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# XJTLU, MTH308 (Cartesian tensors and mathematical models of solids and viscous fluids), Semester 2, 2015

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## Lecture 1, 3rd March 2015: from linear algebra to tensors

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# 1 Tensors and continuous media: motivations and history

Systems of coordinates are mathematical structures used to describe the properties of physical systems in a quantitative way. They depend on the choice of the person who designs the model (for example, a certain choice of axes can make computations easier). However, the physical results should not depend on the choice of coordinates.

• **This independence induces transformation laws on certain quantities.** If one changes the system of coordinates, the quantities we write change as well, but they must do so in a precise way in order for the physical reality to stay the same. Tensors are quantities defined by the way they transform when the system of coordinates changes. The word *Cartesian*<sup>1</sup> refers to the fact that one considers systems of coordinates in which the three axes form an orthogonal base.

• **Continuous media: solids and viscous fluids.** We are interested in scales at which the (discrete, atomic, molecular<sup>2</sup>) microscopic structure of matter is not apparent : distances are large in scale of the size of molecules, and we will treat solids and fluids as continuous media on which we will put coordinates just as if we were on a geometric space. The mathematical framework of continuous media was developed before atomic and molecular physics (for example Euler's equation for the mechanics of fluids was written in 1755<sup>3</sup>). At macroscopic scales it is still valid, because the huge numbers of microscopic constituents have interactions that result in collective behaviours that can be described at larger scales by a relatively small number of parameters (such as elasticity coefficients and density). These parameters are to be determined experimentally, but this course will introduce the mathematical framework that can be described for all continuous media (fluids and solids).

## 2 Elements of linear algebra, notations and conventions

Unless otherwise stated, we will assume in this course that the ambient space is the three-dimensional space  $\mathbf{R}^3$ . This section recalls a few notions of linear algebra and introduces notations.

Let  $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$  be an orthonormal of  $\mathbf{R}^3$ . Let  $O$  be the origin of a coordinate system. Let  $M$  be a point in  $\mathbf{R}^3$ . It can be described by its coordinates  $(x_1, x_2, x_3)$ , the three numbers such that the vector  $\overrightarrow{OM}$  is written as follows:

$$\overrightarrow{OM} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3. \quad (1)$$

### 2.1 Notations: summing over repeated indices

**Notation.** You are familiar with the notation  $\Sigma$  for sums, which saves space:

$$\overrightarrow{OM} = \sum_{i=1}^3 x_i \vec{e}_i. \quad (2)$$

In tensor algebra, it is customary to omit the  $\Sigma$  symbol *when an index is summed over and appears twice and only twice in an expression*:

$$\overrightarrow{OM} = x_i \vec{e}_i. \quad (3)$$

The range of the index  $i$  is not specified anymore (whereas in Eq. it was specified that  $i$  ranges from 1 to 3. So, when an index appears twice in an expression, it is implied that it corresponds to a sum over

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<sup>1</sup>from the name of René Descartes (1596-1650). Under his influence, numbers started being used to describe geometric objects. His name is attached to the laws of geometric optics.

<sup>2</sup>Let us mention that tensors are used for other purposes in atomic and subatomic physics, to address symmetry properties of particles.

<sup>3</sup>Leonhard Euler (1707-1783) founded the discipline of fluid mechanics, and contributed in a decisive way to all branches of mathematics.

all possible values of this index. Since the base  $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$  has three vectors, the sum in Eq. 3 is over  $i = 1, i = 2, i = 3$ .

**Remark (mute indices).** If an index is summed over, it is called a *mute* index. It takes all possible values, one per term in the sum, and does not *say* anything about the components of the result: the result does not have this index. One can therefore use another symbol for it without changing the value of the sum. For example in  $\vec{x} = x_i \vec{e}_i$ , the index  $i$  is a mute index, and one can as well write  $\vec{x} = x_k \vec{e}_k$ .

This notation can be used for indices in matrices as well. From linear algebra, you are already familiar with matrices. Matrices have two indices, one for rows and one for columns. Given the base  $(e) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ , and a linear application  $u$  from  $\mathbf{R}^3$  to  $\mathbf{R}^3$ , a matrix  $U$  can be used to represent the action of  $u$  in the base  $(e)$ :

$$U = \text{Mat}(u, (e)) = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}$$

$$u\left(\sum_{i=1}^3 x_i \vec{e}_i\right) = \sum_{j=1}^3 \left(\sum_{k=1}^3 U_{jk} x_k\right) \vec{e}_j \quad (4)$$

The expression 4 contains two indices ( $j$  and  $k$ ) that appear twice each, and are summed over. The sum convention on repeated indices can be applied to rewrite the expression as follows:

$$u(x_i \vec{e}_i) = U_{jk} x_k \vec{e}_j. \quad (5)$$

## 2.2 Scalar product, Kronecker symbol

Let  $(e) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$  be an orthonormal base. The scalar products between pairs of vectors in the base is as follows:

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}. \quad (6)$$

for which one used the special notation  $\delta_{ij}$  (or Kronecker symbol):

$$\begin{cases} \delta_{ij} = 1 & \text{if } i = j, \\ \delta_{ij} = 0 & \text{if } i \neq j \end{cases}$$

The (Euclidean) scalar product (or dot-product) between two vectors  $x = x_i \vec{e}_i$  and  $y = y_i \vec{e}_i$  is denoted by a dot, which is a bilinear operation:

$$\vec{x} \cdot \vec{y} = (x_i \vec{e}_i) \cdot (y_j \vec{e}_j) = x_i y_j \delta_{ij} = x_i y_i. \quad (7)$$

**Remark.** Whenever repeated indices are summed over, a Kronecker symbol can be inserted (with a new index symbol) without changing the value of the expression:

$$x_i y_i = x_i \delta_{ij} y_j. \quad (8)$$

This trick will be used in the next section when we prove that the scalar product is invariant under change of orthonormal base.

### 3 Transformation rules from changes of orthonormal base

#### 3.1 Transformation of vectors

Consider two different orthonormal bases of  $\mathbf{R}^3$ , denoted by  $(e') = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$  and  $(e) = (\vec{e}'_1, \vec{e}'_2, \vec{e}'_3)$ . Consider a vector  $\vec{x}$  in  $\mathbf{R}^3$ . Let us write the coordinates of this vector in these two bases:

$$\vec{x} = x_i \vec{e}_i, \quad \vec{x} = x'_i \vec{e}'_i. \quad (9)$$

Since the base  $(e)$  is orthonormal, one can decompose the vectors  $(\vec{e}'_1, \vec{e}'_2, \vec{e}'_3)$  over this base by taking dot-products:

$$\forall i \in [1..3], \quad \vec{e}'_i = (\vec{e}'_i, \vec{e}_j) \vec{e}_j. \quad (10)$$

Let us substitute the expression 10 in the expression of  $\vec{x}$  in Eq. 12:

$$\vec{x} = x_i \vec{e}_i, \quad \vec{x} = x'_i (\vec{e}'_i, \vec{e}_j) \vec{e}_j. \quad (11)$$

Now we have two expressions for the vector  $\vec{x}$  in the same base  $(e)$ . Hence the coordinates must be equal. There are two mute indices in the second expression of  $x$ , and we can change the symbols in order to have the symbol  $\vec{e}_i$  at the end:

$$\vec{x} = x_i \vec{e}_i, \quad \vec{x} = x'_k (\vec{e}'_k, \vec{e}_j) \vec{e}_j = x'_k (\vec{e}'_k, \vec{e}_i) \vec{e}_i. \quad (12)$$

Let us write the equality of the components of the vector  $\vec{x}$ :

$$x_i = x'_k (\vec{e}'_k, \vec{e}_i). \quad (13)$$

The r.h.s. of Eq. 14 can be rewritten as the action of a square matrix denoted by  $P$  (the transformation matrix) on the coordinates  $(x'_1, x'_2, x'_3)$ :

$$\boxed{x_i = P_{ik} x'_k}, \quad (14)$$

where

$$\boxed{P_{ik} = \vec{e}'_k \cdot \vec{e}_i}. \quad (15)$$

**Proposition.** *The inverse of the transformation matrix  $P$  is its transpose.*

**Proof.** Consider the product of  $P$  and its transpose  $P^T$ :

$$(PP^T)_{ij} = P_{ik} (P^T)_{kj} = P_{ik} P_{jk} = (\vec{e}'_k \cdot \vec{e}_i) (\vec{e}'_k \cdot \vec{e}_j) = (\vec{e}'_i \cdot \vec{e}_j) \delta_{ij} = (\vec{e}'_i \cdot \vec{e}_j), \quad (16)$$

where we have inserted the Kronecker symbol to rewrite the sum over repeated indices. Let us now rewrite the Kronecker symbol as the dot-product of two vectors of an orthonormal base:

$$\delta_{lp} = (\vec{e}'_l \cdot \vec{e}'_p). \quad (17)$$

Hence

$$(PP^T)_{ij} = (\vec{e}'_i \cdot \vec{e}_j) (\vec{e}'_l \cdot \vec{e}_p) (\vec{e}'_p \cdot \vec{e}_j) = (\vec{e}'_i \cdot \vec{e}_l) (\vec{e}'_l \cdot \vec{e}_p) (\vec{e}'_p \cdot \vec{e}_j) = ((\vec{e}'_i \cdot \vec{e}'_l) \vec{e}'_l) \cdot ((\vec{e}'_p \cdot \vec{e}_j) \vec{e}'_p) \quad (18)$$

where we have used the symmetry of the dot-product. Since  $(e')$  is an orthonormal base, we have

$$\vec{e}_i = (\vec{e}'_i \cdot \vec{e}'_l) \vec{e}'_l, \quad \text{and} \quad \vec{e}_j = (\vec{e}'_p \cdot \vec{e}_j) \vec{e}'_p. \quad (19)$$

Hence

$$\boxed{(PP^T)_{ij} = (\vec{e}_i \cdot \vec{e}_j) = \delta_{ij}}. \quad (20)$$

Hence the inverse of the matrix  $P$  is its transpose:

$$(PP^T) = I_3, \quad P^T P = I_3. \quad (21)$$

We can use this property to obtain an expression of the coordinates  $x'$  in terms of the coordinates  $x$ , simply by multiplying both sides of Eq. 14 by  $P_{ij}$  (and summing over index  $i$ ):

$$P_{ij}x_i = P_{ij}P_{ik}x'_k = (P^t)_{ji}P_{ik}x'_k = (P^T P)_{jk}x'_k = \delta_{jk}x'_k = x'_j, \quad (22)$$

Hence

$$\boxed{x'_i = P_{ji}x_j.} \quad (23)$$

**Exercise.** Verify that the norm of a vector and the scalar product of two vectors are invariant under a change of orthonormal base.

**Solution.** Let us use the same notations as above. The norm of the vector  $\vec{x}$  expressed using the coordinates  $(x_1, x_2, x_3)$  is  $\sqrt{x_i x_i}$ . Using the new coordinates it is expressed as  $\sqrt{x'_i x'_i}$ . Let us use Eq. 14 to write these two expressions in the same system of coordinates:

$$x_i x_i = P_{ik}x'_k P_{il}x'_l \quad (24)$$

The expression  $P_{ik}P_{il}$  can be rewritten as  $(P^T)_{ki}P_{il}^T = (P^T P)_{kl} = \delta_{kl}$ . We therefore have

$$x_i x_i = \delta_{kl}x'_k x'_l = x'_k x'_k. \quad (25)$$

As the indices  $i$  and  $k$  are mute in the above expression, the two quantities are equal, and the norm of  $x$  is invariant under a change of orthonormal base.

As for the scalar product of two vectors  $x$  and  $y$ , we can use the same property to prove the invariance under change of orthonormal base:

$$x_i y_i = P_{ik}x'_k P_{il}y'_l = (P^T)_{ki}P_{il}x'_k y'_l = (P^T P)_{kl}x'_k y'_l = \delta_{kl}x'_k y'_l = x'_k y'_k. \quad (26)$$

### 3.2 Transformation of matrices

Consider the matrix  $U$  of a linear application from  $\mathbf{R}^3$  to  $\mathbf{R}^3$ , presented in an orthonormal base  $(e) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ :

$$U = \text{Mat}(u, (e)) = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \quad (27)$$

For  $i$  in  $[1..3]$ , the image of vector  $e_i$  can be written in the base  $(e)$  using the entries in the  $i$ -th row of the matrix  $U$ :

$$u(e_i) = U_{i1}e_1 + U_{i2}e_2 + U_{i3}e_3 = U_{ij}e_j. \quad (28)$$

Consider another orthonormal base  $(e') = (\vec{e}'_1, \vec{e}'_2, \vec{e}'_3)$ . We want to compute the matrix of  $u$  in the base  $(e')$ . Let us call this matrix  $U'$ :

$$U' = \text{Mat}(u, (e')) = \begin{pmatrix} U'_{11} & U'_{12} & U'_{13} \\ U'_{21} & U'_{22} & U'_{23} \\ U'_{31} & U'_{32} & U'_{33} \end{pmatrix} \quad (29)$$

Again, by definition of the matrix representation of linear applications in base  $(e')$ , we have

$$u(\vec{e}'_i) = U'_{i1}\vec{e}'_1 + U'_{i2}\vec{e}'_2 + U'_{i3}\vec{e}'_3 = U'_{ia}\vec{e}'_a. \quad (30)$$

We are going to express both the l.h.s. and the r.h.s of Eq. 30 on the base  $(e)$ .

The l.h.s of Eq. 30 can be rewritten as follows:

$$u(\vec{e}'_i) = u((\vec{e}'_i, \vec{e}_a)\vec{e}_a) = (\vec{e}'_i, \vec{e}_a)u(\vec{e}_a) = (\vec{e}'_i, \vec{e}_a)U_{ak}\vec{e}_k = P_{ai}U_{ak}\vec{e}_k. \quad (31)$$

where we used the linearity of  $u$  in the second equality, and Eq. 28 in the third equality. The r.h.s of Eq. 30 can be rewritten as follows:

$$U'_{ia} \vec{e}_a = U'_{ia} (\vec{e}_a, \vec{e}_k) \vec{e}_k = U'_{ia} P_{ka} \vec{e}_k. \quad (32)$$

As  $(e)$  is a base of  $\mathbf{R}^3$ , the coefficients of  $\vec{e}_k$  in 31 and 32 must be equal for all  $k$ :

$$U'_{ia} P_{ka} = P_{ai} U_{ak}. \quad (33)$$

Since the inverse of the transformation matrix  $P$  is its transpose  $P^T$ , we can multiply both sides by  $P_{kj}$  and get rid of all  $P$  symbols on the l.h.s.:

$$U'_{ia} P_{ka} P_{kj} = P_{ai} U_{ak} P_{kj}. \quad (34)$$

Since  $P_{ka} P_{kj} = (P^T)_{ak} P_{kj} = (P^T P)_{aj} = \delta_{aj}$ , we have

$$\boxed{U'_{ij} = P_{ai} P_{kj} U_{ak}.} \quad (35)$$

## 4 Definition of tensors

Cartesian tensors are mathematical objects with components in a given orthonormal base. They carry a certain number of indices (the number of indices is called the order of the tensor), which can be used to act on vectors by summing over indices. When the orthonormal base is changed, the components of Cartesian tensors are transformed using transformation matrices.

We have seen examples of low order:

- scalars have no indices, they are invariant under change of base. They are called tensors of order zero (physical examples are mass and temperature). They are called tensors of order 0.
- vectors have one index, they transform as in Eq. 14 under change of base (physical examples include velocities). They are called tensors of order 1.
- matrices have two indices, they. Physical examples include the stress tensor will be introduced in this course. They are called tensors of order 2.

By generalizing the transformation pattern of Eqs. 23 and 35, one defines a tensor  $T$  of order  $n$  to have  $n$  indices, with the transformation rule of the entries:

$$\boxed{T'_{i_1, \dots, i_n} = P_{k_1 i_1} P_{k_2 i_2} \dots P_{k_n i_n} T_{k_1, \dots, k_n}} \quad (36)$$

when one goes from orthonormal base  $(e)$  to orthonormal base  $(e')$ , with the transformation matrix defined by  $P_{ij} = e'_i \cdot e_j$ .

**Exercise.** The Kronecker symbol can be considered as a tensor of order 2. Show that it has the same coefficients in all orthonormal systems of coordinates (one says  $\delta$  is an *isotropic tensor*).

Let us apply the transformation rule of a tensor of order 2, with the above notation for the transformation matrix:

$$(\delta')_{ij} = P_{ki} P_{lj} \delta_{kl} = P_{ki} P_{kj} = (P^T)_{ik} P_{kj} = (P^T P)_{ij} = \delta_{ij}. \quad (37)$$

One can also note that the Kronecker symbol is the matrix of the identity transformation, hence its matrix is  $I_3$  in any base.