Collision of two vortex rings - homework

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Introduction and problem statement

Consider a pair of initially circular vortex sheets in an infinite, incompressible, and inviscid 2D domain. Their geometry is parameterized by $\xi \in [0, 2\pi)$ as:

$$\mathbf{x}_1(\xi, 0) = \begin{bmatrix} \sin(\xi) \\ \cos(\xi) \end{bmatrix}, \qquad \mathbf{x}_2(\xi, 0) = \begin{bmatrix} \sin(\xi) \\ \cos(\xi) + 8 \end{bmatrix}.$$

The initial strengths of the sheets, understood as the tangential velocity jump across the interface, are:

$$\Delta u_{\tau 1}(\xi) = -\sin(\xi), \qquad \Delta u_{\tau 2}(\xi) = \frac{3}{2}\sin(\xi),$$

and the flow is initially irrotational outside the sheet.

Governing Equations

The flow is governed by the streamfunction-vorticity ($\psi - \omega$) formulation. The velocity field $\mathbf{u}$ and vorticity $\boldsymbol{\omega}$ are related via:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}, \qquad \mathbf{u} = \nabla \times \boldsymbol{\psi}.$$

This implies the vector Poisson equation for the streamfunction:

$$\nabla^2 \boldsymbol{\psi} = -\boldsymbol{\omega}.$$

The Green's function for the Poisson equation in an infinite 2D domain is $G = \frac{1}{2\pi} \ln |\mathbf{x}_0 - \mathbf{x}|$, so:

$$\psi(\boldsymbol{x}_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega(\boldsymbol{x}) \ln |\boldsymbol{x}_0 - \boldsymbol{x}| \, dxdy. \tag{1}$$

Therefore, for the u component:

$$u(\boldsymbol{x}_0) = -\frac{\partial \psi}{\partial y_0} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(y_0 - y)\omega(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|^2} \, dxdy.$$

Note that ω is distributed as a Dirac delta across the sheet, which allows the area integral to be reduced to a line integral along the sheet's curve. By applying Stokes theorem to a closed loop tightly enclosing a segment of the vortex sheet and taking the limit as the loop width in the normal direction approaches zero, we get:

$$\omega\, dxdy = d\Gamma = \Delta u_\tau\, dl.$$

Note that for any material segment, by Kelvin's circulation theorem, the product $\Delta u_\tau\, dl$ is conserved, but its length can change due to stretching. It is more convenient to use γ, defined as:

$$\Delta u_\tau\, dl = \gamma\, d\xi,$$

where ξ is a parameter in a fixed range as before. Because ξ is by design conserved in the material segment, so is γ. If we choose ξ to be equal to arc length in the initial condition, then $\gamma = \Delta u_\tau\big|_{t=0}$.

Finally, we can evaluate velocity at any point in the domain $\boldsymbol{x}_0$ by referring to both vortex sheets ($\boldsymbol{x}_1$, $\boldsymbol{x}_2$):

$$u(\boldsymbol{x}_0) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{(y_0 - y_1)\gamma_1}{|\boldsymbol{x}_0 - \boldsymbol{x}_1|^2 + \delta^2}\, d\xi - \frac{1}{2\pi}\int_0^{2\pi} \frac{(y_0 - y_2)\gamma_2}{|\boldsymbol{x}_0 - \boldsymbol{x}_2|^2 + \delta^2}\, d\xi, \tag{2}$$

$$v(\boldsymbol{x}_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(x_0 - x_1)\gamma_1}{|\boldsymbol{x}_0 - \boldsymbol{x}_1|^2 + \delta^2}\, d\xi + \frac{1}{2\pi}\int_0^{2\pi} \frac{(x_0 - x_2)\gamma_2}{|\boldsymbol{x}_0 - \boldsymbol{x}_2|^2 + \delta^2}\, d\xi,$$

where δ is a regularization parameter, limiting the range of scales present in the flow and stabilizing the numerics. To keep the simulation relatively cheap, we will take $\delta^2 = 0.1$.

Being able to evaluate $\boldsymbol{u}$ everywhere, we limit ourselves to what is necessary, i.e., $\boldsymbol{u}(\boldsymbol{x}_1)$ and $\boldsymbol{u}(\boldsymbol{x}_2)$. Having the velocity at the surfaces of both vortex sheets, we advect them in Lagrangian fashion:

$$\frac{d\boldsymbol{x}_1(\xi,t)}{dt} = \boldsymbol{u}_1(\xi,t), \qquad \frac{d\boldsymbol{x}_2(\xi,t)}{dt} = \boldsymbol{u}_2(\xi,t),$$

what basically turns the problem into an N-body problem. The caveat is the stretching of the vortex sheet. To properly resolve its evolution, we need to introduce new nodes, e.g., by linear interpolation. An exemplary criterion is presented below:

`if` $(|\boldsymbol{x}(\xi_i) - \boldsymbol{x}(\xi_{i-1})| > \Delta l_{\max})$:
  `//introduce a new node`
  $\xi_{\text{new}} = \frac{\xi_i + \xi_{i-1}}{2}$

  $\boldsymbol{x}(\xi_{\text{new}}) = \frac{1}{2}\left(\boldsymbol{x}(\xi_i) + \boldsymbol{x}(\xi_{i-1})\right)$
  $\gamma(\xi_{\text{new}}) = \frac{1}{2}\left(\gamma(\xi_i) + \gamma(\xi_{i-1})\right).$

Because δ limits the smallest scales present in the flow, it is reasonable to make $\Delta l_{\max} \sim \delta$. In this particular setup, about 2 or 3 $\Delta l_{\max}$ per δ should be enough.

Tasks:

- Implement the method described above in a programming language of choice. Your implementation should contain the following functions (or methods):
  - `biotSavart` - computes the velocity in one point
  - `computeVelocities` - computes the velocities in all the nodes
  - `refine` - introduces new nodes if needed
  - `timeStep` - updates the positions of the nodes
- Simulate the evolution of the system until at least $t = 20$, preferably $t = 80$.
- Monitor the following quantities, which both should be conserved:

$$\Gamma_{\text{abs}} = \int_0^{2\pi} |\gamma_1| d\xi + \int_0^{2\pi} |\gamma_2| d\xi, \qquad \boldsymbol{P} = \int_0^{2\pi} \gamma_1 \boldsymbol{x}_1 d\xi + \int_0^{2\pi} \gamma_2 \boldsymbol{x}_2 d\xi.$$

Please send the code with selected snapshots (as .png files) to `p.jedrejko@uw.edu.pl`.

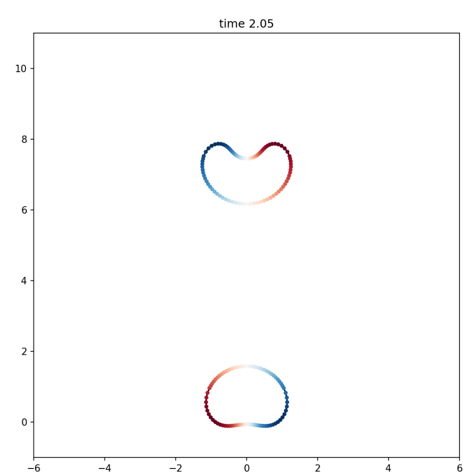

Figure 1: Exemplary snapshot