1 Introduction

Throughout graduate school, my research focused on an area of algebraic topology called unstable homotopy theory. *Stable* homotopy theory studies spaces after taking an infinite number of suspensions and studies generalized (co)homology theories through objects called *spectra*. In contrast, *unstable* homotopy theory focuses on the *spaces* in the Ω-spectrum for a generalized (co)homology theory and uses information about these spaces to get information about the (co)homology theory of interest. For my Ph.D dissertation, I studied the spaces in the truncated Brown-Peterson spectra, $BP\langle n\rangle$, which were first defined and studied by Steve Wilson in his Ph.D. thesis (see [Wil73] and [Wil75]), and I used the Hopf ring structure on $E_*BP\langle n\rangle_*$ to give a new proof of the classic Ando-Hopkins-Strickland Theorem computing the homology of the first three even connective covers of BU (see [AHS01]).

However, over the past three years at Santa Clara University, I have shifted a lot of my focus towards broader interests related to the intersection between mathematics and other fields like philosophy, economics, and education. In particular, I have been working on a variety of different problems within social choice theory, decision theory, philosophy of math, and math education, and I hope to continue my work on these problems in the coming years.

2 Homotopy Theory Background

An Ω-spectrum, E, consists of a sequence of topological spaces, $\{E_n\}_{n\in\mathbb{Z}}$, along with a weak homotopy equivalence $E_n \to \Omega E_{n+1}$ for each n. These Ω-spectra are especially useful because they allow us to describe any generalized homology or cohomology theory just in terms of homotopy classes of maps. More specifically, for any generalized cohomology theory $E^*(-)$, there is an Ω-spectrum E *representing* $E^*(-)$, i.e. such that for any space X, $E^n(X) = [X, E_n]$. Similarly, for any generalized homology theory $E_*(-)$, there is an Ω-spectrum E *representing* $E_*(-)$ such that for any space, X,

$$E_n(X) = \pi_n(E \wedge X) = \varinjlim_m \pi_{n+m}(E_m \wedge X).$$

Furthermore, we can extend this definition to take the generalized (co)homology of spectra instead of spaces, so that for any Ω-spectra E and G we have:

$$E^nG_k = [G_k, E_n]$$
$$E_nG_k = \varinjlim_m \pi_{n+m}(G_k \wedge X)$$
$$E^n(G) = \varprojlim_m E^{n+m}G_m$$
$$E_nG = \varinjlim_m E_{n+m}G_m$$
$$\pi_n(G) = \varinjlim_m \pi_{n+m}(G_m)$$

Of particular interest in my work is the Ω-spectrum, BP, which is called the Brown-Peterson spectrum and is often studied because of its intimate connection with complex cobordism (if MU is the Ω-spectrum representing complex cobordism, then $MU_{(p)} = \vee_i(\Sigma^i BP)$, for a given prime p). The spectrum BP has homotopy groups given by

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, ...]$$

where $|v_i| = 2(p^i - 1)$.

In [Wil75], Wilson used Sullivan's theory of manifolds to construct the *truncated Brown Peterson spectra*, $BP\langle n\rangle$, with homotopy groups given by:

$$\pi_*(BP\langle n\rangle) = \mathbb{Z}_{(p)}[v_1, v_2, ..., v_n].$$

In [Wil73] and [Wil75], Wilson computed $H_*(BP_k; \mathbb{Z}_{(p)})$ for all k and $H_*(BP\langle n\rangle_k; \mathbb{Z}_{(p)})$ for all $k < g(n) = 2(p^n + p^{n-1} + ... + 1)$.

The study of Ω-spectra also naturally gives rise to the notion of a *Hopf ring*, which is an algebraic structure existing on E_*G_* for a wide class of Ω-spectra E and G. Hopf rings are so named because of their intimate connection with both Hopf-spaces (H-spaces) and Hopf algebras. For any generalized homology theory $E_*(-)$ which satisfies the Kunneth isomorphism for all spaces (i.e. $E_*(X \times Y) \cong E_*(X) \otimes E_*(Y)$), $E_*(X)$ is not only a graded vector space, but also a graded coalgebra with coproduct

$$\psi = \Delta_* : E_*(X) \to E_*(X \times X) \cong E_*(X) \otimes E_*(X)$$

induced from the diagonal map $\Delta : X \to X \times X$. Additionally, if X is an H-space with product $\gamma : X \times X \to X$, we get an induced product

$$* = \gamma_* : E_i(X) \otimes E_j(X) \to E_{i+j}(X)$$

which makes $E_*(X)$ into a *graded Hopf algebra* (a group object in the category of graded coalgebras). Furthermore, if we take X to be not just an H-space, but rather the Ω-spectrum associated with a ring spectrum (meaning each of the spaces of X is an H-space), then the product $\mu : X_m \wedge X_n \to X_{m+n}$ gives us even more structure on $E_*(G_*)$. In addition to the Hopf algebra structure given by the maps

$$\psi : E_*(X_n) \to E_*(X_n) \otimes E_*(X_n)$$

and

$$* : E_i(X_n) \otimes E_j(X_n) \to E_{i+j}(X_n)$$

we also get a map

$$\circ = \mu_* : E_i(X_m) \otimes E_j(X_n) \cong E_*(X_m \wedge X_n) \to E_{i+j}(X_{m+n})$$

which makes $E_*(G_*)$ into a graded ring object in the category of graded coalgebras, i.e. a *graded Hopf ring*.

Since the pioneering work of Milgram, Ravenel, and Wilson ([Mil70], [RW77], [RW80]), Hopf rings have been used to compute the (generalized) homology of the spaces in a wide variety of different Ω-spectra. Among other reasons, these computations are particularly useful because E_*G_*, the unstable homology cooperations, are simply the dual of E^*G^*, the unstable cohomology operations.

When we specialize to complex-orientable, multiplicative homology theories, $E_*(-)$ and $G_*(-)$, with associated Ω-spectra E and G, and assume E has a Kunneth isomorphism, Ravenel and Wilson ([RW77]) define special homology elements $b_i^G \in E_*G_*$ and give relations on these elements as follows:

Definition 2.1. For $i \geq 0$, we define

$$b_i^G = (x^G)_*(\beta_i) \in E_{2i}(G_2)$$

where $x^G : \mathbb{CP}^\infty \to G_2$ is the complex orientation for G, $x^E : \mathbb{CP}^\infty \to E_2$ is the complex orientation for E, and β_i is the dual of $(x^E)^i \in E^{2i}(\mathbb{CP}^\infty)$.

Furthermore, we will use the notation

$$b_{(i)}^G = b_{p^i}^G$$

$$b^G(s) = b_0^G + b_1^G s + b_2^G s^2 + ... = \sum_i b_i^G s^i$$

$$b^G(x_1, ..., x_n) = \sum_{i_1,...i_n} (b_{i_1}^G \circ ... \circ b_{i_n}^G) x_1^{i_1} ... x_n^{i_n} = \sum_{i_1,...i_n} (b_{i_1,...,i_n}^G) x_1^{i_1} ... x_n^{i_n}$$

If we let $s +_E t$ be the formal group law for E and $x +_G y = \sum_{i,j} a_{i,j} x^i y^j$ the formal group law for G, then we get the Ravenel-Wilson relations below.

Theorem 2.2 (RW77, Theorem 3.8)**.** *In $E_* G_*[[s, t]]$,*

$$b^G(s +_E t) = b^G(s) +_{[G]} b^G(t)$$

where $b^G(s) +_{[G]} b^G(t) = *_{i,j}([a_{i,j}] \circ b^G(s)^{\circ i} \circ b^G(t)^{\circ j})$

Comparing the coefficients of each $s^i t^j$ in the equality above gives us relations in $E_* G_2$. These relations, in turn, generate relations in the entire Hopf ring $E_* G_*$.

Working from a more algebro-geometric perspective, Ando, Hopkins, and Strickland ([AHS01]) gave relations in $E_* ku_*$ (where ku is the Ω-spectrum for connective complex K-theory) that amount to the following:

Theorem 2.3. *The following relation holds in $E_*(ku_{2k})$:*

$$b^{ku}(x_1, x_2, x_3, ..., x_k) * b^{ku}(x_0, x_1 +_E x_2, x_3, ..., x_k) = b^{ku}(x_0 +_E x_1, x_2, x_3, ...x_k) * b^{ku}(x_0, x_1, x_3, ..., x_k) \tag{2.1}$$

Additionally, for $k \leq 3$, they show that these cocycle relations, along with basic symmetry relations, serve as the defining relations in $E_*(ku_{2k})$, i.e.

$$E_*(ku_{2k}) = E_*[b_{i_1,...,i_k}^{ku}]/\{\sim, S\} \tag{2.2}$$

where $\sim$ indicates the cocycle relations given by (2.1), and S indicates the symmetry relations $b_{i_1,i_2,...i_k}^{ku} = b_{\sigma(i_1),\sigma(i_2),...\sigma(i_k)}^{ku}$ for all permutations σ. For $E = H\mathbb{Q}$, they also show that (2.2) holds for all k.

3 Homotopy Theory Results

The majority of my work in homotopy theory has been related to the (co)homology of the $BP\langle n\rangle_k$ and the Hopf ring, $E_* BP\langle n\rangle_*$, for different n and E. Firstly, I have recovered the aforementioned results of Ando, Hopkins, and Strickland, using the following alternative perspective.

I have a new proof of the Ando-Hopkins-Strickland cocycle relations (2.1), which involves no algebraic geometry, but instead uses the Ravenel-Wilson relations in Theorem 2.2, the Hopf ring structure on $E_*G_*[[x_1, ..., x_k]]$, and general facts about Hopf rings. Combining this with Ravenel and Wilson's work related to $E_*(BP\langle n\rangle_k)$, as well as the fact that the p-localization of connective complex K-theory is $ku_{(p)} = \vee_i(\Sigma^i BP\langle 1\rangle)$, I also prove (2.2) for $k \leq 3$ and complex-orientable, multiplicative $E_*(-)$.

Theorem 3.1.

$$E_*(ku_2) = E_*[b_i^{ku}]$$

Theorem 3.2.

$$E_*(ku_4) = E_*[b_{i,j}^{ku}]/\{\sim\}$$

where $\sim$ indicates the cocycle relations given by (2.1) for $k = 2$.

Theorem 3.3.

$$E_*(ku_6) = E_*[b_{i,j,k}^{ku}]/\{\sim, S\}$$

where $\sim$ indicates the cocycle relations given by (2.1) for $k = 3$, and S indicates the symmetry relations $b_{i,j,k}^{ku} = b_{\sigma(i),\sigma(j),\sigma(k)}^{ku}$ for all permutations σ.

I have also given a nice presentation of the rational homology of ku_{2k} in terms of the generators $b_{i_1,\ldots,i_k}^{ku}$. In particular, I produce the following description:

Theorem 3.4.

$$H_*(ku_{2k}; \mathbb{Q}) \cong \mathbb{Q}[b_{i,1,\ldots,1}^{ku}]$$

This description recovers the result of Ando, Hopkins, and Strickland that (2.2) holds for all k when $E = H\mathbb{Q}$.

Secondly, in studying the homology of the $BP\langle n\rangle_k$, I have been able to give a description of $E_*(BP\langle n\rangle_{g(n)})$ for all n and for complex-orientable, multiplicative E. Although Ravenel and Wilson's Hopf ring description of E_*BP_* ([RW77]) descends to give a description of $E_*(BP\langle n\rangle_k)$ for $k < g(n)$, this description does not descend in the borderline case of $k = g(n)$. So, extending Su's ([Su06]) description of $E_*(BP\langle 1\rangle_6)$ at $p = 2$, I prove the following result for general n and p:

Theorem 3.5. *If $E_*(-)$ is a complex-orientable, multiplicative cohomology theory, then*

$$E_*(BP\langle n\rangle_{g(n)}) = E_*[\lambda_{I,k}, \gamma_{m,k}]/(\lambda_{I,k}^{*p}, \gamma_{m,k}^{*p})$$

where $\lambda_{I,k} = b_{(k)} \circ b_{(k+i_1)}^{\circ p} \circ \ldots \circ b_{(k+i_1+\ldots+i_n)}^{\circ p^n}$, $I = (i_1, ..., i_n)$ is a sequence of nonnegative integers, $\gamma_{m,k} = b_{(k)} \circ \hat{x}_{m,k}^{v_n}$, $\{\hat{x}_m^{v_n}\}$ is the set of all allowable monomials on $[v_1], [v_2], ..., b_{(0)}, b_{(1)}, \ldots$ in $E_*BP\langle n\rangle_{g(n)-2}$ that are not of the form $b_{(k+i_1)}^{\circ p} \circ \ldots \circ b_{(k+i_1+\ldots+i_n)}^{\circ p^n}$, and $\hat{x}_{m,k}^{v_n} = V^{-k}\hat{x}_m^{v_n}$, where V is the Verschiebung map which replaces each $b_{(i)}$ with $b_{(i-1)}$.

4 Non-Homotopy Theory Work

Over the past three years, I have been working on a number of different problems outside of homotopy theory as well. A lot of these problems are fairly accessible, so I have been talking about many of these topics with the undergraduates I teach. Within philosophy of math, I have thinking about the problem of divine aseity and the relationship between abstract, mathematical objects and God. This culminated in a guest lecture I gave on divine aseity in an upper division philosophy course here SCU. Additionally, I have working on addressing certain counterarguments to the Lucas-Penrose argument, which uses Godel's Incompleteness Theorem to argue that human mathematics is fundamentally non-algorithmic (see [Pen94]). I have also been doing some work within social choice theory and decision theory in which I apply Arrow's Impossibility Theorem and the Two Envelopes Problem to the fortified Pascal's Wager argument proposed by Liz Jackson and Andy Rogers in [JR19].

However, my main recent interests have really been within math education and pedagogical theory. Building on my own experience in the classroom, as well as the work of Francis Su, Simone Weil, and Bernard Lonergan, I have developed a new attentive/slothful learning dichotomy which is perpindicular to the traditional active/passive learning dichotomy and better enables us to judge the quality of classroom learning. Truly good learning requires students to be active at certain times and passive at others. It requires them to actively attend to the data and information in front of them while also passively waiting to receive the gift of insight which will bring them up to a higher viewpoint. This new dichotomy, therefore, makes attentiveness the main attitude we should strive to cultivate in our students, rather than mere activity.

Building on the work of Marshall McLuhan and his classic hot/cool media dichotomy, I have also developed a new translucent/opaque media dichotomy by which to judge classroom technologies and the role they play in student learning. Just as hot media tend to cultivate passive learning and cool media tend to cultivate active learning, translucent media tend to cultivate attentive learning and opaque media tend to cultivate slothful learning. Therefore, since attentiveness should be the ultimate attitude we strive to cultivate in our students, translucence should be the ultimate quality we seek out in the technologies and classroom formats we use in our teaching.

5 Future Research Directions

When it comes to my work within homotopy theory, there still remains a lot to be done. We have a long way to go in understanding $E_*BP\langle n\rangle_k$ and the Hopf ring structure on $E_*(BP\langle n\rangle)$ for a general E, n, and k. One potential first step in this direction would be a computation of the Hopf ring $BP\langle 2\rangle_*BP\langle 2\rangle_*$.

Additionally, there is also a class of spectra, $P(m,n)$, which are generalizations of the $BP\langle n\rangle$ and whose homology Hopf rings have yet to be explored in general. These spectra have homotopy groups given by $\pi_*(P(m,n)) = \mathbb{Z}_{(2)}[v_m, v_{m+1}, ...v_n]$ and contain a couple different special cases of note. $P(n,n)$ is the nth Morava K-theory, $P(0,n)$ is $BP\langle n\rangle$, and

$$P(0) = P(0,\infty) = \varinjlim_{n} P(0,n) = BP.$$

Furthermore, just as the BP_k are p-torsion free spaces, the $P(n)_k$ are v_n-torsion free spaces, so there is potentially a lot to discover about them in a $K(n)$-local setting. By studying these spectra and computing their homology Hopf rings, we can gain a better understanding of how $BP\langle n\rangle$, BP, and complex cobordism as a whole, fit into a much bigger picture.

Beyond homotopy theory, I also hope to further develop the translucent/opaque media dichotomy I have been proposing and dig further into the phenomenology of attention in order to better articulate my notion of attentive learning and its relation to active learning and passive learning. I am working on turning my "Attentive Learning" Talk into a paper that I hope to publish this year. I also hope to answer and refute some of the main critiques of the Lucas-Penrose argument put forward by scholars like John Searle. Although my homotopy theory work can be fairly dense and inaccessible to non-specialists, these broader interests of mine are far more accessible. I would be excited to continue pursuing these broader interests in a liberal arts environment and potentially involve undergraduate students in this research as well.

6 References

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