Full Waveform Inversion Adjoint Studies

MATHIAS 2018

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Outline

Outline

The continuous model

Continuous Problem:

$$
\begin{cases}
\dfrac{1}{\rho_0 v_p{}^2}\dfrac{\partial p}{\partial t} + \dfrac{\partial v}{\partial x} = f_p \quad \text{in } \Omega \\
\rho_0 \dfrac{\partial v}{\partial t} + \dfrac{\partial p}{\partial x} = f_v \quad \text{in } \Omega \\
\end{cases}
$$

$$
\begin{cases}
p(t=0) = 0 \\
v(t=0) = 0 \\
\dfrac{\partial p}{\partial t} + v_p \dfrac{\partial p}{\partial x}.\boldsymbol{n} = 0 \quad \text{on } \Gamma
\end{cases}
$$

Figure: 1D Domain Model

The discretized model

Discretized Problem:

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{P}}}{\partial t} = A_{pv}\bar{\boldsymbol{V}} + A_{pp}\bar{\boldsymbol{P}} + \bar{\boldsymbol{F}}_p \\ \dfrac{\partial \bar{\boldsymbol{V}}}{\partial t} = A_{vp}\bar{\boldsymbol{P}} \end{cases}$$

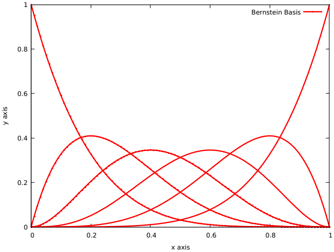
Figure: 1D Discretized Domain

- **Discontinuous Galerkin** space discretization
- Different **time-schemes** (RK4,AB3)
- Two polynomial basis (**Lagrange** and **Bernstein**)
- Constant velocity ($\boldsymbol{v_p}$) per cells
- Constant density ($\boldsymbol{\rho_0}$) per cells

Bernstein/DG Properties

Bernstein formulation:

$$B^N_{ijkl} = C^N_{ijkl}\lambda_0^i\lambda_1^j\lambda_2^k\lambda_3^l \quad \text{with: } C^N_{ijkl} = \frac{N!}{i!j!k!l!}$$



P[X^5] Bernstein basis

Bernstein/DG Properties

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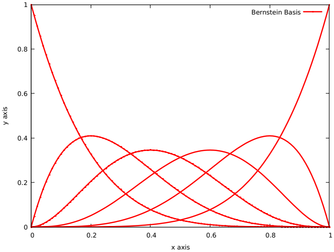
Easy Derivative expression:

$$\frac{\partial B^N_\alpha}{\partial \lambda_p} = N B^{N-1}_{\alpha - e_p} \quad \text{with: } \alpha = (i,j,k,l)$$



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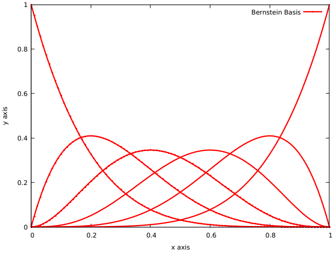
Easy Derivative expression:

$$\frac{\partial B^N_\alpha}{\partial \lambda_p} = N B^{N-1}_{\alpha - e_p} \quad \text{with: } \alpha = (i, j, k, l)$$

Sparse Degree Elevation operator:

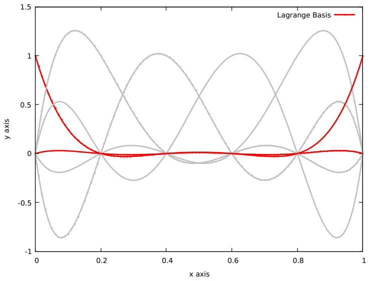
$$B^{N-1}_\alpha = \sum_{p=0}^{d} \frac{\alpha_p + 1}{N} B^N_{\alpha + e_p}$$



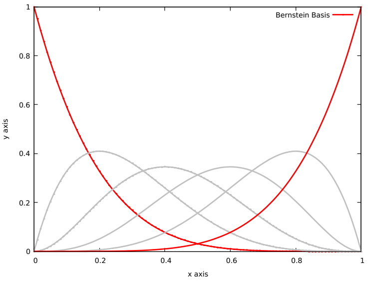
P[X^5] Bernstein basis

Bernstein/DG Properties

Unique boundary condition values :



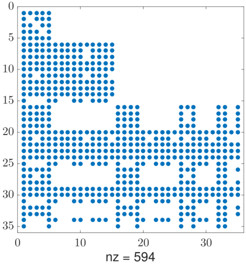
P(X^5) Lagrange basis



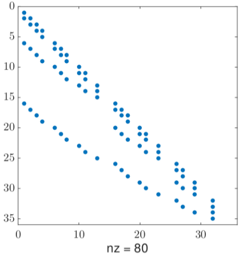
P(X^5) Bernstein basis

$\Rightarrow$ Same Flux Management

Derivative-Operator Analysis



3D Lagrange D matrix



3D Bernstein D matrix

[1] Chan J. and Warburton T.
GPU-Accelerated Bernstein Bézier Discontinuous Galerkin Methods for Wave Problems
SIAM Journal on Scientific Computing 2017

1D Results

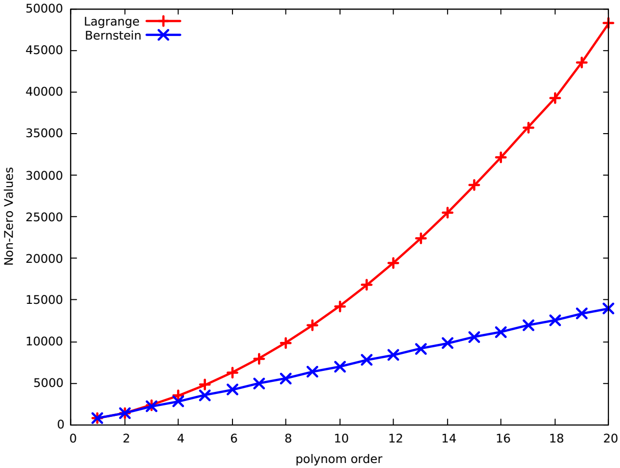
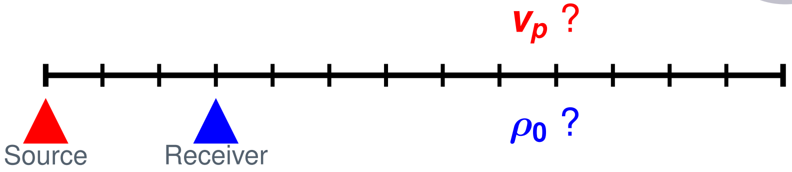


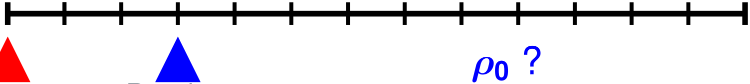
Figure: Operators NZVs as a function of the order

Outline

FWI Introduction

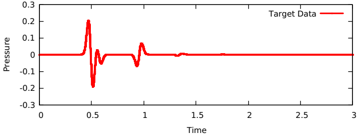


v_p ?

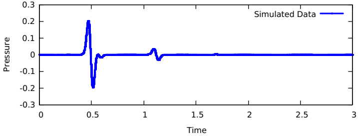


ρ_0 ?

Source   Receiver



Data and Simulation



$$J(p) = \frac{1}{2}||Rp - data||^2$$

- Find v_p and ρ_0 to minimize J

Continuous
Direct Problem

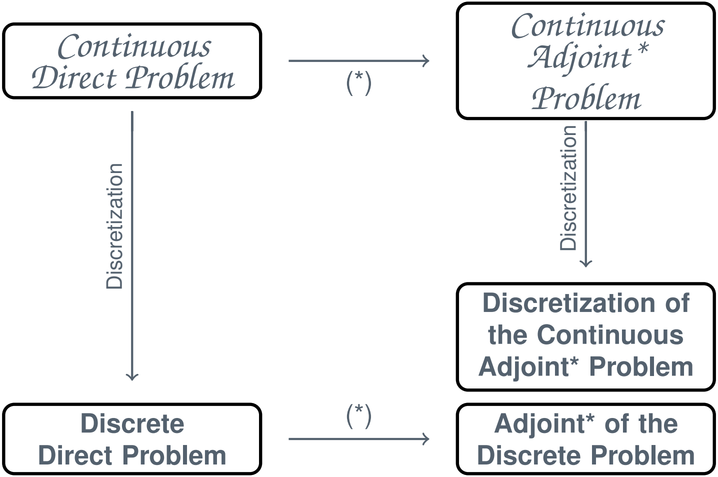
Adjoint Studies

$$\boxed{\text{Continuous Direct Problem}} \xrightarrow[(*)]{} \boxed{\text{Continuous Adjoint}^{*}\text{ Problem}}$$

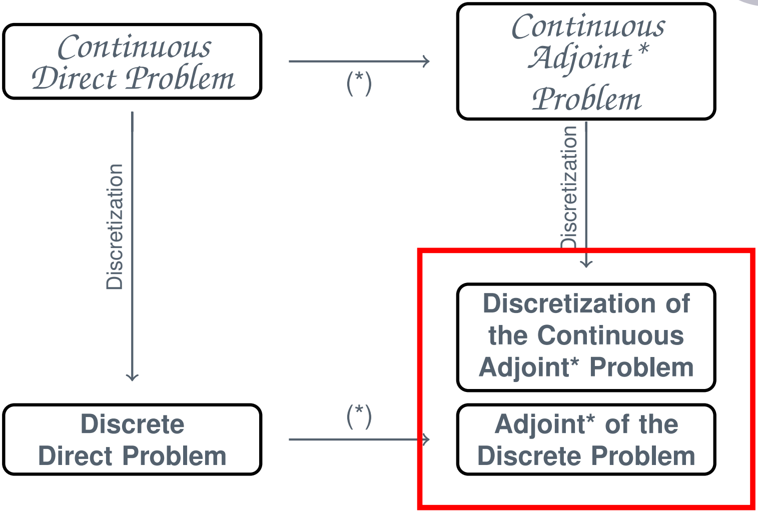
Adjoint Studies

$$\begin{tikzcd}[column sep=large, row sep=huge]
\text{Continuous Direct Problem} \arrow[r, "(*)"'] & \text{Continuous Adjoint* Problem} \arrow[d, "\text{Discretization}"] \\
& \textbf{Discretization of the Continuous Adjoint* Problem}
\end{tikzcd}$$

Adjoint Studies



Adjoint Studies



AtD : Adjoint then Discretized Strategy

$$J(p) = \frac{1}{2}||Rp - data||^2$$

$$\begin{cases} \dfrac{1}{\rho_0 v_p{}^2}\dfrac{\partial p}{\partial t} + \nabla.v = f_p \\ \rho_0 \dfrac{\partial v}{\partial t} + \nabla p = 0 \\ p(t=0) = 0 \\ v(t=0) = 0 \\ \dfrac{\partial p}{\partial t} + v_p \nabla p.\boldsymbol{n} = 0 \quad \text{on } \Gamma \end{cases}$$

$$\begin{cases} \dfrac{1}{\rho_0 v_p{}^2}\dfrac{\partial \lambda_1}{\partial t} + \nabla.\lambda_2 = \dfrac{\partial J}{\partial p} \\ \rho_0 \dfrac{\partial \lambda_2}{\partial t} + \nabla \lambda_1 = 0 \\ \lambda_1(t=T) = 0 \\ \lambda_2(t=T) = 0 \\ \dfrac{\partial \lambda_1}{\partial t} + v_p \nabla \lambda_1.\boldsymbol{n} = 0 \quad \text{on } \Gamma \end{cases}$$

$t \in [0, T]$ $\qquad$ $t \in [T, 0]$

AtD : Adjoint then Discretized Strategy

$$J(\bar{\boldsymbol{P}}) = \frac{1}{2}||R\bar{\boldsymbol{P}} - data||^2$$

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{P}}}{\partial t}^n = A_{pv}\bar{\boldsymbol{V}}^n + A_{pp}\bar{\boldsymbol{P}}^n + \bar{\boldsymbol{F}}_p^n \\ \dfrac{\partial \bar{\boldsymbol{V}}}{\partial t}^n = A_{vp}\bar{\boldsymbol{P}}^n \end{cases}$$

$$\begin{cases} \dfrac{\partial \bar{\boldsymbol{\Lambda}}_{\boldsymbol{1}}}{\partial t}^n = +A_{pv}\bar{\boldsymbol{\Lambda}}_{\boldsymbol{2}}^n + A_{pp}\bar{\boldsymbol{\Lambda}}_{\boldsymbol{1}}^n + \bar{\boldsymbol{D}}_p^n \\ \dfrac{\partial \bar{\boldsymbol{\Lambda}}_{\boldsymbol{2}}}{\partial t}^n = A_{vp}\bar{\boldsymbol{\Lambda}}_{\boldsymbol{1}}^n \end{cases}$$

Time-steps going Forward

Time-steps going Backward

DtA : Discretize then Adjoint Strategy

$$\frac{\partial \bar{\boldsymbol{U}}^n}{\partial t} = A\bar{\boldsymbol{U}}^n + \bar{\boldsymbol{F}}^n \quad \text{With : } \bar{\boldsymbol{U}} = \begin{pmatrix} \bar{\boldsymbol{P}} \\ \bar{\boldsymbol{V}} \end{pmatrix}, A = \begin{pmatrix} A_{pp} & A_p \\ A_v & 0 \end{pmatrix}, \bar{\boldsymbol{F}} = \begin{pmatrix} \bar{\boldsymbol{F}}_p \\ 0 \end{pmatrix}$$

All time scheme can be summed-up such as :

$$L\bar{\boldsymbol{U}} = E\bar{\boldsymbol{F}}$$

We are looking for a Discrete Adjoint state satisfying :

$$L^*\bar{\boldsymbol{\Lambda}} = -R^*(R\bar{\boldsymbol{U}} - data)$$

DtA : Discretize then Adjoint Strategy
Example with RK4

RK4 time-scheme leads to :

$$\bar{\boldsymbol{U}}^{n+1} = B\bar{\boldsymbol{U}}^{n} + C_0\bar{\boldsymbol{F}}^{n} + C_{\frac{1}{2}}\bar{\boldsymbol{F}}^{n+\frac{1}{2}} + C_1\bar{\boldsymbol{F}}^{n+1}$$

$$L\bar{\boldsymbol{U}} = E\bar{\boldsymbol{F}} = \bar{\boldsymbol{G}}$$

$$\begin{pmatrix} I & & & & \\ -B & I & & & \\ & -B & I & & \\ & & \ddots & \ddots & \\ & & & -B & I \end{pmatrix} \begin{pmatrix} \bar{\boldsymbol{U}}^0 \\ \bar{\boldsymbol{U}}^1 \\ \bar{\boldsymbol{U}}^2 \\ \vdots \\ \bar{\boldsymbol{U}}^n \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{G}}^0 \\ \bar{\boldsymbol{G}}^1 \\ \bar{\boldsymbol{G}}^2 \\ \vdots \\ \bar{\boldsymbol{G}}^n \end{pmatrix}$$

So :

$$L^* = \begin{pmatrix} I & -B^* & & & \\ & I & -B^* & & \\ & & \ddots & \ddots & \\ & & & I & -B^* \\ & & & & I \end{pmatrix}$$

Adjoint test

$$< L\bar{\boldsymbol{U}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, L^* \bar{\boldsymbol{\Lambda}} >$$

Adjoint test

$$< L\bar{\boldsymbol{U}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, L^*\bar{\boldsymbol{\Lambda}} >$$

$$\begin{cases} L\bar{\boldsymbol{U}} = E\bar{\boldsymbol{F}} = \bar{\boldsymbol{G}} \\ \bar{\boldsymbol{U}}(t=0) = 0 \end{cases} \qquad \begin{cases} L^*\bar{\boldsymbol{\Lambda}} = -R^*(R\bar{\boldsymbol{U}} - data) = \bar{\boldsymbol{D}} \\ \bar{\boldsymbol{\Lambda}}(t=T) = 0 \end{cases}$$

Time-steps going Forward

Time-steps going Backward

Adjoint test

$$< L\bar{\boldsymbol{U}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, L^* \bar{\boldsymbol{\Lambda}} >$$

$$\begin{cases} L\bar{\boldsymbol{U}} = E\bar{\boldsymbol{F}} = \bar{\boldsymbol{G}} \\ \bar{\boldsymbol{U}}(t = 0) = 0 \end{cases} \qquad \begin{cases} L^* \bar{\boldsymbol{\Lambda}} = -R^*(R\bar{\boldsymbol{U}} - data) = \bar{\boldsymbol{D}} \\ \bar{\boldsymbol{\Lambda}}(t = T) = 0 \end{cases}$$

Time-steps going Forward

Time-steps going Backward

$$< E\bar{\boldsymbol{F}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, -R^*(R\bar{\boldsymbol{U}} - data) >$$

Adjoint test

$$< L\bar{\boldsymbol{U}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, L^* \bar{\boldsymbol{\Lambda}} >$$

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Time-steps going Forward

Time-steps going Backward

$$< E\bar{\boldsymbol{F}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, -R^*(R\bar{\boldsymbol{U}} - data) >$$

$$< \bar{\boldsymbol{G}}, \bar{\boldsymbol{\Lambda}} > = < \bar{\boldsymbol{U}}, \bar{\boldsymbol{D}} >$$

Adjoint test succeeds $\Longleftrightarrow$ operator L^* well established

Outline

Adjoint test

Adjoint test passed for :

- Lagrange Operators
- Bernstein Operators
- Runge Kutta 4 time-scheme
- Adams Bashforth 3 time-scheme
- With a canonical space inner-product ($< u, v >_X = \sum_i u_i v_i$)
- With a M-space inner product ($< u, v >_X^M = < Mu, v >_X$)

Adjoint test

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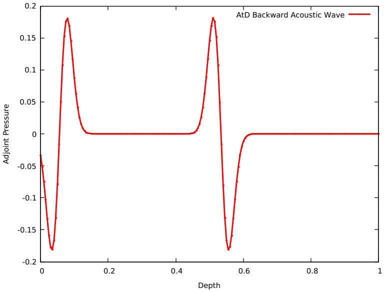
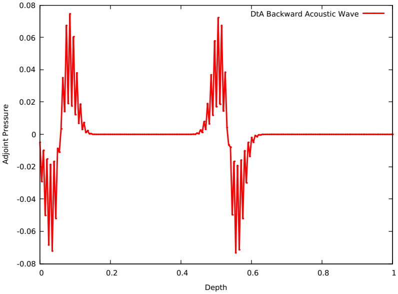
```
./run
--- Adjoint test ----
 inner product UP/DUDP 553123.57586755091
 inner product GPGU/QPQU 553123.57586756046
./run
--- Adjoint test ----
 inner product UP/DUDP -75077.332007383695
 inner product GPGU/QPQU -75077.332007386358
./run
--- Adjoint test ----
 inner product UP/DUDP 125669.89223600870
 inner product GPGU/QPQU 125669.89223600952
./run
--- Adjoint test ----
 inner product UP/DUDP -132852.64215701097
 inner product GPGU/QPQU -132852.64215701059
```

Non consistency of the Adjoint solution





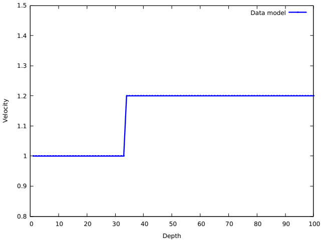
With the AtD strategy

With the DtA strategy using the canonical inner-product (Lagrange+RK4)

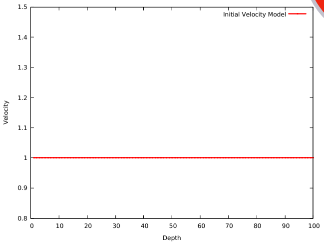
Adjoint test succeeds !

[1] Sei Alain and Symes William
A Note on Consistency and Adjointness for Numerical Schemes
1997

FWI Preliminary test (for all strategies)

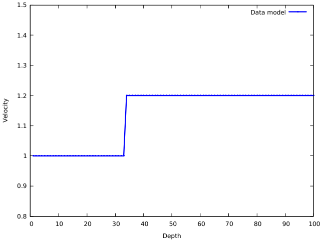


Target model

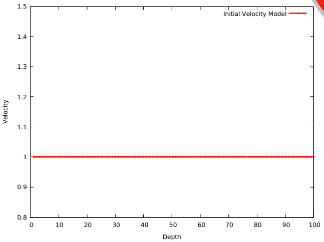


Initial model (iter=0)

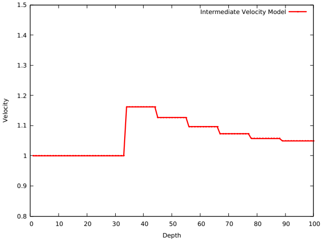
FWI Preliminary test (for all strategies)



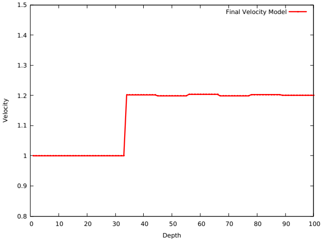
Target model



Initial model (iter=0)



Intermediate model (iter=20)



Final model (iter=50)

Qualitative Cost Function Gradient Study

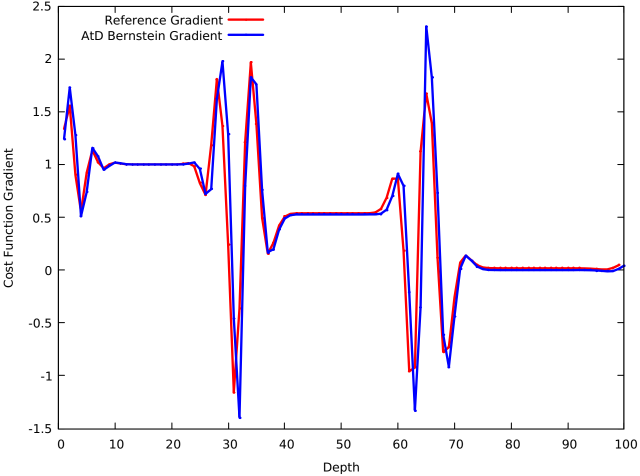


Figure: Comparison between a Reference Gradient and the FWI Gradient with AtD strategy (Bernstein elements and RK4 time scheme)

Qualitative Cost Function Gradient Study

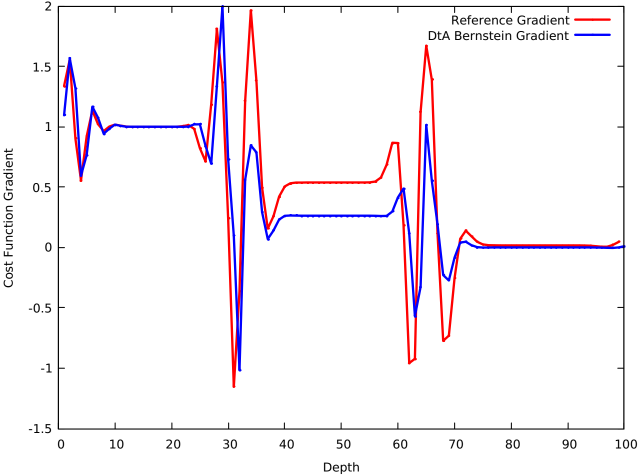
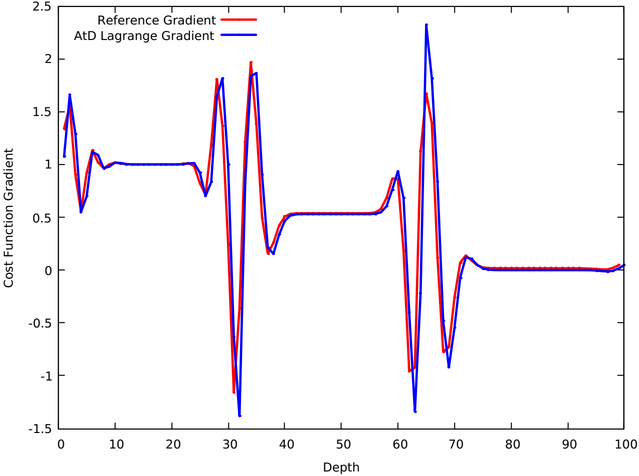


Figure: Comparison between a Reference Gradient and the FWI Gradient with DtA strategy (Bernstein elements and RK4 time scheme)

Qualitative Cost Function Gradient Study



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Qualitative Cost Function Gradient Study

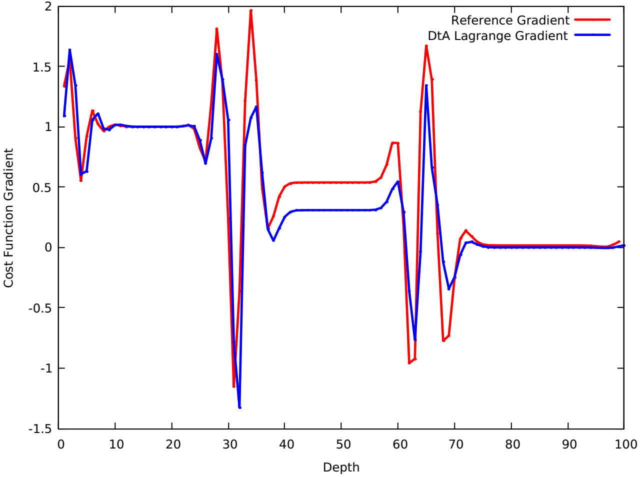

Figure: Comparison between a Reference Gradient and the FWI Gradient with DtA strategy (Lagrange elements and RK4 time scheme)

Conclusion and Perspectives

Conclusion :

- Adjoint then Discretized strategy works
- Discretized then Adjoint strategy has unexpected results (Gradient formulation ? Bug ?)
- The adjoint state is not consistent by using the Discretized and Adjoint strategy (but Adjoint test succeeds)

Conclusion and Perspectives

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- Adjoint then Discretized strategy works
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- The adjoint state is not consistent by using the Discretized and Adjoint strategy (but Adjoint test succeeds)

Perspectives :

- Complementary 1D tests
- 2D FWI + tests
- 3D FWI + tests
- Coupling SEM/DG elements (Aurélien Citrain's thesis)