Résolution de l'équation de Stokes en schéma MAC, par la méthode d'Uzawa et solveur itératif, en Fortran90

Méthodes Numériques Ecoulements Incompressibles

Pierre Jacquet Guillaume Ravel

Enseignant : Y. Coudière

Sommaire

Introduction

Problème de Stokes

$$\begin{cases} \partial_t u - \nu \nabla u + \nabla p = f sur \Omega \\ -\nabla . u = 0 \\ u_{|\Gamma_0} = g \\ u_{|\Gamma_1} = h \end{cases} \quad (1)$$

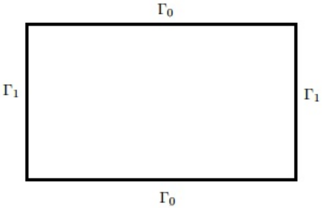


Figure – Domaine Ω

Modélisation mathématique

Schéma MAC

$$
\left\{
\begin{array}{l}
\frac{u^{1,n+1}_{i,j} - u^{1,n}_{i,j}}{dt} - \nu\left(\frac{u^{1,n+1}_{i+1,j} + u^{1,n+1}_{i-1,j} - 2u^{1,n+1}_{i,j}}{dx^2} + \frac{u^{1,n+1}_{i,j+1} + u^{1,n+1}_{i,j-1} - 2u^{1,n+1}_{i,j}}{dy^2} \right) \\
+ \frac{p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} + p^{n+1}_{i+\frac{1}{2},j-\frac{1}{2}} - p^{n+1}_{i-\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}}}{2dx} = f^{1,n}_{i,j} \\
\frac{u^{2,n+1}_{i,j} - u^{2,n}_{i,j}}{dt} - \nu\left(\frac{u^{2,n+1}_{i+1,j} + u^{2,n+1}_{i-1,j} - 2u^{2,n+1}_{i,j}}{dx^2} + \frac{u^{2,n+1}_{i,j+1} + u^{2,n+1}_{i,j-1} - 2u^{2,n+1}_{i,j}}{dy^2} \right) \\
+ \frac{p^{n+1}_{i+\frac{1}{2},j+\frac{1}{2}} + p^{n+1}_{i-\frac{1}{2},j+\frac{1}{2}} - p^{n+1}_{i-\frac{1}{2},j-\frac{1}{2}} - p^{n+1}_{i+\frac{1}{2},j-\frac{1}{2}}}{2dy} = f^{2,n}_{i,j} \\
- \frac{u^{1,n+1}_{i+1,j+1} + u^{1,n+1}_{i+1,j} - u^{1,n+1}_{i,j+1} - u^{1,n+1}_{i,j}}{2dx} - \frac{u^{2,n+1}_{i+1,j+1} + u^{2,n+1}_{i,j+1} - u^{2,n+1}_{i+1,j} - u^{2,n+1}_{i,j}}{2dy} = 0
\end{array}
\right.
\tag{2}
$$

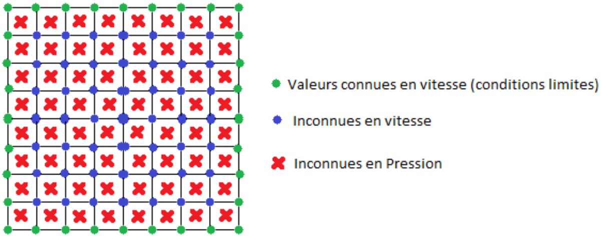


Figure – Illustration du maillage - schéma MAC

Modélisation mathématique

Système matriciel

D'où le système matriciel suivant :

$$\begin{pmatrix} A & 0 & B_1^T \\ 0 & A & B_2^T \\ B1 & B2 & 0 \end{pmatrix} \cdot \begin{pmatrix} U_x^{n+1} \\ U_y^{n+1} \\ P^{n+1} \end{pmatrix} = \begin{pmatrix} F_x^n \\ F_y^n \\ 0 \end{pmatrix} + \frac{1}{dt} \begin{pmatrix} U_x^n \\ U_y^n \\ 0 \end{pmatrix} + CL \tag{3}$$

$$B1 = \begin{pmatrix} S & 0 & 0 \\ S & \ddots & 0 \\ 0 & \ddots & S \\ 0 & 0 & S \end{pmatrix} \quad S = \frac{1}{2dx} \begin{pmatrix} -1 & 0 & 0 \\ 1 & \ddots & 0 \\ 0 & \ddots & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B2 = \begin{pmatrix} -R & 0 & 0 \\ R & \ddots & 0 \\ 0 & \ddots & -R \\ 0 & 0 & R \end{pmatrix} \quad R = \frac{1}{2dy} \begin{pmatrix} 1 & 0 & 0 \\ 1 & \ddots & 0 \\ 0 & \ddots & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Résolution numérique
Code

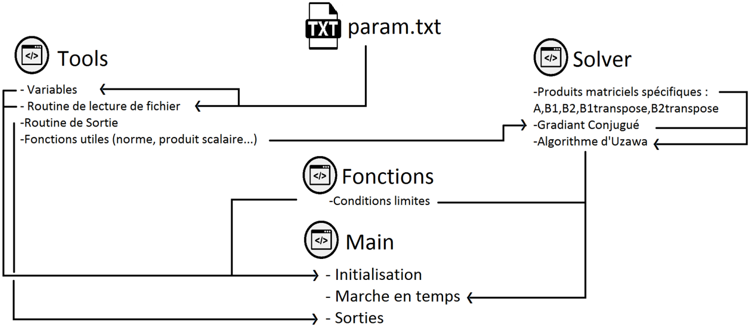


Figure – Organisation du code

Résolution numérique

Algorithme d'Uzawa

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \cdot \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \tag{4}$$

U réalise aussi le minimum de $J(U) = \frac{1}{2} U^T A U - U^T F$ sous la contrainte $BU = 0$
P est un multiplicateur de Lagrange : $< U, B^T P >= 0$
U réalisant le minimum de J convexe, on a alors la relation : $\nabla J = -B^T P$
Lors d'un pas de temps, le système (4) peut alors être résolu avec l'algorithme d'Uzawa :

- Initialisation : on fixe μ et P^0
- Tant que : $\|P^{n+1} - P^n\| = \mu \|BU^n\| > \epsilon$ Faire
  - Résolution de $AU^n = F - B^T P^n$ par l'algorithme du gradient conjugué
  - Mise à jour de P : $P^{n+1} = P^n + \mu B U^n$

Cas tests

Résolution du Laplacien

Cas test 1 :

$$\left\{ \begin{array}{l} u(x,y) = x(1-x)y(1-y) \\ f(x,y) = 2(y - y^2 + x - x^2) \\ g = h = 0 \end{array} \right. \qquad (5)$$

Cas test 2 :

$$\left\{ \begin{array}{l} u(x,y) = sin(x) + cos(y) \\ f(x,y) = sin(x) + cos(y) \\ g = h = f \end{array} \right. \qquad (6)$$

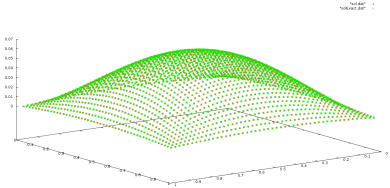


Figure – Visualisation des solutions numérique et exacte - Cas test 1

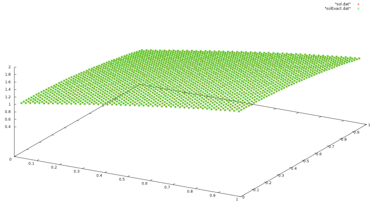


Figure – Visualisation des solutions numérique et exacte - Cas test 2

Cas tests

Rsolution du Laplacien

Cas test 1 :

$$\begin{cases} u(x,y) = x(1-x)y(1-y) \\ f(x,y) = 2(y-y^2+x-x^2) \\ g = h = 0 \end{cases} \quad (7)$$

Cas test 2 :

$$\begin{cases} u(x,y) = sin(x)+cos(y) \\ f(x,y) = sin(x)+cos(y) \\ g = h = f \end{cases} \quad (8)$$

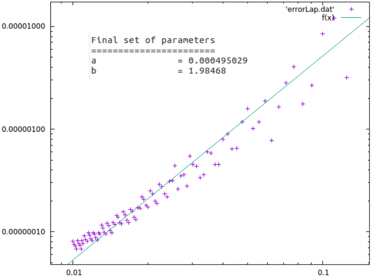


Figure – Comparaison de l'erreur en norme 2 suivant dx avec $f(x) = ax^b$ (échelle log-log)

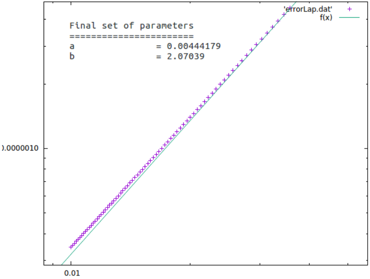


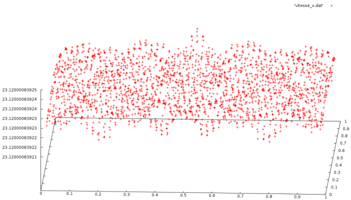
Figure – Comparaison de l'erreur en norme 2 suivant dx avec $f(x) = ax^b$ (échelle log-log)

Cas tests

Cas Test 0a

$$\begin{cases} U = (u_1, u_2) \text{ avec } u_1, u_2 \text{ réels} \\ F = (0, 0) \end{cases} \tag{9}$$

Figure – Vitesse selon x



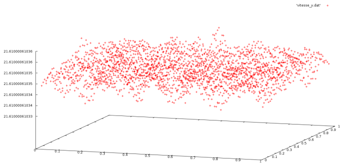


Figure – Vitesse selon y

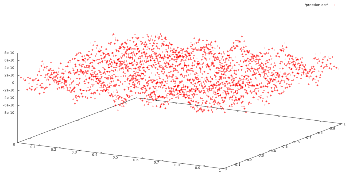


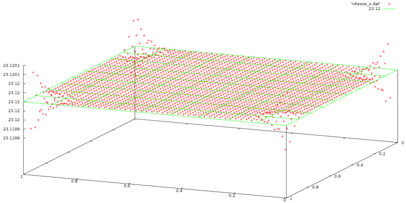
Figure – Pression

Cas tests

Cas Test 0b

$$\begin{cases} U = (u_1, u_2) \text{ avec } u_1, u_2 \text{ réels} \\ F = (0, -g) \end{cases} \tag{10}$$

Figure – Vitesse selon x



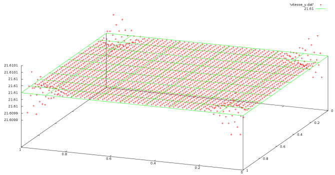


Figure – Vitesse selon y

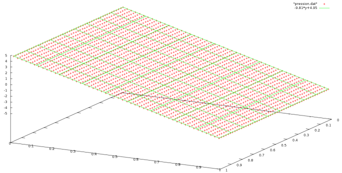


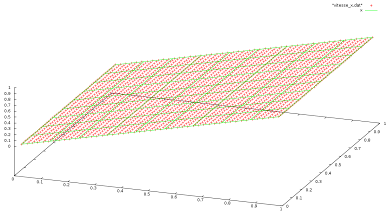
Figure – Pression

Cas tests

Cas Test 1a

$$
\begin{cases}
U = (x, -y) \\
F = (1, 1) \\
G = H = U \\
P = x + y + c
\end{cases}
\tag{11}
$$

Figure – Vitesse selon x



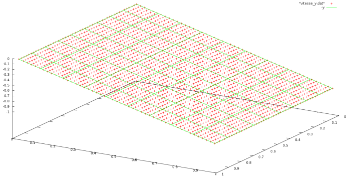


Figure – Vitesse selon y

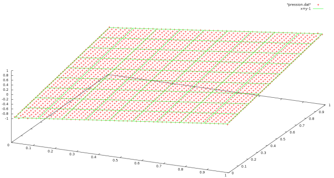


Figure – Pression

Cas tests

Cas Test 1b

$$
\begin{cases}
U = (x, y) \\
F = (1, 1) \\
G = H = U \\
P = x + y + c
\end{cases}
\qquad (12)
$$

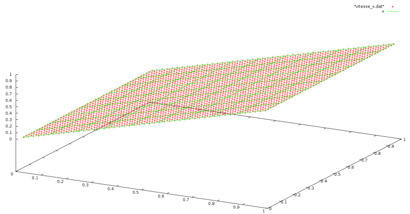


Figure – Vitesse selon x

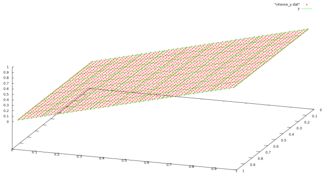


Figure – Vitesse selon y

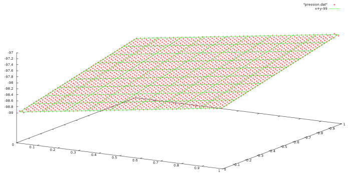


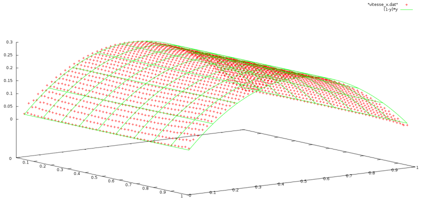
Figure – Pression

Cas tests

Poiseuille

$$\begin{cases} U = (L_y^2(1-\frac{y}{L_y})\frac{y}{L_y}, 0) \\ F = (0,0) \\ G = (1,0) \text{ et } H = (0,0) \\ P = -2x + c \end{cases} \qquad (13)$$

Figure – Vitesse selon x



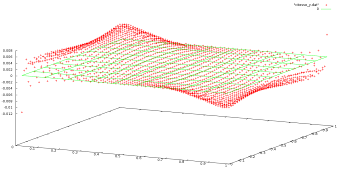


Figure – Vitesse selon y

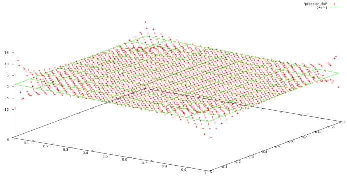


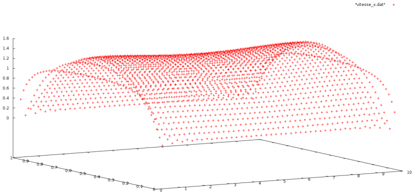
Figure – Pression

Cas tests

Génération d'écoulement de Poiseuille

$$\begin{cases} U = (1,0) \\ F = (0,0) \\ G = (1,0) \text{ et } H = (0,0) \\ P = -ax + c \end{cases} \tag{14}$$

Figure – Vitesse selon x



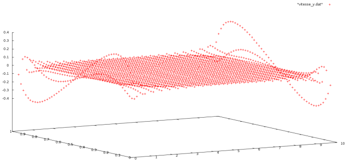


Figure – Vitesse selon y

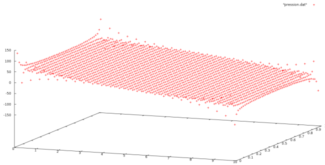


Figure – Pression

Erreurs et ordres

Figure – TC0 : Erreur=f(dx)

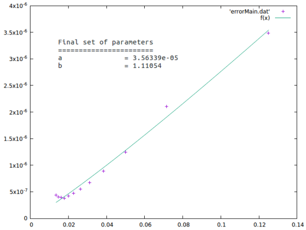


Figure – TC1a : Erreur=f(dx)

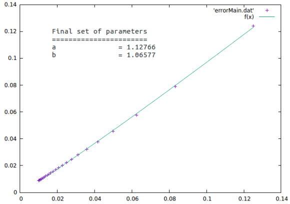


Figure – TC1b : Erreur=f(dx)

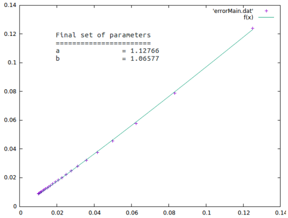
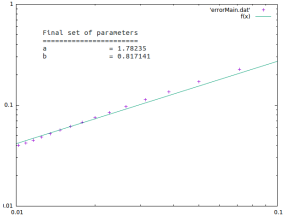
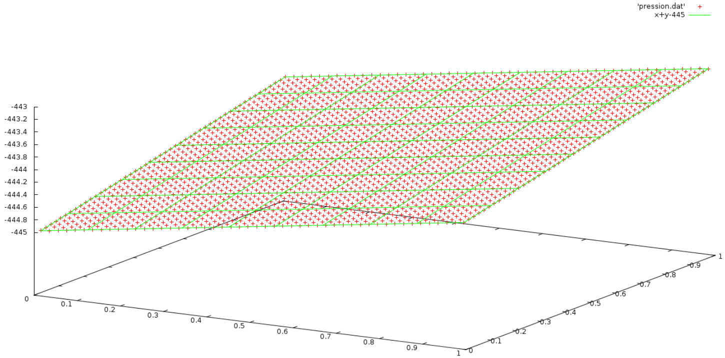


Figure – Poiseuille : Erreur=f(dx)



Ker(B^T)

Observation Cas TC1b



$\mathrm{Ker}(B^T)$

Introduction à la fonction Moy_0

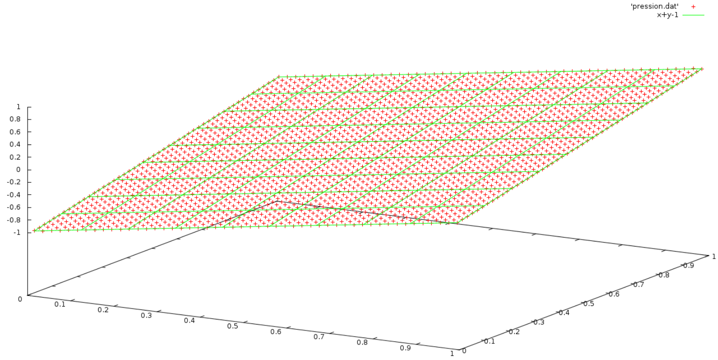
```
FUNCTION Moy_0(V)
  IMPLICIT NONE
  REAL*8,DIMENSION(:),INTENT(IN)::V
  REAL*8,DIMENSION(size(V))::Moy_0
  REAL*8::moy
  INTEGER::i

  moy=SUM(V)/size(V)

  DO i=1,size(V)
    Moy_0(i)=V(i)-moy
  END DO
END FUNCTION
```

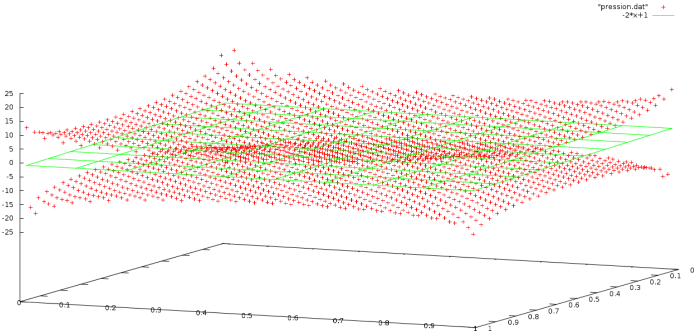
$\mathrm{Ker}(B^T)$

Après Moy_0



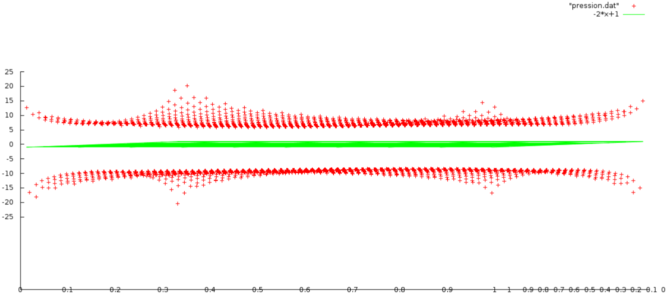
Ker(B^T)

Poiseuille



Ker(B^T)

Poiseuille



Influence de μ

μ	Cas test 0b		Cas test 1a	
-	Nb Iter	Résidu	Nb Iter	Résidu
1	749	10-4	223	10-4
2	1000	2.10-3	1000	3.10-4
3	-	NaN	-	NaN
0.5	970	10-4	292	10-4
0.1	1000	2.10-3	592	10-4
1.5	641	10-4	191	10-4

Conclusion

- Implémentation d'Uzawa et du gradient conjugué
- Gradient conjugué validé sur un problème de Poisson
- Résultats et cas tests probants
- Analyse des phénomènes issus de $\mathrm{Ker}(B^T)$
- Etude sur le critère μ de l'aglorithme d'Uzawa