1 Introduction

Over the past few years my research has been focused on an area of algebraic topology called unstable homotopy theory. Stable homotopy theory studies spaces after taking an infinite number of suspensions and studies generalized (co)homology theories through stable homotopy objects called spectra. In contrast, we can instead just focus on the spaces in the Ω-spectrum for a generalized (co)homology theory and use unstable homotopy information about these spaces to get information about the (co)homology theory of interest. In particular, I have been studying the spaces in the truncated Brown-Peterson spectra, $BP\langle n\rangle$, which were first defined and studied by Steve Wilson in his Ph.D. thesis (see [Wil73] and [Wil75]), in order to glean information about BP (co)homology and $BP\langle n\rangle$-bundles.

Relatedly, I have also been studying Hopf rings (an algebraic structure first defined by Milgram ([Mil70]) and popularized by Ravenel and Wilson ([RW77])) and how they can be used to compute the generalized (co)homology of spaces in different Ω-spectra. I have used the Hopf ring structure on E_*G_* to give a new perspective on the torsion-free part of the homology of the even connective covers of BU, as well as a new proof of Ando, Hopkins, and Strickland's cocycle relations (see [AHS01]) using the Ravenel-Wilson relations (see [RW77]). I also give a new proof of Ando, Hopkins, and Strickland's result that these are the defining relations for the first three even connective covers of BU. Currently, I am continuing this course of study by computing the Hopf ring E_*G_* for a much wider variety of spectra E and G, starting with $H_*(BP\langle 2\rangle_*;\mathbb{F}_p)$ and $BP\langle 2\rangle_*BP\langle 2\rangle_*$. These computations will be of great use within homotopy theory, and beyond, both by shedding light on the nature of $BP\langle n\rangle$-bundles and by providing a greater understanding of the spaces that appear in this wider class of Ω-spectra.

2 Background

An Ω-spectrum, E, consists of a sequence of topological spaces, $\{E_n\}_{n\in\mathbb{Z}}$, along with a weak homotopy equivalence $E_n \to \Omega E_{n+1}$ for each n. These Ω-spectra are especially useful because they allow us to describe any generalized homology or cohomology theory just in terms of homotopy classes of maps. More specifically, for any generalized cohomology theory $E^*(-)$, there is an Ω-spectrum E *representing* $E^*(-)$, i.e. such that for any space X, $E^n(X) = [X, E_n]$. Similarly, for any generalized homology theory $E_*(-)$, there is an Ω-spectrum E *representing* $E_*(-)$ such that for any space, X,

$$E_n(X) = \pi_n(E \wedge X) = \varinjlim_m \pi_{n+m}(E_m \wedge X).$$

Furthermore, we can extend this definition to take the generalized (co)homology of spectra instead of spaces, so that for any Ω-spectra E and G we have:

$$E^n G_k = [G_k, E_n]$$

$$E_n G_k = \varinjlim_m \pi_{n+m}(G_k \wedge X)$$

$$E^n(G) = \varprojlim_m E^{n+m} G_m$$

$$E_n G = \varinjlim_m E_{n+m} G_m$$

$$\pi_n(G) = \varinjlim_m \pi_{n+m}(G_m)$$

Of particular interest in my work is the Ω-spectrum, BP, which is called the Brown-Peterson spectrum and is often studied because of its intimate connection with complex cobordism (if MU is the Ω-spectrum representing complex cobordism, then $MU_{(p)} = \vee_i(\Sigma^i BP)$, for a given prime p). The spectrum BP has homotopy groups given by

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, ...]$$

where $|v_i| = 2(p^i - 1)$.

In [Wil75], Wilson used Sullivan's theory of manifolds to construct the *truncated Brown Peterson spectra*, $BP\langle n\rangle$, with homotopy groups given by:

$$\pi_*(BP\langle n\rangle) = \mathbb{Z}_{(p)}[v_1, v_2, ..., v_n].$$

Although this is not the same as the Postnikov tower for BP (since we are not killing off *everything* above degree $2(p^n - 1)$, only the *generators* above degree $2(p^n - 1)$), it similarly gives us a way to build up BP in successive layers so that $\varinjlim_n BP\langle n\rangle = BP$. In [Wil73] and [Wil75], Wilson computed $H_*(BP_k; \mathbb{Z}_{(p)})$ for all k and $H_*(BP\langle n\rangle_k; \mathbb{Z}_{(p)})$ for all $k < g(n) = 2(p^n + p^{n-1} + ... + 1)$. Later, in [Sin76], Sinkinson computed $H^*(BP\langle n\rangle_k; \mathbb{F}_p)$ for all n and k, giving a complete description of the unstable mod p cohomology of the truncated Brown-Peterson spectra.

The study of Ω-spectra also naturally gives rise to the notion of a *Hopf ring*, which is an algebraic structure existing on E_*G_* for a wide class of Ω-spectra E and G. Hopf rings first appeared in Milgram's 1970 paper "The mod 2 spherical characteristic classes" ([Mil70]) in which he computed the mod 2 homology of the spaces in the sphere spectrum and defined a Hopf ring as a ring object in the category of coalgebras. Ravenel and Wilson later made use of this algebraic machinery in their papers "The Hopf ring for complex cobordism" ([RW77]) and "The Morava K-theories of Eilenberg-MacLane spaces and the Conner-Floyd conjecture." ([RW80]) In these papers they successfully established the usefulness of Hopf rings as a "descriptive and computational tool" in algebraic topology and paved the way for the wide variety of Hopf ring computations that were to follow.

Hopf rings are so named because of their intimate connection with both Hopf-spaces (H-spaces) and Hopf algebras. For any generalized homology theory $E_*(-)$ which satisfies the Kunneth isomorphism for all spaces (i.e. $E_*(X \times Y) \cong E_*(X) \otimes E_*(Y)$), $E_*(X)$ is not only a graded vector space, but also a graded coalgebra with coproduct

$$\psi = \Delta_* : E_*(X) \to E_*(X \times X) \cong E_*(X) \otimes E_*(X)$$

induced from the diagonal map $\Delta : X \to X \times X$. Additionally, if X is an H-space with product $\gamma : X \times X \to X$, we get an induced product

$$* = \gamma_* : E_i(X) \otimes E_j(X) \to E_{i+j}(X)$$

which makes $E_*(X)$ into a *graded Hopf algebra* (a group object in the category of graded coalgebras). Furthermore, if we take X to be not just an H-space, but rather the Ω-spectrum associated with a ring spectrum (meaning each of the spaces of X is an H-space), then the product $\mu : X_m \wedge X_n \to X_{m+n}$ gives us even more structure on $E_*(G_*)$. In addition to the Hopf algebra structure given by the maps

$$\psi : E_*(X_n) \to E_*(X_n) \otimes E_*(X_n)$$

and

$$* : E_i(X_n) \otimes E_j(X_n) \to E_{i+j}(X_n)$$

we also get a map

$$\circ = \mu_* : E_i(X_m) \otimes E_j(X_n) \cong E_*(X_m \wedge X_n) \to E_{i+j}(X_{m+n})$$

which makes $E_*(G_*)$ into a graded ring object in the category of graded coalgebras, i.e. a *graded Hopf ring*.

Since the pioneering work of Milgram, Ravenel, and Wilson, Hopf rings have been used to compute the (generalized) homology of the spaces in a wide variety of different Ω-spectra. Among other reasons, these computations are particularly useful because E_*G_*, the unstable homology cooperations, are simply the dual of E^*G^*, the unstable cohomology operations.

When we specialize to complex-orientable, multiplicative homology theories, $E_*(-)$ and $G_*(-)$, with associated Ω-spectra E and G, and assume E has a Kunneth isomorphism, Ravenel and Wilson ([RW77]) define special homology elements $b_i^G \in E_*G_*$ and give relations on these elements as follows:

Definition 2.1. For $i \geq 0$, we define

$$b_i^G = (x^G)_*(\beta_i) \in E_{2i}(G_2)$$

where $x^G : \mathbb{CP}^\infty \to G_2$ is the complex orientation for G, $x^E : \mathbb{CP}^\infty \to E_2$ is the complex orientation for E, and β_i is the dual of $(x^E)^i \in E^{2i}(\mathbb{CP}^\infty)$.

Furthermore, we will use the notation

$$b^G_{(i)} = b^G_{p^i}$$

$$b^G(s) = b^G_0 + b^G_1 s + b^G_2 s^2 + ... = \sum_i b^G_i s^i$$

$$b^G(x_1, ..., x_n) = \sum_{i_1,...i_n} (b^G_{i_1} \circ ... \circ b^G_{i_n}) x_1^{i_1} ... x_n^{i_n} = \sum_{i_1,...i_n} (b^G_{i_1,...,i_n}) x_1^{i_1} ... x_n^{i_n}$$

If we let $s +_E t$ be the formal group law for E and $x +_G y = \sum_{i,j} a_{i,j} x^i y^j$ the formal group law for G, then we get the Ravenel-Wilson relations below.

Theorem 2.2 (RW77, Theorem 3.8)**.** *In $E_*G_*[[s,t]]$,*

$$b^G(s +_E t) = b^G(s) +_{[G]} b^G(t)$$

where $b^G(s) +_{[G]} b^G(t) = *_{i,j}([a_{i,j}] \circ b^G(s)^{\circ i} \circ b^G(t)^{\circ j})$

Comparing the coefficients of each $s^i t^j$ in the equality above gives us relations in E_*G_2. These relations, in turn, generate relations in the entire Hopf ring E_*G_*.

Working from a more algebro-geometric perspective, Ando, Hopkins, and Strickland ([AHS01]) gave relations in E_*ku_* (where ku is the Ω-spectrum for connective complex K-theory) that amount to the following:

Theorem 2.3. *The following relation holds in $E_*(ku_{2k})$:*

$$b^{ku}(x_1, x_2, x_3, ..., x_k) * b^{ku}(x_0, x_1 +_E x_2, x_3, ..., x_k) = b^{ku}(x_0 +_E x_1, x_2, x_3, ...x_k) * b^{ku}(x_0, x_1, x_3, ..., x_k) \tag{2.1}$$

Additionally, for $k \leq 3$, they show that these cocycle relations, along with basic symmetry relations, serve as the defining relations in $E_*(ku_{2k})$, i.e.

$$E_*(ku_{2k}) = E_*[b^{ku}_{i_1,...,i_k}]/\{\sim, S\} \tag{2.2}$$

where $\sim$ indicates the cocycle relations given by (2.1), and S indicates the symmetry relations $b^{ku}_{i_1,i_2,...i_k} = b^{ku}_{\sigma(i_1),\sigma(i_2),...\sigma(i_k)}$ for all permutations σ. For $E = H\mathbb{Q}$, they also show that (2.2) holds for all k.

3 Completed Research

The majority of the work I have done over the past few years under the guidance of my advisor, Mark Behrens, has been related to the (co)homology of the $BP\langle n\rangle_k$ and the Hopf ring, $E_*BP\langle n\rangle_*$, for different n and E. Firstly, I have recovered the aforementioned results of Ando, Hopkins, and Strickland, using the following alternative perspective.

I have a new proof of the Ando-Hopkins-Strickland cocycle relations (2.1), which involves no algebraic geometry, but instead uses the Ravenel-Wilson relations in Theorem 2.2, the Hopf ring structure on $E_*G_*[[x_1, ..., x_k]]$, and general facts about Hopf rings. Combining this with Ravenel and Wilson's work related to $E_*(BP\langle n\rangle_k)$, as well as the fact that the p-localization of connective complex K-theory is $ku_{(p)} = \vee_i(\Sigma^i BP\langle 1\rangle)$, I also prove (2.2) for $k \leq 3$ and complex-orientable, multiplicative $E_*(-)$, once again, without any reference to an algebro-geometric perspective.

Theorem 3.1.

$$E_*(ku_2) = E_*[b_i^{ku}]$$

Theorem 3.2.

$$E_*(ku_4) = E_*[b_{i,j}^{ku}]/\{\sim\}$$

where $\sim$ indicates the cocycle relations given by (2.1) for $k = 2$.

Theorem 3.3.

$$E_*(ku_6) = E_*[b_{i,j,k}^{ku}]/\{\sim, S\}$$

where $\sim$ indicates the cocycle relations given by (2.1) for $k = 3$, and S indicates the symmetry relations $b_{i,j,k}^{ku} = b_{\sigma(i),\sigma(j),\sigma(k)}^{ku}$ for all permutations σ.

I have also given a nice presentation of the rational homology of ku_{2k} in terms of the generators $b_{i_1,\dots,i_k}^{ku}$. Using the fact that, rationally, infinite loops spaces are products of Eilenberg-MacLane spaces, we have that

$$H_*(BP\langle n\rangle_{2k};\mathbb{Q}) = \mathbb{Q}[[v_1^{j_1}...v_n^{j_n}] \circ (b_1^{BP\langle n\rangle})^{\circ(p-1)j_1+...+(p^n-1)j_n+k}]$$

Specializing to $n = 1$, and using the cocycle relations and the fact that $ku_{(p)} = \vee_i(\Sigma^i BP\langle 1\rangle)$, we can produce the following description:

Theorem 3.4.

$$H_*(ku_{2k};\mathbb{Q}) \cong \mathbb{Q}[b_{i,1,\dots,1}^{ku}]$$

This description also recovers the result of Ando, Hopkins, and Strickland that (2.2) holds for all k when $E = H\mathbb{Q}$.

Secondly, in studying the homology of the $BP\langle n\rangle_k$, I have been able to give a description of $E_*(BP\langle n\rangle_{g(n)})$ for all n and for complex-orientable, multiplicative E. Although Ravenel and Wilson's Hopf ring description of E_*BP_* ([RW77]) descends to give a description of $E_*(BP\langle n\rangle_k)$ for $k < g(n)$, this description does not descend in the borderline case of $k = g(n)$. So, extending Su's ([Su06]) description of $E_*(BP\langle 1\rangle_6)$ at $p = 2$, I prove the following result for general n and p:

Theorem 3.5. *If $E_*(-)$ is a complex-orientable, multiplicative cohomology theory, then*

$$E_*(BP\langle n\rangle_{g(n)}) = E_*[\lambda_{I,k}, \gamma_{m,k}]/(\lambda_{I,k}^{*p}, \gamma_{m,k}^{*p})$$

where $\lambda_{I,k} = b_{(k)} \circ b_{(k+i_1)}^{\circ p} \circ ... \circ b_{(k+i_1+...+i_n)}^{\circ p^n}$, $I = (i_1, ..., i_n)$ *is a sequence of nonnegative integers,* $\gamma_{m,k} = b_{(k)} \circ \hat{x}_{m,k}^{v_n}$, $\{\hat{x}_m^{v_n}\}$ *is the set of all allowable monomials on* $[v_1], [v_2], ..., b_{(0)}, b_{(1)}, ...$ *in* $E_*BP\langle n\rangle_{g(n)-2}$ *that are not of the form* $b_{(k+i_1)}^{\circ p} \circ ... \circ b_{(k+i_1+...+i_n)}^{\circ p^n}$, *and* $\hat{x}_{m,k}^{v_n} = V^{-k}\hat{x}_m^{v_n}$, *where V is the Verschiebung map which replaces each* $b_{(i)}$ *with* $b_{(i-1)}$.

4 Research Agenda

Although some progress has been made on a few fronts, we still have a long way to go in understanding $E_*BP\langle n\rangle_k$ and the Hopf ring structure on $E_*(BP\langle n\rangle)$ for a general E, n, and k. Currently, I am working on making progress on this front by computing the Hopf ring $E_*BP\langle n\rangle_*$ for more combinations of E and n, starting with $BP\langle 2\rangle_*BP\langle 2\rangle_*$.

In general, one of the best methods for computing a Hopf ring E_*G_* is the "trapping" method described by Wilson in [Wil00], often using the bar spectral sequence for a loop space. This method works by using unstable homotopy information, usually E_*G_k, along with stable information, usually E_*G, to "trap" E_*G_* between the two. As a first step toward computing $BP\langle 2\rangle_*BP\langle 2\rangle_*$, I plan on using this trapping method, as well as techniques like those Hara used in his computation of $H_*(BP\langle 1\rangle_*; \mathbb{F}_p)$ for odd primes ([Har91]), in order to compute $H_*(BP\langle 2\rangle_*; \mathbb{F}_p)$. Then, using this unstable information, as well as Culver's recent computation of $BP\langle 2\rangle_*BP\langle 2\rangle$ ([Cul17]), I should be able to compute the Hopf ring $BP\langle 2\rangle_*BP\langle 2\rangle_*$.

Additionally, there is also a class of spectra, $P(m,n)$, which are generalizations of the $BP\langle n\rangle$ and whose homology Hopf rings have yet to be explored in general. These spectra have homotopy groups given by $\pi_*(P(m,n)) = \mathbb{Z}_{(2)}[v_m, v_{m+1}, ...v_n]$ and contain a couple different special cases of note. $P(n,n)$ is the nth Morava K-theory, $P(0,n)$ is $BP\langle n\rangle$, and

$$P(0) = P(0,\infty) = \varinjlim_n P(0,n) = BP.$$

Furthermore, just as the BP_k are p-torsion free spaces, the $P(n)_k$ are v_n-torsion free spaces, so there is potentially a lot to discover about them in a $K(n)$-local setting. By studying these spectra and computing their homology Hopf rings, we can gain a better understanding of how $BP\langle n\rangle$, BP, and complex cobordism as a whole fit into a much bigger picture.

5 References

[AHS01] M. Ando, M. J. Hopkins, and N. P. Strickland. *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. 146 (2001), no. 3, 595-687. MR 2002g:55009

[Cul17] Culver, Dominic. *On $BP\langle 2\rangle$ cooperations.* 2017. https://arxiv.org/pdf/1708.03001.pdf

[Har91] Hara, Shin-ichiro. *The Hopf rings for connective Morava K-theory and connective complex K-theory.*J. Math. Kyoto Univ. 31 (1991), no. 1, 43–70. doi:10.1215/kjm/1250519889. https://projecteuclid.org/euclid.kjm/1250519889.

[Mil70] R.J. Milgram, *The mod* 2 *spherical characteristic classes*, Ann. of Math. 92 (1970) 238-261.

[RW77] Douglas C. Ravenel and W. Stephen Wilson. *The Hopf ring for complex cobordism.* Journal of Pure and Applied Algebra, 9: 241-280, 1977.

[RW80] Douglas C. Ravenel and W. Stephen Wilson. *The Morava K-theories of Eilenberg-MacLane spaces and the Conner-Floyd conjecture.* Amer. J. Math, 102: 691-748, 1980.

[Sin76] Kathleen Sinkinson. *The cohomology of certain spectra associated with the Brown-Peterson spectrum.* Duke Mathematical Journal. 43(3): SEP 1976.

[Su06] Hsin-hao Su. *The $E(1,2)$ Cohomology of the Eilenberg-MacLane Space $K(\mathbb{Z},3)$.* Johns Hopkins University dissertation, May, 2006. http://www.math.jhu.edu/webarchive/grad/Dissertation.Su.pdf (Accessed 8/28/2017)

[Wil73] W. Stephen Wilson. *The Ω-spectrum for Brown-Peterson cohomology, Part I.* Commentarii Mathematici Helvetici, 48:4555, 1973.

[Wil75] W. Stephen Wilson. *The Ω-spectrum for Brown-Peterson cohomology, Part II.* AmericanJournal of Mathematics. 97:101-123, 1975.

[Wil00] W. Stephen Wilson. *Hopf rings in algebraic topology.* 2000. http://www.math.jhu.edu/~wsw/papers2/math/39-hopf-rings-expo-2000.pdf