Transfer Functions

- Convenient representation of a *linear*, dynamic model.
- A transfer function (TF) relates *one* input and *one* output:

$$\begin{matrix} u(t) \\ U(s) \end{matrix} \rightarrow \boxed{\text{system}} \rightarrow \begin{matrix} y(t) \\ Y(s) \end{matrix}$$

The following terminology is used:

$\underline{u}$	$\underline{y}$
input	output
forcing function	response
“cause”	“effect”

Definition of the transfer function:

Let $G(s)$ denote the transfer function between an input, x, and an output, y. Then, by definition

$$G(s) = \frac{Y(s)}{U(s)}$$

where:

$$Y(s) = \mathrm{L}\left[\, y(t) \,\right]$$
$$U(s) = \mathrm{L}\left[\, u(t) \,\right]$$

Development of Transfer Functions

Example: Stirred Tank Heating System

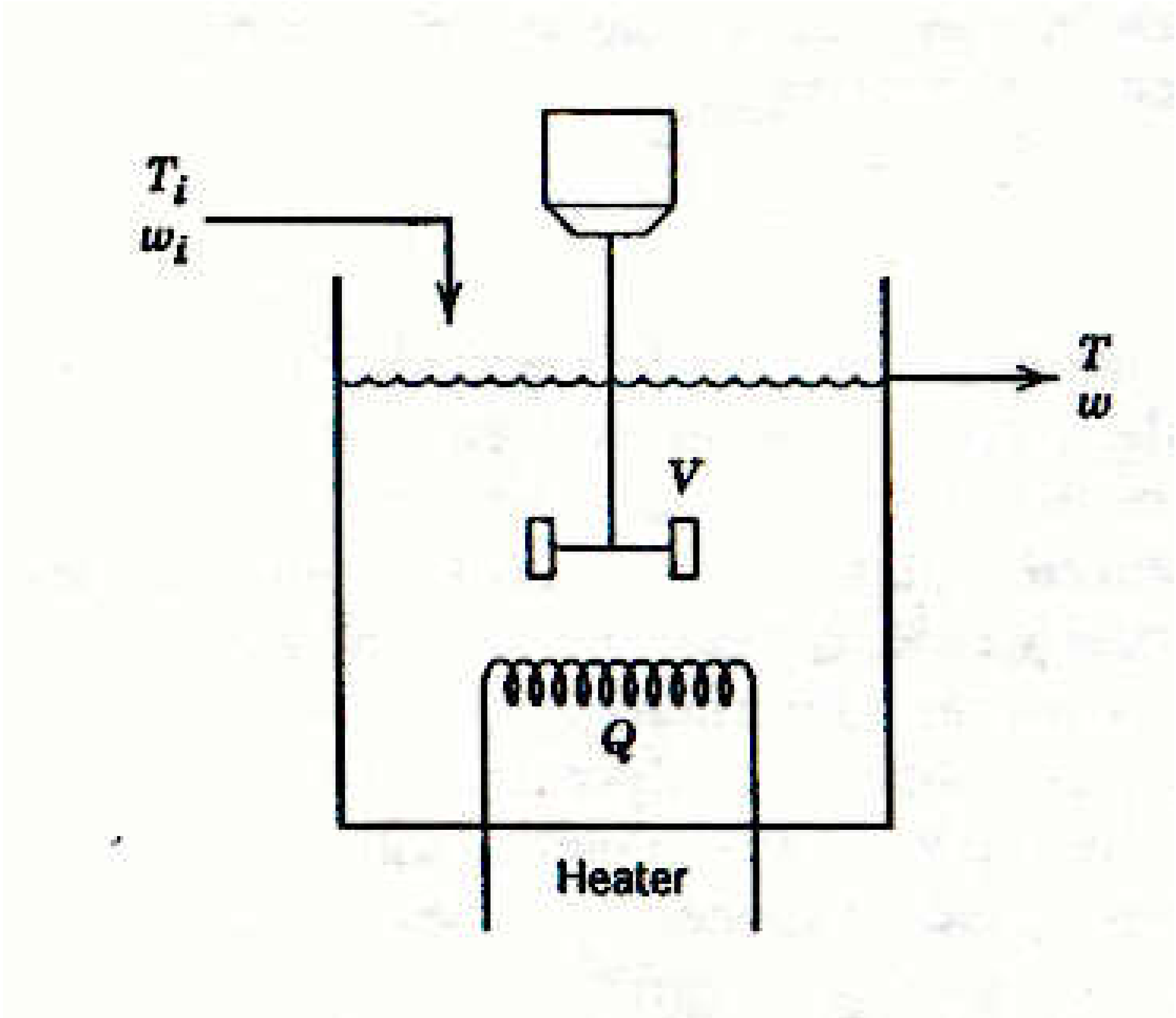

Figure 2.3 Stirred-tank heating process with constant holdup, *V*.

Recall the previous dynamic model, assuming constant liquid holdup and flow rates:

$$V\rho C\frac{dT}{dt} = wC\left(T_i - T\right) + Q \qquad (2\text{-}36)$$

Suppose the process is at steady state:

$$0 = wC\left(\bar{T}_i - \bar{T}\right) + \bar{Q} \qquad (2)$$

Subtract (2) from (2-36):

$$V\rho C\frac{dT}{dt} = wC\left[\left(T_i - \bar{T}_i\right) - \left(T - \bar{T}\right)\right] + \left(Q - \bar{Q}\right) \qquad (3)$$

But,

$$V\rho C\frac{dT'}{dt} = wC\left(T_i' - T'\right) + Q' \qquad (4)$$

where the "*deviation variables*" are

$$T' = T - \overline{T}, \quad T_i' = T_i - \overline{T}_i, \quad Q' = Q - \overline{Q}$$

Take $\mathfrak{L}$ of (4):

$$V\rho C\left[sT'(s) - T'(0)\right] = wC\left[T_i'(s) - T'(s)\right] - Q'(s) \qquad (5)$$

At the initial steady state, T′(0) = 0.

Rearrange (5) to solve for

$$T'(s) = \left(\frac{K}{\tau s + 1}\right) Q'(s) + \left(\frac{1}{\tau s + 1}\right) T_i'(s) \qquad (6)$$

where

$$K = \frac{1}{wC} \text{ and } \tau = \frac{V\rho}{w}$$

$$T'(s) = G_1(s)Q'(s) + G_2(s)T_i'(s)$$

G_1 and G_2 are transfer functions and independent of the inputs, Q' and T_i'.

Note G_1 (process) has gain K and time constant τ.

G_2 (disturbance) has gain=1 and time constant τ.

gain = G(s=0). Both are first order processes.

If there is no change in inlet temperature ($T_i' = 0$), then $T_i'(s) = 0$.

System can be forced by a change in either T_i or Q (see Example 4.3).

Conclusions about TFs

1. Note that (6) shows that the effects of changes in both Q and T_i are *additive*. This always occurs for linear, dynamic models (like TFs) because the Principle of Superposition is valid.

2. The TF model enables us to determine the output response to any change in an input.

3. Use deviation variables to eliminate initial conditions for TF models.

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Example: Stirred Tank Heater

$$K = 0.05 \qquad \tau = 2.0$$

$$T' = \frac{0.05}{2s+1} Q'$$ No change in T_i'

Step change in Q(t): 1500 cal/sec to 2000 cal/sec

$$Q' = \frac{500}{s}$$

$$T' = \frac{0.05}{2s+1} \frac{500}{s} = \frac{25}{s(2s+1)}$$

What is T′(t)? From line 13, Table 3.1

$$T'(t) = 25[1 - e^{-t/\tau}] \longleftarrow T(s) = \frac{25}{s(\tau s + 1)}$$

$$T'(t) = 25[1 - e^{-t/2}]$$

Properties of Transfer Function Models

1. Steady-State Gain

The steady-state of a TF can be used to calculate the steady-state change in an output due to a steady-state change in the input. For example, suppose we know two steady states for an input, *u*, and an output, *y*. Then we can calculate the steady-state gain, *K*, from:

$$K = \frac{\bar{y}_2 - \bar{y}_1}{\bar{u}_2 - \bar{u}_1} \qquad (4\text{-}38)$$

For a linear system, *K* is a constant. But for a nonlinear system, *K* will depend on the operating condition $(\bar{u}, \bar{y})$.

Calculation of *K* from the TF Model:

If a TF model has a steady-state gain, then:

$$\boxed{K = \lim_{s \to 0} G(s)} \tag{14}$$

- This important result is a consequence of the Final Value Theorem
- *Note*: Some TF models do *not* have a steady-state gain (e.g., integrating process in Ch. 5)

2. Order of a TF Model

Consider a general n-th order, linear ODE:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{dy^{n-1}}{dt^{n-1}} + \cdots a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} +$$

$$b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u \qquad \text{(4-39)}$$

Take $\mathfrak{L}$, assuming the initial conditions are all zero. Rearranging gives the TF:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} \qquad \text{(4-40)}$$

Definition:

The order of the TF is defined to be the order of the denominator polynomial.

Note: The order of the TF is equal to the order of the ODE.

Physical Realizability:

For any physical system, $n \geq m$ in (4-38). Otherwise, the system response to a step input will be an impulse. This can't happen.

Example:

$$a_0 y = b_1 \frac{du}{dt} + b_0 u \quad \text{and step change in } u \qquad (4\text{-}41)$$

2nd order process

General 2nd order ODE:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = Ku$$

Laplace Transform: $\left[as^2 + bs + 1\right] \cdot Y(s) = KU(s)$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{as^2 + bs + 1}$$

2 roots $$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

$\frac{b^2}{4a} > 1$: real roots

$\frac{b^2}{4a} < 1$: imaginary roots

Examples

1. $$\frac{2}{3s^2+4s+1} \qquad \frac{b^2}{4a}=\frac{16}{12}=1.333>1$$

$$3s^2+4s+1=(3s+1)(s+1)=3(s+{}^1\!/_3)(s+1)$$

$$transforms\ to\ e^{-t/3}, e^{-t}\ \ (real\ \ roots)$$

(no oscillation)

2. $$\frac{2}{s^2+s+1} \qquad \frac{b^2}{4a}=\frac{1}{4}<1$$

$$s^2+s+1=(s+0.5+\frac{\sqrt{3}}{2}j)(s+0.5-\frac{\sqrt{3}}{2}j)$$

$$transforms\ to\ e^{-0.5t}\cos\frac{\sqrt{3}}{2}t,\ e^{-0.5t}\sin\frac{\sqrt{3}}{2}t$$

(oscillation)

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From Table 3.1, line 17

$$e^{-bt}\sin\omega t \xleftrightarrow{\mathfrak{L}} \frac{\omega}{(s+b)^2+\omega^2}$$

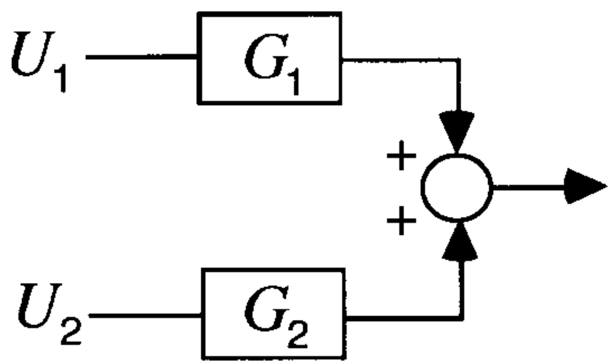
$$\frac{2}{s^2+s+1} = \frac{2}{(s+0.5)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Two IMPORTANT properties (L.T.)

A. Multiplicative Rule

$$Y = G_1 \bullet G_2\, U$$

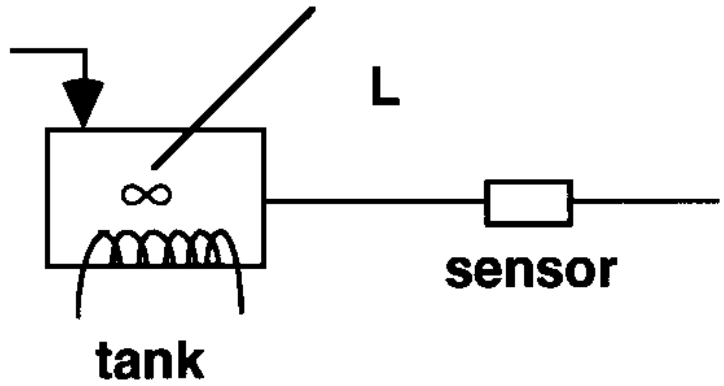
B. Additive Rule



$$Y = G_1 U_1 + G_2 U_2$$

Example 1:

Place sensor for temperature downstream from heated tank (transport lag)



Distance L for plug flow,

Dead time $\theta = \dfrac{L}{V}$

V = fluid velocity

Tank: $$G_1 = \frac{T(s)}{U(s)} = \frac{K_1}{1+\tau_1 s}$$

Sensor: $$G_2 = \frac{T_s(s)}{T(s)} = \frac{K_2 e^{-\theta s}}{1+\tau_2 s}$$ $K_2 \le 1$, τ_2 is very small (neglect)

Overall transfer function:

$$\frac{T_s}{U} = \frac{T_s}{T} \cdot \frac{T}{U} = G_2 \cdot G_1 = \frac{K_1 K_2 e^{-\theta s}}{1+\tau_1 s}$$

Linearization of Nonlinear Models

- Required to derive transfer function.
- Good approximation near a given operating point.
- Gain, time constants may change with
- operating point.
- Use 1st order Taylor series.

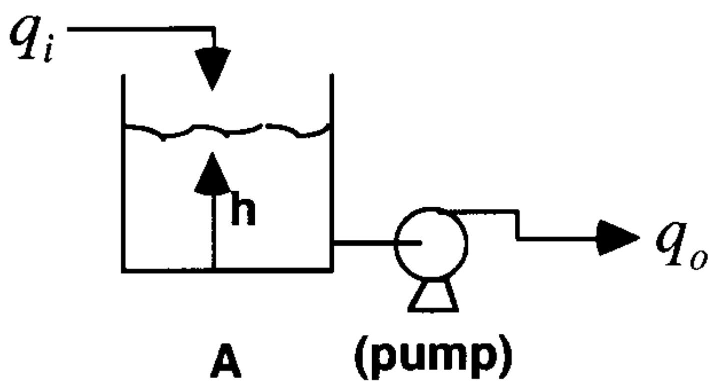
$$\frac{dy}{dt} = f(y,u) \qquad (4\text{-}60)$$

$$f(y,u) \cong f(\bar{y},\bar{u}) + \left.\frac{\partial f}{\partial y}\right|_{\bar{y},\bar{u}} (y-\bar{y}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y},\bar{u}} (u-\bar{u}) \qquad (4\text{-}61)$$

Subtract steady-state equation from dynamic equation

$$\frac{dy'}{dt} = \left.\frac{\partial f}{\partial y}\right|_{s} y' + \left.\frac{\partial f}{\partial u}\right|_{s} u' \qquad (4\text{-}62)$$

Example 3:



q_0: control,
q_i: disturbance

$$A\frac{dh}{dt} = q_i - q_0 \qquad \bar{q}_i = \bar{q}_0 \text{ at s.s.}$$

Use L.T.

$$A\frac{dh'}{dt} = q_i' - q_0'$$

$$AsH'(s) = q_i'(s) - q_0'(s)$$ (deviation variables)

suppose q_0 is constant $\quad q_i' = 0$

$$AsH'(s) = q_i'(s), \quad \frac{H'(s)}{q_i'(s)} = \frac{1}{As}$$

pure integrator (ramp) for step change in q_i

q_0 is manipulated by a flow control valve,

$$q_0 = C_v \sqrt{h}$$

<u>nonlinear element</u>

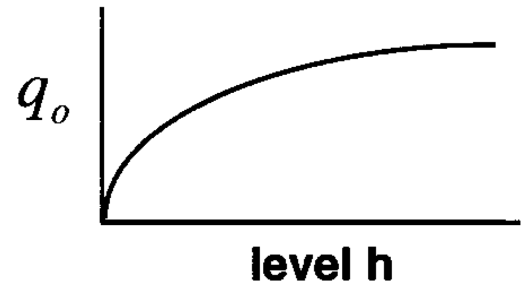
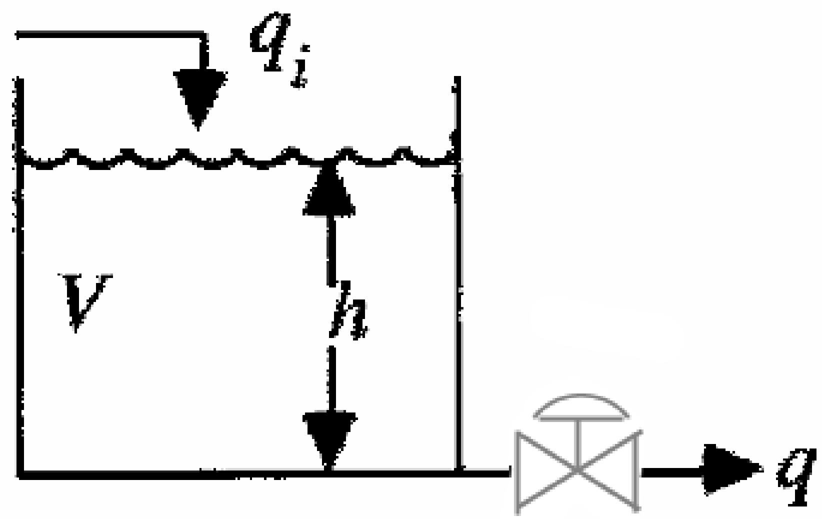


Figure 2.5



Linear model

$$q' = \frac{1}{R} h'$$

R: line and valve resistance

$$A \frac{dh'}{dt} = q_i' - \frac{1}{R} h'$$

<u>linear ODE</u> : eq. (4-74)

$$\text{if } q_0 = C_V \sqrt{h}$$

$$A\frac{dh}{dt} = q_i - C_v \sqrt{h}$$

Perform Taylor series of right hand side

$$A\frac{dh}{dt} = \overline{q}_i - C_v \overline{h}^{\,0.5} + \frac{\partial f}{\partial q_i}(q_i - \overline{q}_i) + \frac{\partial f}{\partial h}(h - \overline{h})$$

$$A\frac{dh'}{dt} = 0 + 1(q_i - \overline{q}_i) - \frac{1}{2} C_v \overline{h}^{\,-0.5}(h - \overline{h}) = q'_i - \frac{1}{2} C_v \overline{h}^{\,-0.5} h'$$

$$A\frac{dh'}{dt} = q_i' - \frac{1}{R} h'$$

$$R = 2\overline{h}^{\,0.5} / C_v$$

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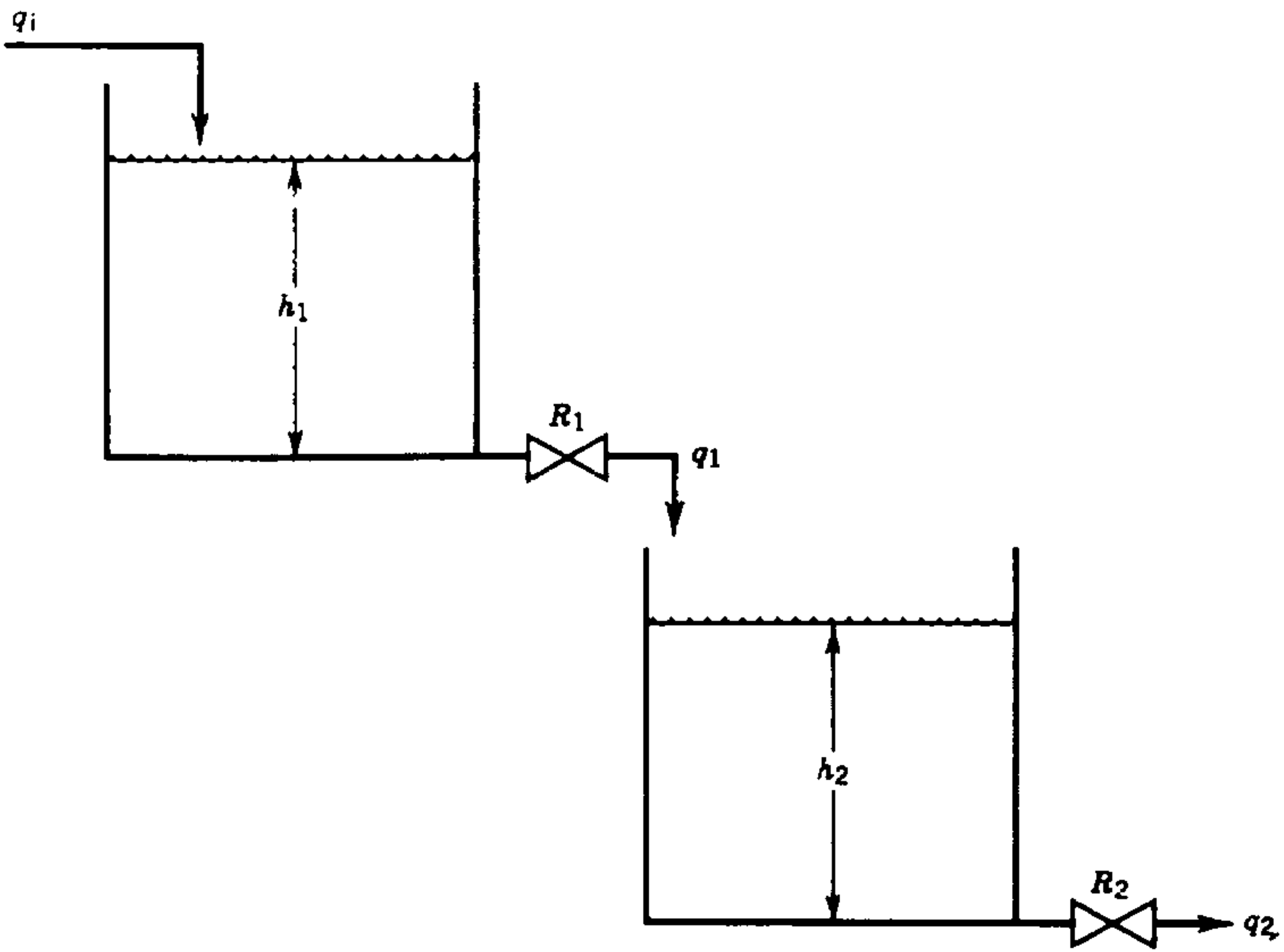

Figure 4.3. Schematic diagram of two liquid surge tanks in series.

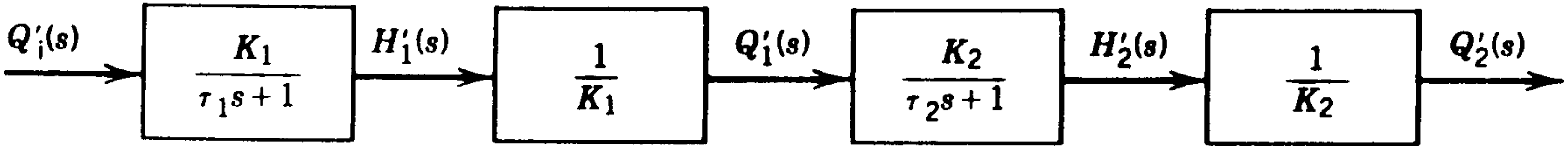

Figure 4.4. Input–output model for two liquid surge tanks in series.

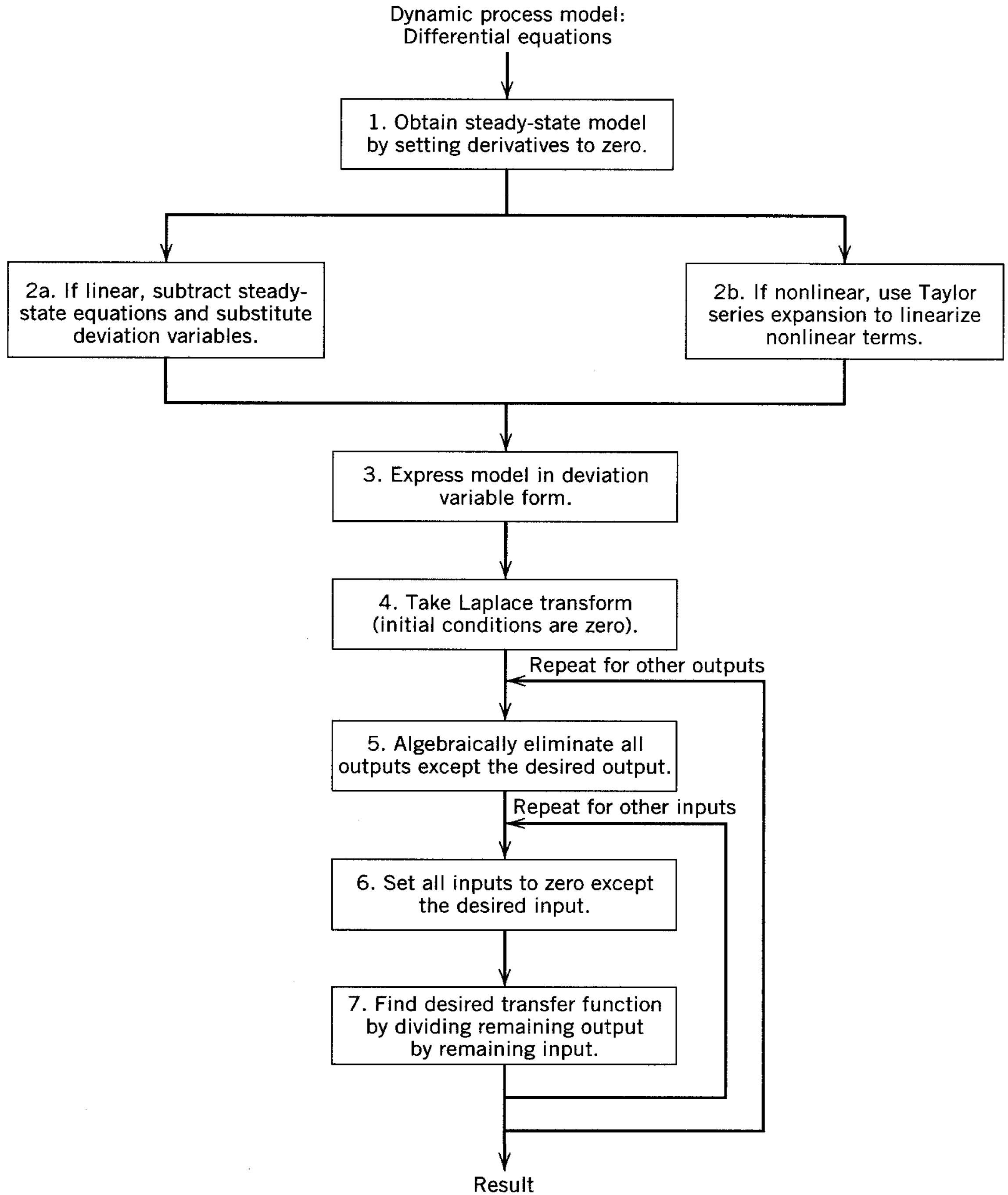

Figure 4.5 Procedure for developing transfer function models.