XJTLU, MTH308 [Cartesian tensors and mathematical models of (elastic) solids and viscous fluids], Semester 2, 2015

Lecture 5, 31st March, 2015: Linear elasticity

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Contents

In this lecture we will be interested in small, reversible deformations of solids. We will use Hooke's law to propose a relation between the strain and stress tensors in terms of two parameters: Young's modulus and Poisson's ratio, which are both measurable from traction experiments on elastic solids.

1 Hooke's law

Traction experiments show that the relative deformation of a cylinder of length L under traction is proportional to the force applied on the ends per unit surface as long as δL is small enough for the transformation to be reversible. This linear relation is known as Hooke's law[1] that has to be applied to

[1] The British physicist and polymath Robert Hooke (1635-1703) first proved this law in the case of springs (in Latin *ut tension sic vis*, or *such tension, such force*, see Fig. 1): for a spring δL is proportional to the tension (*tensio*) of the spring, and F is the force (*vis*). Generalizations of this proportionality law to elastic solids, and their formulation in terms of tensor, are still referred to as Hooke's law.

the spring to produce this tension. The proportionality coefficient E is known as Young's modulus[2] (or elastic modulus), and has the dimension of a pressure:

$$\frac{F}{S} = E\frac{\delta L}{L}. \tag{1}$$

If F is positive one has a *traction*, if it is negative one has a *compression*. For definiteness let us assume that we have a traction. During a traction experiment, the cylinder is going to become thinner, and its diameter can be measured. This effect will be described in the next section by another parameter called Poisson's ratio[3], and denoted by ν. If a denotes the initial radius of the cylinder, and δa the variation of the radius under traction, then ν is the number such that:

$$\frac{\delta a}{a} = -\nu\frac{\delta L}{L}. \tag{2}$$

Linear elasticity describes transformation of solids that are sufficiently small to be reversible: if the force F is decreased to zero after a traction experiment *in the linear regime*, the material comes back to its initial state. The amplitude of the linear regime depends on the material and on the nature of the transformation (for example for steel $\delta L/L$ is of the order of a few thousands in the linear regime, whereas for rubber it can be a few percents), and it has to be determined by experiment, just as Young's modulus. If F is increased beyond the elasticity regime, the is a second regime in which the force can be decreased, while δL decreases without going back to zero (there is a *remanent deformation*). For even larger values of F there is breakage. The scope course of this course is confined to linear elasticity.

Remarks on gravity, linearity and orders of magnitude. In the above discussion we did not specify the direction of the applied forces: we just described them are traction forces, i.e. they are parallel to the axis of the cylinder, i.e. $\vec{F} = F\vec{n}$ at both ends of the cylinder, with $F > 0$ for a traction ($F < 0$ would be a compression), but the normal vector $\vec{n}$ could be horizontal or vertical. This means we neglected gravity (and we will often neglect it in the context of linear elasticity). There are two reasons to neglect gravity:

1. We are interested in small deformations from an initial equilibrium state to a final equilibrium state, and the equilibrium equations in the initial state include gravity and the forces that balance it (for example the reaction of a table if the cylinder rests horizontally on a table, or the reaction of the ceiling if the cylinder is attached to the ceiling). So we begin our work when gravity has already been taken into account, and we look for variations of the stress tensor with respect to this state.
2. For many solids of interest, the value of Young's modulus makes gravity forces smaller than traction forces by orders of magnitude, even for small deformations. In the case of steel, $E \simeq 200$GPa, and the traction regime is linear for $0 \leq \delta L/L \leq 0.5 \times 10^{-3}$, so if we consider a cylinder of steel with mass $m = 1$kg and $S = 10\text{cm}^2$, the effect of gravity by unit of surface is about

$$\frac{mg}{S} = \frac{1 \times 10}{10^{-3}} = 10^4\text{Pa},$$

and that of the traction force per unit surface for $\delta L/L \leq 0.5 \times 10^{-3}$ is about:

$$\frac{F}{S} = E\frac{\delta L}{L} = 2 \times 10^{11} \times 5 \times 10^{-4} = 10^8\text{Pa},$$

or 10^4 times larger.

[2]Named after the British physicist Thomas Young (1773-1829), a universal mind who was also a physician and a linguist, and left his name to the Young slits experiment in optics (which was crucial to the development of the study of interferences of light in the 19th century, and again for the interference between electron and atom beams in the 20th century, leading to the development of quantum mechanics).

[3]Named after Siméon Denis Poisson (1781-1840), French mathematician and physicist who also attached his name to the Poisson law in probability

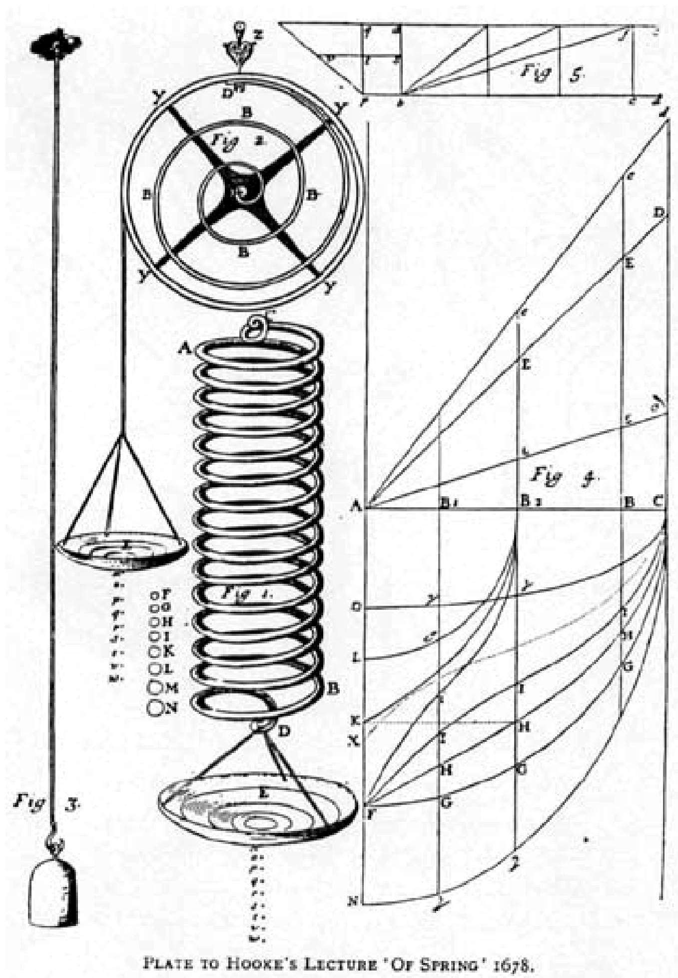

Figure 1: A figure from Hooke's original treaty. The applied force is proportional to the masses put in the plate of the scale. In this chapter we are substituting an elastic solid to the spring, and translating this figure into the language of tensors.

2 Linearized strain tensor

In Lecture 2 we described the kinematics of continuous media in terms of a flow $\vec{\Phi}$ that maps the initial coordinates $X_i\vec{e_i}$ of material points in a continuous medium to a new position at time t, with coordinates $\Phi_i(\vec{X},t)\vec{e_i}$. In the linear regime where Hooke's law (Eq. 1) holds, deformations are small in scale of the original dimensions of the material, so we will express them in terms of the displacement fields u_i defined in terms of the flow by subtracting the identical flow:

$$u_i(\vec{X},t) = \Phi_i(\vec{X},t) - X_i, \tag{3}$$

so that vector fields (see the chapter on kinematics are transformed as follows by the flow:

$$\vec{h} \mapsto (T_{ij}h_j)\,\vec{e_i}, \tag{4}$$

with

$$T_{ij} = \frac{\partial \Phi_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}. \tag{5}$$

We are going to assume that the components u_i for all i are small, and slowly varying over space (i.e. all their derivatives are assumed to be small). Let us expand the deformed dot-product of vectors h and h' in powers of u:

$$\boxed{\begin{array}{lll} \vec{h}.\vec{h'} \mapsto & (T_{ij}h_j\vec{e_i})\,.\,(T_{lk}h'_k\vec{e_l}) & = h_jh'_kT_{ij}T_{lk}\delta_{il} \\ addmyequation & = h & intothislist \\ & & = h_jh'_k\left(\delta_{ij}+\frac{\partial u_i}{\partial X_j}\right)\left(\delta_{ik}+\frac{\partial u_i}{\partial X_k}\right) \\ & & = h_jh'_k\left(\delta_{ij}+\frac{\partial u_i}{\partial X_j}\right)\left(\delta_{ik}+\frac{\partial u_i}{\partial X_k}\right) \\ & & = h_jh'_k\left(\delta_{ij}\delta_{ik}+\delta_{ij}\frac{\partial u_i}{\partial X_k}+\frac{\partial u_i}{\partial X_j}\delta_{ik}+\frac{\partial u_i}{\partial X_j}\frac{\partial u_i}{\partial X_k}\right) \\ & & = h_jh'_k\left(\delta_{jk}+\frac{\partial u_j}{\partial X_k}+\frac{\partial u_k}{\partial X_j}+\frac{\partial u_i}{\partial X_j}\frac{\partial u_i}{\partial X_k}\right). \end{array}}$$

Hence we can express the first-order terms in $\vec{u}$ (and its derivatives) in terms of the deformation tensor ϵ defined as:

$$\boxed{\epsilon_{ij}(\vec{X},t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)}, \tag{6}$$

and we find:

$$\boxed{\vec{h}.\vec{h'} \mapsto \vec{h}.\vec{h'} + 2\epsilon_{ij}h_ih'_j + o(\vec{u}).} \tag{7}$$

Example: traction of a cylinder. In the special case of the traction on a cylinder, the variation of length δL is proportional to the length, so for a cylinder of initial length αL, with the same base surface S and the same traction F, the length would vary by $\alpha\delta L$. One can express this proportionality rule by writing the displacement field in the direction $\vec{e_3}$ as:

$$u_3 = \frac{X_3}{L}\delta L,$$

where the factor X_3/L plays the role of the factor α. Hence we can express one component of the linearized strain tensor:

$$\epsilon_{33} = \frac{\delta L}{L}.$$

If we look at a section of the cylinder by a plane orthogonal to its axis, we have a disk whose radius is $a = \sqrt{S/\pi}$ when $F = 0$. This radius will be $a + \delta a$ when the traction F is applied, with $\delta a < 0$, and by the same reasoning as for the direction X_3 one can convince oneself that the displacement is linear in both X_1 and X_2 directions:

$$u_1 = \frac{X_1}{a}\delta a,$$

$$u_2 = \frac{X_2}{a}\delta a,$$

so that in particular,

$$\epsilon_{11} = \epsilon_{22} = \frac{\delta a}{a}.$$

Instead of describing this situation in terms of the radius a, which depends on the geometry, one chooses to describe it in terms of the ratio between components of the strain tensor. In terms of Poisson's ratio (the parameter ν defined in Eq. 2), for the cylinder in traction, and small deformations, we find:

$$\epsilon_{11} = \epsilon_{22} = -\nu\epsilon_{33}.$$

Poisson's ratio depends on the material, and has no physical dimension. One can compute its value for an incompressible material (exercise, see tutorial), and measure it in traction expreriments as it equals the relative variation of the radius (for example $\nu \simeq 0.28$ for steel, and $\nu = 0.2$ for concrete).

3 Deformations as a function of stress

For traction forces acting on the ends of a cylinder, without volume forces, the following uniaxial stress tensor is statically admissible (see Tutorial 4):

$$\sigma(\vec{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{F}{S} \end{pmatrix}$$

Hooke's law can therefore be rewritten as the following relation between the entry ϵ_{33} of the tensor ϵ and the entry with the same indices of the stress tensor:

$$\epsilon_{33} = \frac{1}{E}\sigma_{33}. \tag{8}$$

But this is not a relation between ϵ and σ as matrices. We can proceed by trial-and-error and ask: *"What if the relation 8 held between all the entries of ϵ and σ?"* We would have

$$\epsilon_{ij} \stackrel{?}{=} \frac{1}{E}\sigma_{ij}, \tag{9}$$

which is the simplest generalization of Eq. 8 to the tensors ϵ and σ. But in that case, the components ϵ_{11} and ϵ_{22} of the deformation tensor would be zero (because σ is uniaxial, Eq. 3), therefore the cylinder would not become thinner under traction, which would be in contradiction with our prediction 2, as we know Poisson's ratio is measured to be different from zero.

To account for the transverse thinning of the cylinder under traction, we therefore need to add other terms to the r.h.s of 8. Linearity and isotropy impose that these terms should have the same eigenvectors as σ (for any choice of σ), and should be a linear function of σ. A natural way to satisfy these two conditions is to add a tensor proportional to the identity matrix, with the trace of σ as a coefficient. We now have two parameters, call them E_1 and E_2:

$$\epsilon_{ij} \stackrel{?}{=} \frac{1}{E_1}\sigma_{ij} + \frac{1}{E_2}(\mathrm{Tr}\sigma)\delta_{ij}. \tag{10}$$

Again we ask: *"What if ϵ was given by Eq. 10?"*
For the cylinder in pure traction, the l.h.s. of 10 equals $F/(ES)$ by Hooke's law and the r.h.s. equals $F/(E_1S) + F/(E_2S)$ from the expression of the uniaxial tensor (Eq. 3). In order to recover Hooke's law (Eq. 8) for the cylinder in pure traction, we must have

$$\frac{1}{E} = \frac{1}{E_1} + \frac{1}{E_2}. \tag{11}$$

Again for the cylinder in pure traction, we can express the diagonal terms of the deformation tensor from Eq. 10 as $\epsilon_{11} = F/(E_2 S)$ and $\epsilon_{22} = F/(E_2 S)$, but from our experimental definition of Poisson's ratio, they must equal $-\nu\epsilon_{33} = -\nu F/(ES)$. Hence the relation

$$-\frac{\nu}{E} = \frac{1}{E_2}. \tag{12}$$

By plugging Eq. 12 into Eq. 11, we can express the parameters E_1 and E_2 in terms of the measurable quantities ν and E, and conclude that the proposed form Eq. 10 is compatible with observations in the linear regime of the traction experiment, provided the stress and strain tensors are related by the following equation:

$$\boxed{\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\left(\mathrm{Tr}\sigma\right)\delta_{ij}}. \tag{13}$$

We will assume from now on that this educated guess gives us the relation between strain and stress in the linear regime of deformation, for all elastic solids, and not only in the case of pure traction but also for more general systems of applied forces (see tutorials for tests of this idea in the cases of shear deformation and isotropic pressure).

4 Stress as a function of deformation

So far we have expressed the deformation tensor ϵ as a function of the stress tensor σ. In order to link dynamics to kinematics we can invert the tensor form of Hooke's law (Eq. 13), and obtain the expression of the stress tensor as a function of the deformation tensor.

As Hooke's law can be rewritten as

$$\sigma_{ij} = \frac{E}{1+\nu}\epsilon_{ij} + \frac{\nu}{1+\nu}\left(\mathrm{Tr}\sigma\right)\delta_{ij}, \tag{14}$$

all we need to do is to express $(\mathrm{Tr}\sigma)$ as a function of ϵ Let us take the trace of Eq. 13:

$$\mathrm{Tr}\epsilon = \frac{1+\nu}{E}\mathrm{Tr}\sigma - \frac{\nu}{E}\left(\mathrm{Tr}\sigma\right)\delta_{ii}. \tag{15}$$

As $\delta_{ii} = 3$, we obtain:

$$\mathrm{Tr}\epsilon = \frac{1-2\nu}{E}\mathrm{Tr}\sigma, \tag{16}$$

hence we obtain the following expression for the stress tensor:

$$\boxed{\sigma_{ij} = \frac{E}{1+\nu}\epsilon_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)}\mathrm{Tr}\epsilon\delta_{ij}.} \tag{17}$$