Discrete Bulk Reconstruction Problem

Compiled Research

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Abstract

The **Discrete Bulk Reconstruction Problem (DBRP)** models the entanglement structure of quantum systems using discrete graphs. It is rooted in the Anti-de Sitter / Conformal Field Theory (AdS/CFT) correspondence, which connects a gravitational theory in AdS space (the bulk) to its quantum field theory on its boundary (the CFT). While this describes a universe different from our own, it offers valuable insights into quantum gravity and spacetime emergence.

Instead of using traditional approaches that rely on continous AdS space, DBRP models the bulk as a graph, where nodes correspond to regions of space and edge weights encode entanglement connectivity. In this framework, geodesics in AdS are approximated by min-cuts in the graph, and the Ryu-Takayanagi (RT) formula determines minimal surfaces in the bulk, allowing for bulk geometry reconstruction from boundary entanglement.

DBRP leverages tensory networks and holography to provide a discrete approach to understanding spacetime, quantum entanglement, and information flow. DBRP could provide profound insight in quantum gravity, improved quantum error correction, and practical quantum computing applications.

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1 Introduction

DBRP aims to reconstruct spacetime (the bulk) from the entanglement structure of quantum systems. This structure is defined by the collection of entanglement entropies of its subsystems, which quantify how much information a given region of the system contains. More precisely, entanglement entropy measures how strongly a subsystem is correlated with the rest of the system.

A key motivation for DBRP comes from the AdS/CFT correspondence, first proposed by Juan Maldacena in 1997. This duality suggests that a (d + 1)-dimensional anti-de Sitter (AdS) space, which includes quantum gravity, is fully encoded by a d-dimensional conformal field theory (CFT) on its boundary. Because the CFT is non-gravitational, computations are easier, making this framework a powerful tool for understanding quantum gravity.

By leveraging insights from AdS/CFT, DBRP offers a discrete, graph-based approach to bulk reconstruction, which allows for a new perspective on how spacetime emerges from entanglement, with potential applications in both holography and quantum information theory.

2 Explanation of "Universe"

AdS (Anti-de Sitter) Space ⇒ The Bulk

- The **bulk** represents a higher-dimensional space in the AdS/CFT framework.
- A **surface** is a geometric object within AdS space that plays a crucial role in holography.
- The **minimal area** surface, also known as the RT surface in AdS/CFT, spans the boundary subregion and has the smallest possible area.
- Since direct calculations in the bulk are challenging due to quantum gravity, computations are instead performed on the CFT boundary.

CFT (Conformal Field Theory) Space ⇒ The Boundary

- The **Boundary** represents a lower-dimensional space the forms "edge" or "boundary" of AdS bulk.
- **Non-gravitational** theory that lives on the boundary of the AdS spacetime.
- The information of boundary is encoded in the bulk.

When dealing with a 5-dimensional AdS bulk, the boundary is typically 4-dimensional, and the CFT lives on this 4-dimensional boundary. This is the idea that the boundary is 1 less dimension that the bulk. The boundary is where we will make our computations. Because of the lack of gravity it is easier to compute and understand. We then can use the CFT computations to reconstruct the gravitational dynamics of the AdS bulk which translates to the geodesics of the hyperbolic disk.

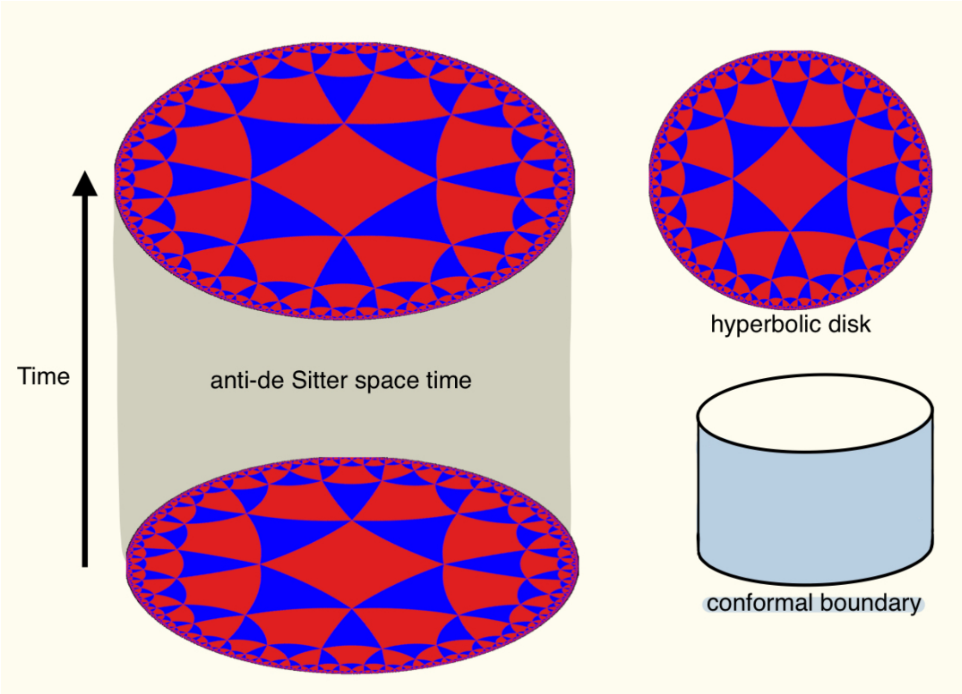


Figure 1: This illustration shows the AdS/ CFT spacetime. Where a geodesic is a straight line across the hyperbolic disk, which is the shortest path between points. In the context of anti-de Sitter spacetime a geodesic represents the path that a particle follows when only affected by gravity.

Quick Notes about AdS and CFT

- DBRP idea in physic terms:
  - Because the CFT is a non-gravitational theory, it is easier to make computations and to understand. And because it lives on the boundary of the AdS space, we can use the CFT to reconstruct the gravitational dynamics of the AdS bulk.
  - Given a CFT state, we can determine the corresponding spacetime that the CFT describes, including its geometry and gravitational dynamics.
  - Then we can use entanglement entropy in the CFT to learn about spacetime geometry in the bulk.

AdS/CFT

- **AdS/CFT correspondence** $\Rightarrow$ The gravitational theory in the bulk is equivalent to the non-gravitational CFT on the boundary
- i.e. Two theories describe the same physics, but from different perspectives
- **Holographic Relationship**: Everything happening in the bulk AdS spacetime can be described by the boundary CFT, and vice versa
- Consists of two theories (theory of quantum gravity and a quantum field theory, with no gravity). The relation between the two theories is called "holographic".

- Theories are "equivalent"? (yet to be proven)
- Meaning, there exists a mapping or "dictionary" that maps all states and observables from one theory to corresponding states and observables in the other
- Currently lack a full dictionary for the AdS side

Quick note about spacetime geometry (physics)

- Gravity is a curvature of spacetime (caused by mass and energy)
- "Geometry" of spacetime refers to this curve:
  - If no gravity, geometry is simple and uncurved

Spacetime Geometry for AdS/CFT

- In AdS space, geometry is negatively curved.
- **CFT can reconstruct the geometry of AdS bulk**: Meaning, data in the CFT can be used to determine how spacetime is shaped or curved in the AdS bulk.
- **Entanglement is directly related to geometry of the bulk spacetime**:
  - Amount of entanglement in different regions of CFT tells us about structure and curvature of the corresponding bulk spacetime in AdS.
  - More entanglement on the boundary implies more curvature or structure in the bulk geometry.
  - Proven with RT formula: more entanglement = larger minimal surface area = more curved and intricate bulk geometry.

Why important?

- Shape of spacetime tells us how gravity behaves.
- If we can reconstruct the bulk geometry from the CFT, we can gain insights into the quantum aspects of gravity and understand how measurements (or experiments) on the boundary affect the information encoded in the bulk.

Entanglement Measurements

- Not needed for DBRP because problem states initially given full AdS/CFT correspondence.

Von Neumann entropy

- Refers to a measure of quantum entanglement, i.e., information contained in a specific region (usually in lower-dimensional CFT).

Ryu-Takayanagi (RT) Formula and Holographic Dictionary

- Key entry in holographic dictionary is *Ryu-Takayanagi (RT) formula.*
- Equates von Neumann entropy of subregion of boundary state with the minimal area among all bulk surfaces that end on that boundary region.
- Meaning: Entropy (amount of quantum information) in a certain part of the boundary is tied to the area of the smallest surface that stretches into the bulk but connects to that boundary (geometrical way of understanding quantum entanglement).

Minimal area of bulk surface

- In higher-dimensional bulk space (AdS) exists a surface whose area corresponds to entanglement in boundary region.
- Theory says: Smallest possible area of this surface (RT surface) is proportional to entropy of boundary subregion.
- Meaning: If you can find RT, you know the entropy (or information) of the subregion inside the boundary.
- The more entangled the boundary region is, the larger the area of corresponding surface in the AdS bulk.

RT surface is proportional to von Neumann entropy which quantifies quantum entanglement or information of subregion.

Discrete Bulk Reconstruction Problem

Official statement initialization:

$$\mathcal{H} = \otimes_{i=1}^{N} \mathcal{H}_i$$

Note about tensor product, Hilbert space, and qubits:

- Tensor Product is a mathematical operation used to describe a system with multiple qubits. It combines two or more Hilbert spaces into a single composite Hilbert space, or one large vector space. The larger space captures the full state space of the combined quantum system.
- For a single qubit: $|0\rangle$, $|1\rangle$ form a basis for a two-dimensional Hilbert space.
- Therefore, an n-qubit system has a Hilbert space of dimension 2^n.

Min-Cut Theory

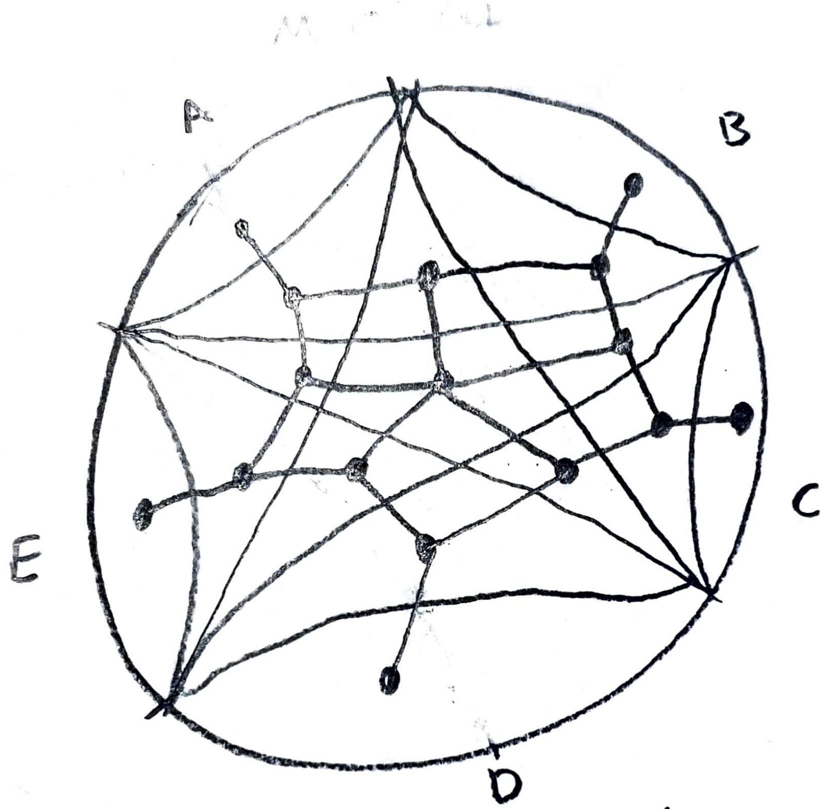
- Feature of AdS/CFT.
- Given a finite weighted undirected graph G with real edge weights $w(e) \geq 0$, as well as two disjoint vertices R, R', a min-cut is a set C of edges with minimum total weight

$$W = \sum_{e \in C} w(e)$$

whose removal disconnects R from R'.

Problem Set-Up

Can model the bulk by a weighted undirected graph where each subregion is an RT (or minimal area) surface, which corresponds to min-cuts in G.



min-cut graph of 5 atomic regions

Official Statement

Given as input a list of atomic boundary regions labeled $1, \ldots, N$, a list of subsets of the regions $R_1, \ldots, R_k \subseteq [N]$, and a real-valued entropy $S(R_i) \geq 0$ for each R_i, the weight of the minimum cut separating R_i from the rest of the boundary vertices (i.e., from $[N] - R_i$) is equal to $S(R_i)$.

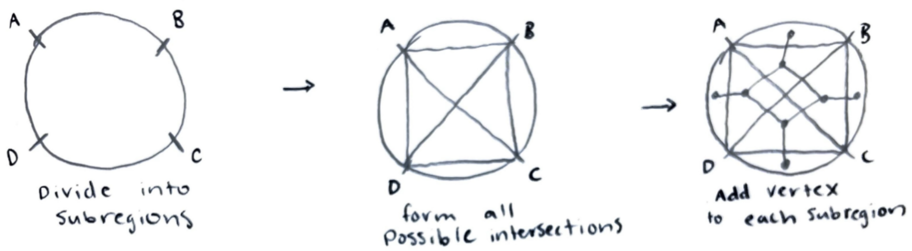
General Idea

If we are given the complete entanglement structure of the boundary, we are therefore provided with a min-cut graph of the boundary entropies. We can then perform an experiment in the bulk space and use min-cut theory on the boundary to reconstruct the entropy graph. Giving us an interpretation of what the outcome would be in the bulk, because the boundary information is encoded in the bulk.

This is basically the inverse min-cut problem. Now we are given min-cut, find graph that gave you min-cut solution, therefore this graph is not unique. While the graph is not unique you can add on additional properties. For example, planar, few vertices. For this problem assume no wormholes. For DBRP to work we need to make ensure all quantum states satisfies properties (more on that later).
It is important to note and prove DBRP is computable, meaning it has an upper bound. It has an upper bound of 2^{2^N}

Proof:

Given $K \leq 2^N$ boundary regions form all $\leq 2^N$ possible intersections of RT regions. At most 1 vertex needed in each region. Using linear programs for each edge weight takes "only" exp(exp(exp(N))) time!



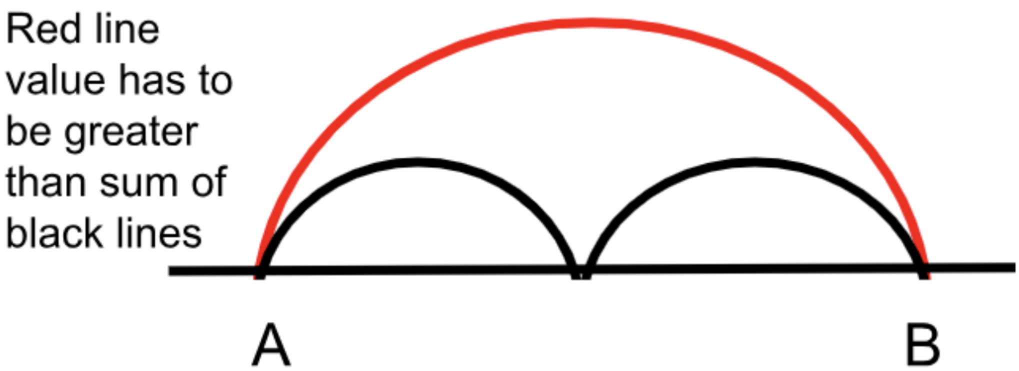
How to determine edge weights?

Use linear programming, for every possibility of what min-cuts could be try to solve some linear equations and equalities. Can we set edge weights so that it will be min-cut? Therefore, gives us algo for deciding any instance of DBRP.

Required Properties

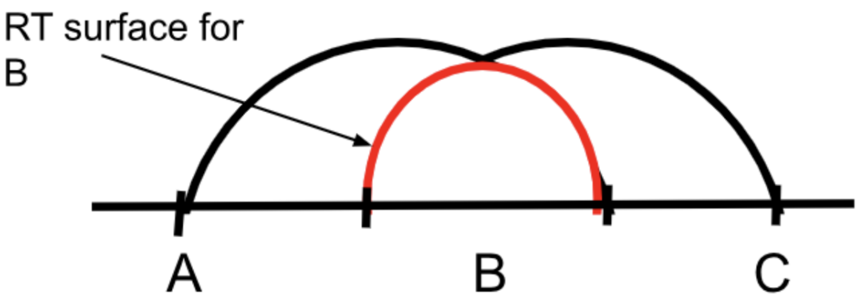
To ensure a solution exists we **need** the following properties:

1. **Subadditivity (SA)**: entropy of A plus entropy of B is $\geq$ entropy of $A \cup B$
$$S(A) + S(B) \geq S(AB)$$



2. **Strong Subadditivity (SSA)**

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$



Holographic Quantum State

- Boundary state in CFT that, due to holography, has a corresponding interpretation in bulk AdS space.
- This allows us to perform computations on a lower-dimensional boundary with no gravity, which then gives us information about higher-dimensional gravity in bulk space.
- Note: Entanglement entropy of a region in boundary can correspond to the area of the minimal surface in the bulk (RT-formula).

All holographic quantum states **must** satisfy the following property:

1. **Monogamy of Mutual Information (MMI)**

$$S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC)$$

Question: Are these properties what give us geometry?

DBRP on 1-dimensional Hilbert Space

- Note: To perform DBRP on a higher-dimensional Hilbert space, take the tensor product of each Hilbert space to create one large Hilbert space.

$$\mathcal{H}_{\text{total}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$$

1. Upper bound for contiguous boundary regions:

$$\frac{N(N-1)}{2}$$

2. Can specify *all* 2^N non-contiguous entropies in terms of $\frac{N(N-1)}{2}$ contiguous entropies.
3. Formula is minimum of 2 different possibilities:

$$S(AC) = \min\{S(A) + S(C), S(B) + S(D)\}$$

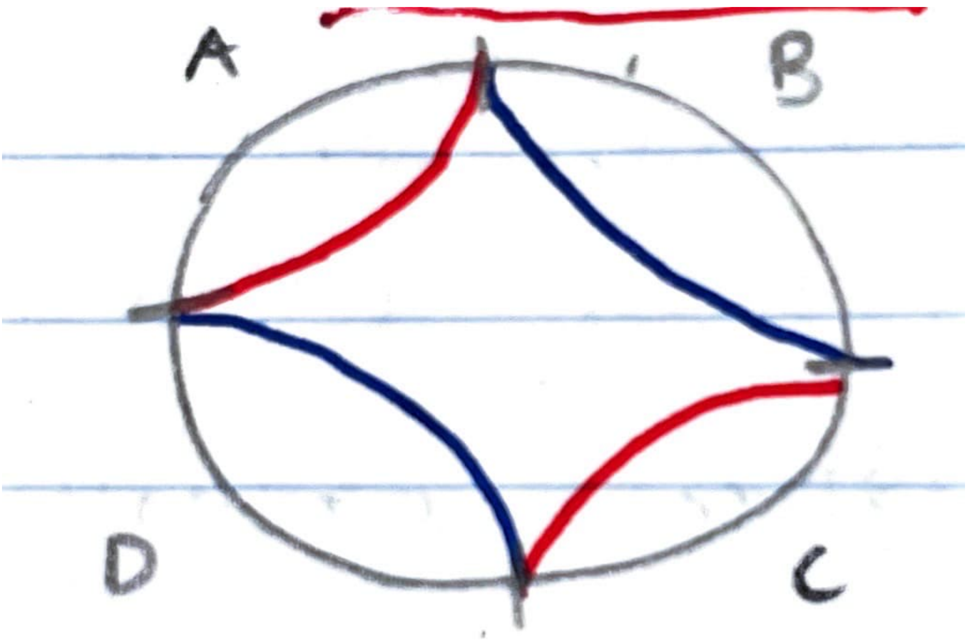


Figure 2: Where the first option is in red and second option is in blue

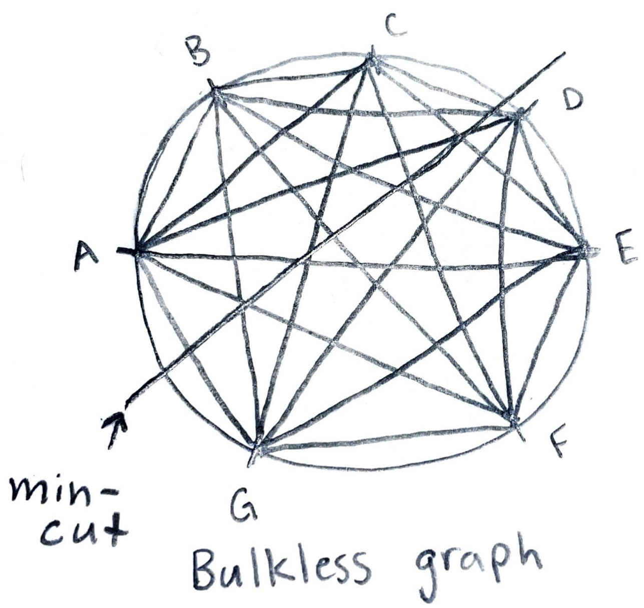
Theorem: 1D Boundary

In special cases of a 1D boundary divided into N parts, given contiguous entropies satisfying SSA, we can always find a graph model of the bulk in linear time $O(N^2)$. The graph model is planar, universal, and has **only** $O(N^2)$ vertices.

Note: Only need to satisfy SSA for 1D.

Proof of 1D, with Examples

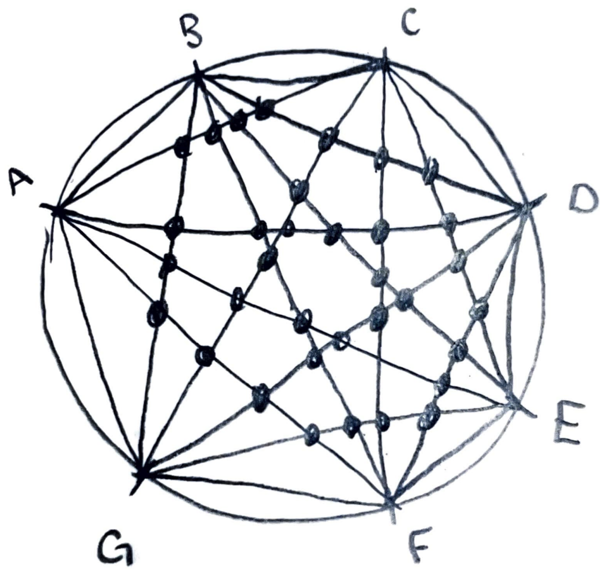
- Any contiguous boundary data that satisfies SSA admits a "bulkless" graph.
- A bulkless graph is a graph of pure edges and no vertices.
- After creating a bulkless graph, we can solve for what the edge weights should be! By using a system of linear equations, SSA ensures nonnegativity.



This will lead to a planar graph model of the bulk. The next step will be to get the model...

Chord Construction

- Get model of graph.
- Take all edges in between atomic boundary regions and add a chord (vertex) where each edge intersects.
- Edge weights = weights from bulkless graph.



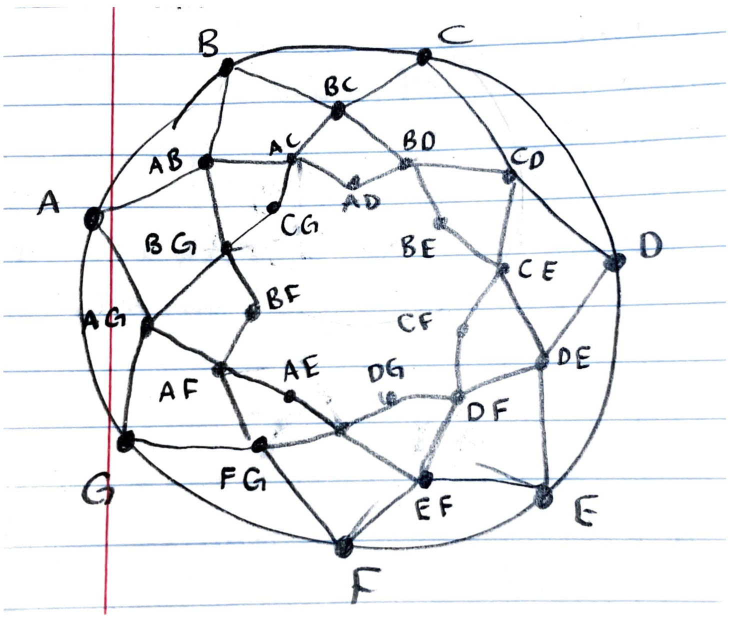
Chord construction for 7 atomic regions

This yields a planar graph with $O(N^4)$ vertices (not optimal) and the same cut structure as the bulkless graph.

Note: Since the bulkless graph has correct cut values, the planar graph will also have correct cut values. However, $O(N^4)$ is not optimal, so a better construction was developed...

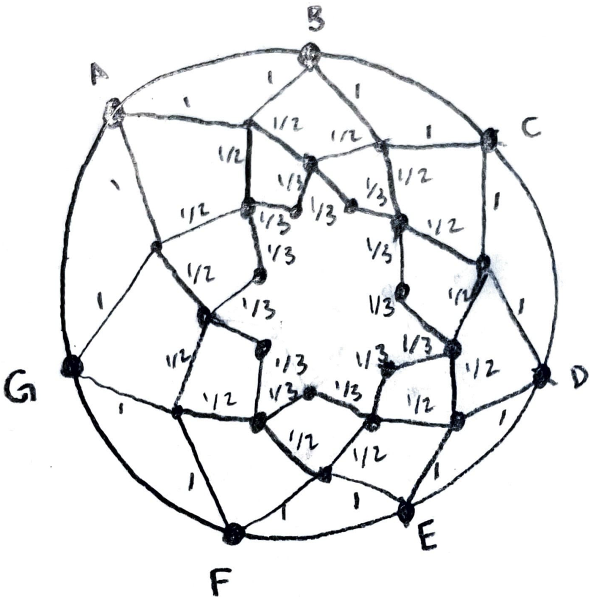
Diamondwork Construction

- Yields $O(N^2)$ vertices and edges.
- Reduce via a planar graph made up of ”overlapping” triangles, each contributing weight from the bulkless graph.
- Overlapping triangles are minimal surfaces that cut deeper into the bulk.
- Add edge weights together to overlap them.
- Min-cuts cut the same as the bulkless graph.



Harmonic Edge Weights

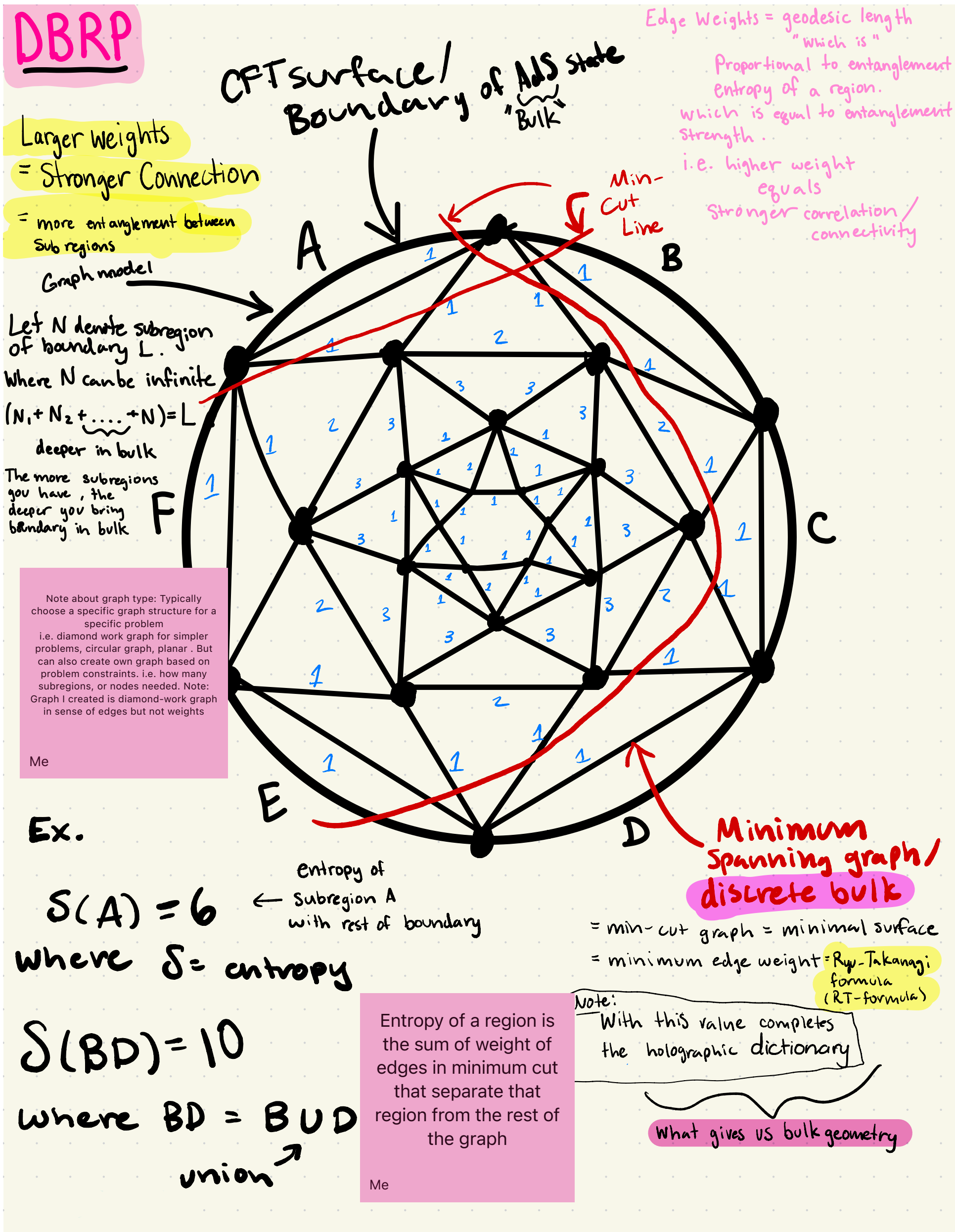
This is the discrete version of AdS geometry. The entropy of a length L boundary region (where $L \leq N/2$) is given as:



$$2\left(1 + 1/2 + 1/3 + \underbrace{\ldots}_{\text{deeper into Bulk}} + 1/L\right) \approx \underline{2 \ln L}$$

Why No Black Holes?

- All edge weights have the same value, therefore the RT minimal surface means nothing.
- There is no advantage to cutting deeper into the bulk.



3 Conclusion

Summarize key findings, limitations, and possible future research directions

References

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