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Piper Jeffries: CS5001 - Homework 2

Instructor: Avah Banerjee

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Problem 1 (Single-qubit rotations)

Consider the single-qubit gates given by the following unitaries:

$$V_1 = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}, \quad V_2 = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

- A. Determine the axis of rotation on the Bloch sphere for each unitary.
- B. Calculate the rotation angle (in radians) for each gate.
- C. Briefly explain the reasoning behind your deductions based on the provided matrices.

(A)

For V_1 :

Break V_1 down to pauli matrices.

$$V_1 = \frac{1+i}{2}I + \frac{1-i}{2}X$$

Only X appears therefore, rotation is around X-axis.

For V_2 :

Break V_2 down to pauli matrices.

V_2 is already in standard form for single qubit rotation. Therefore, we can determine axis and angle of rotation from original form. V_2 only has entries in the diagonal, which corresponds to rotation around the Z-axis

(B)

For V_1 :

Because it is rotating around the x-axis it can be expressed as:

$$R_X(\theta) = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)X$$

Therefore, angle of rotation is $\frac{\theta}{2} = 90^\circ$

For V_2 :

V_2 represents a rotation of $\frac{\pi}{2} = 90^\circ$ around the Z-axis.

(C)

For V_1 :

V_1 is a rotation of 90° around the X-axis that is because only the X matrix shows up when V_1 is broken down into pauli matrix form. This works because:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$V_1 = \frac{1+i}{2}I + \frac{1-i}{2}X \Rightarrow \frac{1}{2} \begin{pmatrix} 1+i & 0 \\ 0 & 1+i \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1-i \\ 1-i & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

For V_2 :

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotation around Z-axis is written as:

$$\begin{aligned} R_z(\theta) &= \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)Z = \begin{pmatrix} \cos(\theta/2) - i\sin(\theta/2) & 0 \\ 0 & \cos(\theta/2) + i\sin(\theta/2) \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \\ V_2 &= \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \end{aligned}$$

Comparing with standard form, note there is a sign change, therefore rotation around Z-axis.

To reconcile phase change to standard form need to factor out $e^{i\pi/2}$

This is because we need to adjust the exponents by $\frac{-\pi}{2}$, therefore we will get matrix with elements $e^{\frac{-\pi}{4}}$ and $e^{\frac{\pi}{4}}$ after factoring out $e^{\frac{\pi}{2}}$

Then comparing with standard form we get, $\frac{\pi}{2} = \frac{\pi}{4}$, meaning $\theta = \frac{\pi}{2} = 90^\circ$

Therefore angle of rotation is $\frac{\pi}{2} = 90^\circ$

Problem 2

Determine the axis and angle of rotation for the two single-qubit unitaries, A and B, such that

$$ABA^\dagger B^\dagger = \sigma_X$$

$\sigma_X \Rightarrow$ Pauli-X matrix: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Therefore, we know A and B gives us X matrix.

We can rewrite A and B to:

$$ABA^\dagger B^\dagger = AB(AB)^\dagger$$

This shows us AB and σ_X only differ by a unitary transformation

Choose A and B as rotations around orthogonal axes.

Let:

A = rotation around Z-axis

B = rotation around Y-axis

We can write these as:

$$A = e^{-i\frac{\theta_A}{2}\sigma_Z}$$

$$B = e^{-i\frac{\theta_B}{2}\sigma_Y}$$

Express A and B as matrices:

For Z-rotation by angle θ_A :

$$A = \begin{pmatrix} e^{-i\frac{\theta_A}{2}\sigma_Z} & 0 \\ 0 & e^{i\frac{\theta_A}{2}\sigma_Z} \end{pmatrix}$$

For Y-rotation by angle θ_B :

$$B = \begin{pmatrix} \cos(\frac{\theta_B}{2}) & -\sin(\frac{\theta_B}{2}) \\ \sin(\frac{\theta_B}{2}) & \cos(\frac{\theta_B}{2}) \end{pmatrix}$$

Next find complex conjugate transpose of A and B (A^\dagger and B^\dagger):

$$A^\dagger = \begin{pmatrix} e^{i\frac{\theta_A}{2}\sigma_Z} & 0 \\ 0 & e^{-i\frac{\theta_A}{2}\sigma_Z} \end{pmatrix}$$

$$B^\dagger = \begin{pmatrix} \cos(\frac{\theta_B}{2}) & \sin(\frac{\theta_B}{2}) \\ -\sin(\frac{\theta_B}{2}) & \cos(\frac{\theta_B}{2}) \end{pmatrix}$$

Calculate $ABA^\dagger B^{dagger}$: First, let's calculate AB:

$$AB = \begin{pmatrix} e^{-i\theta_A/2} \cos(\theta_B/2) & -e^{-i\theta_A/2} \sin(\theta_B/2) \\ e^{i\theta_A/2} \sin(\theta_B/2) & e^{i\theta_A/2} \cos(\theta_B/2) \end{pmatrix}$$

Next, calculate $A^\dagger B^{dagger}$:

$$A^\dagger B^\dagger = \begin{pmatrix} e^{i\theta_A/2} \cos(\theta_B/2) & e^{i\theta_A/2} \sin(\theta_B/2) \\ -e^{-i\theta_A/2} \sin(\theta_B/2) & e^{-i\theta_A/2} \cos(\theta_B/2) \end{pmatrix}$$

Finally, multiply $(AB)(A^\dagger B^{dagger})$. Which will give us a complex expression that needs to equal σ_X :

$$ABA^\dagger B^\dagger = \begin{pmatrix} \cos(\theta_A) \cos(\theta_B) - i \sin(\theta_A) \sin(\theta_B) & \sin(\theta_A) \cos(\theta_B) + i \cos(\theta_A) \sin(\theta_B) \\ \sin(\theta_A) \cos(\theta_B) - i \cos(\theta_A) \sin(\theta_B) & \cos(\theta_A) \cos(\theta_B) + i \sin(\theta_A) \sin(\theta_B) \end{pmatrix}$$

For this result to equal $\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Diagonal elements must equal 0. Therefore,

$$\cos(\theta_A) \cos(\theta_B) - i \sin(\theta_A) \sin(\theta_B) = 0$$

and

$$\cos(\theta_A) \cos(\theta_B) + i \sin(\theta_A) \sin(\theta_B) = 0$$

The only way for these two equations to be true is if:

$$\cos(\theta_A) \cos(\theta_B) = 0$$

$$\sin(\theta_A) \sin(\theta_B) = 0$$

We know that:

$$\cos(\theta) = 0 \text{ when } \theta = \frac{\pi}{2} + n\pi \text{ (because of unit circle)}$$

$$\sin(\theta) = 0 \text{ when } \theta = n\pi$$

(Where n is an integer)

Off diagonal elements must equal 1. Therefore,

$$\sin(\theta_A) \cos(\theta_B) - i \cos(\theta_A) \sin(\theta_B) = 1$$

and

$$\sin(\theta_A) \cos(\theta_B) + i \cos(\theta_A) \sin(\theta_B) = 1$$

If we set $\theta_A = \theta_B = \frac{\pi}{2}$ these constraints are satisfied because:

$$\cos\left(\frac{\pi}{2}\right) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

Therefore,

A is a rotation along the Z-axis with a angle of rotation of 90°

B is a rotation along the Y-axis with a angle of rotation of 90°

Problem 3 (A variation of CHSH game)

If the CHSH game were modified so that Alice and Bob to satisfy $a \vee b = x \oplus y$ instead, what classical and quantum strategies could they employ, and what would be their maximum winning probabilities?

In the standard CHSH game, Alice and Bob need to satisfy $a \otimes b = x \cdot y$

In this modified version, they need to satisfy: $a \vee b = x \oplus y$

Where, $a \vee b$ is the logical OR of their outputs and $x \oplus y$ is the XOR of their inputs The truth table for $a \vee b$ is:

x	y	$a \vee b$
0	0	0
0	1	1
1	0	1
1	1	1

Table 1: Truth table for $a \vee b$

But in order to satisfy $a \vee b = x \oplus y$, $a \vee b$ needs to be:

Classical Strategy for Modified CHSH Game Alice's Strategy: (always output 0 no matter input)

- If $x = 0$, output $a = 0$
- If $x = 1$, output $a = 0$

Bob's Strategy:

x	y	$x \oplus y$	Required $a \vee b$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	0

Table 2: Truth table for the modified CHSH game where $a \vee b = x \oplus y$

- If $y = 0$, output $b = 0$
- If $y = 1$, output $b = y$ (also output 1)

With this strategy:

- When $(x, y) = (0,0)$: $a \vee b = 0 \vee 0 = 0$, matches $x \oplus y = 0$ ✓
- When $(x, y) = (0,1)$: $a \vee b = 0 \vee 1 = 1$, matches $x \oplus y = 1$ ✓
- When $(x, y) = (1,0)$: $a \vee b = 0 \vee 0 = 0$, matches $x \oplus y = 0$ ✗
- When $(x, y) = (1,1)$: $a \vee b = 0 \vee 1 = 1$, matches $x \oplus y = 1$ ✗

This strategy succeeds in 2 out of 4 cases, giving a success probability of $2/4 = 50\%$

Which is the maximum success rate for a classical approach

Quantum Strategy for Modified CHSH Game

In the quantum setting, Alice and Bob can share an entangled quantum state to achieve a higher success probability compared to the classical approach.

Setup:

Alice and Bob share the Bell state:

$$|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$$

Quantum Measurement Strategy:

Alice's Measurements:

- If $x = 0$; Measure in basis rotated by angle 0 (Z-basis)
- $X = 1$; Measure in basis rotated by angle $\frac{\pi}{4}$

Bob's Measurements:

- If $y = 0$; Measure in basis rotated by angle $\frac{-\pi}{8}$
- If $y = 1$; Measure in basis rotated by angle $\frac{3\pi}{8}$

Alice and Bob both use the mapping:

- Measurement outcome $+1 \rightarrow$ output 0
- Measurement outcome $-1 \rightarrow$ output 1

Calculate success for each input pair. Where probability of getting the same outcome is $\cos^2((\theta_a - \theta_\beta)/2)$:

1. $(x,y) = (0,0)$:

- Measurement angles: $\theta_a = 0, \theta_\beta = \frac{-\pi}{8}$
- Want: $a \vee b = 1$ (Meaning at least one output must be 1)
- Probability of at least $+1$: $\cos^2((0 - (\frac{-\pi}{8}))/2) = \cos^2(\frac{\pi}{16}) \approx 0.9616$

2. $(x,y) = (0,1)$:

- Measurement angles: $\theta_a = 0, \theta_\beta = \frac{3\pi}{8}$
- Want: $a \vee b = 1$ (Meaning at least one output must be 1)
- Probability of at least -1 : $1 - \cos^2((0 - (\frac{3\pi}{8}))/2) = 1 - \cos^2(\frac{3\pi}{16}) \approx 0.8536$

3. $(x,y) = (1,0)$:

- Measurement angles: $\theta_a = \frac{\pi}{4}, \theta_\beta = \frac{-\pi}{8}$
- Want: $a \vee b = 1$ (Meaning at least one output must be 1)
- Probability of at least -1 : $1 - \cos^2((\frac{\pi}{4} - (\frac{-\pi}{8}))/2) = 1 - \cos^2(\frac{3\pi}{16}) \approx 0.8536$

4. $(x,y) = (1,1)$:

- Measurement angles: $\theta_a = \frac{\pi}{4}, \theta_\beta = \frac{3\pi}{8}$
- Want: $a \vee b = 1$ (Meaning at least one output must be 1)
- Probability of at least $+1$: $\cos^2((\frac{\pi}{4} - (\frac{3\pi}{8}))/2) = \cos^2(\frac{-\pi}{16}) = \cos^2(\frac{\pi}{16}) \approx 0.9616$

Therefore, average success probability is:

$$\frac{0.9619 + 0.8536 + 0.8536 + 0.9619}{4} \approx 0.9078 \approx 90.87\%$$

The quantum strategy achieves approximately 90.78% success, while the classical approach yields a 50%, demonstrating a quantum advantage for this modified CHSH game.

Problem 4 (Geometric interpretation of controlled rotations)

Consider the controlled X rotation gate $C(R_X(\gamma))$ defined as:

$$C(R_X(\gamma)) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes R_X(\gamma),$$

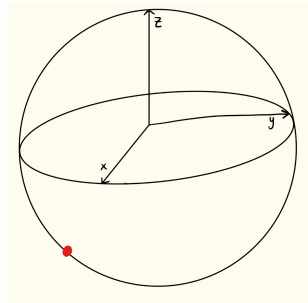
Where $\gamma \neq \pi$. Provide a clear geometric interpretation of the effect of this gate on a two-qubit system. Specifically, describe how the second qubit is rotated on the Bloch sphere depending on the state of the first (control) qubit.

$R_X(\gamma)$ corresponds to a rotation around the X-axis of the Bloch sphere by angle γ . Which is represented as:

$$R_X(\gamma) = \begin{pmatrix} \cos(\frac{\gamma}{2}) & -i \sin(\frac{\gamma}{2}) \\ -i \sin(\frac{\gamma}{2}) & \cos(\frac{\gamma}{2}) \end{pmatrix}$$

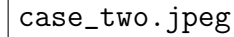
For the controlled version $C(R_X(\gamma))$, the geometric interpretation is:

1. When the control qubit is $|0\rangle$:
 - The operator applies $|0\rangle\langle 0| \otimes I$
 - This means the second qubit (target) remains unchanged
 - Qubit remains stationary on Bloch sphere



2. When the control qubit is $|1\rangle$:

- The operator applies $|1\rangle\langle 1| \otimes R_X(\gamma)$
- The second qubit undergoes a rotation around the X-axis by angle γ
- Target qubit rotates around the X-axis by angle γ



case_two.jpeg

- The rotation looks like this because it is along the X-axis and since $\gamma \neq \pi$, it will not be a full 180-degree rotation
- Therefore, I have represented a 90-degree rotation along the X-axis which would be represented as $R_X(\frac{\pi}{2})$ Where, $\gamma = \frac{\pi}{2}$