lambda \fint_{\mathcal{I}} u'_{1,i}(\boldsymbol{y'}) K_{ijk}(\boldsymbol{\xi}) n_j(\boldsymbol{y}) \mathrm{d}^2\boldsymbol{y'} = \frac{\lambda u'_{1,k}(\boldsymbol{x'})}{2} \quad \boldsymbol{x'} \in \mathcal{I}. \tag{54}$$

Equations 53 and 54 can be added together and combined with equation 43 to obtain

$$\begin{aligned} &\fint_{\mathcal{S}} J_{ik}(\boldsymbol{\xi}) f_{\mathrm{s},i}(\boldsymbol{y'}) \mathrm{d}^2\boldsymbol{y'} + \frac{9}{2D\mathrm{Bo}} \fint_{\mathcal{I}} J_{ik}(\boldsymbol{\xi}) n_i(\boldsymbol{y'}) [\partial'_j n_j(\boldsymbol{y'}) - \hat{z}_j y'_j \mathrm{Bo}] \mathrm{d}^2\boldsymbol{y'} + \\ &(1-\lambda) \fint_{\mathcal{I}} u'_{1,i}(\boldsymbol{y'}) K_{ijk}(\boldsymbol{\xi}) n_j(\boldsymbol{y'}) \mathrm{d}^2\boldsymbol{y'} = \begin{cases} \frac{(1+\lambda) u'_{1,k}(\boldsymbol{x'})}{2} & \boldsymbol{x'} \in \mathcal{I} \\ u'_{s,k} & \boldsymbol{x'} \in \mathcal{S} \end{cases} . \end{aligned} \tag{55}$$

This together with equation 41 completely describes the system in an integral representation.

2.6 Axisymmetric Simplification

We can exploit the axial symmetry of the system to chose the point $\boldsymbol{x'}$ such that it lies in the plane defined by $\phi = 0$. Hence in Cartesian coordinates $\boldsymbol{x'} = (x_r, 0, x_z)$. This also means we can write $\boldsymbol{y'} = (y_r \cos\phi, y_r \sin\phi, y_z)$. On the surface of the sphere $y_{r(z)} = y_{r(z)}(\theta)$ and on the interface $y_{r(z)} = y_{r(z)}(s)$. Additionally $\boldsymbol{f}_{\mathrm{s}} = [f_{\mathrm{s},r}(\theta)\cos\phi, f_{\mathrm{s},r}(\theta)\sin\phi, f_{\mathrm{s},z}(\theta)]$

and $\boldsymbol{n} = [n_r(s)\cos\phi, n_r(s)\sin\phi, n_z(s)]$. Since the system is axisymmetric, it is useful to extract the azimuthal integration from the surface integrals in equations 55 and 41. To achieve this, the Cartesian components of each equation is considered separately. For equation 55, it can be shown that both the left and right hand sides of the 2-component are identically zero. For equation 41 this is true for the 1- and 2-components. To show this, expand J_{ij} and K_{ijk} in terms of the components of $\boldsymbol{x'}$ and $\boldsymbol{y'}$. This leaves three integral equations which can be expressed as

$$\int_{\theta=0}^{\pi} B_{\alpha\beta}(\boldsymbol{x'},\theta)b_\beta(\theta)\sin\theta \mathrm{d}\theta + \int_{s=0}^{\infty}\left(A_{\alpha\beta}(\boldsymbol{x'},s)y_r(s) - \frac{(1+\lambda)\delta_{\alpha\beta}\delta(s-s_0)}{2}\right)a_\beta(s)\mathrm{d}s = -\int_{s=0}^{\infty} C_\alpha(\boldsymbol{x'},s)y_r(s)\mathrm{d}s, \quad \text{when } \boldsymbol{x'}\in\mathcal{I}, \tag{56}$$

$$\int_{\theta=0}^{\pi} B_{\alpha\beta}(\boldsymbol{x'},\theta)b_\beta(\theta)\sin\theta \mathrm{d}\theta + \int_{s=0}^{\infty} A_{\alpha\beta}(\boldsymbol{x'},s)a_\beta(s)y_r(s)\mathrm{d}s - d_\alpha = \int_{s=0}^{\infty} C_\alpha(\boldsymbol{x'},s)y_r(s)\mathrm{d}s, \quad \text{when } \boldsymbol{x'}\in\mathcal{S}, \tag{57}$$

and

$$\int_{\theta=0}^{\pi} b_2(\theta)\sin\theta\mathrm{d}\theta = 3, \tag{58}$$

where the quantities $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{d}$ are defined as:

$$\boldsymbol{A} = (1-\lambda)\int_{\phi=0}^{2\pi}\begin{pmatrix} \begin{gathered} n_r(K_{111}\cos^2\phi + K_{221}\sin^2\phi + 2K_{121}\sin\phi\cos\phi) \\ + n_z(K_{131}\cos\phi + K_{231}\sin\phi)\end{gathered} & \begin{gathered} n_r(K_{131}\cos\phi + K_{231}\sin\phi) \\ + n_z K_{331}\end{gathered} \\ \begin{gathered} n_r(K_{113}\cos^2\phi + K_{223}\sin^2\phi + 2K_{123}\sin\phi\cos\phi) \\ + n_z(K_{133}\cos\phi + K_{233}\sin\phi)\end{gathered} & \begin{gathered} n_r(K_{133}\cos\phi + K_{233}\sin\phi) \\ + n_z K_{333}\end{gathered}\end{pmatrix}\mathrm{d}\phi, \tag{59}$$

$$\boldsymbol{B} = \int_{\phi=0}^{2\pi} \begin{pmatrix} J_{11}\cos\phi + J_{21}\sin\phi & J_{31} \\ J_{13}\cos\phi + J_{23}\sin\phi & J_{33} \end{pmatrix} \mathrm{d}\phi, \tag{60}$$

$$\boldsymbol{C} = \frac{9(\partial_j' n_j - \mathrm{Bo} y_z)}{2D\mathrm{Bo}} \int_{\phi=0}^{2\pi} \begin{pmatrix} n_r(J_{11}\cos\phi + J_{21}\sin\phi) + n_z J_{31} \\ n_r(J_{13}\cos\phi + J_{23}\sin\phi) + n_z J_{23} \end{pmatrix} \mathrm{d}\phi, \tag{61}$$

$$\boldsymbol{a} = \begin{pmatrix} u_{1,r}'(s) \\ u_{1,z}'(s) \end{pmatrix}, \tag{62}$$

$$\boldsymbol{b} = \begin{pmatrix} f_{\mathrm{s},r}(s) \\ f_{\mathrm{s},z}(s) \end{pmatrix}, \tag{63}$$

and

$$\boldsymbol{d} = \begin{pmatrix} 0 \\ u_{\mathrm{s}}' \end{pmatrix} \tag{64}$$

For brevity the function arguments have been dropped from the kernels and the normal vectors but $n_i = n_i[\boldsymbol{y'}(s,\phi)]$ and in equation 59, $K_{ijk} = K_{ijk}[\boldsymbol{x'} - \boldsymbol{y'}(s,\phi)]$, in equation 60, $J_{ij} = J_{ij}[\boldsymbol{x'} - \boldsymbol{y'}(\theta,\phi)]$ and in equation 61, $J_{ij} = J_{ij}[\boldsymbol{x'} - \boldsymbol{y'}(s,\phi)]$.

The aziumthal integrals inside the definitions of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ can be expressed as sums of complete elliptic integrals of the first and second kind (Lee and Leal, 1982; Geller et al., 1985; Graziani, 1989; Pozrikids, 1992; Manga, 1994; Roumeliotis, 2000). Previous authors have used different notations and formulations to present these expressions and we will use a formulation similar to Graziani (1989). First, we define the quantities α and β:

$$\alpha^2 = x_r^2 + y_r^2 + (x_z - y_z)^2 \tag{65}$$

$$\beta^2 = 2x_r y_r \tag{66}$$

A Greens Functions for Stokes Flow

We present here a derivation of equations 46 and 91 following Ladyzhenskaya (1963). First, the Greens function for dynamic pressure $\hat{P}_\alpha(\boldsymbol{\xi})$ is defined such that

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = -\hat{P}_\alpha(\boldsymbol{\xi}) + \Lambda_\alpha[\partial'_i \hat{u}_{\alpha,j}(\boldsymbol{\xi}) + \partial'_j \hat{u}_{\alpha,i}(\boldsymbol{\xi})]. \tag{67}$$

Substituting this into equation 45 and using equation 44 yields

$$-\partial'_j \hat{P}_\alpha(\boldsymbol{\xi}) + \Lambda_\alpha \partial'_i \partial'_i \hat{u}_{\alpha,j}(\boldsymbol{\xi}) + \mathcal{F}_j \delta(\boldsymbol{\xi}) = 0. \tag{68}$$

We also define two further quantities $\bar{P}_{\alpha,i}$ and $\bar{u}_{\alpha,ij}$ such that

$$\hat{P}_\alpha(\boldsymbol{\xi}) = \mathcal{F}_i \bar{P}_{\alpha,i}(\boldsymbol{\xi}), \tag{69}$$

and

$$\hat{u}_{\alpha,j}(\boldsymbol{\xi}) = \mathcal{F}_i \bar{u}_{\alpha,ij}(\boldsymbol{\xi}). \tag{70}$$

Substitution of these expressions into equations 68 and 44 and rearranging results in

$$-\partial'_j \bar{P}_{\alpha,i}(\boldsymbol{\xi}) + \Lambda_\alpha \partial'_k \partial'_k \bar{u}_{\alpha,ij}(\boldsymbol{\xi}) + \delta_{ij}\delta(\boldsymbol{\xi}) = 0, \tag{71}$$

and

$$\partial_i' \bar{u}_{\alpha,ij}(\boldsymbol{\xi}) = 0. \tag{72}$$

To derive functional forms for the Greens functions it is necessary to express equations 71 and 72 in Fourier representation. To do this we need to define the Fourier transformed variables $\tilde{P}_{\alpha,i}$ and $\tilde{u}_{\alpha,ij}$ Riley et al. (2006):

$$\bar{P}_{\alpha,i}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{P}_{\alpha,i}(\boldsymbol{k}) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}, \tag{73}$$

and

$$\bar{u}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{u}_{\alpha,ij}(\boldsymbol{k}) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}. \tag{74}$$

Using these definitions, and equation 109 in appendix B, in equations 71 and 72 gives the Fourier representations of the Stokes equations and the continuity equation. Following some manipulation these can be written as

$$-ik_j \tilde{P}_{\alpha,i}(\boldsymbol{k}) - \Lambda_\alpha k^2 \tilde{u}_{\alpha,ij}(\boldsymbol{k}) + \frac{\delta_{ij}}{(2\pi)^{3/2}} = 0, \tag{75}$$

and

$$k_i \tilde{u}_{\alpha,ij}(\boldsymbol{k}) = 0, \tag{76}$$

where $k = |\boldsymbol{k}|$. By multiplying equation 75 by k_j, substituting in equation 76 and rearranging, it is then possible to obtain the Fourier representation of the Greens function for pressure;

$$\tilde{P}_{\alpha,i}(\boldsymbol{k}) = \frac{-ik_i}{(2\pi)^{3/2} k^2}. \tag{77}$$

A final substitution of this into equation 73 gives the Greens function for pressure;

$$\bar{P}_{\alpha,i}(\boldsymbol{\xi}) = \frac{-i}{(2\pi)^3} \int \frac{k_i \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2}. \tag{78}$$

This integral is evaluated in appendix A.1 and it is shown that

$$\bar{P}_{\alpha,i}(\boldsymbol{\xi}) = -\frac{1}{4\pi} \partial_i' \left(\frac{1}{\xi}\right) = \frac{\xi_i}{4\pi\xi^3} \quad , \quad \xi = \xi_i \xi_i. \tag{79}$$

We also need to find an equivalent expression for $\bar{u}_{\alpha,ij}$. To do so, substitute equation 77 into equation 75 and rearrange;

$$\tilde{u}_{\alpha,ij}(\boldsymbol{k}) = \frac{k^2 \delta_{ij} - k_i k_j}{(2\pi)^{3/2} k^4 \Lambda_\alpha}. \tag{80}$$

Combining this with equation 74 results in an expression for the Greens function for velocity;

$$\bar{u}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^3 \Lambda_\alpha} \left(\delta_{ij} \int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2} - \int \frac{k_i k_j \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^4} \right). \tag{81}$$

These integrals are evaluated in appendix A.2 (equations 99 and 105) and following some manipulation we find

$$\bar{u}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{1}{8\pi\Lambda_\alpha \xi} \left(\delta_{ij} + \frac{\xi_i \xi_j}{\xi^2} \right). \tag{82}$$

We can now substitute equations 79 and 82 into 69 and 70 to obtain

$$\hat{P}_\alpha(\boldsymbol{\xi}) = \frac{\mathcal{F}_i \xi_i}{4\pi\xi^3}, \tag{83}$$

and

$$\hat{u}_{\alpha,j}(\boldsymbol{\xi}) = \frac{\mathcal{F}_i}{8\pi\Lambda_\alpha\xi}\left(\delta_{ij} + \frac{\xi_i\xi_j}{\xi^2}\right). \tag{84}$$

Substitution of equations 83 and 84 into equation 67 results in

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{-3\mathcal{F}_k\xi_i\xi_j\xi_k}{4\pi\xi^5} \tag{85}$$

The kernels $\boldsymbol{J}$ and $\boldsymbol{K}$ are defined as

$$J_{ij} = \frac{1}{8\pi\xi}\left(\delta_{ij} + \frac{\xi_i\xi_j}{\xi^2}\right) \tag{86}$$

$$K_{ijk} = \frac{-3\xi_i\xi_j\xi_k}{4\pi\xi^5} \tag{87}$$

Note that under the interchange $\boldsymbol{\xi} \to -\boldsymbol{\xi}$ the kernels are symmetric and antisymmetric respectively

$$J_{ki}(-\boldsymbol{\xi}) = J_{ki}(\boldsymbol{\xi}) \tag{88}$$

$$K_{jik}(-\boldsymbol{\xi}) = -K_{jik}(\boldsymbol{\xi}) \tag{89}$$

This means we can express the velocity and stress Greens functions as

$$\hat{u}_{\alpha,j}(\boldsymbol{\xi}) = \frac{\mathcal{F}_i J_{ij}(\boldsymbol{\xi})}{\Lambda_\alpha} \tag{90}$$

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = K_{ijk}(\boldsymbol{\xi})\mathcal{F}_k \tag{91}$$

A.1 Integral for Greens Function for Pressure

To evaluate the integral in equation 78 first define a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$. The consider

$$\int_{\mathcal{V}} \nabla^2 \left(\frac{1}{\xi}\right) \mathrm{d}^3\boldsymbol{\xi} = \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \left(\frac{1}{\xi}\right) \mathrm{d}^3\boldsymbol{\xi}. \tag{92}$$

Using the divergence theorem (appendix C) this can be written as

$$\int_{\mathcal{V}} \nabla^2 \left(\frac{1}{\xi}\right) \mathrm{d}^3\boldsymbol{r} = \int_{\mathcal{S}} \boldsymbol{\nabla} \left(\frac{1}{\xi}\right) \cdot \mathrm{d}\mathcal{S}. \tag{93}$$

This can be evaluated for the case that $\mathcal{V}$ is a sphere of radius R;

$$\int_{\mathcal{V}} \nabla^2 \left(\frac{1}{\xi}\right) \mathrm{d}^3\boldsymbol{\xi} = -4\pi = -4\pi \int_{\mathcal{V}} \delta(\boldsymbol{\xi}) \mathrm{d}^3\boldsymbol{\xi}, \tag{94}$$

where the final step follows from comparison with equation 107. Using equation 109 leads to

$$\nabla^2 \left(\frac{1}{\xi}\right) = \frac{-4\pi}{(2\pi)^3} \int \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}. \tag{95}$$

Inspection of this then suggests

$$\partial_i \left(\frac{1}{\xi}\right) = \frac{4i\pi}{(2\pi)^3} \int \frac{k_i \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2}. \tag{96}$$

Hence

$$\frac{-i}{(2\pi)^3} \int \frac{k_i \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2} = -\frac{1}{4\pi} \partial_i \left(\frac{1}{\xi}\right). \tag{97}$$

A.2 Integrals for the Greens Function for Velocity

Equation 81 contains two integrals that need to be evaluated. By inspecting equation 95 it can be seen

$$\frac{1}{\xi} = \frac{4\pi}{(2\pi)^3} \int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2}. \tag{98}$$

Hence

$$\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^2} = \frac{(2\pi)^3}{4\pi\xi}. \tag{99}$$

The second integral requires a bit more work. Firstly, express it in a different form;

$$\int \frac{k_i k_j \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^4} = \partial_i \partial_j \left(\int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^4} \right). \tag{100}$$

To evalaute this, first consider $\nabla^4 \xi = \nabla^2(\nabla^2 \xi)$. Expanding ∇^2 in spherical polar coordinates centred on $\xi = 0$ shows

$$\nabla^4 \xi = 2\nabla^2 \left(\frac{1}{\xi} \right). \tag{101}$$

Combining this with equation 95 we obtain

$$\nabla^4 \xi = \frac{-8\pi}{(2\pi)^3} \int \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}. \tag{102}$$

Insepection of this yields

$$\xi = \frac{-8\pi}{(2\pi)^3} \int \frac{\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k}}{k^4}. \tag{103}$$

Rearrangement of this produces an expression for the integral on the right hand side of equation 100;

$$\int \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} d^3\boldsymbol{k}}{k^4} = -\frac{(2\pi)^3 \xi}{8\pi}. \tag{104}$$

Hence

$$\int \frac{k_i k_j e^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} d^3\boldsymbol{k}}{k^4} = \frac{(2\pi)^3 \partial'_i \partial'_j \xi}{8\pi}. \tag{105}$$

B Dirac Delta Function

In a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$, the Dirac delta function $\delta(\boldsymbol{x}-\boldsymbol{y})$ is defined as (Riley et al., 2006)

$$\int_{\mathcal{V}} f(\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y}) d^3\boldsymbol{y} = \begin{cases} f(\boldsymbol{x}) & \boldsymbol{x} \in \mathcal{V} \\ \frac{f(\boldsymbol{x})}{2} & \boldsymbol{x} \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases}. \tag{106}$$

The result for $\boldsymbol{x} \in \mathcal{S}$ is only valied for the case that the surface is Lyapunov smooth (a local tangent plane exists everywhere) (REFERENCE FOR LYAPUNOV SMOOTH SURFACE). Equation 106 means that

$$\int_{\mathcal{V}} \delta(\boldsymbol{x}-\boldsymbol{y}) d^3\boldsymbol{y} = 1 \quad \boldsymbol{x} \in \mathcal{V}. \tag{107}$$

A key property of the delta function is that it is symmetric under a change of sign of the argument

$$\delta(-\boldsymbol{x}) = \delta(\boldsymbol{x}) \tag{108}$$

It also needs to be noted that the Dirac delta function can be expressed as (Riley et al., 2006)

$$\delta(\boldsymbol{\xi}) = \frac{1}{(2\pi)^3} \int \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\xi}} \mathrm{d}^3\boldsymbol{k} \tag{109}$$

C Divergence Theorem

The divergence theorem states that for a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$, then for a continuous and differentiable vector field $\boldsymbol{a}$ Riley et al. (2006)

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{a} \mathrm{d}\mathcal{V} = \oint_{\mathcal{S}} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} \tag{110}$$

D Lorentz Reciprocal Theorem

Here we present a proof of the Lorentz reciprocal theorem following Kim and Karrila (2005). The dimensionless strain field in fluid alpha is defined as

$$e'_{\alpha,ij}(\boldsymbol{\xi}) = \frac{\partial'_i u'_{\alpha,j}(\boldsymbol{\xi}) + \partial'_j u'_{\alpha,i}(\boldsymbol{\xi})}{2}, \tag{111}$$

and the corresponding Greens function as

$$\hat{e}_{\alpha,ij}(\boldsymbol{\xi}) = \frac{\partial'_i \hat{u}_{\alpha,j}(\boldsymbol{\xi}) + \partial'_j \hat{u}_{\alpha,i}(\boldsymbol{\xi})}{2}. \tag{112}$$

This allows the Greens function for the stress tensor to be expressed as

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi}) = \hat{P}_{\alpha}(\boldsymbol{\xi}) + 2\hat{e}_{\alpha.ij}(\boldsymbol{\xi}). \tag{113}$$

Now consider the double contraction of the dimensionless stress tensor and strain Green function tensor;

$$T'_{\alpha,ij}(\boldsymbol{x})\hat{e}_{\alpha,ij}(\boldsymbol{\xi}) = -P'_{\mathrm{d},\alpha}(\boldsymbol{x})\hat{e}_{\alpha,ii}(\boldsymbol{\xi}) + 2e'_{\alpha,ij}(\boldsymbol{x})\hat{e}_{\alpha,ij}(\boldsymbol{\xi}) = 2e'_{\alpha,ij}(\boldsymbol{x})\hat{e}_{\alpha,ij}(\boldsymbol{\xi}). \tag{114}$$

The last step here follows by using equations 112 and 44 to show that the first term on the right hand side is equal to zero. In an analagous fashion it can be shown that

$$\hat{T}_{\alpha,ij}(\boldsymbol{\xi})e'_{\alpha,ij}(\boldsymbol{x}) = 2\hat{e}_{\alpha,ij}(\boldsymbol{\xi})e'_{\alpha,ij}(\boldsymbol{x}). \tag{115}$$

Since the right hand sides of equations 114 and 115 are identical this means

$$T'_{\alpha,ij}(\boldsymbol{x})\hat{e}_{\alpha,ij}(\boldsymbol{\xi}) = \hat{T}_{\alpha,ij}(\boldsymbol{\xi})e'_{\alpha,ij}(\boldsymbol{x}). \tag{116}$$

Expanding the strain tensors in terms of the velocity gradients and integrating over a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$ with respect to $\boldsymbol{x}$ leads to

$$⨍_{\mathcal{V}} \partial'_j[T'_{ij}(\boldsymbol{x'})\hat{u}_i(\boldsymbol{\xi})]\mathrm{d}\boldsymbol{x'}^3 - ⨍_{\mathcal{V}} [\partial'_j T'_{ij}(\boldsymbol{x'})]\hat{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x'}^3 = ⨍_{\mathcal{V}} \partial'_j[\hat{T}_{ij}(\boldsymbol{\xi})u'_i(\boldsymbol{x'})]\mathrm{d}\boldsymbol{x'}^3 - ⨍_{\mathcal{V}} [\partial'_j \hat{T}_{ij}(\boldsymbol{\xi})]u'_i(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^3. \tag{117}$$

Here the integration has been defined in the sense of the Cauchy Principle Value (CPV) (Riley et al., 2006) to allow for the singular point in the Greens functions at $\boldsymbol{\xi} = \boldsymbol{0}$. Defining $\boldsymbol{n}$ as the normal to the surface $\mathcal{S}$, the divergence theorem (appendix C) can then to applied to the first term on both sides;

$$⨍_{\mathcal{S}} n_j(\boldsymbol{x'})T'_{ij}(\boldsymbol{x'})\hat{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x'}^2 - ⨍_{\mathcal{V}} [\partial'_j T'_{ij}(\boldsymbol{x'})]\hat{u}_i(\boldsymbol{\xi})\mathrm{d}\boldsymbol{x'}^3 = ⨍_{\mathcal{S}} n_j(\boldsymbol{x'})\hat{T}_{ij}(\boldsymbol{\xi})u'_i(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^2 - ⨍_{\mathcal{V}} [\partial'_j \hat{T}_{ij}(\boldsymbol{\xi})]u'_i(\boldsymbol{x'})\mathrm{d}\boldsymbol{x'}^3. \tag{118}$$

This is the Lorentz reciprocal theorem with the additional caveat that the integrals have been defined as CPV integrals so that the singular nature of the Greens functions can be dealt with.

E Elliptic Integrals

The complete elliptic integrals of the first and second kind are defined as (Abramowitz and Stegun, 1972)

$$K(k^2) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1-k^2\sin^2\theta)^{1/2}}, \quad 0 \leq k^2 < 1, \tag{119}$$

and

$$E(k^2) = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{1/2}\mathrm{d}\theta, \quad 0 \leq k^2 < 1. \tag{120}$$

k^2 is defined as the modulus of the integral. Polynomial approximations can be found to evaluate the integrals (Roumeliotis, 2000) and we use the following expressions from Abramowitz and Stegun (1972):

$$K(k^2) = \sum_{i=0}^{4} a_i(1-k^2)^i + \ln\left(\frac{1}{1-k^2}\right)\sum_{i=0}^{4} b_i(1-k^2)^i, \tag{121}$$

$$E(k^2) = 1 + \sum_{i=1}^{4} a_i'(1-k^2)^i + \ln\left(\frac{1}{1-k^2}\right)\sum_{i=1}^{4} b_i'(1-k^2)^i \tag{122}$$

The values of the coefficients in the expansion are in table 2.

Table 2: The coefficients for equations 121 and 122.

a_0	1.38629436112	b_0	0.5
a_1	0.09666344259	b_1	0.12498593597
a_2	0.03590092383	b_2	0.06880248576
a_3	0.03742563713	b_3	0.03328355346
a_4	0.01451196212	b_4	0.00441787012
a'_1	0.44325141463	b'_1	0.24998368310
a'_2	0.06260601220	b'_2	0.09200180037
a'_3	0.04757383546	b'_3	0.04069697526
a'_4	0.01736506451	b'_4	0.00526449639

F Components of $\boldsymbol{A}$

Here we expressions for the components of $\boldsymbol{A}$ in terms of complete elliptic integrals of the first and second kind (appendix E). The expressions are from Graziani (1989) but we use a different notation; the quantities α and β are defined from Manga (1994). K and E are complete elliptic integrals of the first and second kind respectively and they all take $k^2 = 2\beta^2/(\alpha^2 + \beta^2)$ as their modulus. The expressions are:

$$A_{11} = (a_1 n_r + a_2 n_z)K + (a_3 n_r + a_4 n_z)E, \tag{123}$$

$$A_{12} = (a_2 n_r + a_6 n_z)K + (a_4 n_r + a_8 n_z)E, \tag{124}$$

$$A_{21} = (a_9 n_r + a_{10} n_z)K + (a_{11} n_r + a_{12} n_z)E, \tag{125}$$

$$A_{22} = (a_{10} n_r + a_{14} n_z)K + (a_{12} n_r + a_{16} n_z)E. \tag{126}$$

The coefficients a_i are given as

$$a_1 = \frac{(1-\lambda)[x_r\alpha_2(4\alpha^4 - 18x_r^2y_r^2) - x_r(2y_r^2 + x_r^2)(2\alpha^4 - 3\beta^4) - y_r\alpha^2\beta^2(y_r^2 + 2x_r^2) + x_ry_r^2\beta^4]}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)\beta^4} \tag{127}$$

$$a_2 = \frac{(1-\lambda)(x_z - y_z)[2\alpha^4 - 2\beta^4 - \alpha^2(x_z - y_z)^2]}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)\beta^2} \tag{128}$$

$$a_3 = \frac{1-\lambda}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)^2\beta^4}\Bigg(\frac{x_r(-8\alpha^8 + 15\alpha^4\beta^4 - 3\beta^8)}{2}$$
$$-2x_r\alpha^2(2y_r^2 + x_r^2)(-\alpha^4 + 3\beta^4) + y_r\beta^2(y_r^2 + 2x_r^2)(\alpha^4 + 3\beta^4) - 4x_ry_r^2\alpha^2\beta^4\Bigg) \tag{129}$$

$$a_4 = \frac{-(1-\lambda)(x_z - y_z)}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)^2\beta^2}\left(\alpha^4(\alpha^4 - 5\beta^4) + [\alpha^2 - (x_z - y_z)^2](\alpha^4 + 3\beta^4)\right) \tag{130}$$

$$a_6 = \frac{(1-\lambda)(x_z - y_z)^2(2x_r^2 - \alpha^2)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)x_r} \tag{131}$$

$$a_8 = \frac{(1-\lambda)(x_z - y_z)^2(\alpha^4 + 3\beta^4 - 8x_r^2\alpha^2)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)^2x_r} \tag{132}$$

$$a_9 = \frac{(1-\lambda)(x_z - y_z)(-2\alpha^4 + 3\beta^4 - 4y_r^2\alpha^2 + 4y_r^4)}{4\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)y_r} \tag{133}$$

$$a_{10} = \frac{(1-\lambda)(x_z - y_z)^2(\alpha^2 - 2y_r^2)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)y_r} \tag{134}$$

$$a_{11} = \frac{(1-\lambda)(x_z - y_z)(\alpha^6 - 3\alpha^2\beta^4 + 2y_r^2\alpha^4 + 6y_r^2\beta^4 - 8y_r^4\alpha^2)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)^2y_r^2} \tag{135}$$

$$a_{12} = \frac{(1-\lambda)(x_z - y_z)^2(8y_r^2\alpha^2 - \alpha^4 - 3\beta^4)}{2\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)y_r} \tag{136}$$

$$a_{14} = \frac{(1-\lambda)(x_z - y_z)^3}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2-\beta^2)} \tag{137}$$

$$a_{16} = \frac{-4(1-\lambda)(x_z - y_z)^3\alpha^2}{\pi(\alpha^2+\beta^2)^{3/2}(\alpha^2+\beta^2)^2} \tag{138}$$

References

Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions*. Dover, tenth edition, 1972.

G. K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge University Press, first edition, 1967.

J.U. Brackbill, D.B. Kothe, and C. Zemach. A Continuum Method for Modeling Surface Tension. *Journal of Computational Physics*, 100:334 – 354, 1992.

A. S. Geller, S. H. Lee, and L. G. Leal. The creeping motion of a spherical particle normal to a deformable interface. *Journal of Fluid Mechanics*, 169:27–69, 1985.

G. Graziani. A boundary integral equation method for axisymmetric viscous flows. *International Journal of Engineering Science*, 27:855–864, 1989.

Mike Hobson, George Efstathiou, and Anthony Lasenby. *General Relativity: An introduction for physicists*. Cambridge University Press, 4th edition, 2011.

Sangtae Kim and Seppo J. Karrila. *Microhydrodynamics: Principles and Selected Applications*. Dover Publications, second edition, 2005.

O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, first edition, 1963.

S. H. Lee and L. G. Leal. The Motion of a Sphere in the Presence of a Deformable Interface II. A Numerical Study of the Translation of a Sphere Normal to an Interface. *Journal of Colloid and Interface Science*, 87:81–106, 1982.

Michael Manga. *The motion of deformable drops and bubbles at low Reynolds number: Applications to selected problems in geology and geophysics.* PhD thesis, Harvard University, 1994.

C. Pozrikids. *Boundary integral and singularity methods for linearized viscous flows.* Cambridge University Press, first edition, 1992.

Osborne Reynolds. On the Theory of Lubrication and its Application to Mr Beauchamp Tower's Experiments, including an Experimental Determination of the Viscosity of Olive Oil. *Philosophical Transactions of the Royal Society of London*, 177:157–234, 1886.

K. F. Riley, M. P. Hobson, and S. J. Bence. *Mathematical Methods for Physics and Engineering.* Cambridge University Press, third edition, 2006.

John Roumeliotis. *A Boundary Integral Method Applied to Sokes Flow.* PhD thesis, University of New South Wales, 2000.

Frank M. White. *Fluid Mechanics.* Mcgraw Hill, seventh edition, 1999.