Bayes Filtering

AN INTRODUCTION BY EXAMPLE

Introduction

The theme of this course is navigation: we want to figure out where we are relative to other things.

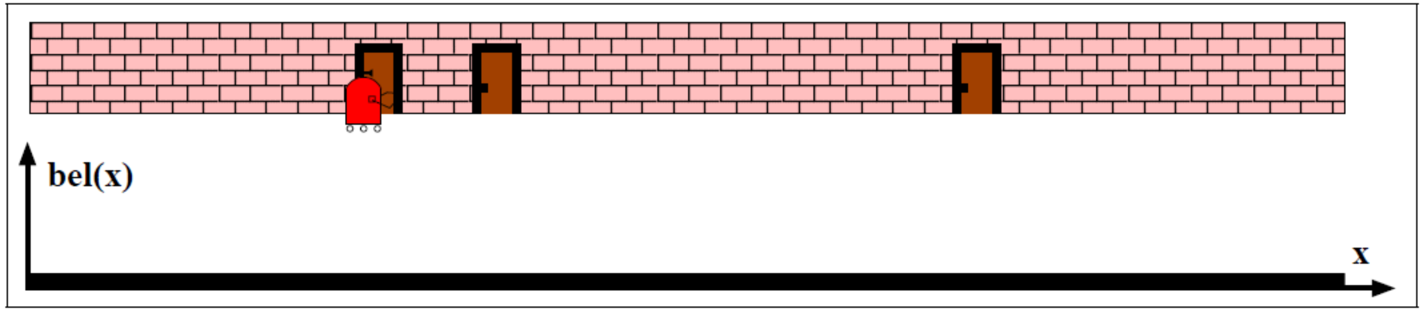
We refer to all possible places you might be as the *state space* and a specific sample of it is referred to as the *state*.

- Typically represented as x

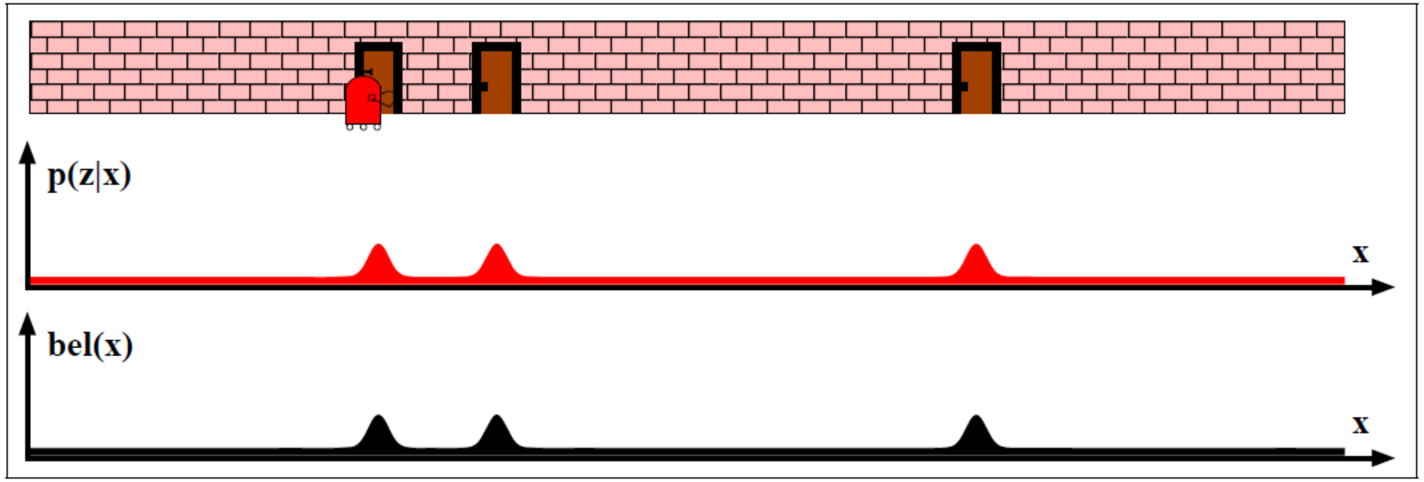
The basic idea behind a probabilistic framework for navigation and the Bayes Filter is to represent your localization as a continuous probability distribution known as the *belief*.

This straightforwardly translates to navigation solutions: you are currently located at the state with the highest belief.

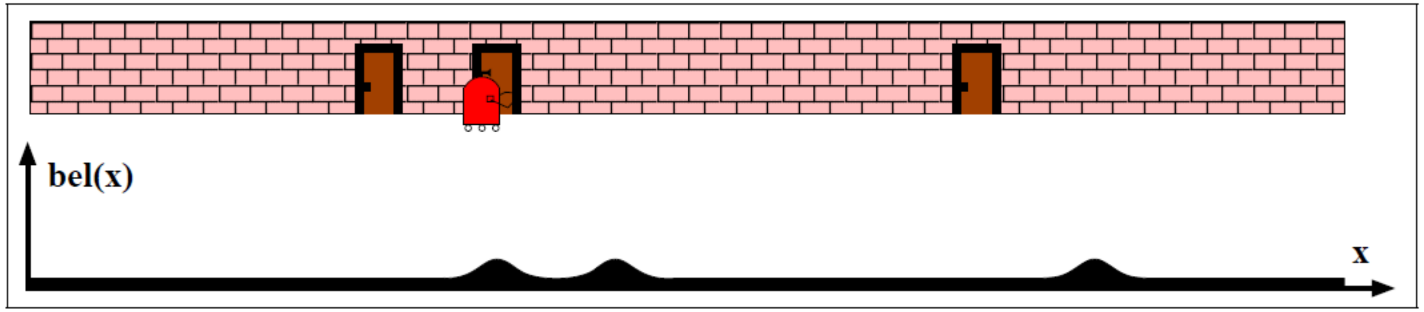
An example in one dimension



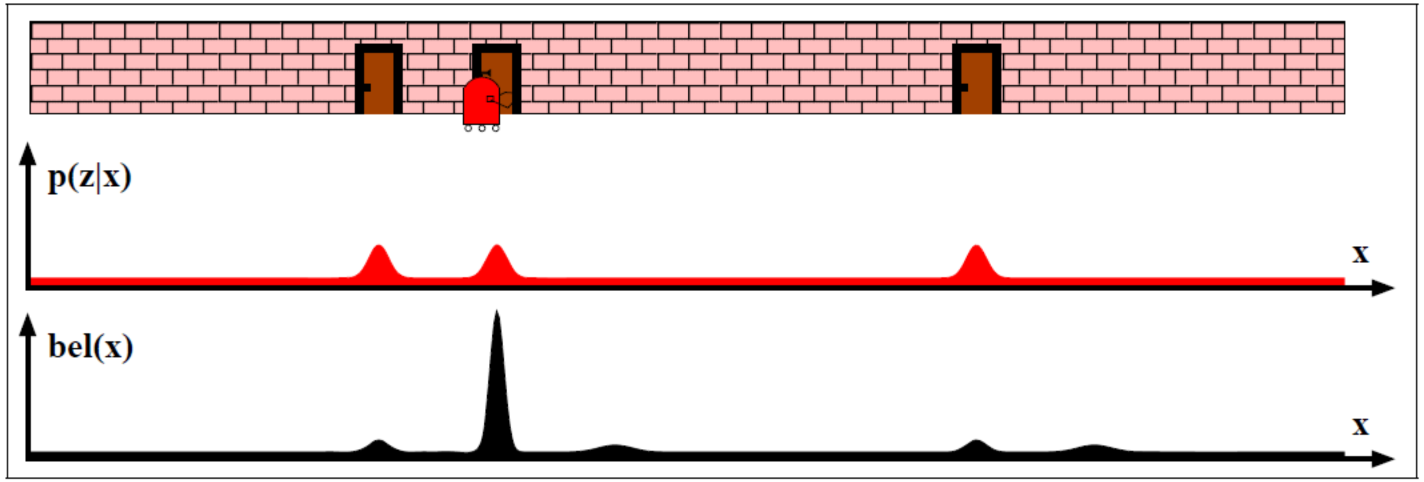
An example in one dimension



An example in one dimension



An example in one dimension



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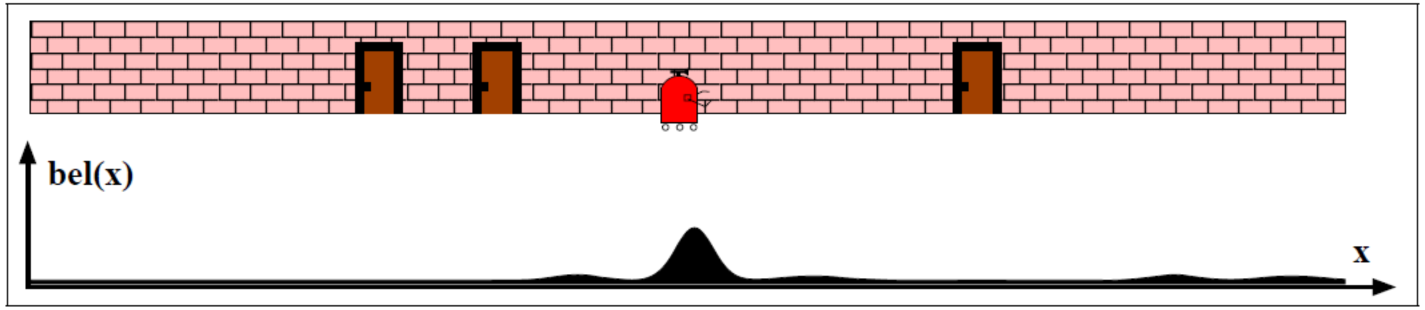
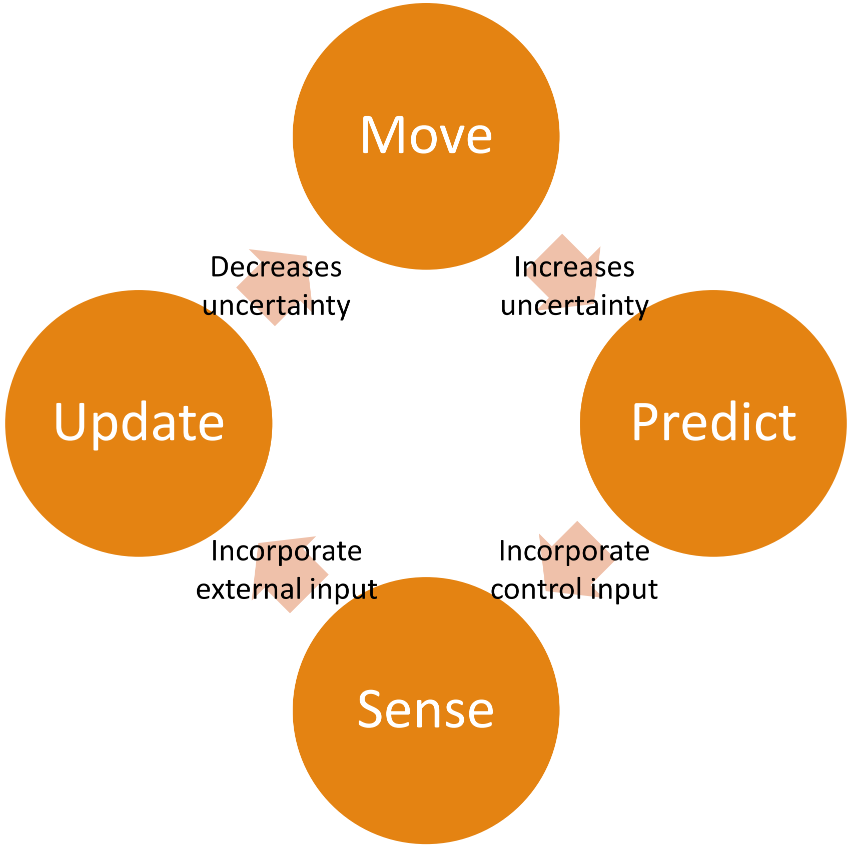


Problem overview



Bayes Filter

PROBABILITY REVIEW

Terminology

Random Variable

- Definition: Variable whose value is subject to change due to randomness or chance
- Properties:
  - Can be continuous (e.g., position in 3D) or discrete (e.g., roll of a die)
  - Observed values of random variables are called realizations
- Example: Pose of a robot, $p(X=x)$, or value of a rolled die, $p(D=d)$

Probability Density Function

- Definition: Function describing the likelihood that a random variable X will take on a particular value x.
- Properties:
  - Total probability is equal to one
    - Continuous: $\int p(X = x)dx = 1$
    - Discrete: $\sum_x p(X = x) = 1$
  - Non-negativity: $p(X = x) \geq 1$
- Example: Pose of the robot, $p(x) = p(X = x)$

Terminology

Expected Value

- Definition: Probability-weighted average value, the center of mass of the probability distribution
  - $E[X] = \int p(X = x)x\, dx$

Joint Probability Distribution

- Definition: The probability density function of a set of two or more random variables (multivariate distribution)
- $p(X = x, Z = z)$

Covariance

- Definition: A measure of two random variables change together
  - $\sigma(X, Y) = E[(X - E[X])(Y - E[Y])]$
- The variance is a special case where the two random variables are identical: $\sigma^2(X) = \sigma(X, X)$
- Can be thought of as the certainty of the distribution
- Frequently represented as a matrix where $X = [\, x_1, x_2, \dots, x_n]^T$

Terminology

Independence

- Definition: Two random variables are independent if the outcome of one has no effect on the outcome of the other
- Example: If X, Z are the outcomes of two dice rolls, $p(X, Z) = p(X)p(Z)$
- Properties:
  - Independent random variables are uncorrelated $\sigma(X, Z) = 0$
  - Uncorrelated random variables are not necessarily independent

Conditional Probability

- Definition: Probability of an event occurring conditioned on another event occurring.
- $p(z|x) = \frac{p(x,z)}{p(x)} \rightarrow p(x, z) = p(z|x)p(x)$

Terminology

Conditional Independence

- Two random variables are conditionally independent if the outcome of one has no effect on the outcome of the other when conditioned on the outcome of a third random variable
- $p(z_1, z_2|x) = p(z_1|x)\, p(z_2|x)$

Marginal Distribution

- The probability distribution of the subset of a collection of random variables
  - $p(z) = \int p(x, z)dx$
- Also known as the Law of Total Probability
  - $p(z) = \sum_i^n p(z|X = x_i)p(X = x_i)$

Bayes Filtering

DERIVATION

Bayes' Theorem: A review

Describes how the belief about a random variable should change to account for the collected evidence (measurements).

$$p(x|z) = \frac{p(z|x)p(x)}{p(z)}$$

For a time-indexed system with a state, measurements, and controls:

$$p(x_t|z_{1:t}, u_{1:t}) = \frac{p(z_t|x_t)p(x_t|z_{1:t-1}, u_{1:t-1})}{p(z_t|z_{1:t-1}, u_{1:t})}$$

Markov assumption and simplification

A Bayes' Filter is recursive and calculates the probability accounting for *all* prior measurements and controls.

Mathematically impossible to implement for any significant duration of time.

Simplify the system by using assuming it is a first-order Markov chain:

- Assume each state is conditionally independent
- Only the previous state and current controls effects the current state

Simplification permits recursion via sequential processing making the filter tractable.

Overview

Basic two-step filter consisting of a prediction step and a measurement step.

- Calculates the probability of a state at time t given all previous measurements and inputs.
- $p(x_t | z_{1:t}, u_{1:t})$

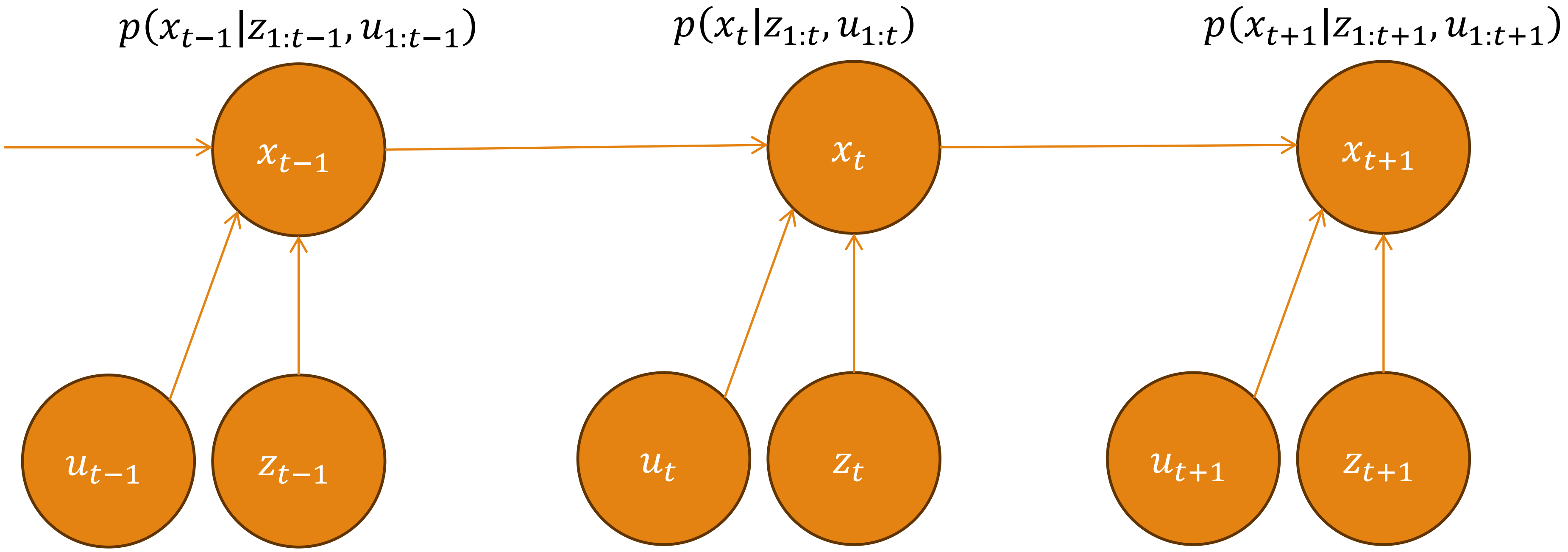
Prediction step

- Takes the prior belief about the state and control input
- Calculates the predicted belief about the state from the process or motion model

Measurement (update) step

- Takes the predicted belief and a measurement
- Calculates the posterior belief about the state according to the measurement model

Overview


$p(x_{t-1}|z_{1:t-1}, u_{1:t-1})$
$p(x_t|z_{1:t}, u_{1:t})$
$p(x_{t+1}|z_{1:t+1}, u_{1:t+1})$
x_{t-1}
x_t
x_{t+1}
u_{t-1}
z_{t-1}
u_t
z_t
u_{t+1}
z_{t+1}

Prediction Step

You can find the current pose via a marginal distribution:

- $p(x_t|z_{1:t-1}, u_{1:t}) = \int p(x_t, x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1}$

Expand the integral terms using a conditional distribution:

- $p(x_t|z_{1:t-1}, u_{1:t}) = \int p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t})p(x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1}$

First order Markov assumption allows for us to state that the current state is conditionally independent of the *past* measurements and controls given the current state.

- Process model: $p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t}) = \ p(x_t|x_{t-1}, u_t)$

The prediction step becomes: $p(x_t|z_{1:t-1}, u_{1:t}) = \int p(x_t|z_{1:t-1}, u_{1:t})\ p(x_{t-1})dx_{t-1}$

Update Step

In the update step, we calculate the posterior distribution by multiplying the likelihood and the predicted distribution, and then normalizing.

- $p(x_t|z_{1:t}, u_{1:t}) = \frac{p(z_T|x_t, z_{1:t-1}, u_{1:t})p(x_t|z_{1:t-1}, u_{1:t})}{p(z_t|z_{1:t-1}, u_{1:t})}$

The denominator can be re-written as a marginal distribution of the numerator

- $p(z_t|z_{1t-1}, u_{1:t}) = \int p(z_t|x_t)p(x_t|z_{1:t-1}, u_{1:t})dx_t$

Markov assumption allows for us to treat the measurement as conditionally independent of the past measurement.

- $p(x_t|z_{1:t-1}, u_{1:t}) = p(z_t|x_t)$

Update step becomes:

$$p(x_t|z_{1:t}, u_{1:t}) = \frac{p(z_t|x_t)\, p(x_t|z_{1:t-1}, u_{1:t})}{\int p(z_t|x_t)\, p(x_t|z_{1:t-1}, u_{1:t})dx_t}$$

Pseudocode

A Bayes filter is only two functions: predict and update

```
for (u, z) in (controls, measurements):
   x = predict(x, u)
   x = update(x, z)
```

Consider a one-dimensional constant velocity system where the state is position, controls are velocity, and the measurement is position. The sensor is modeled as a normal distribution with accuracy σ.

```
predict(state, control, dt):
   return state += control * dt

update(state, measurement):
   return (1 / (σ √2π)) * exp(−0.5 ((state − measurement)/σ)^2)
```