Jacobi method - notes

L - discrete Laplacian matrix
D - diagonal of L of size $N \times N$
u - exact solution

Poisson problem:

$$Lu = b$$

reformulate it as (splitting L):

$$u = D^{-1}\Big(b + (D - L)u\Big) \tag{1}$$

and consider this as a fixed point of some iterative procedure:

$$u^{n+1} = D^{-1}\Big(b + (D - L)u^n\Big). \tag{2}$$

Why? Guessing u is impossible, but guessing some u^n which will lead us to u could be easy. – problem is linear so if u attracts, then globally and any guess will work

More convenient to have an ,,update matrix" on the RHS, so subtract (1) from (2):

$$\underbrace{u^{n+1} - u}_{\epsilon^{n+1}} = \underbrace{D^{-1}(D - L)}_{G}\underbrace{(u^n - u)}_{\epsilon^n},$$

so, like we did with time-stepping schemes:

$$\epsilon^n = G^n \epsilon^0.$$

To get decreasing error ϵ^n we need all[1] eigenvalues of G $|\lambda_i| < 0$.

Eigenproblem:

$$Gv = \lambda v$$

G is not so difficult to compute, just remove diag from L and divide each row by proper diag element.
Put $-I\lambda$ on the diag and:

$$\begin{bmatrix} -1/2 & -\lambda & -1/2 & ... \\ ... & -1/2 & -\lambda & -1/2 \end{bmatrix}$$

it looks kind of Laplacianish $\rightarrow$ we expect similar v as analytical Laplacian. Note:

$$\frac{\pi\kappa x}{L} = \frac{\pi\kappa i\Delta x}{(N-1)\Delta x} = \underbrace{\frac{\pi\kappa i}{N-1}}_{\alpha \text{ (for short)}}, \qquad (i \text{ is just an index here})$$

[1] if L has some $\lambda_i = 0$, say with all Neumann BCs, then corresponding v_i are fine to appear with $\lambda_i = 0$ in G as well. The solution is up to a const anyway and we re just keeping the one from the initial guess.

Now, guessing:

$$v = A\sin(\alpha i) + B\cos(\alpha i)$$

away from boundaries:

$$\sin((i-1)\alpha) + 2\lambda\sin(i\alpha) + \sin((i+1)\alpha) = 0$$

Use some high-school trig identity:

$$\sin(i\alpha)\cos(2\alpha) - \lambda\sin(i\alpha) = 0$$

$$\lambda = \cos(\alpha) = \cos(\frac{\pi\kappa i}{N-1})$$

same with cos; you can match BCs with A and B, but we don't care much now.

Message:

- long waves ($\kappa \to 0$) of **error** can decay very slowly and dominate the overall convergence
- same with very short, but usually they are initially small. (We'll solve that issue in the future)
- how many iterations (n) we expect to reduce initial error from ϵ^0 to ϵ^n? For dominant, long modes ($\kappa \to 0$):

  $$\lambda^n \epsilon^0 = \epsilon^n$$

  $$n\ln\lambda = \ln\frac{\epsilon^n}{\epsilon^0}, \qquad \lambda = \cos(\frac{\pi\kappa i}{N-1})$$

  using Taylor expansion for ln and cos:

  $$n = \mathcal{O}\Big(N^2 \ln\frac{\epsilon^0}{\epsilon^n}\Big)$$

  (keep in mind that this is specific to Poisson problem)