XJTLU, MTH308 [Cartesian tensors and mathematical models of (elastic) solids and viscous fluids], Semester 2, 2015

Lecture 3, 17th March 2015: the stress tensor

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Contents

1 Volume forces and surface forces, the Cauchy postulate

A *volume force* is a force that acts acts on a sample of matter, and is proportional to the volume of the sample. Gravity is an example of volume force. On a small volume dV centered[1] at point $\vec{x}$, a volume

[1]in this chapter, we take a coordinate system on $\mathbf{R}^3$ with an origin O and an orthonormal base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, so we can identify the vector $\vec{x} = x_i \vec{e_i}$ to the point M such that $\overrightarrow{OM} = x_i \vec{e_i}$. Moreover, we

force is written as the product of dV by the intensity $\overrightarrow{f^{vol}}(\vec{x})$ of the force per unit volume:

$$\overrightarrow{f^{vol}}(\vec{x})dV. \tag{2}$$

I want to refer to my last Eq. 2.

In the case of gravity, $\overrightarrow{f^{vol}}(\vec{x}) = \rho(\vec{x})\vec{g}$, where ρ is the density of the material, and $\vec{g}$ is the gravitational field.

A *surface force* is a force acting on the surface of a sample of matter, which is proportional to the area of the surface. Cauchy postulated (in the early 19th century) that the forces that maintain the cohesion of a continuous medium are surface forces. On a small surface[2] centered at point $\vec{x}$, with a unit normal vector $\vec{n}$, such a force is written as

$$\vec{T}(\vec{x}, \vec{n})dS, \tag{3}$$

expressing the fact the force is proportional to the surface, depends on its location (through the variable $\vec{x}$) and on its orientation (through the variable $\vec{n}$). Let us consider a continuous medium at equilibrium occupying the whole space (we are not looking at boundaries yet), and isolate (in thought) a domain $\mathcal{V}$ with boundary $\mathcal{S}$ (oriented towards the exterior of the volume, meaning the normal vector at every point of the surface points towards the exterior of the domain). If the domain $\mathcal{V}$ is at equilibrium, the volume and surface forces add up to zero:

$$\boxed{\vec{0} = \iiint_{\mathcal{V}} \overrightarrow{f^{vol}}(\vec{x})dV + \oint\!\!\!\oint_{\mathcal{S}\to\text{ext}} \vec{T}(\vec{x}, \vec{n})dS.} \tag{4}$$

2 Stokes' theorem: reminder of the scalar version, tensor version

From calculus and physics you are probably familiar with Stokes' theorem written in the following form:

$$\iiint_{\mathcal{V}} \frac{\partial \phi_i(\vec{x})}{\partial x_i} dV = \oint\!\!\!\oint_{\mathcal{S}\to\text{ext}} \phi_i(\vec{x}) n_i(\vec{x}) dS \tag{5}$$

where $\vec{\phi}$ is a vector field (meaning there is a vector $\vec{\phi}(\vec{x})$ at every point $\vec{x}$ in space).

This can be generalised to higher-order tensors. For example, if τ is a tensor with two indices, the following equations (one equation per value of j) hold:

$$\forall j \in \{1, 2, 3\}, \quad \iiint_{\mathcal{V}} \frac{\partial \tau_{ji}(\vec{x})}{\partial x_i} dV = \oint\!\!\!\oint_{\mathcal{S}\to\text{ext}} \tau_{ji}(\vec{x}) n_i(\vec{x}) dS. \tag{6}$$

Fixing index j, one can just apply Stokes' theorem to the vector $\vec{\phi}$ with components $\phi_1 = \tau_{j1}$, $\phi_2 = \tau_{j2}$, $\phi_3 = \tau_{j3}$, and obtain Eq. 6. See the discussion of the Cauchy tetrahedron in the next section for a geometric application.

$$otherequation. \tag{1}$$

use the sum rule over repeated indices.

[2]By *unit normal vector* one means a vector that is orthogonal to the surface and has norm 1, i.e. if $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ is an orthonormal base and $\vec{n} = n_i\vec{e_i}$, one has $n_i n_i = 1$.

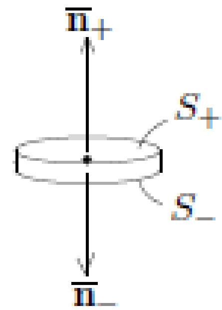

Figure 1: A thin cylinder at equilibrium.

3 Surface forces are a linear function of the normal vector to the surface

In this section we are going to expose the argument due to Cauchy (about 1820), to prove that the vector $\vec{T}(\vec{x}, \vec{n})$ is a linear function of the normal vector $\vec{n} = n_i \vec{e_i}$, meaning:

$$\vec{T}(\vec{x}, \vec{n}) = n_i \vec{T}(\vec{x}, \vec{e_i}). \tag{7}$$

For definiteness let us consider that the volume forces consist of gravity:

$$\overrightarrow{f^{vol}}(\vec{x}) = \rho(\vec{x})\vec{g}, \tag{8}$$

where ρ is the density of the continuous medium (note that it depends on the position $\vec{x}$, since the continuous medium is not necessarily homogeneous), but the result would hold for any bounded volume force.

3.1 Balance equation of a thin cylinder

First we can establish the following lemma:

$$\vec{T}(\vec{x}, -\vec{n}) = -\vec{T}(\vec{x}, \vec{n}). \tag{9}$$

Let us consider a cylinder of height l, and of radius a, cut by a thought experiment inside a continuous medium as in Fig. 1. Let us say that the center of the upper disk is at point M, with $\overrightarrow{OM} = \vec{x}$, and that the unit normal vector to the upper disk pointing towards the exterior coincides with $\vec{n}$. Let $\vec{n}_+$ and $\vec{n}_-$ denote the unit normal vectors to the upper and lower disks. These two unit vectors are opposite to each other:

$$\vec{n}_+ = -\vec{n}_-. \tag{10}$$

If a is small enough (but fixed), the dependence of $\vec{T}$ in its first argument can be neglected, the density can be considered uniform (as $\vec{T}$ and ρ are assumed to be smooth functions), and the equilibrium of the cylinder inside the continuous medium are written as:

$$\vec{0} = \rho \vec{g} \pi a^2 l + \left(\vec{T}(\vec{x}, \vec{n}_+) + \vec{T}(\vec{x} - l\vec{n}_+, \vec{n}_-)\right) \pi a^2 + \oiint_{lateral} \vec{T}(\vec{x}, \vec{n}_{lateral}) dS, \tag{11}$$

where the last term in the r.h.s is the integral over the lateral area of the cylinder, which goes to $\vec{0}$ when l goes to zero (because the integrand is bounded and the lateral surface goes to zero). The first term also goes to zero when l goes to zero, whereas the second term (which is the integral of forces on surfaces

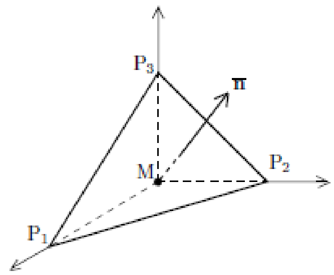

Figure 2: The Cauchy tetrahedron.

S_+ and S_- of Fig. 1) goes to $\left(\vec{T}(\vec{x},\vec{n}_+)+\vec{T}(\vec{x},\vec{n}_-)\right)\pi a^2$. As $\vec{n}_+ = -\vec{n}_-$, the limit of the equilibrium condition of the cylinder when l goes to zero becomes:

$$\vec{T}(\vec{x},-\vec{n}_+) = -\vec{T}(\vec{x},\vec{n}_+), \tag{12}$$

which is just Eq. 9.

3.2 Balance equation of a (small) tetrahedron

Let us call M the point described by vector $\vec{x}$ in our reference system: $\overrightarrow{OM} = \vec{x} = x_i\vec{e_i}$. Let us consider a tetrahedron with one corner at M, and the three other corners P_1, P_2, P_3, such that the vectors $\overrightarrow{MP_i}$ are parallel to the axes of the orthonormal base $(\vec{e_1},\vec{e_2},\vec{e_3})$:

$$\overrightarrow{MP_1} = \epsilon y_1\vec{e_1}, \quad \overrightarrow{MP_2} = \epsilon y_2\vec{e_2}, \quad \overrightarrow{MP_3} = \epsilon y_3\vec{e_3}, \tag{13}$$

and ϵ For commodity the three parmeters y_i, for i in $\{1,2,3\}$ have been chosen to be positive (this can always be ensured upon choosing the axes, see Fig. 2). The parameter ϵ is a positive number which we will eventually let go to zero[3], to use the fact that the volume forces scale as ϵ^3, while the surface forces scale as ϵ^2. The tetrahedron is filled by the continuous medium and it is at equilibrium, so if we take $\mathcal{V}$ to be the tetrahedron and $\mathcal{S}$ to be the surface of the tetrahedron, we have the following equation:

$$\iiint_{\mathcal{V}} \overrightarrow{f^{vol}}(\vec{y})dV + \oiint_{\mathcal{S}\to\text{ext}} \vec{T}(\vec{y},\vec{n}(\vec{y}))dS = \vec{0}. \tag{14}$$

As $\vec{x}$ is reserved as a notation for the fixed vector $\overrightarrow{OM} = \vec{x}$ in this section, we use $\vec{y}$ as the integration variable (describing the position of points inside the tetrahedron). To complete the argument we need to do some geometry.

The surface integral consists of four terms, one per face of the tetrahedron. The unit normal vectors (oriented towards the exterior of the tetrahedron) to the faces MP_1P_2, MP_2P_3 and MP_1P_3 are the vectors $-\vec{e_3}$, $-\vec{e_1}$ and $-\vec{e_2}$ respectively, and the areas of these faces are $\epsilon^2(y_1y_2)/2$, $\epsilon^2(y_2y_3)/2$ and $\epsilon^2(y_1y_3)/2$ respectively. Let us denote by $\vec{n} = n_i\vec{e_i}$ the unit normal vector to the oblique face $P_1P_2P_3$ oriented towards the exterior of the tetrahedron (see Fig. 2), and by S_ϵ its area. By applying Pythagoras theorem one can show that $S_\epsilon = \epsilon^2 S(1)$ (where $S(1)$ is the value of the area of the oblique face when $\epsilon = 1$, and depends only on y_1, y_2, y_3).

Let us denote by C the center of the oblique face $P_1P_2P_3$, and by C_3, C_1, C_2 the centers of faces MP_1P_2, MP_2P_3 and MP_1P_3. The volume $V_{MP_1P_2P_3}$ of the tetrahedron equals a third of the product of

[3] ϵ has no unit, it is just a number, so when ϵ goes to zero the tetrahedron becomes very small, and all its points become very close to M, but the tetrahedron keeps the same shape.

the surface of the base by the height (the numerical factor of 1/3 is not so important, we will be mostly interested in the power of ϵ):

$$V_{MP_1P_2P_3} = \frac{1}{3} \times \frac{\epsilon y_1 \times \epsilon y_2}{2} \times \epsilon y_3 = \epsilon^3 \frac{y_1 y_2 y_3}{6}, \tag{15}$$

We are going to use the continuity of $\overrightarrow{f^{vol}}$ and $\vec{T}$ w.r.t. the variable $\vec{x}$ (all forces are always assumed to have derivatives, in particular they are continuous), as the points C, C_1, C_2 and C_3 all go to M when ϵ goes to zero. At leading order in ϵ, the volume term in the balance equation is therefore:

$$\iiint_{\mathcal{V}} \overrightarrow{f^{vol}}(\vec{y}) dV = \overrightarrow{f^{vol}}(\overrightarrow{OM}) \frac{y_1 y_2 y_3}{6} \epsilon^3 + o(\epsilon^3), \tag{16}$$

while the surface term reads

$$\oiint_{\mathcal{S}\to\text{ext}} \vec{T}(\vec{y}, \vec{n}) dS = \left(\vec{T}(\overrightarrow{OM}, \vec{n}) S(1) + \vec{T}(\overrightarrow{OM}, -\vec{e_1}) \frac{y_2 y_3}{2} + \vec{T}(\overrightarrow{OM}, -\vec{e_2}) \frac{y_1 y_3}{2} + \vec{T}(\overrightarrow{OM}, -\vec{e_3}) \frac{y_1 y_2}{2} \right) \epsilon^2 + o(\epsilon^2). \tag{17}$$

Because the normal vector $\vec{n}$ does not depend on ϵ. So the leading order in ϵ of the balance equation is just a linear combination of values of the function $\vec{T}$ at the same point in space, but different value of the normal vector:

$$\vec{T}(\overrightarrow{OM}, \vec{n}) S(1) + \vec{T}(\overrightarrow{OM}, -\vec{e_1}) \frac{y_2 y_3}{2} + \vec{T}(\overrightarrow{OM}, -\vec{e_2}) \frac{y_1 y_3}{2} + \vec{T}(\overrightarrow{OM}, -\vec{e_3}) \frac{y_1 y_2}{2} = \vec{0}. \tag{18}$$

Let us express the vector $\vec{n}$ in terms of our geometric data. We can apply Stokes' theorem (Eq. 6) to the Kronecker symbol: $\tau_{ij}(\vec{x}) = \delta_{ij}$ (the value is uniform, independent from $\vec{x}$), and to the tetrahedron. The integral over the volume is zero because the tensor has no dependence over the point, and the surface integral is just the integral of the normal vector over the surface, which can readily be expressed in terms of our notations:

$$\vec{0} = S(1)\vec{n} - \frac{1}{2} \left(y_2 y_3 \vec{e_1} + y_1 y_3 \vec{e_2} + y_1 y_2 \vec{e_3} \right). \tag{19}$$

Taking dot-products of this equation with vectors $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$, one obtains the following relations between the components of the vector $\vec{n}$ and the geometric parameters of the problem:

$$S(1) n_1 = \frac{1}{2} y_2 y_3, \quad S(1) n_2 = \frac{1}{2} y_1 y_3, \quad S(1) n_3 = \frac{1}{2} y_1 y_2. \tag{20}$$

Hence we can rewrite Eq. 18, letting components of $\vec{n}$ appear:

$$\vec{0} = \vec{T}(\overrightarrow{OM}, \vec{n}) S(1) + \left(\vec{T}(\overrightarrow{OM}, -\vec{e_1}) S(1) n_1 + \vec{T}(\overrightarrow{OM}, -\vec{e_2}) S(1) n_2 + \vec{T}(\overrightarrow{OM}, -\vec{e_3}) S(1) n_3 \right) \tag{21}$$

Dividing by $S(1)$ and applying Eq. 9 we obtain:

$$\vec{0} = \vec{T}(\overrightarrow{OM}, \vec{n}) - \vec{T}(\overrightarrow{OM}, \vec{e_1}) n_1 - \vec{T}(\overrightarrow{OM}, \vec{e_2}) n_2 - \vec{T}(\overrightarrow{OM}, \vec{e_3}) n_3. \tag{22}$$

Remembering that $\vec{x} = \overrightarrow{OM}$, we can rewrite this as:

$$\vec{T}(\vec{x}, \vec{n}) = n_j \vec{T}(\vec{x}, \vec{e_j}), \tag{23}$$

which is the linearity in the vector $\vec{n}$ we wanted to establish. The matrix of the linear application is called the stress tensor (it depends on the point $\vec{x}$, so it can be called a tensor *field*). It is denoted by $\sigma(\vec{x})$:

$$\boxed{\vec{T}(\vec{x}, n_j \vec{e_j}) = \sigma_{ij}(\vec{x}) n_j \vec{e_i}.} \tag{24}$$

4 Balance equations for a continuous medium

4.1 Forces sum to zero

Even though we have started from an equilibrium equation consisting of a volume integral and a surface integral, we can rewrite it as a volume integral thanks to the linearity of force surfaces in the normal (the existence of the stress tensor), and Stoke's theorem. Indeed, Eq. 24 allows us to rewrite the equilibrium condition of Eq. 4 as

$$\iiint_{\mathcal{V}} f_i^{vol}(\vec{x})dV + \iint_{\mathcal{S}\to\text{ext}} \sigma_{ij}(\vec{x})n_j(\vec{x})dS = 0, \tag{25}$$

where $f_i^{vol} = \overrightarrow{f^{vol}}.\vec{e_i}$ is the component of the volume force density along vector $\vec{e_i}$, and Stoke's theorem yields

$$\iiint_{\mathcal{V}} \left(f_i^{vol}(\vec{x}) + \frac{\partial \sigma_{ij}}{\partial x_j}(\vec{x}) \right) dV \vec{e_i} = \vec{0}. \tag{26}$$

As the above equation holds for any volume V, the integrand is zero, and the following equilibrium equations hold:

$$\boxed{f_i^{vol} + \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \forall i \in \{1,2,3\}} \tag{27}$$

4.2 Momenta of forces sum to zero, hence the stress tensor is symmetric

Let us express write that the momenta of volume forces and surface forces at the origin O sum to zero:

$$\iiint_{\mathcal{V}} \overrightarrow{OM} \wedge \overrightarrow{f^{vol}}(\vec{x})dV + \oiint_{\mathcal{S}\to\text{ext}} \overrightarrow{OM} \wedge \vec{T}(\overrightarrow{OM}, \vec{n})dS = \vec{0}. \tag{28}$$

Let us denote by ϵ the order-three tensor that is totally antisymmetric in all its indices[4] and has $\epsilon_{123} = 1$ (**NB**: in the present context, the quantity ϵ, which is an order-three tensor, is not to be confused with the scaling parameter used in the context of the Cauchy tetrahedron, which is a single number, and was denoted by the same symbol). One can easily check as an exercise that the components of the vector product in three dimensions are expressed by taking the tensor product of vectors with the totally antisymmetric tensor:

$$(x_j\vec{e_j}) \wedge (y_k\vec{e_k}) = (\epsilon_{ijk}x_jy_k)\vec{e_i}. \tag{29}$$

This enables us to rewrite the integrands of Eq. 28 in terms of the components x_i of the vector $\overrightarrow{OM} = x_i\vec{e_i}$. The i-th component of Eq. 28 takes the following form:

$$\iiint_{\mathcal{V}} \epsilon_{ijk}x_j f_k^{vol}(\vec{x})dV + \oiint_{\mathcal{S}\to\text{ext}} \epsilon_{ijk}x_j\sigma_{kl}(\vec{x})n_l dS = 0. \tag{30}$$

Stokes' theorem can be applied to the surface integral:

$$\iiint_{\mathcal{V}} \left(\epsilon_{ijk}x_j f_k^{vol}(\vec{x}) + \frac{\partial}{\partial x_l}\left(\epsilon_{ijk}x_j\sigma_{kl}(\vec{x})\right) \right) dV = 0. \tag{31}$$

The derivative w.r.t. x_l gives rise to two terms:

$$\iiint_{\mathcal{V}} \left(\epsilon_{ijk}x_j f_k^{vol}(\vec{x}) + \epsilon_{ijk}\delta_{jl}\sigma_{kl}(\vec{x}) + \epsilon_{ijk}x_j \frac{\partial \sigma_{kl}(\vec{x})}{\partial x_l} \right) dV = 0, \tag{32}$$

[4] meaning for all values of indices i, j, k, any permutation of two indices flips the sign of the entry: $\epsilon_{ijk} = -\epsilon_{jik}$, $\epsilon_{ijk} = -\epsilon_{ikj}$, $\epsilon_{ijk} = -\epsilon_{jik}$, which implies that all entries of ϵ equal 0 (when at least two indices are equal), 1 (if the indices are obtained from $(1,2,3)$ by an even number of permutations), or -1 (if the indices are obtained from $(1,2,3)$ by an odd number of permutations).

and the equilibrium of forces (Eq. 27) can be used to reduce the integrand to just one term:

$$0 = \iiint_{\mathcal{V}} \left(\epsilon_{ijk} x_j \left(f_k^{vol}(\vec{x}) + \frac{\partial \sigma_{kl}(\vec{x})}{\partial x_l} \right) + \epsilon_{ijk} \sigma_{kj}(\vec{x}) \right) dV = \iiint_{\mathcal{V}} \epsilon_{ijk} \delta_{jl} \sigma_{kl} dV. \tag{33}$$

Since this is true for any domain $\mathcal{V}$, the integrand must be zero at every point $\vec{x}$, hence:

$$\epsilon_{ijk} \sigma_{kj}(\vec{x}) = 0 \tag{34}$$

This equation implies that σ is symmetric. By the definition of the antisymmetric three-tensor ϵ, if one considers $i = 1$ for instance, Eq. 34 only has terms corresponding to $j, k \in \{2, 3\}$, and becomes $\sigma_{23} - \sigma_{32} = 0$. For the same reason, considering $i = 2$ gives rise to $-\sigma_{13} + \sigma_{13} = 0$, and considering $i = 3$ gives rise to $\sigma_{23} - \sigma_{32} = 0$. So we have established the following property as a a consequence of the equilibrium of momenta

$$\boxed{\sigma_{ji} = \sigma_{ij}, \quad \forall i, j \in [1..3].} \tag{35}$$