XJTLU, MTH308 [Cartesian tensors and mathematical models of (elastic) solids and viscous fluids], Semester 2, 2014

Lecture 6, 7th April 2015: the Navier equations of linear elasticity

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Contents

So far we introduced Hooke's law as a material law for elastic solids, but we have not solved the induced PDEs for displacement fields. In this lecture we will write down these equations, called the Navier equations, and study them with boundary conditions in a spherical geometry. Throughout these notes, $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ denotes an orthonormal base of $\mathbf{R}^3$, and the position $\vec{x}$ of a point in space is described through associated Cartesian coordinates defined by $\vec{x} = x_i \vec{e_i}$.

1 The Navier equations

We start with the balance equations

$$\vec{0} = f^{\vec{vol}} + \frac{\partial \sigma_{ij}}{\partial x_j} \vec{e_i}. \tag{1}$$

and the material law relating the stress tensor to the deformation field $\vec{u} = u_i \vec{e_i}$, in the regime of linear elasticity, i.e. when all components of $\vec{u}$ are very small.

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}(\mathrm{Tr}\sigma)\delta_{ij}, \tag{2}$$

where ϵ denotes the linearized strain tensor:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{3}$$

We would like to express Eq. 1 in terms of the field $\vec{u}$, so the first thing we have to do is to 'invert' Hooke's law in order to express σ as a function of ϵ. Since Hooke's law is linear, this is easily done by taking the trace of both sides of Eq. 3:

$$\mathrm{Tr}\epsilon = \frac{1+\nu}{E}\mathrm{Tr}\sigma - 3\frac{\nu}{E}\mathrm{Tr}\sigma = \frac{1-2\nu}{E}\mathrm{Tr}\sigma, \tag{4}$$

hence Hooke's law becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{E}\mathrm{Tr}\sigma\delta_{ij}\right) = \frac{E}{1+\nu}\epsilon_{ij} + \frac{E\nu}{(1-2\nu)(1+\nu)}(\mathrm{Tr}\epsilon)\delta_{ij}. \tag{5}$$

It is usual to rewrite this equation in terms of the Lamé coefficients μ and λ defined (note the factor of 2) as

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda(\mathrm{Tr}\epsilon)\delta_{ij}, \tag{6}$$

from which we read off

$$\boxed{\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}.} \tag{7}$$

To describe the elastic properties of a solid, one can either choose Young's modulus and Posson's ratio or the Lamé coefficients. Navier's equation are often written using the Lamé coefficients. Substituting the form of the material law written in Eq. 6, we obtain for all i in [1..3]:

$$0 = f_i^{vol} + \mu \triangle u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right), \tag{8}$$

which are called the Navier equations of linear elasticity.

2 Solution in the case of a spherical shell

2.1 Explicit form of the Navier equations (with spherical symmetry)

Consider an elastic spherical shell (meaning the region between concentric spheres of radius R_1 and R_2, with $R_1 \leq R_2$, is filled with an elastic material). A uniform pressure P_2 is applied on the outside, a uniform pressure P_1 is applied on the inside. Volume forces are neglected (or reather they have been compensated by the reaction of some support in the reference configuration). The spherical shell contracts under the influence of the pressure. Compute the displacement fields.

- **Use spherical symmetry.** As we have a spherical symmetry (because of the spherical geometry and the uniform pressure), the displacement field will also have spherical symmetry, so we look for solutions of the form

$$\vec{u}(x_1, x_2, x_3) = \phi(\sqrt{x_i x_i})\vec{x}, \tag{9}$$

where we expressed the coordinates in Cartesian form, in order to avoid having to look up the expression of differential operators in spherical coordinates. All we have to do is therefor to compute the function

ϕ.

• **Work out all terms in the Navier equations in terms of the unknown function ϕ.** We have to solve the Navier equations

$$0 = \mu \triangle u_i + (\lambda + \mu) \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right). \tag{10}$$

To that end, let us rewrite them as a set of three differential equations in the function ϕ. We will need the expression of the derivatives of the displacement fields:

$$\frac{\partial u_i}{\partial x_j}(x_1, x_2, x_3) = \delta_{ij}\phi(\sqrt{x_k x_k}) + x_i \frac{\partial}{\partial x_j}(\phi(\sqrt{x_k x_k})) = \delta_{ij}\phi(\sqrt{x_k x_k}) + \frac{x_i x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}), \tag{11}$$

from which we can compute the divergence of the displacement field needed in Navier's equations:

$$\frac{\partial u_k}{\partial x_k}(x_1, x_2, x_3) = 3\phi(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k}). \tag{12}$$

The second term in Navier's equation is obtained by taking one more derivative:

$$\frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right) (x_1, x_2, x_3) \tag{13}$$

$$= \frac{\partial}{\partial x_i} \left(3\phi(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k}) \right) \tag{14}$$

$$= 3\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\frac{x_i}{\sqrt{x_k x_k}}\phi''(\sqrt{x_k x_k}) \tag{15}$$

$$= 4\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}). \tag{16}$$

On the other hand, we can express the Laplacian term as follows:

$$\triangle u_i = \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} \right) (x_1, x_2, x_3) = \frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \frac{3x_i + \delta_{ij}x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i x_j \frac{\partial}{\partial x_j} \left(\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) \right), \tag{17}$$

but

$$x_i x_j \frac{\partial}{\partial x_j} \left(\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) \right) = -x_i x_j \frac{x_j}{(x_k x_k)^{\frac{3}{2}}}\phi'(\sqrt{x_k x_k}) + x_i x_j \frac{1}{\sqrt{x_k x_k}} \frac{x_j}{\sqrt{x_k x_k}}\phi''(\sqrt{x_k x_k}) \tag{18}$$

$$= -\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}). \tag{19}$$

Putting all the terms together we find exactly the same form for the Laplacian term as in Eq. 16:

$$\triangle u_i = 4\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}), \tag{20}$$

hence the Navier equations take the form:

$$0 = (\lambda + 2\mu)x_i \left(4\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \phi''(\sqrt{x_k x_k}) \right), \tag{21}$$

for all i in [1..3]. These three equations are identical (up to a multiplication by x_i so we obtain the following scalar differential equation for ϕ:

$$0 = \frac{4}{r}\phi'(r) + \phi''(r). \tag{22}$$

We can look for solutions of the forl $\phi' = Ar^{\alpha} + B$, and obtain $\alpha = -4$ and $B = 0$. One more integration gives rise to

$$\boxed{\phi(r) = -\frac{A}{3r^3} + C.} \tag{23}$$

2.2 Determination of the integration constants using boundary conditions

We have to express the boundary conditions on the internal and external spheres using Hooke's law. From our computations of derivatives of displacements (Eq. 11) we obtain:

$$\epsilon_{ij} = \delta_{ij}\phi(x_1, x_2, x_3) + \frac{x_i x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}). \tag{24}$$

whose trace gives us

$$\mathrm{Tr}(\epsilon) = 3\phi + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k}), \tag{25}$$

which we can express in sperical coordinates as

$$\mathrm{Tr}(\epsilon) = 3\phi + r\phi'(r) = 3C \tag{26}$$

Consider the following two points, one on the internal sphere $\vec{a} = R_1\vec{e_1}$, and one on the outer sphere, $\vec{a} = R_2\vec{e_1}$. The outward-pointing unit normal vector to the shell is $-\vec{e_1}$ at $\vec{a}$ and $+\vec{e_1}$ at $\vec{b}$, hence

$$\sigma_{ij} n_j \vec{e_i} = +P_1\vec{e_1} \text{ on the inside,} \tag{27}$$

and

$$\sigma_{ij} n_j \vec{e_i} = -P_2\vec{e_1} \text{ on the outside.} \tag{28}$$

Using Hooke's law we therefore obtain (for components of indices $i = j = 1$):

$$-P_1 = 2\mu\left(-2\frac{A}{3R_1^3} + C\right) + 3\lambda C, \tag{29}$$

$$-P_2 = 2\mu\left(-2\frac{A}{3R_2^3} + C\right) + 3\lambda C. \tag{30}$$

from which one obtains the constants:

$$\boxed{A = \frac{3(P_1 - P_2)R_1^3 R_2^3}{4\mu(R_2^3 - R_1^3)},} \tag{31}$$

$$\boxed{C = \frac{1}{3\lambda + 2\mu}\frac{P_1 R_1^3 - P_2 R_2^3}{R_2^3 - R_1^3}.} \tag{32}$$