HOPF RINGS AND THE ANDO-HOPKINS-STRICKLAND THEOREM

A Dissertation

Submitted to the Graduate School

of the University of Notre Dame

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

by

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Notre Dame, Indiana

July 2018

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Abstract

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In this dissertation, we give a new proof of the main results of Ando, Hopkins, and Strickland regarding the generalized homology of the even connective covers of BU. In particular, we prove that the so-called "symmetry" and "cocycle" relations hold in $E_*(BU\langle 2k\rangle)$ for any complex-orientable E and that these relations are the defining relations whenever $E = H\mathbb{Q}$ or $k = 1, 2,$ or 3. This new proof avoids the algebro-geometric perspective of Ando, Hopkins, and Strickland and instead uses the work of Ravenel, Wilson, and Yagita on the unstable homology of the truncated Brown-Peterson spectra, as well as the relationship between these spectra and $BU\langle 2k\rangle$. This approach allows for a somewhat simpler proof of the classic Ando-Hopkins-Strickland Theorem, clarifies its relation to the Hopf ring E_*ku_*, and shows how it fits into a broader algebraic picture.

to Notre Dame, Our Lady of the Lake, at whose university and under whose mantle I have been blessed to study for the past five years.

CONTENTS

SYMBOLS

C_p	the cyclic group of order p
$\mathbb{N}$	the natural numbers $\{0, 1, ...\}$
$\mathbb{Z}$	the ring of integers $\{..., -1, 0, 1, ...\}$
$\mathbb{Q}$	the ring of rational numbers
$\mathbb{F}_p$	the ring of integers mod p, i.e. $\{0, 1, ..., p-1\}$
$\mathbb{Z}_{(p)}$	the ring of integers localized at a prime p
$X_{(p)}$	the p-localization of the space X
x^G	the complex orientation of a complex-oriented spectrum G
$x +_G y$	the formal group law of a complex-oriented spectrum G
E_*	the coefficient ring of the spectrum E, i.e. the graded ring $\pi_*(E)$
E_*G_*	the graded Hopf ring constituted by the homologies $E_m(G_n)$

ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor, Mark Behrens, whose patient guidance and support have been vital to my development as a mathematician. This dissertation would not exist without his help, and I am immensely grateful for his advice and example over the past four years. I would also like to thank the Arthur J. Schmitt Foundation for its generous financial and professional support throughout my entire time at Notre Dame, as well as the National Science Foundation for its support of my work through NSF grant DMS-1547292.

Thank you to Steve Wilson for helpful conversations, for the suggestion of possible future directions of study related to $BP\langle 2\rangle_* BP\langle 2\rangle_*$ and the $K(n)$-local setting, and for publishing such a plethora of work on Hopf rings. Thank you to my fellow graduate students as well, especially Jeremy Mann, J.D. Quigley, Jens Kjaer, and Laura Wells, for so many helpful mathematical conversations over the years.

I would also like to thank Alan Liddell, Brian and Kathryn Mulholland, Peter and Nicole Deak, Elena Royal, Laura Wells, Whitney Liske, and Doan Le, whose presence and friendship have made the hard work, confusion, and turmoil of graduate school much more bearable.

Finally, I would like to thank my parents, Gary and Bridget, my brothers, Ben and Dylan, my sister-in-law Amy, and my nieces, Juliet and Lucy, for their constant love and support and for reminding me of all the goodness, truth, and beauty that exist beyond the world of mathematics.

CHAPTER 1

INTRODUCTION

In this dissertation, we will use Hopf rings to prove a number of different results in unstable homotopy theory, most notably the Ando-Hopkins-Strickland Theorem computing the generalized homology of $BU\langle 2\rangle$, $BU\langle 4\rangle$, and $BU\langle 6\rangle$, the first three connected covers of BU, the classifying space of the unitary group U. One of the advantages of this approach is the fact that it avoids the algebro-geometric perspective that Ando, Hopkins, and Strickland use and instead shows how the homology of these even-connected covers fits into a wider algebraic and topological picture.

The cohomology of these even-connected covers, $BU\langle 2k\rangle$, was studied and understood much earlier than their homology. In the 1960s, Stong and Singer computed $H^*(BU\langle 2k\rangle;\mathbb{F}_p)$ for all primes p (see [17] and [15]), giving us a complete understanding of the ordinary cohomology of these spaces. In particular, the cohomology is fairly easy to describe for the first few even-connected covers. $BU\langle 2\rangle$, which is simply equal to BU since BU is simply connected, has cohomology given by $H^*(BU\langle 2\rangle;\mathbb{Z}) = \mathbb{Z}[c_1, c_2, ...]$ where c_i is the ith Chern class, which lies in dimension $2i$. From this, we get the cohomology of $BU\langle 4\rangle$, which is often called BSU, by simply killing off c_1, i.e. $H^*(BU\langle 4\rangle;\mathbb{Z}) = \mathbb{Z}[c_2, c_3, ...]$. This pattern does not continue as one might hope, though. In fact, Singer showed that for all $k \geq 3$, the map $H^*(BU;\mathbb{Z}) \to H^*(BU\langle 2k\rangle;\mathbb{Z})$ is not surjective.

The homology of these spaces turned out to be harder to compute than the cohomology. Though the Splitting Principle tells us that for any complex orientable spectrum E, $E_*(BU\langle 2\rangle) = E_*BU = E_*[b_1, b_2, ...]$, with generators b_i in degree $2i$, this easy description breaks down for $E_*(BU\langle 4\rangle)$, even in the case of $E = H\mathbb{Z}$. Ando, Hopkins, and Strickland greatly advanced our understanding of the homology of these spaces in their 2001 paper "Elliptic spectra, the Witten genus and the theorem of the cube" ([1]). In this paper they computed $E_*(BU\langle 4\rangle)$ and $E_*(BU\langle 6\rangle)$ for all E complex-orientable and $H_*(BU\langle 2k\rangle;\mathbb{Q})$ for all k. In particular, they constructed generators $b_{i_1,\ldots,i_k}$ and proved symmetry and cocycle relations in $E_*(BU\langle 2k\rangle)$ for all complex-orientable E and k, and they showed that these relations are the defining relations when E is the rational Eilenberg-MacLane spectrum $H\mathbb{Q}$, or $k = 1, 2$, or 3.

In the 1970s, in seemingly unrelated work, Ravenel and Wilson began to popularize an algebraic construction called Hopf rings, using them to compute the homology of spaces that appear in certain Ω-spectra. In [11], they used Hopf rings to compute and describe the generalized homology of the even spaces in the Ω-spectra for MU (the complex cobordism spectrum) and BP (the Brown-Peterson spectrum). They also proved general relations (Ravenel-Wilson relations) that hold in E_*G_* for any complex-orientable spectra E and G. Furthermore, in [14], Ravenel, Wilson, and Yagita showed that this computation of E_*BP_* restricts to give a similar description of the generalized homology of the even spaces in $BP\langle n\rangle$, the nth truncated Brown-Peterson spectrum (These spectra will be discussed in greater detail in Section 2.2).

In fact, it turns out that the work of Ravenel, Wilson, and Yagita is closely related to Ando, Hopkins, and Strickland's results on $E_*BU\langle 2k\rangle$, since the Ω-spectrum for connective complex K-theory has even spaces given by $ku_{2k} = BU\langle 2k\rangle$, and $ku_{(p)} \simeq \bigvee_{0\le i\le p-2}\Sigma^{2i}BP\langle 1\rangle$ for all primes p. In this dissertation, we explore the relationship between these two bodies of work, using the results of Ravenel, Wilson and Yagita to prove the main theorems of Ando, Hopkins, and Strickland. In particular, we show that the symmetry relations follow from basic Hopf ring facts, the cocycle relations follow from the Ravenel-Wilson relations for the spectrum ku, and the Ando-Hopkins-Strickland description of $E_*BU\langle 2k\rangle$ follows from the Ravenel-Wilson-

Yagita description of $E_*BP\langle 1\rangle_{2k}$.

In Chapter 2, we will give some preliminary definitions and background information that will be helpful in understanding and following the proofs throughout the rest of the dissertation. In particular, we will discuss the important spectra that will be used, describe how they are related to each other, and construct the homology generators defined by Ando, Hopkins, and Strickland, as well as those defined by Ravenel and Wilson.

In Chapter 3, we will give a brief overview of Hopf rings and the properties of Hopf rings that will be used later on. We will focus on the Hopf ring E_*G_*, which is the most common and motivating example of a Hopf ring, as well as the formal power series ring $E_*G_*[[x_1, ..., x_n]]$, a certain subring of which is a Hopf ring. The Hopf ring properties which we prove in $E_*G_*[[x_1, ..., x_n]]$ turn out to be the key properties allowing us to prove the cocycle relations later on.

In Chapter 4, we will discuss the relevant results of Ravenel, Wilson, and Yagita which we will apply in our computation of E_*ku_*. We will start by defining and giving examples of the Ravenel-Wilson relations, and then we will list the major results in E_*BP_* and $E_*BP\langle n\rangle_{2k}$ that come into play in our later proofs.

In Chapter 5, we will discuss the relationship between the homology generators used by Ravenel and Wilson and those used by Ando, Hopkins, and Strickland. In particular, we will extend the work of Hara to compute the image in E_*ku_* of the Ravenel's and Wilson's generators in $E_*BP\langle 1\rangle_*$ after applying the homology map induced by the inclusion $BP\langle 1\rangle \to ku_{(p)}$.

Finally, in Chapter 6, we will prove the main theorems of Ando, Hopkins, and Strickland, also showing how the Hopf ring perspective leads to several different alternate descriptions of E_*ku_{2k} along the way. In Chapter 7, we will discuss how things change in the periodic case and show how the results of Hopkins and Hunton lead to easy computations of $E_*E(1)_*$ and E_*KU_*. And, in Chapter 8, we will discuss possible directions of future work related to $E_*BP\langle 2\rangle_*$ and $E_*P(m,n)_*$.

CHAPTER 2

PRELIMINARIES

2.1 Stable Homotopy Theory, Unstable Homotopy Theory, and Ω-Spectra

The basic objects of study in stable homotopy theory are a certain generalization of topological spaces called spectra.

Definition A *spectrum* is a sequence of spaces $E = \{E_n\}_{n\in\mathbb{N}}$ with basepoint, along with a basepoint-preserving structure map $i_n : S^1 \wedge E_n \to E_{n+1}$ for each n.

Example One of the most common spectra used is the sphere spectrum $\mathbb{S}$, which has spaces $\mathbb{S}_n = S^n$ and structure maps $id_{S^{n+1}} : S^1 \wedge S^n = S^{n+1} \to S^{n+1}$.

Definition A *map of spectra* $f : E \to G$ is a collection of maps $f_n : E_n \to G_n$, one for each $n \in \mathbb{N}$, such that the diagram below commutes for all n.

$$\begin{tikzcd}
S^1 \wedge E_n \arrow[r, "id\wedge f_n"] \arrow[d, "i_n"] & S^1 \wedge G_n \arrow[d, "i_n"] \\
E_{n+1} \arrow[r, "f_{n+1}"] & G_{n+1}
\end{tikzcd}$$

In particular, the class of spectra most often used are Ω-spectra, which are defined below.

Definition An Ω-*spectrum* is a spectrum E each of whose adjoint structure maps $E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence.

Definition A *ring spectrum* is a spectrum, E, along with a multiplication $\mu : E \wedge E \to E$ and a unit $\eta : S \to E$ for which the associativity and unitality diagrams below commute up to homotopy.

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mu^E \wedge id"] \arrow[d, "id \wedge \mu^E"] & E \wedge E \arrow[d, "\mu^G"] \\ E \wedge E \arrow[r, "\mu^E"] & E \end{tikzcd} \qquad \begin{tikzcd} E \wedge \mathbb{S} \simeq E \simeq \mathbb{S} \wedge E \arrow[r, "\eta \wedge id"] \arrow[d, "id \wedge \eta"] \arrow[dr, "id"] & E \wedge E \arrow[d, "\mu^E"] \\ E \wedge E \arrow[r, "\mu^E"] & E \end{tikzcd}$$

We say that such a ring spectrum is commutative if the following diagram commutes up to homotopy, where s is the swap map.

$$\begin{tikzcd} E \wedge E \arrow[d, "s"] \arrow[dr, "\mu^E"] & \\ E \wedge E \arrow[r, "\mu^E"] & E \end{tikzcd}$$

Definition If E is a commutative ring spectrum, then a *complex orientation* on E is a cohomology element $x^E \in \tilde{E}^2(\mathbb{CP}^\infty)$ such that $i^*(x^E) = 1 \in \tilde{E}^2(\mathbb{CP}^1)$, where $i : \mathbb{CP}^1 \to \mathbb{CP}^\infty$ is the canonical inclusion. We call a spectrum for which such a x^E exists *complex-orientable*, and if a particular orientation x^E has been chosen, we call E complex-oriented.

From this point forward, unless otherwise noted, we will assume that all spectra are complex-oriented Ω-spectra. It is also noteworthy that every space G_k in an Ω-spectrum G is an infinite loop space, a fact which we will use in the next chapter.

Definition A space X is said to be an *infinite loop space* if for each $n > 0$, there is a space Y_n (the n-fold delooping of X) such that $\Omega^n Y_n \simeq X$.

Ω-spectra are especially useful because they allow us to describe any generalized homology or cohomology theory simply in terms of homotopy classes of maps. More specifically, for any generalized cohomology theory $E^*(-)$, there is an Ω-spectrum

E *representing* $E^*(-)$, i.e. such that for any CW-complex X, $E^n(X) = [X, E_n]$. Similarly, for any generalized homology theory $E_*(-)$, there is an Ω-spectrum E *representing* $E_*(-)$, i.e. such that for any space, X,

$$E_n(X) = \pi_n(E \wedge X) = \varinjlim_m \pi_{n+m}(E_m \wedge X).$$

Accordingly, if we want to work unstably, as is done in unstable homotopy theory, we can take the homotopy groups, generalized homology, or generalized cohomology of the spaces in any Ω-spectrum G,

$$\pi_n(G_k)$$

$$E^n G_k = [G_k, E_n]$$

$$E_n G_k = \varinjlim_m \pi_{n+m}(E_m \wedge G_k).$$

Furthermore, if we want to work stably, as is done in stable homotopy theory, we can extend these definitions to take the homotopy groups and generalized (co)homology of Ω-spectra instead of spaces, so that for any Ω-spectra E and G we have:

$$E^n G = [G, \Sigma^n E]$$

$$E_n G = \varinjlim_m E_{n+m} G_m$$

$$\pi_n(G) = \varinjlim_m \pi_{n+m}(G_m).$$

If we restrict ourselves to the unstable homotopy theory setting, we will use the notation $E_* G_*$ to refer to the bigraded module of E-homologies of the spaces in the spectrum G. As we will see in the next chapter, after a small restriction on E

this bigraded module will inherit a coproduct structure and two compatible product structures, making it into a Hopf ring.

2.2 Important Spectra

We will be using a variety of different spectra, so in this section we will give a brief overview of the spectra at play throughout the rest of the work. One of the most widely used and studied spectra in homotopy theory is the spectrum MU, which represents complex cobordism and is the universal example of a complex-orientable spectrum. MU has homotopy groups given by

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, ...]$$

with generators $x_i \in \pi_{2i}(MU)$. The Hopf ring MU_*MU_* was computed by Ravenel and Wilson in [11].

Of more direct use for our purposes are the periodic complex K-theory spectrum, KU, and the connective complex K-theory spectrum, ku. The spectrum ku, which is also sometimes denoted bu, is vital to our new proof of the Ando-Hopkins-Strickland Theorem because its even spaces are precisely the connected covers of BU whose homology they computed, i.e. $ku_{2k} = BU\langle 2k\rangle$. The homotopy groups of ku are given by

$$\pi_*(ku) = \mathbb{Z}[v]$$

where $v \in \pi_2(ku)$. The spectrum KU is closely related to ku, and its homotopy groups are given by

$$\pi_*(KU) = \mathbb{Z}[v, v^{-1}]$$

where once again $v \in \pi_2(KU)$, and therefore $v^{-1} \in \pi_{-2}(KU)$. KU is periodic, meaning that the spaces in KU repeat according to a certain frequency. In fact,

$KU_k = KU_{k+2}$ for all k.

To prove our results in E_*ku_*, we will often work in a p-local setting for an arbitrary prime p. When doing so, there are a number of relevant p-local spectra that we will use. The most fundamental p-local spectrum is the Brown-Peterson spectrum BP, which is the universal example of a complex-orientable p-local spectrum and has homotopy groups given by

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, ...]$$

where $v_i \in \pi_{2(p^i-1)}(BP)$. Note that there is a separate Brown-Peterson spectrum for each prime p, even though the notation BP is used to denote each of these spectra. Since BP and MU are both the universal example of a complex-orientable spectrum in their appropriate contexts, it should not be surprising that they are intimately related. In fact, for any prime p, we have that

$$MU_{(p)} = \bigvee_i \Sigma^{2n_i} BP.$$

Of particular use to us will be the truncated Brown-Peterson spectra $BP\langle n\rangle$ which Wilson constructs for each $n \geq 0$ in [20] using Sullivan's theory of manifolds with singularities. The truncated Brown-Peterson spectra are essentially obtained by killing off the generators $v_i \in \pi_*BP$ for $i > n$. Its homotopy groups are therefore given by:

$$\pi_*(BP\langle n\rangle) = \mathbb{Z}_{(p)}[v_1, v_2, ..., v_n].$$

Although this is not the same as the nth space in the Postnikov tower for BP (since we are not killing off *everything* above degree $2(p^n - 1)$, only the *generators* above degree $2(p^n - 1)$), it similarly gives us a way to build up BP in successive layers so that $\varprojlim_n BP\langle n\rangle = BP$. In [19, 20], Wilson computed $H_*(BP_k; \mathbb{Z}_{(p)})$ for all k

and $H_*(BP\langle n\rangle_k;\mathbb{Z}_{(p)})$ for all $k < g(n) = 2(p^n + p^{n-1} + ... + 1)$. Later, in [16], Sinkinson computed $H^*(BP\langle n\rangle_k;\mathbb{F}_p)$ for all n and k, giving a complete description of the unstable mod p cohomology of the truncated Brown-Peterson spectra.

Just as with complex K-theory, where there were both connective and periodic spectra, ku and KU, there is also a periodic version of $BP\langle n\rangle$, which is the Johnson-Wilson spectrum $E(n)$. Its homotopy groups are similarly given by

$$\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, v_2, ..., v_n, v_n^{-1}]$$

where $v_i \in \pi_{2(p^i-1)}(E(n))$.

These p-local spectra are all closely related to their counterparts in the non-local setting. Just as we could write $MU_{(p)}$ as a wedge sum of suspensions of BP, we can do the same thing for KU and ku to get that

$$KU_{(p)} = \bigvee_{0\leq i\leq p-2} \Sigma^{2i}E(1)$$

and

$$ku_{(p)} = \bigvee_{0\leq i\leq p-2} \Sigma^{2i}BP\langle 1\rangle.$$

This last equation is the key relation which will allow us to compute E_*ku_* using Ravenel's and Wilson's computations in $E_*BP\langle 1\rangle_*$ at each prime p. We also note the fact that MU, $MU_{(p)}$, KU, $KU_{(p)}$, ku, $ku_{(p)}$, BP, $BP\langle n\rangle$, and $E(n)$ are all complex-orientable, which we will use later on.

2.3 Homology Generators

For any complex-oriented Ω-spectra E and G, one can use the complex orientations x^E and x^G to construct canonical homology generators in E_*G_*. In [1], Ando, Hopkins, and Strickland define homology generators $\bar{b}_{i_1,\ldots,i_k} \in E_{2i}ku_{2k}$. In [11], Ravenel and Wilson define a broader class of homology generators $b_i^G \in E_{2i}G_2$. As we will see in Lemma 3.3.2, the $\bar{b}_i$ are actually a special case of the b_i^G, namely $\bar{b}_i = b_i^{ku}$ and $\bar{b}_{i_1,\ldots,i_k} = b_{i_1}^{ku} \circ \ldots \circ b_{i_k}^{ku}$ (The $\circ$-product will be defined in Section 3.2).

If we think of the complex orientation of G as a map

$$x^G : \mathbb{CP}^\infty \to G_2,$$

then we can consider its induced E-homology map:

$$x_*^G : E_*(\mathbb{CP}^\infty) \to E_*G_2.$$

Furthermore, we know that for any complex-orientable E,

$$E^*(\mathbb{CP}^n) = E^*[[c_1]]/(c_1^{n+1})$$

and

$$E^*(\mathbb{CP}^\infty) = E^*[[c_1]]$$

where $c_1 \in E^2(\mathbb{CP}^n) \subset E^2(\mathbb{CP}^\infty)$ is the first Chern class. If we let $\beta_i \in E_{2i}(\mathbb{CP}^n)$ be the homology element dual to $c_1^i \in E^{2i}(\mathbb{CP}^n)$, we can similarly define $\beta_i \in E_{2i}(\mathbb{CP}^\infty)$ to be the image of $\beta_i \in E_{2i}(\mathbb{CP}^n)$ under the inclusion $E^*(\mathbb{CP}^n) \subset E^*(\mathbb{CP}^\infty)$. Using these β_i, Ravenel and Wilson define the generators b_i^G as follows.

Definition For each $i \geq 0$, let

$$b_i^G = (x^G)_*(\beta_i) \in E_{2i}(G_2).$$

In the case of $i = 0$, this gives $b_0^G = 1$, which is the $*$-product identity in E_0G_2 (The $*$-product is the usual Pontryagin product on the H-space G_2 and will be defined in

Section 3.1). For any $k \geq 1$, we can similarly define a map

$$\alpha_k : (\mathbb{CP}_+^\infty)^{\wedge k} \xrightarrow{(x^G)^{\wedge k}} (G_2)^{\wedge k} \xrightarrow{\mu^G} G_{2k}$$

where μ^G is the ring spectrum multiplication for G. This induces an E-homology map

$$(\alpha_k)_* : \tilde{E}_*\left(\left(\mathbb{CP}_+^\infty\right)^{\wedge k}\right) = \left(E_*\left(\mathbb{CP}^\infty\right)\right)^{\otimes k} \to E_*(G_{2k}).$$

Using this map for $G = ku$, Ando, Hopkins, and Strickland's definition of the $\bar{b}_{i_1,\ldots,i_k} \in E_{2(i_1+\ldots+i_k)}(ku_{2k}) = E_{2(i_1+\ldots+i_k)}(BU\langle 2k\rangle)$ amounts to the following.

Definition For all $i_1, \ldots, i_k, k \geq 1$, let

$$\bar{b}_{i_1,i_2,\ldots i_k} = (\alpha_k)_*(\beta_{i_1} \otimes \ldots \otimes \beta_{i_k}) \in E_*BU\langle 2k\rangle.$$

In the case of $k = 1$, it is clear that $\bar{b}_i = b_i^{ku}$. In the next chapter, we will additionally show that $\bar{b}_{i_1,i_2,\ldots,i_k} = \bar{b}_{i_1} \circ \bar{b}_{i_2} \circ \ldots \circ \bar{b}_{i_k} = b_{i_1}^{ku} \circ b_{i_2}^{ku} \circ \ldots \circ b_{i_k}^{ku}$ for all $i_1, \ldots, i_k$, meaning that the Ando-Hopkins-Strickland generators can be written in terms of the Ravenel-Wilson generators. As a result, we will also get that $\bar{b}_{i_1,i_2,\ldots i_k} = \bar{b}_{\sigma(i_1),\sigma(i_2),\ldots\sigma(i_k)}$ for any permutation σ, which will become one of the main relations used in the Ando-Hopkins-Strickland Theorem.

Furthermore, we will use the notation b_i to denote $b_i^{BP\langle 1\rangle}$ and $\bar{b}_i$ to denote b_i^{ku}. As we will show in Chapter 5, the relation $ku_{(p)} \simeq \bigvee_{0\leq i\leq p-2} \Sigma^{2i} BP\langle 1\rangle$ gives us a close relationship between the b_i and the $\bar{b}_i$. Finally, when localized at a particular prime p, we will also use the short-hand notation $b_{(i)}^G = b_{p^i}^G$ (and therefore, $b_{(i)} = b_{p^i}$ and $\bar{b}_{(i)} = \bar{b}_{p^i}$).

CHAPTER 3

HOPF RINGS

The notion of a Hopf ring was first defined and used by Milgram in [9], in which he computed the mod 2 homology of the spaces in the sphere spectrum and defined a Hopf ring as a ring object in the category of coalgebras. This algebraic construction of a Hopf ring arises naturally within unstable homotopy theory from the study of the generalized homology of the spaces in a wide class of Ω-spectra. Ravenel and Wilson later made use of this algebraic machinery in [11] and [12], in which they successfully established the usefulness of Hopf rings as a "descriptive and computational tool" in algebraic topology and paved the way for the wide variety of Hopf ring computations that were to follow. Hopf rings have become a tremendously valuable tool for computing and describing the generalized homology of the spaces in Ω-spectra. In this chapter, therefore, we will give a brief overview of Hopf rings and how they arise and are used in the study of infinite loop spaces and Ω-spectra.

3.1 H-Spaces and Hopf Algebras

Hopf rings are so named because of their intimate connection with both Hopf-spaces (H-spaces) and Hopf algebras, which arise when computing generalized homology. If E is a ring spectrum, and X is a space, $E^*(X)$ has the structure of a graded algebra with product (the cup product) given by

$$\nabla : E^*(X) \otimes_{E^*} E^*(X) \to E^*(X \times X) \to E^*(X),$$

where the first map is the Kunneth map and the second map is induced by $\Delta : X \to X \times X$, the diagonal map on X. If we furthermore suppose that the Kunneth map $E_*(X) \otimes_{E_*} E_*(X) \to E_*(X \times X)$ is an isomorphism, then $E_*(X)$ similarly gets the structure of a graded coalgebra with coproduct given by

$$\psi = \Delta_* : E_*(X) \to E_*(X \times X) \cong E_*(X) \otimes_{E_*} E_*(X).$$

We will now define these terms more explicitly. Note that whenever we speak of (co)commutativity, we mean (co)commutativity in the graded sense.

Definition A *graded coalgebra* is a ($\mathbb{Z}$-graded) vector space C over a commutative, associative ring with unit, R, along with R-linear maps $\psi : C \to C \otimes_R C$ (coproduct) and $\epsilon : C \to R$ (counit) such that the coassociativity and counitality diagrams below commute.

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "\psi"] & C \otimes_R C \arrow[d, "id\otimes\psi"] \\ C \otimes_R C \arrow[r, "\psi\otimes id"] & C \otimes_R C \otimes_R C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\psi"] \arrow[d, "\psi"] \arrow[drr, "id"] & & C \otimes_R C \arrow[d, "id\otimes\epsilon"] \\ C \otimes_R C \arrow[r, "\epsilon\otimes id"] & R \otimes_R C \cong & C \cong C \otimes_R R \end{tikzcd}$$

We say that such a graded coalgebra is *cocommutative* if the following diagram commutes.

$$\begin{tikzcd} & C \otimes_R C \arrow[d, "s"] \\ C \arrow[ur, "\psi"] \arrow[r, "\psi"] & C \otimes_R C \end{tikzcd}$$

where s is the map sending $x \otimes y$ to $(-1)^{|x||y|} y \otimes x$ for all x, y. More explicitly, the coproduct of $x \in E_*(X)$ will be denoted $\psi(x) = \sum x' \otimes x''$.

Definition A *graded algebra* is a ($\mathbb{Z}$-graded) vector space A over a commutative, associative ring with unit, R, along with R-linear maps $\nabla : A \otimes_R A \to A$ (product) and $\eta : R \to A$ (unit) such that the associativity and unitality diagrams below commute.

$$\begin{tikzcd}
A & A \otimes_R A \arrow[l, "\nabla"'] \\
A \otimes_R A \arrow[u, "\nabla"] & A \otimes_R A \otimes_R A \arrow[l, "\nabla \otimes id"] \arrow[u, "id \otimes \nabla"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A & & A \otimes_R A \arrow[ll, "\nabla"'] \\
A \otimes_R A \arrow[u, "\nabla"] & R \otimes_R A \cong A \cong A \otimes_R R \arrow[l, "\eta \otimes id"'] \arrow[ul, "id"] \arrow[ur, "id \otimes \eta"']
\end{tikzcd}$$

We say that such a graded algebra is *commutative* if the following diagram commutes.

$$\begin{tikzcd}
 & A \otimes_R A \arrow[dl, "\nabla"'] \arrow[d, "s"] \\
A & A \otimes_R A \arrow[l, "\nabla"]
\end{tikzcd}$$

where s is again the map sending $x \otimes y$ to $(-1)^{|x||y|} y \otimes x$.

Note that since algebras are just the dual of coalgebras, these diagrams are the same as the diagrams defining a coalgebra, but with the arrows reversed and the coproduct and counit replaced with the product and unit.

If we have additional structure on the space X, namely if X is a grouplike, homotopy associative H-space, then $E_*(X)$ and $E^*(X)$ both get the additional structure of a graded Hopf algebra over E_*.

Definition A topological space X is said to be an *H-space* if there is a continuous map $\gamma : X \times X \to X$, called multiplication, and an element $e \in X$, called the identity, such that for all $x \in X$ the diagram below commutes up to homotopy.

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, "id"] & X \times X \arrow[d, "\gamma"] \\
X \times X \arrow[r, "\gamma"] & X
\end{tikzcd}$$

Here, i_1 is the map which takes x to (x, e) and i_2 is the map which takes x to

(e, x).

We say that an H-space is *homotopy associative* if the following diagram commutes up to homotopy.

$$\begin{tikzcd} X \times X \times X \arrow[r, "id\times\gamma"] \arrow[d, "\gamma\times id"] & X \times X \arrow[d, "\gamma"] \\ X \times X \arrow[r, "\gamma"] & X \end{tikzcd}$$

We say that a homotopy associative H-space is *grouplike* if there is also a homotopy inverse, i.e. a map $i : X \to X$ such that the following diagram commutes up to homotopy.

$$\begin{tikzcd} X \arrow[r, "i\times id"] \arrow[d, "id\times i"] \arrow[dr, "id"] & X \times X \arrow[d, "\gamma"] \\ X \times X \arrow[r, "\gamma"] & X \end{tikzcd}$$

Here $i \times id$ and $id \times i$ are the maps taking $x \in X$ to $(i(x), x)$ and $(x, i(x))$, respectively.

Note that this definition is similar to that of a topological group, except we do not necessarily require associativity or inverses, and the identity relation only has to hold up to homotopy.

H-spaces are fairly common in algebraic topology. Examples of H-spaces include topological groups, the spheres S^0, S^1, S^3, and S^7 (but no other spheres), and loop spaces. In particular, we will make use of the lemma below.

Lemma 3.1.1 *Every infinite loop space is a grouplike, homotopy associative H-space.*

Definition A *(graded) Hopf algebra over R* is a group object in the category $CoAlg_R$ of (graded) cocommutative coalgebras over an associative, commutative ring with unit, R.

As Ravenel and Wilson have noted, since a Hopf algebra is simply a group object in a particular category, it might more appropriately be called a Hopf group. More concretely, a Hopf algebra is essentially a coalgebra with some additional compatible structure, namely a product (corresponding to the group product) and an antipode map (corresponding to the inverse map).

With these definitions in place, we can see how an H-space structure on X makes $E_*(X)$ into a Hopf algebra over E_*. If X is a group-like, homotopy associative H-space with product $\gamma : X \times X \to X$, we get an induced product

$$* = \gamma_* : E_i(X) \otimes E_j(X) \to E_{i+j}(X),$$

which along with a naturally defined unit $\eta : E_* \to E_*(X)$ and antipode $S : E_*(X) \to E_*(X)$ (induced by the homotopy inverse i) makes $E_*(X)$ into a graded Hopf algebra over E_*.

Proposition 3.1.2 *If X is a group-like, homotopy associative H-space, then $E_*(X)$ and $E^*(X)$ are graded Hopf algebras over E_*.*

This essentially follows from the fact that generalized (co)homology takes group objects in the homotopy category of topological spaces to group objects in $CoAlg_{E_*}$. One of the benefits of working with this Hopf algebra structure on $E_*(X)$ is that it allows us to study all of the homology groups of X all at once as a single object, rather than studying them separately, just as the cup product allows us to do for $E^*(X)$.

3.2 Ring Spectra and Hopf Rings

If we consider the generalized homology of a spectrum G instead of a space, we get even more structure on E_*G_*, making it into what will be called a Hopf ring. Suppose E and G are complex-oriented spectra, and E has a Kunneth isomorphism. Then, since every space in an Ω-spectrum is an infinite loop space, and every infinite loop space is an group-like, homotopy associative H-space, each G_m has a product and coproduct:

$$* : E_i(G_m) \otimes E_j(G_m) \to E_{i+j}(G_m)$$

$$\psi : E_*(G_m) \to E_*(G_m) \otimes E_*(G_m).$$

Furthermore, as mentioned in the previous chapter, since G is complex-oriented, it automatically inherits a ring spectrum structure with multiplication $\mu^G : G_m \wedge G_n \to G_{m+n}$. If we take its induced homology map, we get another product on E_*G_*, which adds the degrees of the spaces in addition to the homological degrees:

$$\circ = \mu_*^G : E_i(G_m) \otimes E_j(G_n) \to E_{i+j}(G_{m+n}).$$

This additional product acts just like the product in a ring, and in fact it makes $E_*(G_*)$ into precisely a ring object in the category $CoAlg_R$. Therefore, following Ravenel and Wilson, we define a Hopf ring.

Definition A *(graded) Hopf ring over R* is a ring object in the category $CoAlg_R$ of (graded) cocommutative coalgebras over an associative, commutative ring with unit, R.

Therefore, in contrast with a Hopf algebra, a Hopf ring has *two* different products, denoted $*$ and $\circ$, which are compatible but interact in non-trivial ways. This definition is fairly abstract, but fortunately Ravenel and Wilson give a comprehensive list of the defining properties of a Hopf ring in Lemma 1.12 of [11]. We will list a few of the particularly useful properties here, but for a more in depth treatment of these properties, refer to their paper. Note that all of the powers of -1 and $[-1]$ (which is defined in the next section) will disappear in our setting since we are only working with even spaces with homologies concentrated in even degree.

Lemma 3.2.1 (Ravenel-Wilson, Lemma 1.12) *If $H_*(*)$ is a graded Hopf ring (where the subscript denotes the "homological" degree) with $a \in H_i(n)$, $b \in H_j(k)$,* *and* $c \in H_q(k)$*, then*

(i) $b * c = (-1)^{jq} c * b \in H_{i+j}(k)$

(ii) $\psi(b * c) = \psi(b) * \psi(c)$

(iii) $\psi(a \circ b) = \psi(a) \circ \psi(b)$

(iv) $a \circ b = (-1)^{ij}[-1]^{\circ nk} b \circ a$

(v) $a \circ (b * c) = \sum (-1)^{|a''||b|}(a' \circ b) * (a'' \circ c)$*, where* $\psi(a) = \sum a' \otimes a''$*.*

3.3 The Hopf Ring E_*G_*

Although Hopf rings can be defined abstractly as ring objects in $CoAlg_R$, their real computational usefulness lies in the particular case of the bigraded module E_*G_*, which, inherits a Hopf ring structure for a very broad class of spectra E and G. Among other reasons, studying the unstable homology cooperations, E_*G_*, can be useful because they are simply the dual of the unstable cohomology operations, which are more widely studied.

Proposition 3.3.1 (Ravenel-Wilson, Lemma 1.13) *If* E *and* G *are complex-oriented spectra, and* E *has a Kunneth isomorphism, then* E_*G_* *is a graded Hopf ring over* E_**.*

By this proposition, we know that we can apply all of the identities in Lemma 3.2.1 whenever E has a Kunneth isomorphism. Below we will note some additional important properties in the Hopf ring E_*G_* that we will use later on.

Lemma 3.3.2

$$\bar{b}_{i_1,i_2,\ldots i_k} = b^{ku}_{i_1} \circ b^{ku}_{i_2} \circ \ldots \circ b^{ku}_{i_k} = \bar{b}_{i_1} \circ \bar{b}_{i_2} \circ \ldots \circ \bar{b}_{i_k}$$

for all $i_1, \ldots, i_k \geq 1$*.*

Proof Both $\bar{b}_{i_1,i_2,...i_k}$ and $b^{ku}_{i_1} \circ b^{ku}_{i_2} \circ ... \circ b^{ku}_{i_k}$ are defined to be the image of $\beta_{i_1} \otimes ... \otimes \beta_{i_k}$ under the map

$$\alpha_k : (\mathbb{CP}^\infty_+)^{\wedge k} \xrightarrow{(x^G)^{\wedge k}} (G_2)^{\wedge k} \xrightarrow{\mu^G} G_{2k}.$$

So, the result follows by definition. □

Therefore, by the commutativity of the $\circ$-product (see Lemma 3.2.1(iv)), we get the following theorem.

Theorem 3.3.3 (Symmetry) $\bar{b}_{i_1,i_2,...i_k} = \bar{b}_{\sigma(i_1),\sigma(i_2),...\sigma(i_k)}$ *for any permutation* σ*.*

For each $y \in G^i$ there is a distinguished Hopf ring element $[y] \in E_0G_i$ given by $[y] = y_*(1)$, where y is thought of as a map $S^0 \to G_i$, and 1 is the generator of E_0S^0.

Lemma 3.3.4 (Ravenel-Wilson, Lemma 1.14(a)) $[yz] = [y] \circ [z]$ *for all* $y, z \in G^*$*.*

Lemma 3.3.5 (Ravenel-Wilson, Lemma 1.14(b)) $[y+z] = [y] * [z]$ *for all* $y, z \in G^*$*.*

Lemma 3.3.6 (Ravenel-Wilson, Lemma 1.14(c)) $\psi([z]) = [z] \otimes [z]$ *for all* $z \in G^*$*.*

Lemma 3.3.7 *If* x *is an element of* E_iG_m*, and* $1 \in E_0G_m$ *is the identity, then* $1 * x = x * 1 = x$*.*

Lemma 3.3.8

$$\psi(b^G_i) = b^G_i \otimes 1 + b^G_{i-1} \otimes b^G_1 + ... + 1 \otimes b^G_i$$

Proof This follows from the fact that $\psi(\beta_i) = \beta_i \otimes 1 + \beta_{i-1} \otimes \beta_1 + ... + 1 \otimes \beta_i$, $b^G_0 = (x^G)_*(\beta_0) = (x^G)_*(1) = 1$, and x^G is a map of coalgebras. □

Lemma 3.3.9 *If $x \in E_iG_m$, and $1 \in G^*$ is the identity, then $[1] \circ x = x \circ [1] = x$.*

Proof This follows easily from the properties in Lemma 1.12 of [11]. $\square$

Lemma 3.3.10 *If $R = \mathbb{Z}, \mathbb{Z}_{(p)}, \mathbb{F}_p$, or $\mathbb{Q}$, then for all $c \in R$ and $i \geq 0$, $[c] \circ b_i^G = cb_i^G$, modulo $*$-decomposables.*

Proof This follows from the properties in Lemma 1.12 of [11] and the fact that $\psi(b_i^G) = b_i^G \otimes 1 + b_{i-1}^G \otimes b_1^G + ... + 1 \otimes b_i^G$. $\square$

Lemma 3.3.11 *$b_0^G \circ b_i^G = 0$ for all $i > 0$.*

Proof This follows easily from the properties in Lemma 1.12 of [11] since $b_0^G = 1 \in E_0G_2$. $\square$

The previous lemma is essentially what allows us to account for the index boundary discrepancy for the b_i^G, which were defined for $i \geq 0$, and the $\bar{b}_{i_1,...,i_k} = b_{i_1}^{ku} \circ ... \circ b_{i_k}^{ku}$, which were defined for $i_1, ..., i_k \geq 1$. If $\{i_1, ..., i_k\}$ contains both 0 and a nonzero index, then $b_{i_1}^{ku} \circ ... \circ b_{i_k}^{ku} = 0 \in E_0G_{2k}$.

If we only look at E_*G_* modulo decomposable elements (with respect to the $*$-product), then the $\circ$ product is still well-defined by the following lemma.

Lemma 3.3.12 *If E is connective, $x, y, z \in E_*G_*$, and $y = z$ in QE_*G_*, then $x \circ y = x \circ z$ in QE_*G_*.*

Proof Suppose $y = z$ in QE_*G_*. Then, we must have $y = z + \sum_i D_i * \bar{D}_i$ in E_*G_* for some $D_i, \bar{D}_i \in E_{>0}G_*$. By Lemma 3.2.1(v), and the fact that the $\circ$-product distributes across addition, we get that

$$x \circ y = x \circ z + \sum_i \sum (-1)^{|x''||D_i|}(x' \circ D_i) * (x'' \circ \bar{D}_i).$$

Furthermore, each $x' \circ D_i$ and $x'' \circ \bar{D}_i$ must have positive homological degree, since the $D_i, \bar{D}_i$ have positive homological degree, and x' and x'' have nonnegative homological degree (since E is connective). Therefore, each $(x' \circ D_i) * (x'' \circ \bar{D}_i)$ is decomposable, and the lemma follows. $\square$

Lemma 3.3.13 *If $x, y, z \in E_*G_*$, and x is grouplike, i.e. $\psi(x) = x \otimes x$, then*

$$x \circ (y * z) = (-1)^{|x||y|}(x \circ y) * (x \circ z)$$

The previous two lemmas are vital to our work in the coming chapters. Since E_*G_* is only a Hopf ring when E has a Kunneth isomorphism, these lemmas do not necessarily hold for any complex-oriented E and G. However, since these identities are stated without reference to the coproduct structure (which is what we need the Kunneth isomorphism for), and the identities hold in E_*MU_* (By Corollary 4.7 of [11]), we can prove that the same identities hold for all elements $x, y, z \in E_*G_*$ that are in the image of $f_*^G : E_*MU_* \to E_*G_*$, the homology map induced by the ring spectrum map $f^G : MU \to G$ associated to the complex orientation x^G. Since f^G is a ring spectrum map, f_*^G commutes with both the $\circ$-product and the $*$-product.

If we let E and G be any complex-oriented spectra, then we get the following corollaries.

Corollary 3.3.14 *For any $x, y, z \in im(f_*^G) \subset E_*G_*$,*

$$x \circ (y * z) = \sum (-1)^{|x'||y|}(x' \circ y) * (x'' \circ z).$$

Corollary 3.3.15 *For any $x, y, z \in im(f_*^G) \subset E_*G_*$, where x is the image of a grouplike element,*

$$x \circ (y * z) = (-1)^{|x||y|}(x \circ y) * (x \circ z).$$

Corollary 3.3.16 *For any $x, y, z \in im(f_*^G) \subset E_*G_*$ such that $y = z$ in QE_*G_*,*

$$x \circ y = x \circ z$$

in QE_*G_*.

3.4 Polynomial Hopf Rings

Since the relations between the different products and generators in E_*G_* can be quite complex, it is often useful to work in the formal power series ring $E_*G_*[[x_1, ..., x_n]]$ instead, where these relations can be written in a much simpler form. This is particularly true for the Ravenel-Wilson relations which we will discuss in the next chapter. In this section, we will prove that a certain subring of this formal power series ring is actually a Hopf ring, and we will note some important identities in this Hopf ring which we will use later.

If E has a Kunneth isomorphism, and $E_*(G_*)$ is thus a graded Hopf ring over E_*, there is a bigrading on $E_*(G_*)$ given by $deg(a) = (i, j)$ for $a \in E_i(G_j)$. The formal power series ring $E_*G_*[[x_1, ...x_n]]$ therefore inherits a bigrading given by taking $deg(x_i) = (-2, 0)$ and then extending linearly to all of $E_*G_*[[x_1, ...x_n]]$. We make this choice of degree for x_i because we will essentially be using these formal variables to keep track of the degree of even-dimensional elements in E_*G_*. However, to extend the coproduct and products on E_*G_* to $E_*G_*[[x_1, ..., x_n]]$ in a well-defined way, we have to do an extension of scalars so that we are working over $E_*[[x_1, ..., x_n]]$ instead of E_*. We extend the coproduct and products as follows.

Coproduct:

$$\psi\left(\sum_{i_1,\ldots,i_n} a_{i_1,\ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}\right) = \sum_{i_1,\ldots,i_n} \psi\left(a_{i_1,\ldots i_n}\right) x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

Additive product:

$$\left(\sum_{i_1,\dots,i_n} a_{i_1,\dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}\right) * \left(\sum_{j_1,\dots j_n} c_{j_1,\dots,j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}\right) =$$

$$\sum_{i_1,\dots,i_n,j_1,\dots,j_n} \left(a_{i_1,\dots i_n} * c_{j_1,\dots,j_n}\right) x_1^{i_1+j_1} x_2^{i_2+j_2} \dots x_n^{i_n+j_n}$$

Multiplicative product:

$$\left(\sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}\right) \circ \left(\sum_{j_1,\dots,j_n} c_{j_1,\dots,j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}\right) =$$

$$\sum_{i_1,\dots,i_n,j_1,\dots,j_n} \left(a_{i_1,\dots,i_n} \circ c_{j_1,\dots,j_n}\right) x_1^{i_1+j_1} x_2^{i_2+j_2} \dots x_n^{i_n+j_n}$$

In [11], Ravenel and Wilson also define the following element of $(E_*(G_*)[[x]])_{(0,2)}$.

$$b^G(x) = \sum_{i\geq 0} b_i^G x^i = b_0^G + b_1^G x + b_2^G x^2 + \dots$$

To generalize this, let us define the following element of $(E_*(G_*)[[x_1, x_2, \dots, x_n]])_{(0,2n)}$.

Definition $b^G(x_1, x_2, \dots, x_n) = \sum_{i_1,\dots,i_n\geq 0} (b_{i_1}^G \circ \dots \circ b_{i_n}^G) x_1^{i_1} \dots x_n^{i_n}$

We will also use the shorthand notation:

$$\bar{b}(x_1, \dots, x_n) = b^{ku}(x_1, \dots, x_n)$$

$$b(x_1, \dots, x_n) = b^{BP\langle 1\rangle}(x_1, \dots, x_n).$$

Alternatively, we could define $\bar{b}(x_1, \dots, x_n)$ to be $1 + \sum_{i_1,\dots,i_n\geq 1} \bar{b}_{i_1,\dots,i_n} x_1^{i_1} \dots x_n^{i_n}$. These two definitions are equivalent by Lemma 3.3.11. We also have the following result.

Lemma 3.4.1 $b^G(x_1, x_2, \dots, x_n) = b^G(x_1) \circ b^G(x_2) \circ \dots \circ b^G(x_n)$

Proof It will suffice to show that for all $i_1, ..., i_n$ the coefficient of $x_1^{i_1}...x_n^{i_n}$ is the same on both sides of the equation. By definition, the coefficient on the each side is $b_{i_1}^G \circ ... \circ b_{i_n}^G$. □

Proposition 3.4.2 *For any complex-oriented E and G such that E has a Kunneth isomorphism, $E_*G_*[[x_1, ..., x_n]]$ is a graded Hopf ring over $E_*[[x_1, ..., x_n]]$*

Proof This follows from the aforementioned definitions and the fact that E_*G_* is a graded Hopf ring over E_*. □

From this proposition, we then get the corollaries below, which are the key corollaries allowing us to prove that the Ando-Hopkins-Strickland cocycle relations hold in E_*ku_*.

Corollary 3.4.3 *If E has a Kunneth isomorphism, then*

$$b^G(x_i) \circ (y * z) = (b^G(x_i) \circ y) * (b^G(x_i) \circ z)$$

for all $y, z \in E_*G_*[[x_1, ...x_n]]$.

Proof Using the fact that the coproduct of b_i^G is $\psi(b_i^G) = b_i^G \otimes 1 + ...b_{i-1}^G \otimes b_1^G + ... + 1 \otimes b_i^G$, we get that

$$\psi(b^G(x_i)) = b^G(x_i) \otimes b^G(x_i).$$

Therefore, since $b^G(x_i)$ is grouplike, the formula in Lemma 3.2.1(v) gives us the desired result. □

Since we only used the Kunneth isomorphism for E to define the coproduct in $E_*G_*[[x_1, ..., x_n]]$, not the $\circ$-product or $*$-product, and since the above identity does not reference the coproduct, we can prove a similar result for a general complex-orientable E. Note that the naturally defined scalar extension of f_*^G, which is denoted

$\bar{f_*^G} : E_*MU_*[[x_1, ..., x_n]] \to E_*G_*[[x_1, ..., x_n]]$ and takes $yx_1^{i_1}...x_n^{i_n}$ to $f_*^G(y)x_1^{i_1}...x_n^{i_n}$, commutes with $\circ$ and $*$. Thus, we get the following corollary.

Corollary 3.4.4 *For any complex-oriented E and G,*

$$b^G(x_i) \circ (y * z) = (b^G(x_i) \circ y) * (b^G(x_i) \circ z)$$

holds for all $y, z \in im(f_*^G)[[x_1, ...x_n]]$.

Proof Since both $\circ$ and $*$ commute with addition and scalar multiplication, it will suffice to prove the identity for arbitrary monomials $y = f_*^G(\tilde{y})x_1^{i_1}...x_n^{i_n}$ and $z = f_*^G(\tilde{z})x_1^{j_1}...x_n^{j_n}$ in $im(f_*^G)[[x_1, ...x_n]]$. By the previous corollary, we know that the formula holds in $E_*MU_*[[x_1, ..., x_n]]$, and thus

$$b^{MU}(x_i) \circ ((\tilde{y}x_1^{i_1}...x_n^{i_n}) * (\tilde{z}x_1^{j_1}...x_n^{j_n})) = (b^{MU}(x_i) \circ (\tilde{y}x_1^{i_1}...x_n^{i_n})) * (b^{MU}(x_i) \circ (\tilde{z}x_1^{j_1}...x_n^{j_n})).$$

Now, if we apply $\bar{f_*^G}$ to this equation, we get

$$b^G(x_i) \circ ((f_*^G(\tilde{y})x_1^{i_1}...x_n^{i_n}) * (f_*^G(\tilde{z})x_1^{j_1}...x_n^{j_n}))$$

$$= (b^G(x_i) \circ (f_*^G(\tilde{y})x_1^{i_1}...x_n^{i_n})) * (b^G(x_i) \circ (f_*^G(\tilde{z})x_1^{j_1}...x_n^{j_n}))$$

since $\bar{f_*^G}(b^{MU}(x_i)) = b^G(x_i)$, and $\bar{f_*^G}$ commutes with $\circ$ and $*$. This can be rewritten as

$$b^G(x_i) \circ (y * z) = (b^G(x_i) \circ y) * (b^G(x_i) \circ z),$$

which is exactly what we wanted to prove. $\square$

CHAPTER 4

THE HOPF RING E_*BP_*

The results that we prove in this dissertation, and in particular our new proof of the Ando-Hopkins-Strickland Theorem for E_*ku_*, rely heavily on Ravenel's and Wilson's computation of the Hopf ring BP_*BP_* in [11], as well as Ravenel's, Wilson's, and Yagita's computation of the Hopf ring $E_*BP\langle n\rangle_*$ in [14]. In this chapter, we will give a brief overview of these results and discuss how they can be applied to the Hopf ring E_*ku_* using the relation $ku_{(p)} \simeq \bigvee_{0\leq i\leq p-2} \Sigma^{2i}BP\langle 1\rangle$.

4.1 Ravenel-Wilson Relations

One of the most important parts of their computation of BP_*BP_* (and by extension E_*BP_*) is the proof of the so-called Ravenel-Wilson relations, which hold in E_*G_*, even when E_*G_* is not necessarily a Hopf ring. If we consider the elements $b(s), b(t) \in E_*G_*[[s,t]]$ defined as in Section 3.4, and we use the notation

$$b(s) +_{[G]} b(t) = \ast_{i,j\geq 0}[a_{i,j}] \circ b(s)^{\circ i} \circ b(t)^{\circ j},$$

where $x +_G y = \sum_{i,j\geq 0} a_{i,j}x^i y^j$ is the formal group law for G, then Ravenel and Wilson proved the following.

Theorem 4.1.1 (Ravenel-Wilson, Theorem 3.8) *For any complex-orientable spectra E and G, the equality*

$$b(s +_E t) = b(s) +_{[G]} b(t)$$

holds in $E_*G_*[[s,t]]$.

If we consider the coefficient of s^it^j in this equation, we get a relation $R^E_{i,j}$ in E_*G_2. The set of *Ravenel-Wilson relations* for E_*G_* is the set $R^E = \{R^E_{i,j}\}_{i,j\in\mathbb{N}}$.

Definition We define $E^R_*G_*$ to be the free E_*-Hopf ring generated by the b_i, for all i, and the $[w] \in E_0G_*$, for all $w \in \pi_*G$, modulo the relations generated by R^E (i.e. generated under both the $*$-product and the $\circ$-product).

We note below some examples of these Ravenel-Wilson relations which are especially relevant for our purposes.

Corollary 4.1.2 *At* $p=2$*, we have the following relation in* $H_*BP_*[[s,t]]$.

$$b(s+t) = b(s) +_{[BP]} b(t) = b(s) * b(t) * ([-v_1] \circ b(s) \circ b(t)) * \left([v_1^2] \circ b(s)^{\circ 2} \circ b(t)\right) * \ldots$$

Corollary 4.1.3 *At* $p=2$*, we have the following relation in* $BP_*BP_*[[s,t]]$.

$$b\left(s+t-v_1st+v_1^2s^2t+\ldots\right) = b(s) * b(t) * ([-v_1] \circ b(s) \circ b(t)) * \left([v_1^2] \circ b(s)^{\circ 2} \circ b(t)\right) * \ldots$$

Corollary 4.1.4 *In* $E_*(ku_*)[[s,t]]$,

$$\bar{b}(s +_E t) = \bar{b}(s) +_{[ku]} \bar{b}(t) = \bar{b}(s) * \bar{b}(t) * \left([v] \circ \bar{b}(s) \circ \bar{b}(t)\right).$$

The following corollary was also noted in Section 4.5 of [7] for $E = H\mathbb{F}_2$.

Corollary 4.1.5 *For* E *connective, then the following relation holds in* $QE_*(ku_2)/E_{>0}$.

$$\binom{i+j}{i}\bar{b}_{i+j} = [v] \circ \bar{b}_i \circ \bar{b}_j$$

Proof This is exactly the coefficient of s^it^j modulo decomposables, modulo $E_{>0}$ in the equation given by Corollary 4.1.4. $\square$

4.2 E_*BP_*

In [11], Ravenel and Wilson compute the homology of the even spaces in BP by showing that the Ravenel-Wilson relations are actually the defining relations for the Hopf ring $E_*(BP_*)$.

Proposition 4.2.1 (Ravenel-Wilson, Corollary 4.7) *The natural map*

$$E_*^R(BP_{2*}) \to E_*(BP_{2*})$$

is an isomorphism of Hopf rings.

Hopkins and Hunton prove the same thing for the integral homology of a much broader class of Landweber exact ring spectra (See Chapter 7 for more details). However, these results do not apply to $E_*(ku)$ or $E_*(ku_{(p)})$ since neither ku nor $ku_{(p)}$ is Landweber exact.

Although it is not necessarily true that $E_*(ku_{(p)*}) \simeq E_*^R(ku_{(p)*})$, it turns out that these homologies coincide for the first few spaces. As we will see later on, $E_*(\underline{ku_{(p)}}_{2k}) = E_*^R(\underline{ku_{(p)}}_{2k})$ for $k = 1, 2, 3$.

Using these Ravenel-Wilson relations and a bar spectral sequence argument, Ravenel and Wilson also explicitly compute Hopf algebra generators for E_*BP_*. Since $E_*^R(BP_{2k}) \simeq E_*(BP_{2k})$ for all k, and π_*BP is generated by the v_i, we know that E_*BP_{2k} must be generated as a Hopf algebra by $\circ$-monomials on the $[v_i]$ and b_i. Ravenel and Wilson further show that E_*BP_{2k} is generated by monomials of the form $v^Ib^J = [v_1^{i_1}...v_n^{i_n}] \circ b_{(j_1)} \circ ... \circ b_{(j_{k+\frac{1}{2}(i_1|v_1|+...+i_n|v_n|)})}$. Since there are still many of these monomials that get equated under the Ravenel-Wilson relations, they define an *allowability* condition for these monomials which allows us to describe E_*BP_{2k} using a minimal set of generators.

Definition Given a prime p, we say that a monomial v^Ib^J is *allowable* if the following implication holds for all $s > 0$:

($v^I b^J$ can be written as $[v_1^{i_1} v_2^{i_2} ... v_m^{i_m}] \circ (b^G_{(j_1)})^{\circ k_1} \circ (b^G_{(j_2)})^{\circ k_2} \circ ... \circ (b^G_{(j_n)})^{\circ k_n}$ with $k_1 \geq p$, $k_2 \geq p^2$, ..., $k_s \geq p^s$, $s \leq n$, and $k_1 \leq k_2 \leq ... \leq k_s$) $\Rightarrow$ ($i_s = 0$)

Proposition 4.2.2 (Ravenel-Wilson, 5.3 and 5.8) *$H_*(BP_{2k}; \mathbb{Z}_{(p)})$ is a polynomial algebra and is generated by the allowable monomials $v^I b^J$ in $H_*(BP_{2k}; \mathbb{Z}_{(p)})$.*

4.3 $E_* BP\langle n \rangle_*$

Ravenel's and Wilson's description of $E_* BP_*$ becomes useful to us for the purposes of this dissertation because it descends to a description of $E_* BP\langle n \rangle_{2k}$ for some k. Using this fact from Ravenel, Wilson, and Yagita, as well as the relationship we already know to exist between $BP\langle 1 \rangle$ and ku, we can prove useful information about $E_* ku_*$.

In [14], Ravenel, Wilson, and Yagita prove the following theorem, where $g(n) = 2(p^n + ... + p + 1)$ and $I(n) = ([v_{n+1}], [v_{n+2}], ...)$.

Proposition 4.3.1 (Ravenel-Wilson-Yagita 8.2.1)

$$E^R_*(BP_{2k})/I(n) \simeq E_*(BP\langle n \rangle_{2k})$$

for $0 < 2k \leq g(n)$.

Corollary 4.3.2 *For all $2k \leq g(n)$, $E_*(BP\langle n \rangle_{2k})$ is generated as an E_*-algebra by the monomials $[v_1^{i_1} ... v_n^{i_n}] \circ b_{(j_1)} \circ ... \circ b_{(j_{k+\frac{1}{2}(i_1|v_1|+...+i_n|v_n|)})}$.*

Corollary 4.3.3 *For all $2k \leq g(1)$, $H_*((ku_{2k})_{(p)}; \mathbb{Z})$ is generated as a $\mathbb{Z}$-algebra by the monomials $[v^i] \circ b_{(j_1)} \circ ... \circ b_{(j_{k+i})}$.*

Proof We know that additively

$$H_*\left((ku_k)_{(p)}\right) = H_*\left(ku_k; \mathbb{Z}_{(p)}\right) = H_*\left(\left(\bigvee_{0 \leq i \leq p-2} \Sigma^{2i} BP\langle 1 \rangle\right)_k\right)$$

$$= \bigotimes_{0\leq i\leq p-2} H_*\left(\left(\Sigma^{2i} BP\langle 1\rangle\right)_k\right)$$

for all $k > 0$. Furthermore, we know that the homotopy groups of $ku_{(p)}$ are

$$\pi_* ku_{(p)} = (\pi_* BP\langle 1\rangle)[v]/(v^{p-1} = v_1).$$

We also know from Lemmas 3.3.5 and 3.3.4 that when we move from $\pi_* ku$ to $H_*\left(ku_*\right)$, addition becomes the $*$-product and multiplication becomes the $\circ$-product. Therefore, since we get $\pi_* ku_{(p)}$ by adjoining v to $\pi_* BP\langle 1\rangle$ as an additive generator and then imposing $v^{p-1} = v_1$, we should similarly get $H_*(ku_*; \mathbb{Z}_{(p)})$ by adjoining $[v]$ to $H_*(BP\langle 1\rangle_*)$ as a $*$-product generator (an indecomposable) and then imposing the relation $[v]^{\circ p-1} = [v^{p-1}] = [v_1]$. So,

$$H_*\left(ku_*; \mathbb{Z}_{(p)}\right) = (H_* BP\langle 1\rangle_*)[\,[v]\,]/([v]^{\circ p-1} = [v_1])$$

since this matches the size of what we know $H_*\left(ku_k; \mathbb{Z}_{(p)}\right)$ to be additively. Now, for $k \leq 3$, if we take an arbitrary element of $QH_*(ku_{2k}; \mathbb{Z}_{(p)})$, it should therefore be equal to $\sum_{0\leq i\leq p-2}[v^i] \circ B_i$ for some $B_i \in QH_*(BP\langle 1\rangle_{2k+2i})$. Since $i \leq p-2$, we have that

$$2k + 2i \leq 2k + 2(p-2) = 2(k+p-2) \leq 2(3+p-2) = 2(p+1) = g(1).$$

Therefore, by Corollary 4.3.2, B_i must be the sum of monomials of the form $[v_1^{i_1}] \circ b_{(j_1)} \circ ... \circ b_{(j_{k+i+i_1(p-1)})}$, and thus our arbitrary element of $QH_*(ku_{2k}; \mathbb{Z}_{(p)})$ must be a sum of monomials of the form

$$[v^i] \circ [v_1^{i_1}] \circ b_{(j_1)} \circ ... \circ b_{(j_{k+i+i_1(p-1)})} = [v^{i+i_1(p-1)}] \circ b_{(j_1)} \circ ... \circ b_{(j_{k+i+i_1(p-1)})}.$$

Replace $i + i_1(p-1)$ with i and you get the desired result. $\square$

Here $b_{(i)}$ really denotes $\iota_*(b_{(i)})$, the image of $b_{(i)}$ in $H_*((ku_{(p)})_2; \mathbb{Z})$. See the discussion in Chapter 5 for more details.

Remark These corollaries show that for $2k \leq g(1)$, both $BP\langle 1\rangle_{2k}$ and $(ku_{2k})_{(p)}$ have ordinary homology concentrated in even degrees. This also means that for $k \leq 3$, ku_{2k} has ordinary homology concentrated in even degrees, since $g(1)$ is smallest at $p = 2$, where it is equal to 6.

CHAPTER 5

COMPARING THE b_i AND $\bar{b}_i$

The key insight allowing for a new proof of the Ando-Hopkins-Strickland Theorem is the identification $(BU\langle 2k\rangle)_{(p)} = (ku_{2k})_{(p)} = \left(\bigvee_{0\leq i\leq p-2} \Sigma^{2i} BP\langle 1\rangle\right)_{2k}$ for each prime p. This allows us to translate Ravenel, Wilson, and Yagita's description of $E_*BP\langle 1\rangle_{2k}$ into a description of $E_*BU\langle 2k\rangle$. However, a significant obstacle to this translation still remains in the fact that the generators $b_i = b_i^{BP\langle 1\rangle} \in E_*BP\langle 1\rangle_2$ and $\bar{b}_i = b_i^{ku} \in E_*((ku_2)_{(p)})$ are defined using different complex orientations, and therefore are not necessarily equal to each other in $E_*((ku_{2k})_{(p)})$. This chapter overcomes this obstacle by computing the images of the b_i in $E_*((ku_2)_{(p)})$ in terms of the $\bar{b}_i$.

In section 4.8 of [7], Hara computes the relationship between the $b_i \in H_*(BP\langle 1\rangle_{2i}; \mathbb{F}_p)$ and the $\bar{b}_i \in H_*(ku_{2i}; \mathbb{F}_p)$ for odd primes p. In this section, we will use a similar schema to show that the same general relationship holds at $p = 2$ and for homology with $\mathbb{Z}_{(p)}$-coefficients.

Consider the inclusion and projection maps

$$\iota : BP\langle 1\rangle \rightleftarrows ku_{(p)} : \rho$$

where we take ι to be the inclusion into the first summand of $ku_{(p)} = BP\langle 1\rangle \vee \Sigma^2 BP\langle 1\rangle \vee ... \vee \Sigma^{2(p-2)} BP\langle 1\rangle$ and ρ to be the projection onto the first summand. These induce maps

$$\underline{\iota} : BP\langle 1\rangle_2 \rightleftarrows (ku_2)_{(p)} : \underline{\rho}$$

and their corresponding (co)homology maps $\underline{\iota}_*$, $\underline{\rho}_*$, $\underline{\iota}^*$, and $\underline{\rho}^*$. Since ι is a map of ring spectra (see Section 2.1 of [2]), it follows that $\underline{\iota}_*$ is a Hopf ring map and therefore commutes with $*$-products and $\circ$-products. Therefore, it will be enough for us to compute the $\underline{\iota}_*(b_i)$ in terms of the $\bar{b}_i$, and then we can easily extend this to get a description of the $\underline{\iota}_*(b_{i_1,\ldots,i_k}) = \underline{\iota}_*(b_{i_1} \circ \ldots \circ b_{i_k}) = \underline{\iota}_*(b_{i_1}) \circ \ldots \circ \underline{\iota}_*(b_{i_k})$ in terms of the $\bar{b}_{i_1,\ldots,i_k}$. For ease of notation, we will sometimes refer to $\underline{\iota}_*(b_i)$ just as b_i when it is clear that we are working in $E_*((ku_2)_{(p)})$ instead of $E_*(BP\langle 1\rangle_{2k})$, or when $p = 2$ and these two homologies are equal.

5.1 $p = 2$

In this section, we will show that the same general relationship holds at $p = 2$, and we will compute this relationship explicitly. Since $\bar{b}_i$ and b_i are defined to be the image of β_i under $x_*^{ku_{(2)}}$ and $x_*^{BP\langle 1\rangle}$, respectively, we should first compute the relationship between $x^{ku_{(2)}}$ and $x^{BP\langle 1\rangle}$. If we let $x = x^{BP<1>}$ for ease of notation, then

$$x = exp^{BP\langle 1\rangle}\left(log^{ku_{(2)}}\left(x^{ku_{(2)}}\right)\right),$$

and

$$x^{ku_{(2)}} = exp^{ku_{(2)}}\left(log^{BP<1>}(x)\right).$$

Using the expansions

$$exp^{ku_{(2)}}(y) = y + \frac{v}{2!}y^2 + \frac{v^2}{3!}y^3 + \ldots + \frac{v^{n-1}}{n!}y^n + \ldots$$

and

$$log^{BP<1>}(y) = y + \frac{v_1}{2}y^2 + \frac{v_1^3}{4}y^4 + \ldots + \frac{v_1^{2^n-1}}{2^n}y^{2^n} + \ldots,$$

as well as the fact that $v = v_1$ at $p = 2$, we get

$$x^{ku_{(2)}} = \left(x + \frac{v_1}{2}x^2 + \frac{v_1^3}{4}x^4 + \ldots\right) + \frac{v}{2!}\left(x + \frac{v_1}{2}x^2 + \frac{v_1^3}{4}x^4 + \ldots\right)^2 + \ldots$$

$$= x + v_1x^2 + \frac{2}{3}v_1^2x^3 + \frac{2}{3}v_1^3x^4 + \frac{7}{15}v_1^4x^5 + \frac{16}{45}v_1^5x^6 + \frac{67}{315}v_1^6x^7 + \frac{88}{315}v_1^7x^8 + ...$$
$$= \sum a_i v_1^{i-1}x^i.$$

Note that the coefficients, a_i, must all be elements of $\mathbb{Z}_{(2)}$. Now, by the definitions of b_i and $\bar{b}_i$, we get that

$$\bar{b}_i = (x^{ku_{(2)}})_*(\beta_i) = \left(x + v_1x^2 + \frac{2}{3}v_1^2x^3 + ...\right)_*(\beta_i),$$

and

$$b_i = x_*^{BP\langle 1\rangle}(\beta_i) = x_*(\beta_i).$$

However, to further simplify the expression for $\bar{b}_i$, we need to know a little bit more about the relationship between these cohomology elements and the β_i. By Lemma 3.3.8, we know that the coproducts of the b_i and the $\bar{b}_i$ are

$$\psi(b_i) = b_i \otimes 1 + b_{i-1} \otimes b_1 + ... + 1 \otimes b_i$$

and

$$\psi(\bar{b}_i) = \bar{b}_i \otimes 1 + \bar{b}_{i-1} \otimes \bar{b}_1 + ... + 1 \otimes \bar{b}_i.$$

Using these coproducts, we can prove the following lemmas, which, along with Lemmas 3.3.4 and 3.3.10, allow us to give an explicit expression for $\bar{b}_i$ in terms of the b_i. As will become clear in their proofs, these lemmas essentially result from the fact that addition in $G^*\mathbb{CP}^\infty$ and $*$-multiplication in E_*G_* are both defined using the H-space structure on each G_k, and multiplication in $G^*\mathbb{CP}^\infty$ and $\circ$-multiplication in E_*G_* are both defined using the ring spectrum multiplication μ^G.

Lemma 5.1.1 *For any $y, z \in G^j(\mathbb{CP}^\infty)$,*

$$(y+z)_*(\beta_i) = \sum_{i_1+i_2=i} (y_*(\beta_{i_1})) * (z_*(\beta_{i_2}))$$

in E_*G_*.

Proof If we consider the map representing $y+z$

$$\mathbb{CP}^\infty \xrightarrow{\psi} (\mathbb{CP}^\infty)^{\times 2} \xrightarrow{y\times z} (G_j)^{\times 2} \xrightarrow{\gamma^G} G_j$$

and its induced E_*-homology map

$$E_*(\mathbb{CP}^\infty) \xrightarrow{\psi_*} (E_*(\mathbb{CP}^\infty))^{\otimes 2} \xrightarrow{(y\times z)_*=y_*\otimes z_*} (E_*G_j)^{\otimes 2} \xrightarrow{\gamma_*^G} E_*G_j,$$

we see that

$$(y+z)_*(\beta_i) = \gamma_*^G((y_*\otimes z_*)((\psi_*)(\beta_i))) = \left(\gamma_*^G\right)\left((y_*\otimes z_*)\left(\sum_{i_1+i_2=i}\beta_{i_1}\otimes\beta_{i_2}\right)\right)$$

$$= \gamma_*^G\left(\sum_{i_1+i_2=i} y_*(\beta_{i_1})\otimes z_*(\beta_{i_2})\right) = \sum_{i_1+i_2=i} y_*(\beta_{i_1}) * z_*(\beta_{i_2}).$$

□

Lemma 5.1.2 *For any $a \in G^*$ and $y \in G^*(\mathbb{CP}^\infty)$,*

$$(ay^j)_*(\beta_i) = \sum_{i_1+i_2+\ldots i_j=i} [a]\circ y_*(\beta_{i_1})\circ \ldots \circ y_*(\beta_{i_j})$$

in E_*G_*.

Proof If we consider the map representing ay^j

$$\mathbb{CP}^\infty = \mathbb{CP}^\infty \wedge S^0 \xrightarrow{\psi^j\wedge id} (\mathbb{CP}^\infty)^{\wedge j}\wedge S^0 \xrightarrow{y^j\wedge a} (G_{|y|})^{\wedge j}\wedge G_{|a|} \xrightarrow{\mu^G} G_{|ay^j|}$$

and its induced E_*-homology map

$$E_*(\mathbb{CP}^\infty) \longrightarrow E_*(\mathbb{CP}^\infty)^{\otimes j} \otimes E_*(S^0) \longrightarrow E_*(G_{|y|})^{\otimes j} \otimes E_*(G_{|a|}) \longrightarrow E_*G_{|ay^j|},$$

we see that

$$(ay^j)_*(\beta_i) = \mu_*^G \left(\left((y_*)^{\otimes j} \otimes a_* \right) \left(\left((\psi_*)^{\otimes j} \otimes id_* \right) (\beta_i) \right) \right)$$

$$= \left(\mu_*^G\right) \left(\left((y_*)^{\otimes j} \otimes a_* \right) \left(\sum_{i_1+i_2+\ldots i_j=i} a \otimes \beta_{i_1} \otimes \ldots \otimes \beta_{i_j} \right) \right)$$

$$= \mu_*^G \left(\sum_{i_1+i_2+\ldots i_j=i} [a] \otimes y_*(\beta_{i_1}) \otimes \ldots \otimes y_*(\beta_{i_j}) \right)$$

$$= \sum_{i_1+i_2+\ldots i_j=i} [a] \circ y_*(\beta_{i_1}) \circ \ldots \circ y_*(\beta_{i_j}).$$

□

Corollary 5.1.3 *For any $j > i$ and $a_j \in \mathbb{Z}_{(p)}$, we have that*

$$(a_j v^{j-1} (x^{BP\langle 1 \rangle})^j)_*(\beta_i) = 0$$

in $E_*(ku_{(p)})$.

Proof This follows from Lemma 5.1.2, as well as the fact that $\bar{b}_0 \circ \bar{b}_i = 1 \circ \bar{b}_i = 0$ for all $i > 0$ (By Lemma 3.3.11). □

Using these lemmas and corollary, we then get that

$$\bar{b}_i = \left(x + v_1 x^2 + \frac{2}{3} v_1^2 x^3 + \ldots \right)_* (\beta_i)$$

$$= \left(x + v_1 x^2 + \frac{2}{3} v_1^2 x^3 + \ldots + a_i v_1^{i-1} x^i \right)_* (\beta_i)$$

$$= \sum_{m_1+....+m_i=i} (x_*(\beta_{m_1})) * \left(\left(v_1x^2\right)_*(\beta_{m_2})\right) * ... * \left(\left(a_iv_1^{i-1}x^i\right)_*(\beta_{m_i})\right)$$

$$= \sum_{m_1+....+m_i=i} \left((b_{m_1}) * ... * \left(\sum_{m_{i,1}+...+m_{i,i}=m_i} [a_iv_1^{i-1}] \circ b_{m_{i,1}} \circ ... \circ b_{m_{i,i}}\right)\right)$$

$$= \sum_{m_1+....+m_j=i} \left((b_{m_1}) * ([v_1] \circ b_{m_2} \circ b_{m_3}) * ... * \left([a_iv_1^{i-1}] \circ b_{m_{j-i+1}} \circ ... \circ b_{m_j}\right)\right)$$

where $j = \frac{i(i+1)}{2}$.

This gives us each $\bar{b}_i$ as a sum of $*$-products of elements $[a_jv_1^{j-1}] \circ b_{j_1} \circ ... \circ b_{j_j} \in E_*((ku_2)_{(2)})$. In fact, we can generalize this to rewrite our relations as a single relation on $\bar{b}(s)$ and $b(s)$.

Proposition 5.1.4

$$\bar{b}(s) = b(s) * \left([v_1] \circ (b(s))^{\circ 2}\right) * \left([\frac{2}{3}v_1^2] \circ (b(s))^{\circ 3}\right) * ... = \left(exp^{ku_{(2)}} \circ log^{BP<1>}\right)_{[]}(b(s))$$

where $\left(exp^{ku_{(2)}} \circ log^{BP<1>}\right)_{[]}(x) = \ast [a_iv_1^{i-1}] \circ x^{\circ i}$.

Proof It will suffice to show that the coefficient of s^i is the same on both sides of the equation. On the left-hand-side, the coefficient of s^i is $\bar{b}_i$, and on the right-hand-side the coefficient is

$$\sum_{m_1+....+m_j=i} \left((b_{m_1}) * ([v_1] \circ b_{m_2} \circ b_{m_3}) * ... * \left([a_iv_1^{i-1}] \circ b_{m_{j-i+1}} \circ ... \circ b_{m_j}\right)\right).$$

The result follows since we have already shown these to be equal. □

Examining the s^i coefficient of each side of this equation gives us an expression for $\bar{b}_i$ in terms of $[v_1]$ and the b_j. For example, if we write out these relations, using Lemmas 5.1.1 and 5.1.2 to simplify, we get:

$\bar{b}_1 = b_1$

$$
\begin{aligned}
\bar{b}_2 &= b_2 + [v_1] \circ b_1^{\circ 2} \\
\bar{b}_3 &= b_3 + 2[v_1] \circ b_1 \circ b_2 + [\frac{2}{3}v_1^2] \circ b_1^{\circ 3} + b_1 * ([v_1] \circ b_1^{\circ 2}) \\
\bar{b}_4 &= b_4 + 2[v_1] \circ b_1 \circ b_3 + [v_1] \circ b_2^{\circ 2} + 3[\frac{2}{3}v_1^2] \circ b_1^{\circ 2} \circ b_2 + [\frac{2}{3}v_1^3] \circ b_1^{\circ 4} + 2b_1 * ([v_1] \circ b_1 \circ b_2) \\
&\quad + b_1 * \left([\frac{2}{3}v_1^2] \circ b_1^{\circ 3} \right) + b_2 * ([v_1] \circ b_1^{\circ 2}) + b_1^{*2} * ([v_1] \circ b_1^{\circ 2}) \\
\bar{b}_5 &= b_5 + 2[v_1] \circ b_1 \circ b_4 + 2[v_1] \circ b_2 \circ b_3 + 3[\frac{2}{3}v_1^2] \circ b_1^{\circ 2} \circ b_3 + 3[\frac{2}{2}v_1^2] \circ b_1 \circ b_2^{\circ 2} + 4[\frac{2}{3}v_1^3] \circ b_1^{\circ 3} \circ b_2 \\
&\quad + [\frac{7}{15}v_1^4] \circ b_1^{\circ 5} + 2b_1 * ([v_1] \circ b_1 \circ b_3) + b_1 * ([v_1] \circ b_2^{\circ 2}) + 3b_1 * \left([\frac{2}{3}v_1^2] \circ b_1^{\circ 2} \circ b_2 \right) \\
&\quad + b_1 * \left([\frac{2}{3}v_1^3] \circ b_1^{\circ 4} \right) + 2b_2 * ([v_1] \circ b_1 \circ b_2) + b_2 * \left([\frac{2}{3}v_1^2] \circ b_1^{\circ 3} \right) + b_3 * ([v_1] \circ b_1^{\circ 2}) \\
&\quad + \left([\frac{2}{3}v_1] \circ b_1^{\circ 2} \right) * ([v_1^2] \circ b_1^{\circ 3}) .
\end{aligned}
$$

Remark This computation establishes that $\underline{\iota}_*(b_1) = \bar{b}_1$, and it is not hard to see that $\rho_*(\bar{b}_1) = b_1$ as well.

Remark This process allows us to get an easy formula for $\{b_i^{G_1}\}$ in terms of $\{b_i^{G_2}\}$ for any two complex orientations of G, x^{G_1} and x^{G_2}. If $x^{G_2} = \sum_i a_i x^{G_1} \in G^2(\mathbb{CP}^\infty) = G^*[[x^{G_1}]]$, then

$$
b^{G_1}(s) = \mathop{\Large *} \left([a_i] \circ \left(b^{G_2}(s) \right)^{\circ i} \right) .
$$

We could use the Ravenel-Wilson relations to further reduce the expressions we have for the $\bar{b}_i$ and write each of them as a $*$-polynomial on the b_i without any $[v_1]$'s, but this becomes computationally difficult very quickly. So, we will simply prove the proposition below, which turns out to be all we need for our purposes.

Proposition 5.1.5 *If E is connective, then for all $i \geq 0$ we have that*

$$
b_i = c_i \bar{b}_i
$$

in $QE_*((ku_2)_{(2)})/E_{>0}$, where $c_i \in \mathbb{Z}_{(2)}$ is a unit.

Proof We will use Proposition 5.1.4 and Corollary 4.1.5 to prove the theorem by induction on i. To ground our induction at $i = 0$, we note that $b_0 = 1 = \bar{b}_0$. Now, for the inductive step, we examine the s^i-coefficient on both sides of the equation in Proposition 5.1.4, modulo decomposables, modulo $E_{>0}$. On the left-hand-side we get $\bar{b}_i$, and on the right-hand-side we get b_i plus additional terms, each of which looks like $c[v_1^{j-1}] \circ b_{i_1} \circ ... \circ b_{i_j}$ with $j > 1$ and $i_1, ..., i_j > 0$. In fact, we will show that we can rewrite each of these terms as an even multiple of $\bar{b}_i$, which, when combined with the $\bar{b}_i$ on the left-hand-side, gives that an odd (congruent to $1 \ (mod\, 2)$) multiple of $\bar{b}_i$ is equal to b_i. Since the coefficient is odd, it is also a unit in $\mathbb{Z}_{(2)}$.

To perform this rewrite/reduction, we first replace each b_{i_k} in $c[v^{j-1}] \circ b_{i_1} \circ ... \circ b_{i_j}$ with $c_{i_k}\bar{b}_{i_k}$ (which can be done by the inductive hypothesis, since $i_k < i$ for degree reasons), and then apply the identity $[v] \circ \bar{b}_m \circ \bar{b}_n = \binom{m+n}{m}\bar{b}_{m+n}$ (see Corollary 4.1.5) over and over again until there are no more copies of $[v]$, and we are left with just a multiple of $\bar{b}_i$. If any two of the $i_1, ..., i_j$ are equal to the same thing, say r, then we will choose to reduce these together and the resulting coefficient, $\binom{2r}{r}$, will be even. Therefore, the final coefficient of $\bar{b}_i$ will also be even. If no two of the $i_1, ..., i_j$ are equal, then $c = j!$, which is even since $j > 1$, so the final coefficient of $\bar{b}_i$ will also be even. $\square$

5.2 General Prime, p

For the most part, we can use the same process we used at $p = 2$ to compute each $\underline{\iota}_*(b_{(i)})$ in terms of the $\bar{b}_{(i)}$ at an arbitrary prime p. Hara did essentially the same thing when he proved that $\underline{\iota}_*(b_{(i)}) = \bar{b}_{(i)}$ in $H_*((ku_2)_{(p)}; \mathbb{F}_p)$, for all odd primes p (See 4.8 in [7]).

If we go through the same translation process for a general prime p we will similarly get that

$$\underline{\iota}_*(b_i) = \underline{\iota}_*(x_*^{BP\langle 1\rangle}(\beta_i)) = (\underline{\iota}^* x^{BP\langle 1\rangle})_*(\beta_i) = (exp^{BP\langle 1\rangle}(log^{ku_{(p)}}(x^{ku_{(p)}})))_*(\beta_i),$$

and thus

Proposition 5.2.1

$$\underline{\iota}_*(b(s)) = (exp^{BP\langle 1\rangle} \circ log^{ku_{(p)}})_{[]}(\bar{b}(s)) = \bar{b}(s) * ([a_2] \circ \bar{b}(s)^{\circ 2}) * ([a_3] \circ \bar{b}(s)^{\circ 3})...$$

where

$$(exp^{BP\langle 1\rangle} \circ log^{ku_{(p)}})(y) = \sum_i a_i y^i.$$

Note that the coefficients in the expansion of $exp^{BP\langle 1\rangle} \circ log^{ku_{(p)}}$ must all be elements of $\mathbb{Z}_{(p)}$.

This allows us to prove a more general version of Proposition 5.1.5 in the theorem below.

Theorem 5.2.2 *If E is connective and p is an arbitrary prime, then for all $i \geq 0$ we have that*

$$\underline{\iota}_*(b_{(i)}) = c_{p^i}\bar{b}_{(i)}$$

in $QE_*((ku_2)_{(p)})/E_{>0}$, where $c_{p^i} \in \mathbb{Z}_{(p)}$ is a unit.

Proof The proof of this theorem is similar to the proof of Proposition 5.1.5. The $i = 0$ case is easy since $\underline{\iota}_*(b_{(0)}) = \underline{\iota}_*(b_1) = \bar{b}_1 = \bar{b}_{(0)}$ is exactly the s-coefficient in the equation in Proposition 5.2.1. For the $i \geq 1$ case, we examine the s^i-coefficient on both sides of the equation, modulo decomposables, modulo $E_{>0}$. On the left-hand-side we get $\underline{\iota}_*(b_{(i)})$ and on the right-hand-side we get $\bar{b}_{(i)}$ plus additional terms, each of which looks like $ca_j[v^{j-1}] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_j}$, modulo decomposables, with $j \geq 2$, $i_1 + ... + i_j = p^i$, and $c \in \mathbb{Z}_{(p)}$. We will show that each of these terms is equal to

$p\bar{c}a_j\bar{b}_{(i)}$ for some $\bar{c} \in \mathbb{Z}_{(p)}$, meaning that when we combine them with the single copy of $\bar{b}_{(i)}$ we will get the desired result.

Using Corollary 4.1.5 repeatedly, we get that

$$ca_j[v^{j-1}] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_j} = a_j \binom{i_1 + i_2}{i_1} \binom{i_1 + i_2 + i_3}{i_1 + i_2} ... \binom{i_1 + ... + i_j}{i_1 + ... + i_{j-1}} \bar{b}_{(i)}.$$

We will show that one of these binomial coefficients is a multiple of p (and thus the entire coefficient is a multiple of p) when there are at least p repeats among $\bar{b}_{i_1}, ..., \bar{b}_{i_j}$ and that c is a multiple of p otherwise.

Suppose that there are at least p repeats among $\bar{b}_{i_1}, \bar{b}_{i_2}, ..., \bar{b}_{i_j}$, i.e. there is some k such that $\bar{b}_{i_k} = \bar{b}_{i_{k+1}} = ... = \bar{b}_{i_{k+p-1}}$. We will assume without loss of generality that $\bar{b}_{i_1} = \bar{b}_{i_2} = ... = \bar{b}_{i_p}$ (If this is not the case, then rearrange the $\bar{b}_{i_1}, \bar{b}_{i_2}, ..., \bar{b}_{i_j}$ so that it is the case before performing the Corollary 4.1.5 reduction). This means that the first p binomial coefficients in the reduction equation are $\binom{2i_1}{i_1}\binom{3i_1}{i_1}...\binom{pi_1}{i_1}$. Let m be the unique integer such that $p^m \leq i_1 < p^{m+1}$. We know, therefore, that $i_1 < p^{m+1} \leq pi_1$, and thus $(d-1)i_1 < p^{m+1} \leq di_1$ for some d between 2 and p. But, combinatorially, this means that $\binom{di_1}{i_1}$ must be a multiple of p, and thus the entire coefficient must be a multiple of p.

Now, suppose that there are not at least p repeats among $\bar{b}_{i_1}, \bar{b}_{i_2}, ..., \bar{b}_{i_j}$. Then, $c = \frac{(p^i)!}{(m_1)!...(m_r)!}$ where $m_1, ..., m_r < p$ are the multiplicities of the different subscripts among $i_1, ..., i_j$. This means that there is a multiple of p in the numerator, but not the denominator, so c is a multiple of p. □

Corollary 5.2.3 *If E is connective and p is an arbitrary prime, then for all $i_1, ..., i_k \geq 0$ we have that*

$$\underline{\iota}_*(b_{(i_1)} \circ ... b_{(i_k)}) = c_{p^{i_1}} ... c_{p^{i_k}} \bar{b}_{(i_1)} \circ ... \circ \bar{b}_{(i_k)}$$

in $QE_*((ku_{2k})_{(p)})/E_{>0}$, where $c_{p^{i_1}}, ..., c_{p^{i_k}} \in \mathbb{Z}_{(p)}$ are units.

Proof This follows directly from Theorem 5.2.2 and the fact that $\underline{\iota}_*$ is a map of Hopf rings. $\square$

CHAPTER 6

MAIN THEOREMS

In this chapter, we will prove the main theorems of Ando, Hopkins, and Strickland computing the E-homology of $ku_{2k} = BU\langle 2k\rangle$ when $k = 1, 2, 3$ or $E = H\mathbb{Q}$. These proofs make essential use of the work of Ravenel, Wilson, and Yagita on E_*BP_* and $E_*BP\langle n\rangle_*$ to avoid the algebro-geometric perspective of Ando, Hopkins, and Strickland. Throughout this chapter we will assume that E is complex-oriented. In [1], Ando, Hopkins, and Strickland further assume that E is an even periodic spectrum, but the methods they use also prove the theorem for any complex-oriented spectrum, so we will work without the added even periodicity assumption.

6.1 Statement of the Theorems

In [1], after Ando, Hopkins, and Strickland construct the homology generators $\bar{b}_{i_1,\ldots,i_k}$ in $E_*(BU\langle 2k\rangle)$ for each even periodic E, they prove the following symmetry relations on them.

Theorem 6.1.1 (Symmetry Relations) $\bar{b}_{i_1,i_2,\ldots i_k} = \bar{b}_{\sigma(i_1),\sigma(i_2),\ldots\sigma(i_k)}$ *for any permutation* σ.

When we consider the fact that $\bar{b}_{i_1,\ldots,i_k} = b^{ku}_{i_1}\circ\ldots\circ b^{ku}_{i_k}$, this theorem follows directly from Theorem 3.3.3, which itself follows directly from the graded commutativity of Hopf rings. We will use S^E_k (or just S if k and E are clear) to refer to the set of all these relations.

They also prove the following theorem establishing the main set of relations (cocycle relations) in E_*ku_*.

Theorem 6.1.2 (Cocycle Relations) *The following relation in* $E_*(\underline{ku}_{2n})[[x_0, ..., x_k]]$ *holds for every* $k \geq 2$*.*

$$\bar{b}\,(x_1, x_2, ..., x_k) * \bar{b}\,(x_0, x_1 +_E x_2, x_3, ..., x_k) = \bar{b}\,(x_0 +_E x_1, x_2, ...x_k) * \bar{b}\,(x_0, x_1, x_3, ..., x_k)$$

In particular, for $k = 2$ this gives us the theorem below.

Theorem 6.1.3 *The following relation in* $E_*(\underline{ku}_4)[[x_0, x_1, x_2]]$ *holds.*

$$\bar{b}\,(x_1, x_2) * \bar{b}\,(x_0, x_1 +_E x_2) = \bar{b}\,(x_0 +_E x_1, x_2) * \bar{b}\,(x_0, x_1)$$

For each monomial $x_0^{i_0}...x_k^{i_k}$, we can equate its coefficients on the left and right sides of the equation in Theorem 6.1.2 to give a relation in $E_*(\underline{ku}_{2k})$. We will call this relation $C^E_{k,i_0,...,i_k}$. We will also use $C^E_k = \{C^E_{k,i_0,...,i_k}\}$ (or just C if n and E are clear) to refer to the set of all of these cocycle relations in $E_*(\underline{ku}_{2k})$.

Given that these symmetry relations and cocycle relations hold, for all $k > 0$ we can therefore define an E_*-algebra map,

$$\varphi^E_k : E_*[\bar{b}_{i_1,...,i_k}]/(S, C) \to E_*(ku_{2k}),$$

which takes each E_*-polynomial on the $\bar{b}_{i_1,...,i_k}$ to the E_*-polynomial of the same name in $E_*(ku_{2k})$. Here $E_*[\bar{b}_{i_1,...,i_k}]/(S, C)$ refers to the free E_*-algebra generated by the $\bar{b}_{i_1,...,i_k}$, modulo the ideal generated by the symmetry and cocycle relations. We also assume this ideal to be empty when $k = 1$. The existence of this map simply means that the elements $\bar{b}_{i_1,...,i_k}$ exist in $E_*(ku_{2k})$ and that the symmetry and cocycle relations hold.

Furthermore, in the special cases of $E = H\mathbb{Q}$ and $k = 1, 2, 3$, they prove that φ_k^E is an injection (i.e. the symmetry and cocycle relations are the *only* relations on the $\bar{b}_{i_1,\ldots,i_k}$) and a surjection (i.e. the $\bar{b}_{i_1,\ldots,i_k}$ generate all of $E_*(ku_{2k})$ as a polynomial algebra), thereby establishing the following theorems.

Theorem 6.1.4 *The map*

$$\varphi_k^{H\mathbb{Q}} : \mathbb{Q}[\bar{b}_{i_1,\ldots i_k}]/(S, C) \to H_*(ku_{2k}; \mathbb{Q})$$

is an isomorphism for all $k > 0$.

Theorem 6.1.5 *The map*

$$\varphi_k^E : E_*[\bar{b}_{i_1,\ldots i_k}]/(S, C) \to E_*(ku_{2k})$$

is an isomorphism for $k = 1, 2, 3$.

In the following sections, we will prove these results without the algebro-geometric perspective of Ando, Hopkins, and Strickland. However, we will make use of the following algebraic fact, which follows directly from Propositions 3.1, 3.9, and 3.28 in [1].

Proposition 6.1.6 (Ando, Hopkins, Strickland) *The algebra $\mathbb{Z}[\bar{b}_{i_1,\ldots,i_k}]/(S, C)$ is torsion-free for $k = 1, 2, 3$.*

6.2 Proof of the Cocycle Relations

We can now prove the cocycle relations using the Ravenel-Wilson relations in E_*ku_* given by Corollary 4.1.4. In fact, it will turn out that these two sets of relations give us essentially the same information about $E_*(ku_{2k})$ for $k \geq 2$.

The following proves Theorem 6.1.3. The proof essentially makes use of the Ravenel-Wilson relations and the Hopf ring identities holding in $E_*(ku_*)[[x_0, x_1, x_2]]$.

Proof of Theorem 6.1.3

$$\bar{b}\,(x_1, x_2) * \bar{b}\,(x_0, x_1 +_E x_2) = \big(\bar{b}\,(x_1) \circ \bar{b}\,(x_2)\big) * \big(\bar{b}\,(x_0) \circ \bar{b}\,(x_1 +_E x_2)\big)$$

$$= \big(\bar{b}\,(x_1) \circ \bar{b}\,(x_2)\big) * \Big(\bar{b}\,(x_0) \circ \Big(\bar{b}\,(x_1) * \bar{b}\,(x_2) * \big([v] \circ \bar{b}\,(x_1) \circ \bar{b}\,(x_2)\big)\Big)\Big)$$

(By Corollary 4.1.4)

$$= \big(\bar{b}\,(x_1) \circ \bar{b}\,(x_2)\big) * \big(\bar{b}\,(x_0) \circ \bar{b}\,(x_1)\big) * \big(\bar{b}\,(x_0) \circ \bar{b}\,(x_2)\big) * \big([v] \circ \bar{b}\,(x_0) \circ \bar{b}\,(x_1) \circ \bar{b}\,(x_2)\big)$$

(By Theorem 3.4.4 and the associativity of $*$)

$$= \big(\bar{b}\,(x_0) \circ \bar{b}\,(x_1)\big) * \Big(\bar{b}\,(x_2) \circ \Big(\bar{b}\,(x_0) * \bar{b}\,(x_1) * \big([v] \circ \bar{b}\,(x_0) \circ \bar{b}\,(x_1)\big)\Big)\Big)$$

(By Theorem 3.4.4 and the commutativity and associativity of $*$ and $\circ$)

$$= (\bar{b}(x_0) \circ \bar{b}(x_1)) * (\bar{b}(x_2) \circ \bar{b}(x_0 +_E x_1))$$

(By Corollary 4.1.4)

$$= \bar{b}\,(x_0, x_1) * \bar{b}\,(x_0 +_E x_1, x_2)$$

$$= \bar{b}\,(x_0 +_E x_1, x_2) * \bar{b}\,(x_0, x_1)$$

(By commutativity of $*$). $\square$

We can also prove the general case of Theorem 6.1.2 by making use of the proof of 6.1.3.

Proof of Theorem 6.1.2

$$\bar{b}\left(x_1, x_2, x_3, ..., x_k\right) * \bar{b}\left(x_0, x_1 +_E x_2, x_3, ..., x_k\right)$$

$$= \left(\bar{b}\left(x_1\right) \circ \bar{b}\left(x_2\right) \circ ... \circ \bar{b}\left(x_k\right)\right) * \left(\bar{b}\left(x_0\right) \circ \bar{b}\left(x_1 +_E x_2\right) \circ ... \circ \bar{b}\left(x_k\right)\right)$$

$$= \left(\left(\bar{b}\left(x_1\right) \circ \bar{b}\left(x_2\right)\right) * \left(\bar{b}\left(x_0\right) \circ \bar{b}\left(x_1 +_E x_2\right)\right)\right) \circ \bar{b}\left(x_3\right) \circ ... \circ \bar{b}\left(x_k\right)$$

(By Theorem 3.4.4)

$$= \left(\bar{b}\left(x_1, x_2\right) * \bar{b}\left(x_0, x_1 +_E x_2\right)\right) \circ \bar{b}\left(x_3\right) \circ ... \circ \bar{b}\left(x_k\right)$$

$$= \left(\bar{b}\left(x_0 +_E x_1, x_2\right) * \bar{b}\left(x_0, x_1\right)\right) \circ \bar{b}\left(x_3\right) \circ ... \circ \bar{b}\left(x_k\right)$$

(By Theorem 6.1.3)

$$= \left(\left(\bar{b}\left(x_0 +_E x_1\right) \circ \bar{b}\left(x_2\right)\right) * \left(\bar{b}\left(x_0\right) \circ \bar{b}\left(x_1\right)\right)\right) \circ \bar{b}\left(x_3\right) \circ ... \circ \bar{b}\left(x_k\right)$$

$$= \left(\left(\bar{b}\left(x_0 +_E x_1\right) \circ \bar{b}\left(x_2\right) \circ \bar{b}\left(x_3\right) \circ ... \circ \bar{b}\left(x_k\right)\right) * \left(\bar{b}\left(x_0\right) \circ \bar{b}\left(x_1\right) \circ \bar{b}\left(x_3\right) \circ ... \circ \bar{b}\left(x_k\right)\right)\right)$$

(By Theorem 3.4.4)

$$= \bar{b}\left(x_0 +_E x_1, x_2, x_3, ..., x_k\right) * \bar{b}\left(x_0, x_1, x_3, ..., x_k\right).$$

□

The cocycle relations that these equations yield in $E_*(ku_{2k})$ are fairly complicated to write out. However, if we only work rationally and restrict ourselves to the case where E is connective, they become a little bit easier to work with.

In particular, for E connective the cocycle relations imply the following corollary.

Corollary 6.2.1 *For E connective, the relation*

$$\bar{b}_{i,j} = \frac{(i+1)!}{j!(i+j-2)!}\bar{b}_{i+j-1,1}$$

holds in $\big(QE_*[\bar{b}_{i_1,i_2}]/(S,C,E_{>0})\big) \otimes \mathbb{Q}$.

Proof Looking at the coefficient of $x_0^i x_1 x_2^{j-1}$ in the cocycle relation above, modulo decomposables, modulo $E_{>0}$, we get the relation

$$j\bar{b}_{i,j} = (i+1)\bar{b}_{i+1,j-1}$$

for all $i \geq 1$ and $j \geq 2$. Applying this $j-1$ times and multiplying by $j!$ gives the corollary. □

In the special case of $E = H\mathbb{Q}$, this corollary shows that each $\bar{b}_{i_1,i_2}$ in $\mathbb{Q}[\bar{b}_{i_1,i_2}]/(S,C)$ is a rational multiple of $\bar{b}_{i_1+i_2-1,1}$. Therefore, $\mathbb{Q}[\bar{b}_{i_1,i_2}]/(S,C)$ really only contains one indecomposable in each even degree.

If we similarly write out these cocycle relations in $(QE_*(ku_{2k})/(E_{>0})) \otimes \mathbb{Q}$ more explicitly for a general $k \geq 2$, we can show that $\mathbb{Q}[\bar{b}_{i_1,\ldots i_k}]/(S,C)$ only contains one indecomposable in each even degree.

Corollary 6.2.2 *The following relation holds in* $(QE_*(ku_{2k})/(E_{>0})) \otimes \mathbb{Q}$.

$$\bar{b}_{i_0,i_1,\ldots,i_{k-1}} = \frac{(i_0+1)!(i_0+i_1)!\ldots(i_0+\ldots+i_{k-2}-k+3)!}{i_1!\ldots i_{k-1}!(i_0+i_1-2)!\ldots(i_0+\ldots+i_{k-1}-k)!}\bar{b}_{i_0+i_1+\ldots+i_{k-1}-k+1,1,\ldots,1}$$

Proof Looking at the coefficient of $x_0^{i_0} x_1 x_2^{i_1-1} x_3^{i_2} \ldots x_k^{i_{k-1}}$ in the cocycle relation above, modulo decomposables, modulo $E_{>0}$, we get the relation

$$i_1\bar{b}_{i_0,i_1,\ldots i_{k-1}} = (i_0+1)\bar{b}_{i_0+1,i_1-1,\ldots i_{k-1}}$$

for all $i_0 \geq 1$ and $i_1 \geq 2$. Applying the relation $i_1 - 1$ times reduces the second subscript to 1, and gives us

$$\bar{b}_{i_0,i_1,\ldots i_{k-1}} = \frac{(i_0+1)!}{i_1!(i_0+i_1-2)!}\bar{b}_{i_0+i_1-1,1,i_2,\ldots,i_{k-1}}.$$

If we use the symmetry condition (Theorem 3.3.3) we can reduce all the subsequent subscripts to 1 as well. Keeping track of the coefficients along the way, this gives the corollary. □

As a consequence of this corollary, we get that the inclusion map

$$\mathbb{Q}[\bar{b}_{i,1,\ldots,1}] \xrightarrow{\psi_k^{H\mathbb{Q}}} \mathbb{Q}[\bar{b}_{i_1,i_2,\ldots,i_k}]/(S,C)$$

is not only a $\mathbb{Q}$-algebra map but also a surjection.

Corollary 6.2.3 *The map $\psi_k^{H\mathbb{Q}}$ is a surjection of $\mathbb{Q}$-algebras for all $k > 0$.*

6.3 Computation of $H_*(ku_{2k};\mathbb{Q})$

In this section, we will use what we have done so far to compute $H_*(ku_{2k};\mathbb{Q})$, thereby proving Theorem 6.1.4. Since at $p = 2$ we have the equality $ku_{(2)} = BP\langle 1\rangle$, we know that $H_*(ku_{2k};\mathbb{Q}) = H_*(ku_{(2)_{2k}};\mathbb{Q}) = H_*(BP\langle 1\rangle_{2k};\mathbb{Q})$. Therefore, it will suffice to compute $H_*(BP\langle 1\rangle_{2k};\mathbb{Q})$ at $p = 2$ in terms of the b_i. Then, we will simply have to translate this description in terms of the b_i to a description in terms of the $\bar{b}_i$.

From Section 2.4 of [11], we have the following result. The corollary for $BP\langle n\rangle$ follows easily.

Lemma 6.3.1 (Ravenel-Wilson 2.4) *The map*

$$\circ\, b_1^{MU} : H_*(MU_*;\mathbb{F}_p) \to H_{*+2}(MU_{*+2};\mathbb{F}_p)$$

is the double iteration of the homology suspension homomorphism.

Corollary 6.3.2 *The map* $\circ\, b_1^{BP\langle n\rangle} : H_*(BP\langle n\rangle_*;\mathbb{Q}) \to H_{*+2}(BP\langle n\rangle_{*+2};\mathbb{Q})$ *is the double iteration of the homology suspension homomorphism.*

Furthermore, we know that rationally every space $BP\langle 1\rangle_{2k}$ is a product of Eilenberg-MacLane spaces, since it is the $2k$-th space in the Ω-spectrum $BP\langle 1\rangle$, and therefore an infinite loop space. So, if we let $\psi : H_*(BP\langle 1\rangle_*;\mathbb{Q}) \to H_{*+1}(BP\langle 1\rangle_{*+1};\mathbb{Q})$ be the homology suspension homomorphism, then we have isomorphisms

$$\pi_*(BP\langle 1\rangle;\mathbb{Q}) \cong PH_*(BP\langle 1\rangle_{2k};\mathbb{Q}) \cong QH_*(BP\langle 1\rangle_{2k};\mathbb{Q}),$$

the first of which takes each $v_1^n \in \pi_*(BP\langle 1\rangle;\mathbb{Q}) = \mathbb{Q}\{1, v_1, v_1^2, v_1^3, ...\}$ to $\psi^{2k+2n}([v_1^n]) \in PH_*(BP\langle 1\rangle_{2k};\mathbb{Q})$, which, in turn, the second isomorphism takes to $\psi^{2k+2n}([v_1^n]) \in QH_*(BP\langle 1\rangle_{2k};\mathbb{Q})$. Since $\psi^2 = \circ b_1$ by the corollary above, we get the following theorem.

Theorem 6.3.3 $H_*(BP\langle 1\rangle_{2k};\mathbb{Q}) = \mathbb{Q}[[v_1^n] \circ b_1^{\circ n+k}]$.

This gives us what we need to prove the main theorem for rational homology.

Proof of Theorem 6.1.4 The previous sections of this chapter, in proving that the cocycle relations and symmetry relations hold in $E_*(ku_{2k})$, establish the existence of the map we are interested in:

$$\varphi_k^{H\mathbb{Q}} : \mathbb{Q}[\bar{b}_{i_1,i_2,\dots i_k}]/(S,C) \to H_*(ku_{2k};\mathbb{Q}).$$

Surjectivity

By Theorem 6.3.3, we know that

$$H_*(ku_{2k};\mathbb{Q}) = H_*(BP\langle 1\rangle_{2k};\mathbb{Q}) = \mathbb{Q}[[v_1^n] \circ b_1^{\circ n+k}],$$

so it will suffice to show that each $[v_1^n] \circ b_1^{\circ n+k}$ is in the image of $\varphi_k^{H\mathbb{Q}}$ (since, by definition, $\varphi_k^{H\mathbb{Q}}$ is a map of $\mathbb{Q}$-algebras).

Regarding $ku_{\mathbb{Q}}$ as $BP\langle 1\rangle_{\mathbb{Q}}$ for $p = 2$, ι_* and ρ_* are both isomorphisms, and we have $b_1 = \bar{b}_1$ and $v_1 = v$. Therefore, applying Corollary 4.1.5 for $E = H\mathbb{Q}$, we get that

$$[v_1^n] \circ b_1^{\circ n+k} = [v^n] \circ \bar{b}_1^{\circ n+k} = ([v^n] \circ \bar{b}_1^{\circ n+1}) \circ \bar{b}_1^{\circ k-1} = \binom{2}{1}\binom{3}{1} \cdots \binom{n+1}{1} \bar{b}_{n+1} \circ \bar{b}_1^{\circ k-1}$$

$$= (n+1)! \bar{b}_{n+1,1,\ldots,1}$$

in $QH_*(ku_{2k}; \mathbb{Q})$. So, $[v_1^n] \circ b_1^{\circ n+k}$ is in the image since $\bar{b}_{n+1,1,\ldots,1}$ is in the image, and $(n+1)!$ is invertible in $\mathbb{Q}$.

Injectivity

Since $H_*(ku_{2k}; \mathbb{Q}) = \mathbb{Q}[[v_1^n] \circ b_1^{\circ n+k}]$, we know that no rational polynomial on the $[v_1^n] \circ b_1^{\circ n+k}$ can be zero. Furthermore, since $[v_1^n] \circ b_1^{\circ n+k}$ is a nonzero multiple of $\bar{b}_{n+1,1,\ldots,1}$, no polynomial on the $\bar{b}_{n+1,1,\ldots,1}$ can be equal to zero either. But, we know by Corollary 6.2.2 that $H\mathbb{Q}_*[\bar{b}_{i_1,i_2,\ldots i_k}]/(S, C)$ is generated as a graded vector space by the polynomials on the $\bar{b}_{n+1,1,\ldots,1}$. Therefore, since $\varphi_k^{H\mathbb{Q}}$ is a map of $\mathbb{Q}$-algebras, it cannot take any nonzero element to zero. $\square$

Corollary 6.3.4 *For all $k > 0$, $\psi_k^{H\mathbb{Q}}$ is an isomorphism, so*

$$H_*(ku_{2k}; \mathbb{Q}) \cong \mathbb{Q}[\bar{b}_{i,1,\ldots,1}].$$

Proof Combining Theorem 6.1.4 and Proposition 6.2.3 gives us that the $\mathbb{Q}$-algebra map

$$\mathbb{Q}[\bar{b}_{i,1,\ldots,1}] \longrightarrow H_*(ku_{2k}; \mathbb{Q})$$

taking each $\bar{b}_{i,1,\ldots,1}$ to $\bar{b}_{i,1,\ldots,1}$ is a surjection. Furthermore, we know from Theorem 6.3.3 that $H_*(ku_{2k}; \mathbb{Q})$ is a free polynomial algebra with a single indecomposable generator in each even degree greater than or equal to $2k$. Therefore, if any $*$-polynomial on the $\bar{b}_{i,1,...,1}$ were sent to zero by this map, this would mean that $H_*(ku_{2k};\mathbb{Q})$ is not a polynomial algebra. So, the map must also be an injection. $\square$

Proposition 6.3.5 *The homology elements* $[v_1]$ *and* b_1 *freely generate the Hopf ring* $H_*(BP\langle 1\rangle_*;\mathbb{Q})$.

Using the same arguments, we also get the analogous results below for $BP\langle n\rangle$.

Proposition 6.3.6

$$H_*(BP\langle n\rangle_{2k};\mathbb{Q}) = \mathbb{Q}[[v_1^{j_1}...v_n^{j_n}] \circ b_1^{\circ j_1+3j_2...+(2^n-1)j_n+k}]$$

Proposition 6.3.7 *The homology elements* $[v_1],...,[v_n]$, *and* b_1 *freely generate the Hopf ring* $H_*(BP\langle n\rangle_*;\mathbb{Q})$.

Question 6.3.8 *Do the elements* $b^{BP\langle n\rangle}_{j_1,...j_k}$ *generate* $H_*(BP\langle n\rangle_{2k};\mathbb{Q})$ *as a Hopf algebra for* $k \geq n$*?*

6.4 Computation of $E_*(ku_{2k})$ for $E = H\mathbb{Z}$, $k \leq 3$

In this section, we will use the results from the previous sections to prove the Ando, Hopkins, Strickland theorem for ordinary homology. This proof makes essential use Proposition 6.1.6, as well as the results of Ravenel, Wilson, and Yagita on the structure of $H_*BP\langle 1\rangle_{2k}$. In particular, we will use Corollary 4.3.3, which followed fairly easily from the corresponding propositions that Ravenel, Wilson, and Yagita proved for $BP\langle 1\rangle$.

Proof of Theorem 6.1.5 for $E = H\mathbb{Z}$

Injectivity Consider the diagram below.

$$\begin{tikzcd}
\mathbb{Z}[\bar{b}_{i_1,\dots,i_k}]/(S,C) \arrow[r, "\varphi_k^{H\mathbb{Z}}"] \arrow[d, "\otimes\mathbb{Q}"] & H_*(ku_{2k};\mathbb{Z}) \arrow[d, "\otimes\mathbb{Q}"] \\
\mathbb{Z}[\bar{b}_{i_1,\dots,i_k}]/(S,C)\otimes\mathbb{Q} \arrow[r, "\varphi_k^{H\mathbb{Z}}\otimes\mathbb{Q}"] \arrow[d, "\cong"] & H_*(ku_{2k};\mathbb{Z})\otimes\mathbb{Q} \arrow[d, "\cong"] \\
\mathbb{Q}[\bar{b}_{i_1,\dots,i_k}]/(S,C) \arrow[r, "\varphi_k^{H\mathbb{Q}}"] & H_*(ku_{2k};\mathbb{Q})
\end{tikzcd}$$

The horizontal maps exist by our proof of the cocycle and symmetry relations in $E_*(ku_{2k})$. The isomorphism in the bottom right is trivial, and the isomorphism in the bottom left holds since the cocycle and symmetry relations for $E = H\mathbb{Q}$ only have integer coefficients, and therefore the cocycle and symmetry relations are exactly the same for $E = H\mathbb{Z}$ and $E = H\mathbb{Q}$, even though they generate slightly different ideals. The top half of the diagram clearly commutes by definition of the rationalization, and the commutativity of the bottom half follows from the fact that $\varphi_k^{H\mathbb{Z}} \otimes \mathbb{Q}$ and $\varphi_k^{H\mathbb{Q}}$ are both $\mathbb{Q}$-algebra maps which take $\bar{b}_{i_1,\dots,i_k}$ to $\bar{b}_{i_1,\dots,i_k}$.

By Proposition 6.1.6, we know that $\mathbb{Z}[\bar{b}_{i_1,\dots,i_k}]/(S,C)$ is torsion-free, meaning that

$$\otimes\mathbb{Q} : \mathbb{Z}[\bar{b}_{i_1,\dots,i_k}]/(S,C) \to \mathbb{Q}[\bar{b}_{i_1,\dots,i_k}]/(S,C)$$

is injective. Furthermore, by Theorem 6.1.4, we know that $\varphi_k^{H\mathbb{Q}}$ is an isomorphism, which implies that

$$\varphi_k^{H\mathbb{Q}} \circ (\otimes\mathbb{Q}) \text{ is injective} \Rightarrow (\otimes\mathbb{Q}) \circ \varphi_k^{H\mathbb{Z}} \text{ is injective} \Rightarrow \varphi_k^{H\mathbb{Z}} \text{ is injective.}$$

Surjectivity It will suffice to show that $\varphi_k^{H\mathbb{Z}} \otimes \mathbb{Z}_{(p)}$ is surjective for all primes p. Fix a prime p, and consider the diagram below.

$$\begin{tikzcd}
\mathbb{Z}[\bar{b}_{i_1,\ldots,i_k}]/(S,C)\otimes\mathbb{Z}_{(p)} \arrow[r, "\varphi_k^{H\mathbb{Z}\otimes\mathbb{Z}_{(p)}}"] \arrow[d, "\cong"] & H_*(ku_{2k};\mathbb{Z})\otimes\mathbb{Z}_{(p)} \arrow[d, "\cong"] \\
\mathbb{Z}_{(p)}[\bar{b}_{i_1,\ldots,i_k}]/(S,C) \arrow[r, "\varphi_k^{H\mathbb{Z}_{(p)}}"] & H_*(ku_{2k};\mathbb{Z}_{(p)})
\end{tikzcd}$$

The isomorphism on the left holds since the cocycle and symmetry relations for $E = H\mathbb{Z}_{(p)}$ only have integer coefficients, and therefore the cocycle and symmetry relations are exactly the same for $E = H\mathbb{Z}$ and $E = H\mathbb{Z}_{(p)}$, even though they generate slightly different ideals. The isomorphism on the right follows from the Universal Coefficient Theorem and the fact that $H_*(ku_{2k};\mathbb{Z}_{(p)})$ is concentrated in even degree (by Corollary 4.3.3 for $E = H\mathbb{Z}_{(p)}$). The commutativity of the diagram follows from the fact that $\varphi_k^{H\mathbb{Z}}\otimes\mathbb{Z}_{(p)}$ and $\varphi_k^{H\mathbb{Z}_{(p)}}$ are both $\mathbb{Z}_{(p)}$-algebra maps which take $\bar{b}_{i_1,\ldots,i_k}$ to $\bar{b}_{i_1,\ldots,i_k}$. Because of this diagram, we simply have to show that $\varphi_k^{H\mathbb{Z}_{(p)}}$ is surjective.

We also know by Corollary 4.3.3 that $H_*(ku_{2k};\mathbb{Z}_{(p)}) = H_*((ku_{2k})_{(p)};\mathbb{Z}_{(p)})$ is a $\mathbb{Z}_{(p)}$-algebra generated by the monomials $[v^i]\circ b^J$. Therefore, since $\varphi_k^{H\mathbb{Z}_{(p)}}$ is a map of $\mathbb{Z}_{(p)}$-algebras by definition, it will suffice to show that each such monomial is in the image of $\varphi_k^{H\mathbb{Z}_{(p)}}$.

If we work in $QH_*((ku_{2k})_{(p)};\mathbb{Z}_{(p)})$, then for any monomial $[v^{n-k}]\circ b_{(j_1)}\circ b_{(j_2)}\circ\ldots\circ b_{(j_n)}$ we have

$$[v^{n-k}]\circ b_{(j_1)}\circ b_{(j_2)}\circ\ldots\circ b_{(j_n)} = c_{j_1}c_{j_2}\ldots c_{j_n}[v^{n-k}]\circ\bar{b}_{(j_1)}\circ\bar{b}_{(j_2)}\circ\ldots\circ\bar{b}_{(j_n)}$$

for nonzero integers $c_{j_1},\ldots c_{j_n}$, by Corollary 5.2.3. We also implicitly used Lemma 3.3.16 in this equation to get that the $\circ$-product is a well-defined operation on $QH_*((ku_{2k})_{(p)};\mathbb{Z}_{(p)})$.

Next, if we repeatedly apply Corollary 4.1.5 to $c_{j_1}c_{j_2}\ldots c_{j_n}[v^{n-k}]\circ\bar{b}_{(j_1)}\circ\bar{b}_{(j_2)}\circ\ldots\circ\bar{b}_{(j_n)}$

to get rid of the powers of $[v]$, again working in $QH_*((ku_{2k})_{(p)};\mathbb{Z}_{(p)})$, we get that

$$c_{j_1}...c_{j_n}[v^{n-k}]\circ\bar{b}_{(j_1)}\circ\bar{b}_{(j_2)}\circ...\circ\bar{b}_{(j_n)} = c_{j_1}c_{j_2}...c_{j_n}\binom{p^{j_1}+p^{j_2}}{p^{j_2}}[v^{n-k-1}]\circ\bar{b}_{p^{j_1}+p^{j_2}}\circ\bar{b}_{(j_3)}\circ...\circ\bar{b}_{(j_n)}$$

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$$= c_{j_1}...c_{j_n}\binom{p^{j_1}+p^{j_2}}{p^{j_2}}...\binom{p^{j_1}+...+p^{j_{n-k}}}{p^{j_{n-k}}}\bar{b}_{p^{j_1}+...+p^{j_{n-k}}}\circ\bar{b}_{(j_{n-k+1})}\circ...\circ\bar{b}_{(j_n)}$$

$$= c_{j_1}...c_{j_n}\binom{p^{j_1}+p^{j_2}}{p^{j_2}}...\binom{p^{j_1}+...+p^{j_{n-k}}}{p^{j_{n-k}}}\bar{b}_{p^{j_1}+...+p^{j_{n-k}},p^{j_{n-k+1}},...p^{j_n}}$$

which is a nonzero multiple of an element of $\bar{b}_{i_1,...i_k}$ since the $c_{j_1}, c_{j_2}, ..., c_{j_n}$ and the $\binom{p^{j_1}+p^{j_2}}{p^{j_2}}$, $\binom{p^{j_1}+p^{j_2}+p^{j_3}}{p^{j_3}}$, ..., $\binom{p^{j_1}+...+p^{j_{n-k}}}{p^{j_{n-k}}}$ are all nonzero. Therefore, $[v^{n-k}]\circ b_{(j_1)}\circ b_{(j_2)}\circ ...\circ b_{(j_n)} + \sum_i D_i$ is in the image of $\varphi_k^{H\mathbb{Z}_{(p)}}$, for some decomposable elements, D_i. Finally, then, if we use induction on the homological degree, we get that each $[v^{n-k}]\circ b_{(j_1)}\circ b_{(j_2)}\circ ...\circ b_{(j_n)}$ is strictly in the image of $\varphi_k^{H\mathbb{Z}_{(p)}}$, since the inductive step will give us that each D_i is in the image, as it is the product of indecomposable elements of smaller degree. □

This more explicit, computational approach to the Ando-Hopkins-Strickland Theorem allows us to move beyond the scope of what they do to constuct an isomorphism $\chi_k^E : E_*^R(ku_{2k}) \to E_*(ku_{2k})$ as well. For any complex-oriented spectrum, E, and $k \geq 1$, define χ_k^E to be the map of E_*-algebras which takes each $[v^j]\circ\bar{b}_{i_1}^E...\circ\bar{b}_{i_{j+k}}^E$ to $[v^j]\circ\bar{b}_{i_1}^E...\circ\bar{b}_{i_{j+k}}^E$.

For all $k \geq 1$, we also define a map $\kappa_k^E : E_*[\bar{b}_{i_1,...,i_k}^E]/(S_k^E, C_k^E) \to E_*^R(ku_{2k})$ to be the map of E_*-algebras which takes each $\bar{b}_{i_1,...,i_k}^E$ to $\bar{b}_{i_1,...,i_k}^E = \bar{b}_{i_1}^E\circ...\circ\bar{b}_{i_k}^E \in E_*^R(ku_{2k})$. This E_*-algebra map exists since the symmetry and cocycle relations, S_k^E and C_k^E, follow from the Ravenel-Wilson relations (Corollary 4.1.4), as shown in the proof of

Theorems 6.1.2 and 6.1.3.

For $E = H\mathbb{Z}$, this gives us the diagram below.

$$\begin{tikzcd}
H_*^R(ku_{2k};\mathbb{Z}) \arrow[drr, "\chi_k^{H\mathbb{Z}}"] & & \\
\mathbb{Z}[\bar{b}_{i_1,...,i_k}]/(S_k^{H\mathbb{Z}}, C_k^{H\mathbb{Z}}) \arrow[u, "\kappa_k^{H\mathbb{Z}}"] \arrow[rr, "\varphi_k^{H\mathbb{Z}}"] & & H_*(ku_{2k};\mathbb{Z})
\end{tikzcd}$$

This diagram commutes because $\varphi_k^{H\mathbb{Z}}$, $\kappa_k^{H\mathbb{Z}}$, and $\chi_k^{H\mathbb{Z}}$ are all $\mathbb{Z}$-algebra maps which take $\bar{b}_{i_1,...,i_k}$ to $\bar{b}_{i_1,...,i_k}$.

Corollary 6.4.1 *The maps $\chi_k^{H\mathbb{Z}}$ and $\kappa_k^{H\mathbb{Z}}$ are isomorphisms for $k = 1, 2, 3$.*

Proof It will suffice to show that $\kappa_k^{H\mathbb{Z}}$ is an isomorphism, since $\varphi_k^{H\mathbb{Z}}$ is an isomorphism, and the diagram above commutes. Injectivity is easy since the injectivity of $\varphi_k^{H\mathbb{Z}}$ implies the injectivity of $\kappa_k^{H\mathbb{Z}}$. To prove that $\kappa_k^{H\mathbb{Z}}$ is surjective, it will suffice to show that each algebra generator $[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}$ is in the image of $\kappa_k^{H\mathbb{Z}}$, since it is a map of $\mathbb{Z}$-algebras. We will prove this by induction on homological degree. The base case follows from the fact that $\bar{b}_1$ is the only algebra generator in $H_2^R(ku_{2k};\mathbb{Z})$, and it is in the image of $\kappa_k^{H\mathbb{Z}}$, by definition. Now, suppose that each generator of homological degree less than $i = i_1 + ... + i_{j+k}$ is in the image of $\kappa_k^{H\mathbb{Z}}$. We know by Corollary 4.1.5 for $E = H\mathbb{Z}$ that $[v] \circ \bar{b}_r \circ \bar{b}_s = \binom{r+s}{r}\bar{b}_{r+s}$ in $QH_*^R(ku_{2k};\mathbb{Z})$, since this corollary follows from the Ravenel-Wilson relations. Applying this repeatedly to $[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}$, and using Lemma 3.3.16, we get that

$$[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}} = \binom{i_1 + i_2}{i_2}\binom{i_1 + i_2 + i_3}{i_3} ... \binom{i_1 + ... + i_{j+1}}{i_{j+1}}\bar{b}_{i_1+...+i_{j+1}, i_{j+2}, ..., i_{j+k}}$$

$$+ \sum_m (D_m * \hat{D}_m)$$

where each D_m and $\hat{D}_m$ has homological degree greater than 1 and less than i. Therefore, since $\bar{b}_{i_1+...+i_{j+1}, i_{j+2}, ..., i_{j+k}}$ and the D_m, $\hat{D}_m$ are all in the image of $\kappa_k^{H\mathbb{Z}}$, and

$\mathsf{\kappa}_k^{H\mathbb{Z}}$ is a $\mathbb{Z}$-algebra map, we get that $[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}$ is in the image of $\mathsf{\kappa}_k^{H\mathbb{Z}}$. $\square$

6.5 Computation of $E_*(ku_{2k})$ for E Complex-Oriented, $k \leq 3$

Proof of Theorem 6.1.5 We have already proven the theorem for $E = H\mathbb{Z}$, thereby computing the structure of $H_*(ku_{2k};\mathbb{Z})$. To prove the theorem for a general complex-oriented E, we can simply use the Atiyah-Hirzebruch spectral sequence to extend this result from ordinary homology to E-homology.

Let E be a complex-oriented Ω-spectrum. The Atiyah-Hirzebruch spectral sequence for E is a spectral sequence of algebras and has E^2-page

$$E^2_{p,q} = H_p(ku_{2k};\pi_q E) \Rightarrow E_{p+q}(ku_{2k}).$$

Since $H_*(ku_{2k};\mathbb{Z})$ is torsion-free by Proposition 6.1.6 and Section 6.4, we have that

$$E^2_{*,*} = H_*(ku_{2k};\pi_* E) = H_*(ku_{2k};\mathbb{Z}) \otimes E_* = \left(\mathbb{Z}[\bar{b}_{i_1,\ldots i_k}]/\left(S_k^{H\mathbb{Z}}, C_k^{H\mathbb{Z}}\right)\right) \otimes E_*$$

$$= E_*[\bar{b}_{i_1,\ldots i_k}]/\left(S_k^{H\mathbb{Z}}, C_k^{H\mathbb{Z}}\right).$$

Moreover, we can identify what each element in $E^2_{*,*} = E_*[\bar{b}_{i_1,\ldots i_k}]/\left(S_k^{H\mathbb{Z}}, C_k^{H\mathbb{Z}}\right)$ detects in $E_*(ku_{2k})$ and therefore conclude that all elements in the E^2-page must persist, meaning that all of the differentials in this spectral sequence must be zero. In particular, each $\bar{b}_{i_1,\ldots,i_k} \in E^2_{*,*}$ detects $\bar{b}_{i_1,\ldots,i_k} \in E_*(ku_{2k})$, and since the AHSS is a spectral sequence of algebras, this means that every nonzero E_*-polynomial on the $\bar{b}_{i_1,\ldots,i_k} \in E^2_{*,*}$ detects the corresponding polynomial on the $\bar{b}_{i_1,\ldots,i_k} \in E_*(ku_{2k})$. Therefore, the associated graded must be

$$E^\infty_{*,*} = E_*[\bar{b}_{i_1,\ldots i_k}]/\left(S_k^{H\mathbb{Z}}, C_k^{H\mathbb{Z}}\right).$$

Note that $E_*[\bar{b}_{i_1,\ldots i_k}]/\left(S_k^{H\mathbb{Z}}, C_k^{H\mathbb{Z}}\right)$ and $E_*[\bar{b}_{i_1,\ldots i_k}]/\left(S_k^E, C_k^E\right)$ are not necessarily equal as algebras, since the cocycle relations are potentially different for E and for $H\mathbb{Z}$. More specifically, $C^E_{k,i_0,\ldots,i_k} = C^{H\mathbb{Z}}_{k,i_0,\ldots,i_k} + \hat{C}^E_{k,i_0,\ldots,i_k}$, where $\hat{C}^E_{k,i_0,\ldots,i_k}$ is the sum of all the monomials in $C^E_{k,i_0,\ldots,i_k}$ whose coefficients are not integers, but rather are elements of $E_{>0}$. Thus, $C^E_{k,i_0,\ldots,i_k}$ and $C^{H\mathbb{Z}}_{k,i_0,\ldots,i_k}$ are only equal modulo $E_{>0}$.

In fact, if we consider the fact that $S_k^{H\mathbb{Z}} = S_k^E$ and $C^{H\mathbb{Z}}_{k,i_0,\ldots,i_k} = -\hat{C}^E_{k,i_0,\ldots,i_k}$ in $E_*(ku_{2k})$, we can see that

$$E_*(ku_{2k}) \cong E_*[\bar{b}_{i_1,\ldots,i_k}]/\left(S_k^{H\mathbb{Z}}, C_k^{H\mathbb{Z}} + \hat{C}_k^E\right) \cong E_*[\bar{b}_{i_1,\ldots,i_k}]/\left(S_k^E, C_k^E\right).$$

Furthermore, this isomorphism is a E_*-algebra map and takes each $\bar{b}_{i_1,\ldots,i_k}$ to $\bar{b}_{i_1,\ldots,i_k}$, so it must be equal to φ_k^E. □

As with the $E = H\mathbb{Z}$ case, we are also able to give an alternate description of $E_*(ku_{2k})$ by considering the commutative diagram below.

$$\begin{tikzcd}
E_*^R(ku_{2k}) \arrow[drr, "\chi_k^E"] & & \\
E_*[\bar{b}_{i_1,\ldots,i_k}]/(S_k^E, C_k^E) \arrow[u, "\kappa_k^E"] \arrow[rr, "\varphi_k^E"] & & E_*(ku_{2k})
\end{tikzcd}$$

Proposition 6.5.1 *For any complex-oriented spectrum E, χ_k^E and κ_k^E are isomorphisms for $k = 1, 2, 3$.*

Proof By the commutativity of the diagram and the fact that φ_k^E is an isomorphism, it will suffice to prove that χ_k^E is an isomorphism. The proof of this fact is essentially the same as the proof of Theorem 6.1.5 above. We similarly get that

$$E^2_{*,*} = H_*(ku_{2k}; \pi_* E) = H_*(ku_{2k}; \mathbb{Z}) \otimes E_* = H_*^R(ku_{2k}; \mathbb{Z}) \otimes E_*$$

$$= E_*[[v^j] \circ \bar{b}_{i_1} \circ \ldots \circ \bar{b}_{i_{j+k}}]/\left(R_k^{H\mathbb{Z}}\right)$$

by Corollary 6.4.1 and the fact that $H_*(ku_{2k};\mathbb{Z})$ is torsion-free.

Each $[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}} \in E^2_{*,*}$ detects $[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}} \in E_*(ku_{2k})$, and since the AHSS is a spectral sequence of algebras, every nonzero E_*-polynomial on the $[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}$ must detect the corresponding polynomial on the $[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}$. Therefore, all elements in the E^2-page must persist, and all of the differentials in this spectral sequence must be zero. So,

$$E^{\infty}_{*,*} = E_*[[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}]/\left(R^{H\mathbb{Z}}_k\right).$$

Once again, however, there are algebraic extensions. Observe that $R^E_{k,i} = R^{H\mathbb{Z}}_{k,i} + \hat{R}^E_{k,i}$, where $\hat{R}^E_{k,i}$ is the sum of all the monomials in $R^E_{k,i}$ whose coefficients are not integers, but rather are elements of $E_{>0}$. Therefore, $R^{H\mathbb{Z}}_{k,i} = -\hat{R}^E_{k,i}$ in $E_*(ku_{2k})$, and thus

$$E_*(ku_{2k}) \cong E_*[[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}]/\left(R^{H\mathbb{Z}}_k + \hat{R}^E_k\right) \cong E_*[[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}]/\left(R^E_k\right)$$

$$= E^R_*(ku_{2k}).$$

Furthermore, this isomorphism is a E_*-algebra map and takes each $E_*[[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}]$ to $E_*[[v^j] \circ \bar{b}_{i_1} \circ ... \circ \bar{b}_{i_{j+k}}]$, so it must be equal to χ^E_k. $\square$

CHAPTER 7

RELATION TO THE PERIODIC CASE

The difficulty of studying $E_*BP\langle n\rangle_*$ and E_*ku_* results from the fact that neither $BP\langle n\rangle$ nor ku is Landweber exact. In [6], Hopkins and Hunton give an implicit description of E_*G_* for all Landweber exact G.

Proposition 7.0.1 (Hopkins-Hunton, Theorem 2.2 and Remark 2.6) *If E and F are both complex-oriented spectra and F is Landweber exact, then E_*F_k is an exterior algebra if k is odd and a polynomial algebra if k is even, and the natural map*

$$E_*^R F_* \to E_*F_*$$

is an isomorphism of Hopf rings.

Using this fact, we can make our Hopf ring computations much simpler by simply replacing $BP\langle n\rangle$ and ku with the closely related Landweber exact spectra $E(n)$ (the Johnson-Wilson spectrum) and KU (periodic complex K-theory), which have homotopy groups given by

$$\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, v_2, ..., v_n, v_n^{-1}]$$

$$\pi_*(KU) = \mathbb{Z}[v, v^{-1}]$$

where once again $v_i \in \pi_{2(p^i-1)}(E(n))$ and $v \in \pi_2(KU)$. These two spectra are related in exactly the same way that $BP\langle 1\rangle$ and ku are related to each other. Namely,

$$KU_{(p)} = \bigvee_{0 \le i \le p-2} \Sigma^{2i} E(1).$$

In fact, just as in previous chapters we computed part of $E_* ku_*$ using our knowledge of $E_* BP\langle 1\rangle_*$ at each prime, we can similarly compute $E_* KU_*$ by first computing $E_* E(1)_*$ at each prime and then piecing these computations together. If we apply Proposition 7.0.1 to $E(1)$ and write out the Ravenel-Wilson relations explicitly, we get the following explicit description of the Hopf ring $E_* E(1)_*$. Unlike $E_* BP\langle 1\rangle_{2k}$, which is only a polynomial algebra for $k < 3$, $E_* E(1)_{2k}$ is a polynomial algebra for all k.

Theorem 7.0.2

$$E_* E(1)_* = \bigotimes \{E([v_1^i] \circ e_1 \circ b_{(j_1)}^{\circ k_1} \circ ... \circ b_{(j_m)}^{\circ k_m})\} \otimes \{P([v_1^i] \circ b_{(j_1)}^{\circ k_1} \circ ... \circ b_{(j_m)}^{\circ k_m})\}$$

where the tensor products are over all $[v_1^i] \circ b^J$ such that $0 \le j_1 < j_2 < ... < j_m$, $i \in \mathbb{Z}$, and $k_1, ..., k_m < p$.

Proof The result follows somewhat easily from Proposition 7.0.1. We simply have to show that $E_*^R E(1)_*$ is a polynomial algebra generated by the $[v_1^i] \circ b^J$ satisfying the conditions in the theorem. Since the formal group laws for $E(1)$ and $BP\langle 1\rangle$ are identical, their Ravenel-Wilson relations are identical as well. The only difference between $E_*^R E(1)_*$ and $E_*^R BP\langle 1\rangle_*$ is that $E_*^R E(1)_*$ has an extra element $[v_1^{-1}]$ adjoined with the relation $[v_1^{-1}] \circ [v_1] = [1]$, meaning that we can always make a copy of $[v_1]$ present in an equation by $\circ$-producting with $[v_1^{-1}] \circ [v_1]$. So, just as the Ravenel-Wilson relations for $BP\langle 1\rangle$ eliminated copies of $[v_1] \circ b_{(i)}^{\circ p}$, the Ravenel-Wilson relations for $E(1)$ eliminate all copies of $b_{(i)}^{\circ p}$ whether or not a $[v_1]$ is present at all. So, all generators are eliminated other than the $[v_1^i] \circ b^J$ containing no powers of p, and there can be no relations on these generators since all of the Ravenel-Wilson relations only kill expressions involving powers of p. $\square$

In particular, if we start writing out $E_*E(1)_*$, we see that

$$E_*(E(1)_2) = P[[v_1^{\frac{\alpha_p(1+m(p-1))-1}{p-1}}] \circ b^{J_{1+m(p-1)}}]$$

$$E_*(E(1)_3) = E[[v_1^{\frac{\alpha_p(1+m(p-1))-1}{p-1}}] \circ e_1 \circ b^{J_{1+m(p-1)}}]$$

$$E_*(E(1)_4) = P[[v_1^{\frac{\alpha_p(2+m(p-1))-2}{p-1}}] \circ b^{J_{2+m(p-1)}}]$$

$$E_*(E(1)_5) = E[[v_1^{\frac{\alpha_p(2+m(p-1))-2}{p-1}}] \circ e_1 \circ b^{J_{2+m(p-1)}}]$$

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$$E_*(E(1)_{2p-2}) = P[[v_1^{\frac{\alpha_p(p-1+m(p-1))-(p-1)}{p-1}}] \circ b^{J_{p-1+m(p-1)}}]$$

$$E_*(E(1)_{2p-1}) = E[[v_1^{\frac{\alpha_p(p-1+m(p-1))-(p-1)}{p-1}}] \circ e_1 \circ b^{J_{p-1+m(p-1)}}],$$

where $\alpha_p(i)$ is the number of nonzero digits in the p-adic expansion of i, the p-adic expansion of i is $i_0 + pi_1 + p^2 i_2 + ...$, and $b^{J_i} = b_{(0)}^{\circ i_0} \circ b_{(1)}^{i_1} \circ ....$ Furthermore, $E_*E(1)_*$ is $|v_1| = 2(p-1)$-periodic, so we see that

Corollary 7.0.3 *For any $k \geq 0$ and prime p,*

$$E_*(E(1)_{2k}) = P[[v_1^{\frac{\alpha_p(k+m(p-1))-k}{p-1}}] \circ b^{J_{k+m(p-1)}}]$$

$$E_*(E(1)_{2k+1}) = E[[v_1^{\frac{\alpha_p(k+m(p-1))-k}{p-1}}] \circ e_1 \circ b^{J_{k+m(p-1)}}]$$

where m ranges over all possible integers such that $k + m(p-1) \geq 0$.

Note that at $p = 2$ this gives us

$$E_*(E(1)_{2k}) = P[[v_1^{\alpha_2(i)-k}] \circ b^{J_i}] = P[[v_1^{1-k}] \circ b_i].$$

Using the description of $E_*E(1)_*$ in Theorem 7.0.2 and an argument similar to the one used in the proof of 4.3.3, we can thus compute E_*KU_*. Unlike in the case of E_*ku_* this allows us to compute the entire Hopf ring structure, instead of just the homology of the first three spaces.

Theorem 7.0.4

$$E_*(KU_{2k}) = P[[v^{1-k}] \circ \iota_*(b_i)] = P[[v^{1-k}] \circ \bar{b}_i]$$

$$E_*(KU_{2k+1}) = E[[v^{1-k}] \circ e_1 \circ \iota_*(b_i)] = E[[v^{1-k}] \circ e_1 \circ \bar{b}_i]$$

This result was already proven by Morton and Strickland in [10]. Our use of $E(1)$ allows for a slightly different approach, but in the end the arguments are not fundamentally different, since Morton and Strickland's proof uses the same sort of bar spectral sequence argument that Ravenel and Wilson used in their proof of the relevant facts about $H_*BP\langle n\rangle_*$ which we made use of. A perhaps even more direct approach to computing E_*KU_* would simply use the fact that KU is Landweber exact to get that $E_*KU_* = E_*^R KU_*$ and then write out the Ravenel-Wilson relations for KU more explicitly to get a description of E_*KU_*.

CHAPTER 8

FUTURE DIRECTIONS

8.1 $E_*(ku_{2k})$ for $k > 3$

Since Ravenel's and Wilson's computation of E_*BP_* essentially comes down to a bar spectral sequence computation, a slightly more direct approach to computing E_*ku_* would be to compute the bar spectral sequence for E_*ku_* directly instead of going through BP. In [7], Hara makes progress towards this by computing all of $H_*(BP\langle 1\rangle_*;\mathbb{F}_p)$ and $H_*(ku_{(p)};\mathbb{F}_p)$ for odd primes p, using the same bar spectral sequence approach. In particular, in Theorem 4.23 of [7] he proves that $H_*((ku_{(p)})_*)$ is a certain tensor product of exterior, polynomial, and truncated polynomial algebras:

$$H_*((ku_{(p)})_*;\mathbb{F}_p) = \bigotimes\{E([v^k]\circ e\circ \bar{b}^J)\otimes P([v^k]\circ \bar{b}^J)\}$$
$$\bigotimes\{E([v^k]\circ e\circ \partial\alpha^I\circ \bar{b}^J)\otimes TP_1([v^k]\circ \partial\alpha^I\circ \bar{b}^J)\}.$$

If we could also give a similar description of $H_*(BP\langle 1\rangle_*;\mathbb{F}_2)$ and $H_*((ku_{(2)})_*;\mathbb{F}_2)$, then we could combine this with our understanding of $H_*(ku_*;\mathbb{Q})$ to give a complete understanding of $H_*(ku_*;\mathbb{Z})$. We could then use an Atiyah-Hirzebruch spectral sequence argument to compute the Hopf ring E_*ku_* for all complex-orientable E. Among other things, this could give us yet another different approach to proving the Ando-Hopkins-Strickland theorem, because it turns out that the generators containing copies of $\partial\alpha^I$ only appear in the $g(1)$th space and above, and at $p = 2$, these $\partial\alpha^I$ in $H_*((ku_{(2)})_{g(1)};\mathbb{F}_2) = H_*((ku_{(2)})_6;\mathbb{F}_2)$ can be written in terms of the $\bar{b}_{(i)}$.

8.2 $E_*BP\langle n\rangle_*$ for $n > 1$

Although some progress has been made on a few fronts, we still have a long way to go in understanding $E_*BP\langle n\rangle_k$ and the Hopf ring structure on $E_*(BP\langle n\rangle_*)$ for a general n and k. The work of Ravenel, Wilson, and Yagita allows us to nicely describe $E_*BP\langle n\rangle_k$ for $k < g(n)$, but for $k \geq g(n)$ we have to take things on a more case-by-case basis. In [16], Sinkinson computed $H^*(BP\langle n\rangle_k)$ for all k, which gives us exactly what $H_*(BP\langle n\rangle_k)$ should be additively for all k. But, the Hopf algebra structure of $H_*(BP\langle n\rangle_k)$ and the Hopf ring structure of $H_*(BP\langle n\rangle_*)$ do not follow from this cohomology calculation, so they are more difficult to compute.

One fruitful and approachable computation that could be done in the near future is to compute the Hopf ring $BP\langle 2\rangle_*BP\langle 2\rangle_*$. One of the best general methods for computing a Hopf ring E_*G_* is the "trapping" method described by Wilson in [18], which often uses the bar spectral sequence for a loop space. This method works by using unstable homotopy information, usually E_*G_k, along with stable information, usually E_*G, to "trap" E_*G_* between the two. As a first step toward computing $BP\langle 2\rangle_*BP\langle 2\rangle_*$, one could use this trapping method, as well as techniques like those Hara used in his computation of $H_*(BP\langle 1\rangle_*;\mathbb{F}_p)$ ([7]), to compute $H_*(BP\langle 2\rangle_*;\mathbb{F}_p)$. Then, using this unstable information, as well as Culver's recent computation of $BP\langle 2\rangle_*BP\langle 2\rangle$ ([5]), one should be able to compute the Hopf ring $BP\langle 2\rangle_*BP\langle 2\rangle_*$.

8.3 $E_*P(m,n)_*$

Additionally, there is also a class of Ω-spectra, $P(m,n)$ and $P(n)$, which are generalizations of the $BP\langle n\rangle$ and whose homology Hopf rings have yet to be explored in general (See [4, 3, 13] for more details). These Ω-spectra have homotopy groups given by

$$\pi_*(P(m,n)) = \mathbb{Z}_{(2)}[v_m, v_{m+1}, ...v_n]$$

$$\pi_*(P(n)) = \mathbb{Z}_{(2)}[v_n, v_{n+1}, ...]$$

and contain a couple of different special cases of note. $P(n,n)$ is the nth Morava K-theory $K(n)$, $P(0,n)$ is $BP\langle n\rangle$, and

$$P(0) = P(0,\infty) = \varinjlim_n P(0,n) = BP.$$

Furthermore, just as BP_k and $BP\langle n\rangle_k$ are p-torsion-free spaces for $k \leq g(n)$, $P(n)_k$ and $P(m,n)_k$ are v_n-torsion free-spaces for $k \leq g(m,n) = 2(p^m + p^{m+1} + ... + p^n)$, so there is potentially a lot to discover about them in a $K(n)$-local setting, especially since we have already been given a description of $K(q)_*BP\langle n\rangle_*$ in [8]. By studying these Ω-spectra and computing their homology Hopf rings, we can gain a better understanding of how $BP\langle n\rangle$, BP, and complex cobordism as a whole fit into a much bigger picture.

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