State Estimation - Assignment 1

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Abstract

The first assignment of the directed studies course has the two following questio.

1 Question 1

x is a random variable of length K:

$$x = N(0,1)$$

a) What type of random variable is the following random variable?

$$y = x^T \times x$$

b) Calculate the mean and variance of y. c) Using Python, plot the PDF of y for K=1, 2, 3, 10, 100. Hint: see Sec 2.2.2

1.1 Part a)

After doing the vector product we will end up with the chi-squared of order K (based on the definition) $y = x_1^2 + x_2^2 + \cdots + x_k^2 = \sum_{i=1}^{k} x_i^2 \Rightarrow$ sum of k random variables

1.2 Part b)

To find the mean of y, we need to take its expectation and we will have:
$E[y] = \sum_{i=1}^{k} E\left[x_i^2\right] = k$

To calculate the variance, we need to calculate the below expectation:
$E\left[(y-k)^2\right] = E\left[y^2 - 2yk + k^2\right] = E\left[y^2\right] - 2E\left[yk\right] + k^2 = E[y^2] - k^2$
now we need to find the $E[y^2]$
$E[y^2] = E[(x_1^2 + ... + x_k^2)^2] = E[\sum_{i=1}^{k}\sum_{j=1}^{k} x_i^2 x_j^2] = k^2 + 2k$
substituting $E[y^2]$ yields:
$E[(y-k)^2] = 2k$

1.3 Part c)

When $K = 1$ our X is a random variable but when $K > 1$ our X is a random vector. But since each element of the random vector is a random variable, we can actually calculate the y by performing a summation for K times.
$K = 1 : X = [x_1] = x_1$

$y = X^T X = x_1^2$

$K = 2 : X = [x_1, x_2]$

$y = X^T X = [x_1, x_2]^T \times [x_1, x_2] = x_1^2 + x_2^2$
The figures are shown in Figs.1-5.

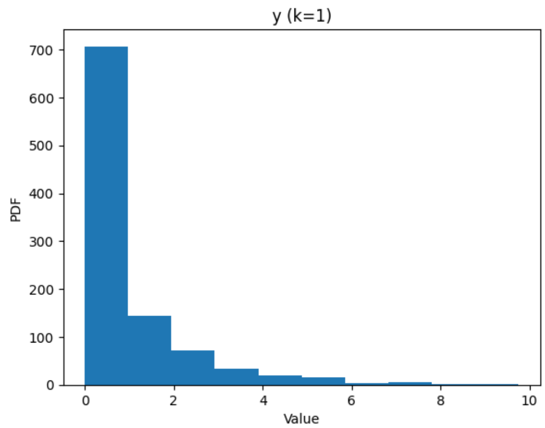

Figure 1: $K = 1$

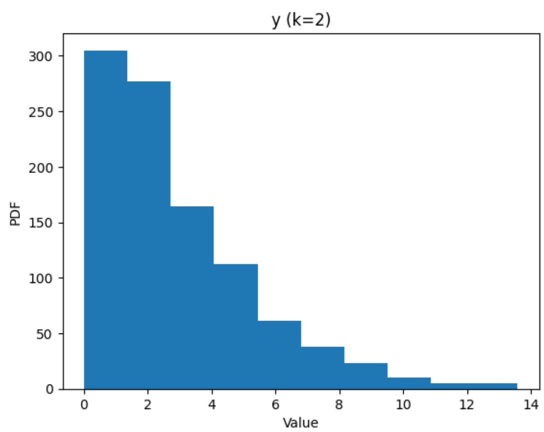

Figure 2: $K = 2$

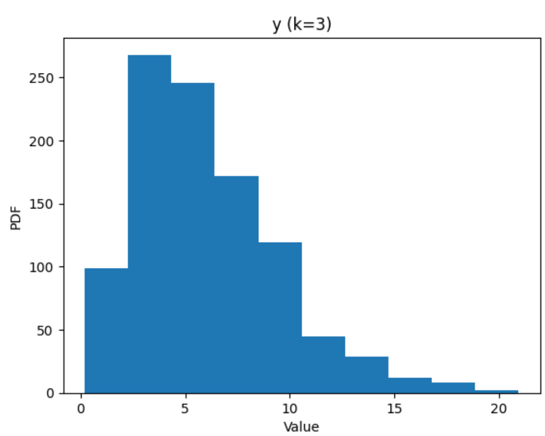

Figure 3: $K = 3$

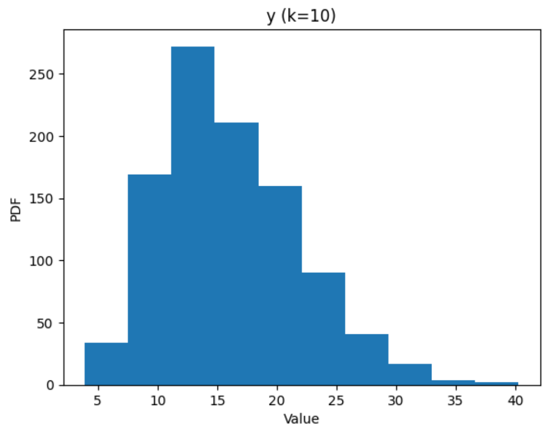

Figure 4: $K = 10$

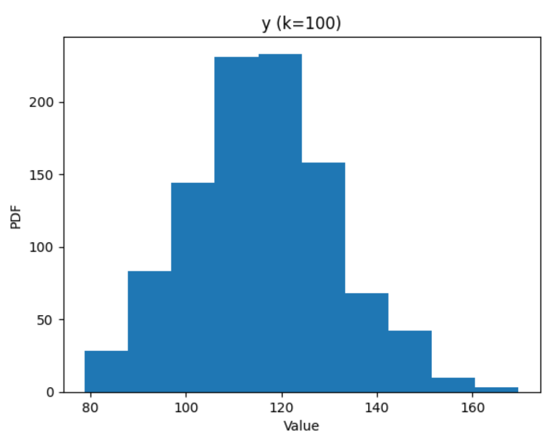

Figure 5: $K = 100$

2 Question 2

x is a random variable of length N:

$$x = N(\mu, \Sigma)$$

a) Assume x is transformed linearly, i.e., $y = Ax$, where A is an $N \times N$ matrix. Calculate the mean and covariance of y. Show the derivations.

b) Repeat a) when $y = A_1x + A_2x$.

c) If x is transformed by a nonlinear differentiable function, i.e. $y = f(x)$, compute the covariance matrix of y. Show the derivation.

d) Apply c) when:

$$x = \begin{bmatrix} \rho \\ \theta \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_{\rho\rho}^2 & \sigma_{\rho\theta}^2 \\ \sigma_{\rho\theta}^2 & \sigma_{\theta\theta}^2 \end{bmatrix}$$

$$y = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix}$$

Compute the covariance of y analytically. This models how range-bearing measurements in the polar coordinate frame are converted to a Cartesian coordinate frame.

e) Simulate d) using the Monte Carlo simulation, i.e. assume

$$x = \begin{bmatrix} 1m \\ 0.5° \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.005 \end{bmatrix}$$

Sample 1000 points from this distribution and plot the transformed results on x-y coordinates. Plot the uncertainty ellipse, calculated from part d). Overlay the ellipse on the point samples.

2.1 Part a)

Knowing the fact that a constant coefficient can be factorized from the expectation we can write:

$\mu_y = E[Ax] = AE[x] = A\mu_x$

Substituting the second moment to calculate the covariance yields:

$\sum_{yy} = E\left[(y - \mu_y)(y - \mu_y)^\top\right] = E\left[(Ax - A\mu_x)\left(x^\top A^\top - \mu_x A^\top\right)\right]$
$= AE\left[(x - \mu_x)(x - \mu_x)^\top\right] A^\top = A\Sigma A^\top$

2.2 Part b)

Following the same procedure for the mean we would have:

$E\left[A_1x + A_2x\right] = A_1E[x] + A_2E[x] = A_1\mu_x + A_2\mu_x$

And for the covariance we would have:

$E\left[(A_1 + A_2)x - (A_1 + A_2)\mu_x)((A_1 + A_2)x - (A_1 + A_2)\mu_x)^\top\right]$
$= (A_1 + A_2)E\left[(x - \mu_x)(x - \mu_x)^T\right](A_1 + A_2)^T = (A_1 + A_2)\sum_{yy}(A_1 + A_2)^T$

2.3 Part c)

To calculate the covariance matrix of $y = f(x)$ we assume $E[f(x)] = \mu_y$, and start by implementing the second moment formula for y : $\sum_{yy} = E[(y -_y)(y -_y)^T] = E[(f(x) -_y)(f(x) -_y)^T] = E[f(x)f(x)^T - f(x)_y^T -_y f(x)^T +_y {}_y^T]$ Now, we need to take the expectation of each term in the above equation and sum them up together in the end. For the first expectation, We need to represent the function in terms of its Jacobian $E[f(x)f(x)^T] = E[Mxx^T]$ and $M_{ij} = \partial f_i/\partial x_j$ Since μ_y is a mean and is constant, we can take it out of the expectation and write: $E[f(x)_y^T] =_y^T E[f(x)]$ and $E[_y f(x)^T] =_y E[f(x)^T]$ Finally, the last

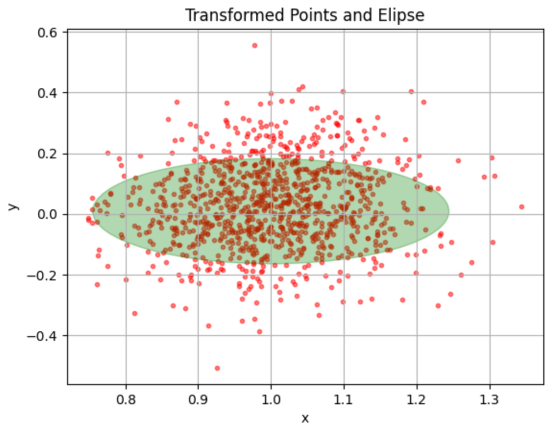

Figure 6: Monte-Carlo

term is the outer product of two constant vectors and comes out of the expectation. Now we can write the covariance as: $\sum_{yy} = E[Mx(Mx)^T] -_y^T E[f(x)] -_y E[f(x)]^T +_y {}_y^T = E[Mx(Mx)^T] -_y^T {}_y -_y {}_y^T +_y {}_y^T$ $\sum_{yy} = E[Mx(Mx)^T] -_y {}_y^T = ME[xx^T]M^T -_y {}_y^T$ $\sum_{yy} = M \sum_{xx} M^T -_y {}_y^T$

2.4 Part d)

First we find the Jacobian of y: $J(y) = \begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix}$ then, substitute them in the equation obtained in Part c) and we will have: $\sum_{yy} = \begin{bmatrix} {}^2cos^2 - 2^2 sincos +^2 sin^2 & {}^2cossin -^2 sincos +^2 cos^2 \\ {}^2cossin -^2 sincos +^2 cos^2 & {}^2sin^2 + 2^2 sincos +^2 cos^2 \end{bmatrix}$

2.5 Part e)

Running the Monte-Carlo algorithm for the given values yields the transformed points and elipse inFig.6.