XJTLU, MTH308 (Cartesian tensors and mathematical models of (elastic) solids and viscous fluids), Semester 2, 2015

Lecture 2, 10th March, 2015: Kinematics and deformations

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Convention. Unless otherwise stated, the sum rule over repeated indices is observed.

Contents

In this chapter we will develop the mathematical the description of movements of continuous media (which can be elastic solids or fluids) without studying their causes. This is the kinematics of continuous media. A function of *initial position and time* is used to describe the motion (or *flow*) of the continuous medium. To decide whether the medium is deformed, we have to compute how scalar products of vectors change under the flow (see Fig. 1.1 for a physical example on a solid: we are going to describe mathematically how the grid on the l.h.s. of the figure is deformed).
Technically, the first chapter was entirely based on linear algebra (we learned how the components of a fixed object such as a vector or the matrix of a linear application transform under a change of base). This chapter will be mostly based on differential geometry: we will work with a fixed base, but study the tangent vectors to moving curves.

1 Lagrangian description of flows

1.1 Definitions and notations

To avoid boundary effects, let us study a sample of continuous medium (solid or fluid) that "fills the entire space" (meaning we are describing matter at scales that are large compared to the size of molecules, but we are far away from the boundaries of the sample).

The space $\mathbf{R}^3$ is endowed with an orthonormal base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ (which will be kept fixed in this chapter) and a fixed origin O, so the position of a material point M can be described by its coordinates, $\overrightarrow{OM} = \vec{x} = x_i \vec{e_i}$. We are interested in the way classical particles (small lumps of continuous medium, which could be visualized using tracers in a fluid, or simply by marking the surface by a grid on a solid, as in Fig. 1.1) move over time. Let us say that a particle is at position $\vec{X}$ at time $t = 0$ (one says that $\vec{X}$ is the initial position of the particle). Let us say we are interested in the deformation of the medium between time $t = 0$ and some final time T (the duration of an experiment for instance).

The position of a material particle at time $t > 0$ can be described by a vector which depends on t and on the initial position. Let us denote this vector by $\Phi_1(\vec{X}, t)\vec{e_1} + \Phi_2(\vec{X}, t)\vec{e_2} + \Phi_3(\vec{X}, t)\vec{e_3}$. Since we can consider any initial position $\vec{X}$, we have just defined an application $\vec{\Phi}$ that describes the movements of the continuous medium over a time between $t = 0$ and $t = T$:

$$\mathbf{R}^3 \times [0, T] \longrightarrow \mathbf{R}^3 \tag{1}$$

$$(\vec{X}, t) \mapsto \vec{\Phi}(\overrightarrow{X}, t) = \Phi_1(\vec{X}, t)\vec{e_1} + \Phi_2(\vec{X}, t)\vec{e_2} + \Phi_3(\vec{X}, t)\vec{e_3}. \tag{2}$$

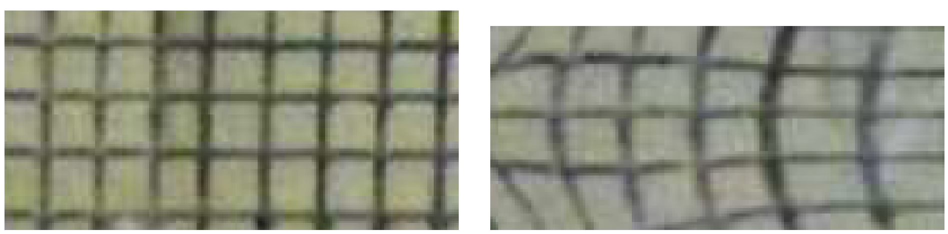

Figure 1: A sample of perspex with orthogonal lines drawn with a pen at $t = 0$, and at at some time $t > 0$ after traction.

We will assume that the application $\vec{\Phi}$ is continuous (and has as many continuous derivatives as we need) in all the variables, as we want to study smooth deformations. Continuity w.r.t. time at time $t = 0$ implies that for all $\vec{X}$ in $\mathbf{R}^3$, we have the property $\vec{\Phi}(\vec{X}, 0) = \vec{X}$.

We want to ask the following question (explicit examples will be given in the exercises): **given the application $\vec{\Phi}$**, and two vectors in the initial state, *can we compute the evolution of the scalar product of these two vectors?*

1.2 Computing the tangent to a curve in three dimensions

• **Let us parametrize a straigth line at time $t = 0$.** With the system of coordinates described by the origin O and the base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, consider the particle which is at position $\vec{X} = O\vec{M}$ in the continuous medium, at time $t = 0$. We can move around this vector by adding some vector $\vec{h} \in \mathbf{R}^3$ to it. We will need to do differential calculus (in order to compute tangent vectors), so let us introduce a scalar parameter $s \in \mathbf{R}$, which we will eventually let go to 0.

If we consider the set of all possible values of the parameter s, we get the straight line going through the point M in direction $\vec{h}$. The vector $\vec{h}$ is the tangent vector of the straight line that goes through M in direction $\vec{h}$, as we have

$$\frac{1}{s}\left((\vec{X}+s\vec{h})-\vec{X}\right) \longrightarrow_{s\to 0} \vec{h}. \tag{3}$$

NB: the parameter s is not time, it is just a geometric parameter describing a straight line at a fixed time ($t=0$).

- **What happens to this line through time? It is mapped to a line defined through $\vec{\Phi}$.**

What does this situation look like at time $t>0$?
By definition of the flow, the particle that was at $\vec{X}$ is now at $\vec{\Phi}(\vec{X},t)$ and the particle that was at $\vec{X}+s\vec{h}$ is now at $\vec{\Phi}(\vec{X}+s\vec{h},t)$, so the quantity we studied in Eq. 3 has now become

$$\frac{1}{s}\left(\vec{\Phi}(\vec{X}+s\vec{h},t)-\vec{\Phi}(\vec{X},t)\right) \tag{4}$$

Let us study its limit when s goes to zero, which is[1] the tangent to the curve $s\in\mathbf{R}\mapsto\vec{\Phi}(\vec{X}+s\vec{h},t)$.

Let us write all the components in matrix form in the base $(\vec{e_1},\vec{e_2},\vec{e_3})$ to compute this tangent:

$$\vec{\Phi}(\vec{X}+s\vec{h},t)-\vec{\Phi}(\vec{X},t)=\begin{pmatrix}\Phi_1(\vec{X}+s\vec{h},t)-\Phi_1(\vec{X},t)\\ \Phi_2(\vec{X}+s\vec{h},t)-\Phi_2(\vec{X},t)\\ \Phi_3(\vec{X}+s\vec{h},t)-\Phi_3(\vec{X},t)\end{pmatrix} \tag{5}$$

Let us assume that the components of $\vec{\Phi}$ are smooth functions (they posess derivatives at all orders). One can write a first-order Taylor expansion of each of the components of the vector in Eq. 5. For example:

$$\Phi_1(\vec{X}+s\vec{h},t)=\Phi_1(\vec{X},t)+sh_1\frac{\partial\Phi_1}{\partial X_1}(\vec{X},t)+sh_2\frac{\partial\Phi_1}{\partial X_2}(\vec{X},t)+sh_3\frac{\partial\Phi_1}{\partial X_3}(\vec{X},t)+o(s). \tag{6}$$

Hence the quantity

$$\frac{1}{s}\left(\Phi_1(\vec{X}+s\vec{h},t)-\Phi_1(\vec{X},t)\right)=h_1\frac{\partial\Phi_1}{\partial X_1}(\vec{X},t)+h_2\frac{\partial\Phi_1}{\partial X_2}(\vec{X},t)+h_3\frac{\partial\Phi_1}{\partial X_3}(\vec{X},t)+\frac{o(s)}{s}, \tag{7}$$

and its limit when s goes to zero is given by:

$$h_i\frac{\partial\Phi_1(\vec{X})}{\partial X_i}, \tag{8}$$

where the sum rule over doubled indices is applied. The computation can be repeated for Φ_2 and Φ_3, and one obtains the following limit:

$$\frac{1}{s}\left(\vec{\Phi}(\vec{X}+s\vec{h},t)-\vec{\Phi}(\vec{X},t)\right)\longrightarrow_{s\to 0}\begin{pmatrix}h_i\frac{\partial\Phi_1(\vec{X},t)}{\partial X_i}\\ h_i\frac{\partial\Phi_2(\vec{X},t)}{\partial X_i}\\ h_i\frac{\partial\Phi_3(\vec{X},t)}{\partial X_i}\end{pmatrix}. \tag{9}$$

- **A line is mapped to a line, tangent vectors are mapped to tangent vectors.** Hence the vector $\vec{h}=h_j\vec{e_j}$pointing from $\vec{X}$ is mapped to the vector

$$h_i\frac{\partial\Phi_j(\vec{X})}{\partial X_i}\vec{e_j}=h_iT_{ji}(\vec{X},t)\vec{e_j}, \tag{10}$$

[1]As an exercise, the reader should put the symbols $\vec{X},\vec{h},\vec{\Phi}(\vec{X},t),\vec{\Phi}(\vec{X}+\vec{h},t)$ on Fig. 1.1, and draw the two curves we are talking about.

by the flow, where we have defined the matrix T (whose value depends on the initial position $\vec{X}$ by:

$$T_{ij}(\vec{X},t) = \frac{\partial \Phi_i(\vec{X},t)}{\partial X_j}, \tag{11}$$

which maps vectors from position $\vec{X}$ to position $\vec{\Phi}(\vec{X},t)$. T is called the *strain tensor*. It depends on the initial point $\vec{X}$ so one could (should) call it *strain tensor field*.

2 How does the scalar product change under the flow?

Taking a pair of vectors $\vec{h} = h_i \vec{e_i}$ and $\vec{u} = u_i \vec{e_i}$ starting from position $\vec{X}$, one can transport both vectors by the flow using Eq. 10, and compute the dot-product of the resulting vectors:

$$\begin{aligned} \left(h_i \tfrac{\partial \Phi_j(\vec{X},t)}{\partial X_i} \vec{e_j}\right) \cdot \left(u_k \tfrac{\partial \Phi_l(\vec{X},t)}{\partial X_k} \vec{e_l}\right) &= h_i u_k \tfrac{\partial \Phi_j(\vec{X},t)}{\partial X_i} \tfrac{\partial \Phi_l(\vec{X},t)}{\partial X_k} \delta_{jl} \\ &= h_i u_k \tfrac{\partial \Phi_j(\vec{X},t)}{\partial X_i} \tfrac{\partial \Phi_j(\vec{X},t)}{\partial X_k} \\ &= h_i u_k T_{ji}(\vec{X},t) T_{jk}(\vec{X},t). \end{aligned} \tag{12}$$

One could be tempted to say that there is "no deformation" if the matrix T is the identity matrix. However this condition is too restrictive. Indeed we can see from Eq. 12 that if $T_{ji}T_{jk} = \delta_{ik}$, the scalar product is preserved (hence the grid drawn on Fig. 1.1 is not deformed, just transported). Hence we see that the local deformations are measured by the measured by the difference $T_{ji}T_{jk} - \delta_{ik}$.