Notes on Cartesian Tensors and Mathematical Models of Elastic Solids and Viscous Fluids

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Preface

This document contains lecture notes and tutorials for the one-semester module Cartesian Tensors and Mathematical Models of Elastic Solids and Viscous Fluids, taught in the Applied Mathematics programme at Xi'an Jiaotong-Liverpool University between 2014 and 2017. The format of lecture notes and tutorials (with solutions in most places) has been kept and roughly reflects the pace of weekly delivery of the module. The organisation into chapters aims to guide the reader through the main themes of the course. General concepts of continuum mechanics are developed in the first four chapters, Chapter 6 and Chapter 5 are independent from each other and can be studied in any order.

Each tutorial sheet starts with Exercise Zero, which invites the reader to edit the present text and to report typos, errors or expository issues of any kind. These notes are therefore by nature a work in progress, but they benefitted from the answers given to Exercise Zero by several cohorts of students. Substantial contributions by Zuqian Huang, Jin Yan, Ran Bi, Zhiwei Cheng, Qier Yu, Henger Li, Chang Liu and Chenxia Gu are gratefully acknowledged.

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Chapter 1

Cartesian coordinates, vectors, tensors

Lecture 1 : from linear algebra to tensors

Keywords. Linear algebra, vectors, matrices, scalar product, change of orthonormal basis, transformation matrix, tensor, isotropic tensors.

1.1 Tensors and continuous media: motivations and history

Systems of coordinates are mathematical structures used to describe the properties of physical systems in a quantitative way. They depend on the choice of the person who designs the model (for example, a certain choice of axes can make computations easier). However, the physical results should not depend on the choice of coordinates.

• **This independence induces transformation laws on certain quantities.** If one changes the system of coordinates, the quantities we write change as well, but they must do so in a precise way in order for the physical reality to stay the same. Tensors are quantities defined by the way they transform when the system of coordinates changes. The word *Cartesian*[1] refers to the fact that one considers systems of coordinates in which the three axes form an orthogonal basis.

• **Continuous media: solids and viscous fluids.** We are interested in scales at which the (discrete, atomic, molecular[2]) microscopic structure of matter is not apparent: distances are large in scale of the size of molecules, and we will treat solids and fluids as continuous media on which we will put coordinates just as if we were on a geometric space. In some cases, such as concrete and most rocks (think of granit and marble), heterogeneities are detectable by the naked eye but quantitative modeling by a continuum proves satisfactory. The mathematical framework of continuous media was developed before atomic and molecular physics (for example the Euler equation for the mechanics of fluids was written in 1755[3]). At macroscopic scales it is still valid, because the huge numbers of microscopic constituents have interactions that result in collective

[1]from the name of René Descartes (1596-1650). Under his influence, numbers started being used to describe geometric objects. His name is also attached to the laws of geometric optics.

[2]Let us mention that tensors are used for other purposes in atomic and subatomic physics, to address symmetry properties of particles.

[3]Leonhard Euler (1707-1783) founded the discipline of fluid mechanics, and contributed in a decisive way to all branches of mathematics.

behaviours that can be described at larger scales by a relatively small number of parameters (such as elasticity coefficients and density). These parameters are to be determined experimentally, but this course will introduce the mathematical framework that can be described for all continuous media (fluids and solids).

1.2 Linear algebra reminders, notations and conventions

Unless otherwise stated, we will assume in this course that the ambient space is the three-dimensional space $\mathbf{R}^3$ (but tensors can be defined in any finite dimension, using the reasoning ofthe present document). This section recalls a few notions of linear algebra and introduces notations.

Let $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ be an orthonormal of $\mathbf{R}^3$. Let O be the origin of a coordinate system. Let M be a point in $\mathbf{R}^3$. It can be described by its coordinates (x_1, x_2, x_3), the three numbers such that the vector $O\vec{M}$ is written as follows:

$$\overrightarrow{OM} = x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3}. \tag{1.1}$$

1.2.1 Notations: summing over repeated indices

Notation. We are familiar with the notation Σ for sums, which saves space:

$$\overrightarrow{OM} = \sum_{i=1}^{3} x_i\vec{e_i}. \tag{1.2}$$

In tensor algebra, it is customary to omit the Σ symbol *when an index is summed over and appears twice and only twice in an expression*:

$$\overrightarrow{OM} = x_i\vec{e_i}. \tag{1.3}$$

The range of the index i is not specified anymore (whereas in Eq. 1.2 it was specified that i ranges from 1 to 3). So, when an index appears twice in an expression, it is implied that it corresponds to a sum over all possible values of this index. Since the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ has three vectors, the sum in Eq. 1.3 is over $i = 1, i = 2, i = 3$.

Remark (mute indices). If an index is summed over, it is called a *mute* index. It takes all possible values, one per term in the sum, and does not *say* anything about the components of the result: the result does not have this index. One can therefore use another symbol for it without changing the value of the sum. For example in $\vec{x} = x_i\vec{e_i}$, the index i is a mute index, and one can as well write $\vec{x} = x_k\vec{e_k}$.

This notation can be used for indices in matrices as well. From linear algebra, you are already familiar with matrices. Matrices have two indices, one for rows and one for columns. Given the basis $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$, and a linear application u from $\mathbf{R}^3$ to $\mathbf{R}^3$, a matrix U can be used to represent the action of u in the basis (e):

$$U = \mathrm{Mat}(u,(e)) = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}$$

$$u\left(\sum_{i=1}^{3} x_i \vec{e_i}\right) = \sum_{j=1}^{3}\left(\sum_{k=1}^{3} U_{jk} x_k\right)\vec{e_j} \tag{1.4}$$

The expression 1.4 contains two indices (j and k) that appear twice each, and are summed over. The sum convention on repeated indices can be applied to rewrite the expression as follows:

$$u(x_i \vec{e_i}) = U_{jk} x_k \vec{e_j}. \tag{1.5}$$

1.2.2 Scalar product, Kronecker symbol

Let $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ be an orthonormal basis of $\mathbf{R}^3$. The scalar products between pairs of vectors in the basis is as follows:

$$\vec{e_i}.\vec{e_j} = \delta_{ij}. \tag{1.6}$$

for which one used the special notation δ_{ij} (or Kronecker symbol):

$$\begin{cases} \delta_{ij} = 1 \text{ if } i = j, \\ \delta_{ij} = 0 \text{ if } i \neq j \end{cases}$$

The (Euclidean) scalar product (or dot-product) between two vectors $x = x_i\vec{e_i}$ and $y = y_i\vec{e_i}$ is denoted by a dot, which is a bilinear[4] operation:

$$\vec{x}.\vec{y} = (x_i\vec{e_i}).(y_j\vec{e_j}) = x_i y_j \delta_{ij} = x_i y_i. \tag{1.7}$$

[4]A bilinear application on a vector space E is an application B with two arguments that is linear in both arguments, meaning for all vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ and all scalars λ and μ we have

$$B(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda B(\vec{x}, \vec{z}) + \mu B(\vec{y}, \vec{z})$$

and

$$B(\vec{z}, \lambda\vec{x} + \mu\vec{y}) = \lambda B(\vec{z}, \vec{x}) + \mu B(\vec{z}, \vec{y}).$$

In particular, if B is a bilinear map, knowing the values of $B(\vec{e_i}, \vec{e_j})$ for all the vectors in a base is enough to know all the values of B, just as knowing the values of a linear application on the vectors of a basis is enough to characterize the linear application completely.

Remark. Whenever repeated indices are summed over, a Kronecker symbol can be inserted (with a new index symbol) without changing the value of the expression:

$$x_i y_i = x_i \delta_{ij} y_j. \tag{1.8}$$

This trick will be used in the next section when we prove that the scalar product is invariant unde change of orthonormal basis.

1.3 Transformation rules from changes of orthonormal basis

1.3.1 Transformation of vectors

Consider two orthonormal bases of $\mathbf{R}^3$, denoted by $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ and $(e') = (\vec{e'_1}, \vec{e'_2}, \vec{e'_3})$. Consider a vector $\vec{x}$ in $\mathbf{R}^3$. Let us write the coordinates of this vector in these two bases:

$$\vec{x} = x_i \vec{e_i}, \quad \vec{x} = x'_i \vec{e'_i}. \tag{1.9}$$

Since the basis (e) is orthonormal, one can decompose the vectors (e'_1, e'_2, e'_3) over this basis by taking dot-products:

$$\forall i \in [1..3], \;\; \vec{e'_i} = (\vec{e'_i}.\vec{e_j})\vec{e_j}. \tag{1.10}$$

Let us substitute the expression 1.10 in the expression of $\vec{x}$ in Eq. 1.12:

$$\vec{x} = x_i \vec{e_i}, \quad \vec{x} = x'_i (\vec{e'_i}.\vec{e_j})\vec{e_j}. \tag{1.11}$$

Now we have two expressions for the vector $\vec{x}$ in the same basis (e). Hence the coordinates must be equal. There are two mute indices in the second expression of $\vec{x}$, and we can rewrite the indices in the sum in order to have the symbol $\vec{e_i}$ at the end:

$$\vec{x} = x_i \vec{e_i}, \quad \vec{x} = x'_k (\vec{e'_k}.\vec{e_j})\vec{e_j} = x'_k (\vec{e'_k}.\vec{e_i})\vec{e_i}. \tag{1.12}$$

Let us write the equality of the components of the vector $\vec{x}$:

$$x_i = x'_k (\vec{e'_k}.\vec{e_i}). \tag{1.13}$$

The r.h.s. of Eq. 1.14 can be rewritten as the action of a square matrix denoted by P (the transformation matrix) on the coordinates (x'_1, x'_2, x'_3):

$$\boxed{x_i = P_{ik} x'_k,} \tag{1.14}$$

where

$$\boxed{P_{ik} = \vec{e'_k}.\vec{e_i}.} \tag{1.15}$$

Proposition. *The inverse of the transformation matrix P is its transpose.*

Proof. Consider the product of P and its transpose P^T:

$$(PP^T)_{ij} = P_{ik}(P^T)_{kj} = P_{ik}P_{jk} = (\vec{e'_k}.\vec{e_i})(\vec{e'_k}.\vec{e_j}) = (\vec{e'_l}.\vec{e_i})\delta_{lp}(\vec{e'_p}.\vec{e_j}), \tag{1.16}$$

where we have inserted the Kronecker symbol to rewrite the sum over repeated indices. Let us now rewrite the Kronecker symbol as the dot-product of two vectors of an orthonormal basis:

$$\delta_{lp} = (\vec{e'_l}.\vec{e'_p}). \tag{1.17}$$

Hence

$$(PP^T)_{ij} = (\vec{e'_l}.\vec{e_j})(\vec{e'_l}.\vec{e'_p})(\vec{e'_p}.\vec{e_j}) = (\vec{e_i}.\vec{e'_l})(\vec{e'_l}.\vec{e'_p})(\vec{e_j}.\vec{e'_p}) = \Big((\vec{e_i}.\vec{e'_l})\vec{e'_l}\Big).\Big((\vec{e_j}.\vec{e'_p})\vec{e'_p}\Big) \tag{1.18}$$

where we have used the *bilinearity* of the dot-product. Since (e') is an orthonormal basis, we have

$$\vec{e_i} = (\vec{e_i}.\vec{e'_l})\vec{e'_l}, \text{ and } \vec{e_j} = (\vec{e_j}.\vec{e'_p})\vec{e'_p}. \tag{1.19}$$

Hence

$$\boxed{(PP^T)_{ij} = (\vec{e_i}, \vec{e_j}) = \delta_{ij}.} \tag{1.20}$$

Hence the inverse of the matrix P is its transpose:

$$(PP^T) = I_3, \quad P^TP = I_3. \tag{1.21}$$

We can use this property to obtain an expression of the "new" coordinates x'_1, x'_2, x'_3 in terms of the coordinates x_1, x_2, x_3, simply by multiplying both sides of Eq. 1.14 by P_{ij} (and summing over index i):

$$P_{ij}x_i = P_{ij}P_{ik}x'_k = (P^t)_{ji}P_{ik}x'_k = (P^TP)_{jk}x'_k = \delta_{jk}x'_k = x'_j, \tag{1.22}$$

Hence

$$\boxed{x'_i = P_{ji}x_j.} \tag{1.23}$$

Exercise. Verify that the norm of a vector and the scalar product of two vectors are invariant under a change of orthonormal basis.

Solution. Let us use the same notations as a above. The norm of the vector $\vec{x}$ expressed using the coordinates (x_1, x_2, x_3) is $\sqrt{x_ix_i}$. Using the new coordinates it is expressed as $\sqrt{x'_ix'_i}$. Let us use Eq. 1.14 to write these two expressions in the same system of coordinates:

$$x_ix_i = P_{ik}x'_kP_{il}x'_l \tag{1.24}$$

The expression $P_{ik}P_{il}$ can be rewritten as $(P^T)_{ki}P^T_{il} = (P^TP)_{kl} = \delta_{kl}$. We therefore have

$$x_i x_i = \delta_{kl} x'_k x'_l = x'_k x'_k. \tag{1.25}$$

As the indices i and k are mute in the above expression, the two quantities are equal, and the norm of x is invariant under a change of orthonormal basis.
As for the scalar product of two vectors x and y, we can use the same property to prove the invariance under change of orthonormal basis:

$$x_i y_i = P_{ik} x'_k P_{il} y'_l = (P^T)_{ki} P_{il} x'_k y'_l = (P^TP)_{kl} x'_k y'_l = \delta_{kl} x'_k y'_l = x'_k y'_k. \tag{1.26}$$

1.3.2 Transformation of the matrix of an endomorphism under a change of orthonormal basis

Consider the matrix U of a linear application from $\mathbf{R}^3$ to $\mathbf{R}^3$, presented in an orthonormal basis $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$:

$$U = \mathrm{Mat}(u, (e)) = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \tag{1.27}$$

For i in $[1..3]$, the image of vector e_i can be written in the basis (e) using the entries in the i-th row of the matrix U:

$$u(e_i) = U_{i1}\vec{e_1} + U_{i2}\vec{e_2} + U_{i3}\vec{e_3} = U_{ij}\vec{e_j}. \tag{1.28}$$

Consider another orthonormal basis $(e') = (\vec{e'_1}, \vec{e'_2}, \vec{e'_3})$. We want to compute the matrix of u in the basis (e'). Let us call this matrix U':

$$U' = \mathrm{Mat}(u, (e')) = \begin{pmatrix} U'_{11} & U'_{12} & U'_{13} \\ U'_{21} & U'_{22} & U'_{23} \\ U'_{31} & U'_{32} & U'_{33} \end{pmatrix} \tag{1.29}$$

Again, by defnition of the matrix representation of linear applications in basis (e'), we have

$$u(\vec{e'_i}) = U'_{i1}\vec{e'_1} + U'_{i2}\vec{e'_2} + U'_{i3}\vec{e'_3} = U'_{ia}\vec{e'_a}. \tag{1.30}$$

We are going to express both the l.h.s. and the r.h.s of Eq. 1.30 on the basis (e).
The l.h.s of Eq. 1.30 can be rewritten as follows:

$$u(\vec{e'_i}) = u((\vec{e'_i}, \vec{e_a})\vec{e_a}) = (\vec{e'_i}, \vec{e_a})u(\vec{e_a}) = (\vec{e'_i}, \vec{e_a})U_{ak}\vec{e_k} = P_{ai}U_{ak}\vec{e_k}. \tag{1.31}$$

where we used the linearity of u in the second equality, and Eq. 1.28 in the third equality.
The r.h.s of Eq. 1.30 can be rewritten as follows:

$$U'_{ia}\vec{e'_a} = U'_{ia}(\vec{e'_a}, \vec{e_k})\vec{e_k} = U'_{ia}P_{ka}\vec{e_k}. \tag{1.32}$$

As (e) is a basis of $\mathbf{R}^3$, the coefficients of $\vec{e_k}$ in 1.31 and 1.32 must be equal for all k:

$$U'_{ia}P_{ka} = P_{ai}U_{ak}. \tag{1.33}$$

Since the inverse of the transformation matrix P is its transpose P^T, we can multiply both sides by P_{kj} and get rid of all P symbols on the l.h.s.:

$$U'_{ia}P_{ka}P_{kj} = P_{ai}U_{ak}P_{kj}. \tag{1.34}$$

Since $P_{ka}P_{kj} = (P^T)_{ak}P_{kj} = (P^TP)_{aj} = \delta_{aj}$, we have

$$\forall i, j \in [1..3], \quad \boxed{U'_{ij} = P_{ai}P_{kj}U_{ak}.} \tag{1.35}$$

1.4 Definition of tensors

Cartesian tensors are mathematical objects with components in a given orthonormal basis. They carry a certain number of indices (the number of indices is called the order of the tensor), which can be used to act on vectors by summing over indices. When the orthonormal basis is changed, the components of Cartesian tensors are transformed using transformation matrices.

We have seen examples of low order:
- scalars have no indices, they are invariant under change of basis. They are called tensors of order zero (physical examples are mass and temperature). They are called tensors of order 0.
- vectors have one index, they tansform as in Eq. 1.14 under change of basis (physical examples include velocities). They are called tensors of order 1.
- matrices have two indices, they. Physical examples include the stress tensor will be introduced in this course. They are called tensors of order 2.
By generalizing the transformation pattern of Eqs. 1.23 and 1.35, one defines a tensor T of order n to have n indices, with the transformation rule of the entries:

$$\boxed{\forall i_1, i_2, \ldots, i_n \in [1..3], \quad T'_{i_1,\ldots,i_n} = P_{k_1i_1}P_{k_2i_2}\ldots P_{k_ni_n}T_{k_1,\ldots,k_n}} \tag{1.36}$$

when one goes from orthonormal basis (e) to orthonormal basis (e'), with the transformation matrix defined by $P_{ij} = e'_i.e_j$.

Exercise. Consider a tensor of order 2 whose components in some orthonormal basis are Kronecker symbols. Show that it has the same coefficients in all orthonormal bases (one says δ is an *isotropic tensor*).

Let us apply the transformation rule of a tensor of order 2, with the above notation for the transformation matrix:

$$(\delta')_{ij} = P_{ki}P_{lj}\delta_{kl} = P_{ki}P_{kj} = (P^T)_{ik}P_{kj} = (P^TP)_{ij} = \delta_{ij}. \tag{1.37}$$

One can also note that the Kronecker symbol is the matrix of the identity transformation, hence its matrix is I_3 in any basis.

Tutorial Sheet 1 : from linear algebra to tensors

Exercise 0. Report by email issues of any kind (typos, inconsistencies, lack of clarity) you found in the lecture notes.

Exercise 1. Transformation of matrices. Consider two orthonormal bases of $\mathbf{R}^3$, and the matrices U and U' of an endomorphism in these two bases. Find the relation between the matrices U and U' in terms of the transformation matrix (section 3.2 of the notes).

Exercise 2. Invariances. Show that the dot-product of two vectors is invariant under a change of orthonormal basis. Show that the trace of a matrix does depend on the basis.

Exercise 3. A geometric example. Consider the case where the basis $(e') = \left(\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\right)$ is obtained from $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ by a rotation of axis $\vec{e_3}$ and of angle θ.
a) Draw the two bases, and use trigonometry to write the components of vectors $\left(\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\right)$ in the basis (e).
b) Consider a vector $\vec{x}$ in the two-dimensional space spanned by $(\vec{e_1}, \vec{e_2})$. Add this vector to your drawing. Call ϕ the angle between $\vec{e_1}$ and $\vec{x}$. Write the components of $\vec{x}$ in the basis (e) and in basis (e'), in terms of the angles θ and ϕ, again using trigonometry (not the formulas from the course).
c) Now use the formulas from the course: write the transformation matrix in terms of θ. Take the coordinates of $\vec{x}$ in the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ (written in terms of ϕ), and apply the transformation rules of tensors of order 1 to find the coordinates of $\vec{x}$ in basis $\left(\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\right)$. Compare with the result of b).

Exercise 4. Euclidean geometry: change of orthonormal basis and orthogonal transformations. Let u be an endomorphism of $\mathbf{R}^N$ (we will be interested in the case $N = 3$, but the results of this exercise hold for any finite dimension N). Let $(e) = (\vec{e_1}, \vec{e_2}, \ldots, \vec{e_N})$ be an orthonormal basis of $\mathbf{R}^N$, and let the scalar product be the bilinear application defined by the relations

$$\forall i, j \in [1..N], \quad \vec{e_i}.\vec{e_j} = \delta_{ij}. \tag{1.38}$$

Define the norm $N(\vec{x})$ of any vector $\vec{x}$ as follows:

$$N : \mathbf{R}^N \longrightarrow \mathbf{R}_+, \quad N(\vec{x}) = \sqrt{\vec{x}.\vec{x}} \tag{1.39}$$

1) Prove that if u preserves the norm, then it preserves the scalar product, meaning: if $N(u(\vec{x})) = N(\vec{x})$ for all vectors $\vec{x}$, then $u(\vec{x}).u(\vec{y}) = \vec{x}.\vec{y}$ for all vectors $\vec{x}$ and $\vec{y}$.
2) Show that u preserves the norm if and only if it maps an orthonormal basis to an orthonormal base.
3) Show that u preserves the norm if and only the inverse of its matrix U in the orthonormal basis (e) is its transpose, $U^{-1} = U^T$.
4) Consider the special case $N = 3$, two bases (e) and (e') of $\mathbf{R}^3$, and the transformation matrix P described in the course. Interpret this matrix geometrically (to what endomorphism of $\mathbf{R}^3$ does P correspond, and in what base?). Show that it preserves the norm, and deduce that the inverse of P is P^T.

Exercise 5 (exam question, June 2015, one of five questions in the three-hour exam). In this question all tensors are defined on the space $\mathbf{R}^3$ endowed with an orthonormal basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$. The position of a point M is described in Cartesian coordinates by a vector $\vec{x} = \overrightarrow{OM} = x_i \vec{e_i}$.

(i) [**10 marks**] Compute the following quantities :
(a) δ_{ii}
(b) $\partial_k x_k$
(c) $\partial_k x_k \partial_j x_j$
(d) $\partial_k x_j \partial_j x_k$

(ii) [**5 marks**] Which of the following expressions (if any) are inconsistent with the sum rule over repeated indices?
(a) $A_i = \epsilon_{ijk} C_j D_k$
(b) $A_i B_k = \epsilon_{ijk} C_j D_k$
(c) $C_{ij} C_{ij} C_{ai}$

(iii) [**5 marks**] Recall the definition of an isotropic Cartesian tensor of rank n (in dimension three). Prove that there are no non-zero isotropic Cartesian tensors of rank one.