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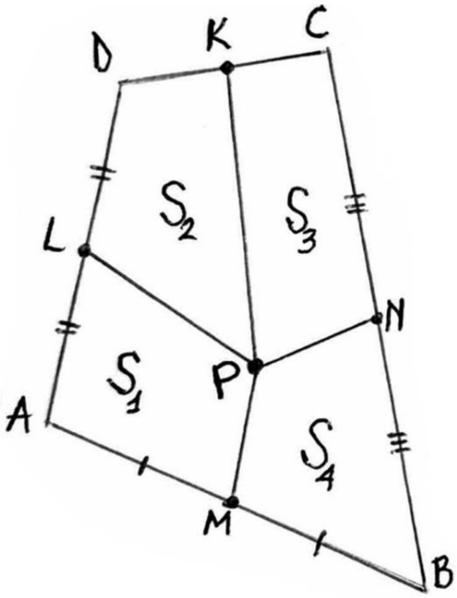
TIRANA, ALBANIA on July 6, 2025

TOPIC: A PROBLEM from PLANE GEOMETRY

Given any quadrilateral ABCD. Let there be any point P inside this quadrilateral. We join the point P with the midpoints of the sides of the quadrilateral and thus 4 quadrilaterals are formed. We denote by S_1,S_2,S_3,S_4 the surfaces of these quadrilaterals that were formed, taking them in turn, in a sequence (according to the movement of the clock). The problem is: Can we find a connection between the surfaces S_1,S_2,S_3,S_4?



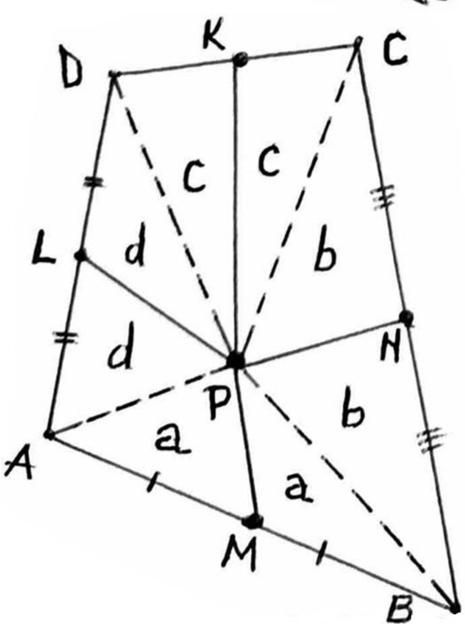
• We are reasoning as follows:
First we are making auxiliary constructions, we join the point P with the vertices ABCD. In the triangle PAB we have the two triangles PAM and PMB where AM=MB which means that these two triangles have equal areas. We denote $a=S_{PAM}=S_{PMB}$

In triangle PBC we have two triangles PBN and PNC where BN=NC means that these two triangles have equal areas.

We denote $b=S_{PBN}=S_{PNC}$

We denote $c=S_{PCK}=S_{PKD}$

We denote $d=S_{PDL}=S_{PLA}$



We evaluate $S_1+S_3=(a+d)+(b+c)=a+b+c+d$

We evaluate $S_2+S_4=(c+d)+(a+b)=a+b+c+d$

It is clear that the amounts: $\mathbf{S_1+S_3=S_2+S_4}$

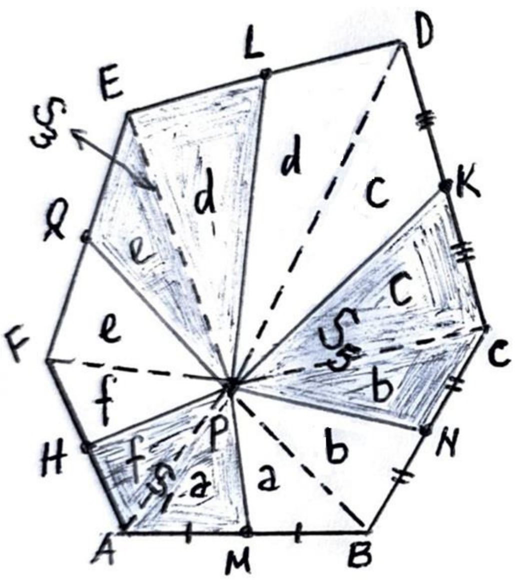
So we have a True Statement, we have proven a THEOREM.

Now I am formulating the THEOREM that I PROVEN.

THEOREM: If any point P inside any concave quadrilateral, is joined to the midpoints of the sides of the quadrilateral, 4 quadrilaterals with surfaces by S_1, S_2, S_3, S_4 (marked in a direction of rotation) are formed, then the equation is true: $\mathbf{S_1+S_3=S_2+S_4}$.

ATTENTION: We are continuing the study of the PROBLEM, now in the general case, for any concave polygon with an even number of sides n=6,n=8,n=10,…. And we reason as follows:

•Let's have any concave hexagon:



Point P anywhere inside the hexagon, and M,N,K,L,Q,H are the midpoints of the sides.

We connect the point P with the midpoints of the sides and 6 quadrilaterals will be formed as in the figure. Auxiliary construction, we connect the point P with all the vertices of the hexagon ABCDEF.

In triangle PAB we have AM=MB, which means:

Surface $S_{PAM}=S_{PMB}$=a square unit.

We continue the reasoning as above and in triangle PFA we have FH=HA, leading to:

$S_{PFH}=S_{PHA}$=f square unit.

We denote: $S_1=S_{PMAH}$, $S_2=S_{PHFQ}$, $S_3=S_{PQEL}$, $S_4=S_{PLDK}$, $S_5=S_{PKCN}$, $S_6=S_{PNBM}$

We evaluate: $S_1+S_3+S_5=(a+f)+(e+d)+(c+b)=a+b+c+d+e+f$

We also have $S_2+S_4+S_6=(f+e)+(d+c)+(b+a)=a+b+c+d+e+f$

• It is clear that the sums: $\mathbf{S_1+S_3+S_5 = S_2+S_4+S_6}$

It is clear that the Theorem we proved above for n=4 and n=6 is true when the number of sides n is even for n=4,6,8,10,12,…..

So it is true that: $\mathbf{S_1+S_3+S_5+S_7+….+S_{n-1} = S_2+S_4+S_6+S_8+….+S_n}$

• I believe that I DISCOVERED a truth in PLANE GEOMETRY, that is, I PROVEN a THEOREM, and I wish and believe that I have the right to give it a NAME, I am naming it:

THEOREM PJETËR NEÇAJ

We are formulating the Theorem for the general case:

In any concave polygon where the number of sides n is an even number, so n=4,6,8,10… and any point P inside this polygon is joined to the midpoints of the sides of the polygon, n quadrilaterals will be formed, and if we mark their surfaces with $S_1,S_2,S_3,S_4,S_5,S_6,\ldots,S_{n-1},S_n$ (marked in a sense of rotation), then it is shown that the equation is true: $\mathbf{S_1+S_3+S_5+S_7+\ldots+S_{n-1} = S_2+S_4+S_6+S_8+\ldots+S_n}$

Note:
This THEOREM was proven by teacher Pjetër Neçaj and published on July 6, 2025.

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