# XJTLU, MTH308 (Cartesian tensors and mathematical models of solids and viscous fluids), Semester 2, 2015

## Lecture 10, 12th May, 2015: the Couette flow as a viscosimeter

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## Contents



# 1 The Navier–Stokes equations for the Couette flow

Let us remind the affine form of the material law of Newtonian fluids.

$$\sigma_{ij}(\vec{x},t) = -P(\vec{x},t)\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{1}$$

In the case of thin layers of fluid, this model corresponds to friction forces that are proportional to the relative velocity between thin layers of fluid. For incompressible fluids, the divergence of the velocity field is zero, hence the equations of motion become the Navier–Stokes equation:

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}.\vec{\nabla})\vec{v}\right)=f^{\vec{vol}}-\vec{\nabla}P+\mu\triangle\vec{v}. \tag{2}$$

In this lecture we will encounter one more example of solution of the Navier–Stokes equations with cylindrical symmetry, but this time the direction of the velocity field (and not only the norm) varies from point to point.

# 2 The Couette flow

The Couette flow[1] is a permanent flow of an incompressible fluid occupying the space between two coaxial cylinders of radii $a$ and $b$, with $a<b$. The inner cylinder is fixed, while the external cylinder is rotating at constant speed $\omega$ radians per second. As a viscous fluid adheres to boundaries, the velocity field is zero on the inner cylinder, while it equals the velocity of the external cylinder on $r=b$.

Let us look for a velocity field that take the following form in cylindrical coordinates:

$$\vec{v}(r,\theta)=v(r)\vec{e_\theta}, \tag{3}$$

where $v$ is a scalar function we will have to determine, which means that the fluid flows in the orthoradial direction, and the norm of the velocity respects the cylindrical symmetry (it depends only on the distance from the axis). We also assume that the length of the cylinder is large compared to $a$ and $b$, so we did not include any dependence on $z$ in the form (this is equivalent to considering an "infinite cylinder"). Moreover there is no explicit dependence in time in Eq. 2, which means that we are interested in a steady flow.

# 3 Solution of the equations

## 3.1 Cylindrical coordinates (see tutorial for detailed derivations)

For a scalar function of a point in $\mathbf{R}^3$ described by cylindrical coordinates $f:(r,\theta,z)\mapsto f(r,\theta,z)$, the gradient in cylindrical coordinates is expressed as

$$\boxed{\vec{\nabla}f=\frac{\partial f}{\partial r}\vec{e_r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\vec{e_\theta}+\frac{\partial f}{\partial z}\vec{e_z}.} \tag{4}$$

We can compute the Laplacian of $f$ in cylindrical coordinates by applying Stokes' theorem to the flux of $\vec{\nabla}f$ through the boundary of an elementary volume oriented towards the exterior. Consider a vector field $\vec{u}=u_r\vec{e_r}+u_\theta\vec{e_\theta}+u_z\vec{e_z}$ and apply Stokes' theorem to the flux of $\vec{u}$ through an elementary volume centered at point $\vec{x}$ and spanned by the three vectors $dr\vec{e_r}, rd\theta\vec{e_\theta}, dz\vec{e_z}$ (see tutorial for details):

$$dV=rd\theta drdz, \tag{5}$$

$$\begin{aligned}\mathrm{div}\vec{u}(r,\theta,z)dV &= (u_z(r,\theta,z+dz)-u_z(r,\theta,z))rd\theta dr\\ &+((r+dr)u_r(r+dr,\theta,z)-ru_r(r,\theta,z))d\theta dz\\ &+(u_\theta(r,\theta+d\theta,z)-u_\theta(r,\theta,z))drdz.\\ &= rdrd\theta dz\left(\tfrac{1}{r}\tfrac{\partial(ru_r)}{\partial r}+\tfrac{1}{r}\tfrac{\partial u_\theta}{\partial\theta}+\tfrac{\partial u_z}{\partial z}\right),\end{aligned} \tag{6}$$

[1]named after the French physicist Maurice Couette (1858-1943), who built the first viscosimeter.

hence

$$\boxed{\mathrm{div}\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}} \tag{7}$$

Substituting the gradient of $f$ (Eq. 4) to $\vec{u}$, we obtain:

$$\boxed{\triangle f(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial^2 f}{\partial z^2}.} \tag{8}$$

### 3.2 Rewriting the Navier–Stokes equations in cylindrical coordinates

The time derivative of the velocity field is zero, as usual for permanent flows:

$$\boxed{\frac{\partial \vec{v}}{\partial t} = \vec{0}.} \tag{9}$$

Consider the expression proposed in Eq. 2 for the velocity field in cylindrical coordinates. As usual we write down the differential operator used to compute the convection term, which from Eq. 4:

$$\vec{v}.\vec{\nabla} = \frac{v(r)}{r}\frac{\partial}{\partial \theta}. \tag{10}$$

We compute the convection term by acting with this differential on the velocity field, without forgetting that $\vec{e_\theta}$ is a not constant vector but depends on $\theta$:

$$(\vec{v}.\vec{\nabla})\vec{v} = \frac{v(r)}{r}\frac{\partial}{\partial \theta}(v(r)\vec{e_\theta}) = \frac{v(r)^2}{r}\frac{\partial}{\partial \theta}\vec{e_r} = -\frac{v(r)^2}{r}\vec{e_r}. \tag{11}$$

Of course we obtain the same result if we use the base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ to express the velocity field, because the vectors $\vec{e_1}$ and $\vec{e_2}$ are constant:

$$\boxed{(\vec{v}.\vec{\nabla})\vec{v} = \frac{v(r)}{r}\frac{\partial}{\partial \theta}\left(v(r)(-\sin\theta\vec{e_x} + \cos\theta\vec{e_y})\right) = \frac{v(r)^2}{r}(-\cos\theta\vec{e_x} - \sin\theta\vec{e_y}) = -\frac{v(r)^2}{r}\vec{e_r}.} \tag{12}$$

The pressure term is a simple application of the differential operator to the scalar function $P$:

$$\boxed{-\vec{\nabla}P = -\frac{\partial P}{\partial r}\vec{e_r} - \frac{1}{r}\frac{\partial P}{\partial \theta}\vec{e_\theta} - \frac{\partial P}{\partial z}\vec{e_z}.} \tag{13}$$

The Laplacian term can be computed in the same way, acting with the differential operator on the vector $\vec{v}$, not forgetting to differentiate the vectors of the base:

$$\begin{aligned}\triangle\vec{v}(r,\theta) &= \tfrac{1}{r}\tfrac{\partial}{\partial r}\left(r\tfrac{\partial(v(r)\vec{e_\theta})}{\partial r}\right) + \tfrac{1}{r}\tfrac{\partial}{\partial \theta}\left(\tfrac{1}{r}\tfrac{\partial(v(r)\vec{e_\theta})}{\partial \theta}\right)\\ &= \left(\tfrac{1}{r}\tfrac{d}{dr}(rv'(r))\right)\vec{e_\theta} + \tfrac{1}{r}\tfrac{\partial}{\partial \theta}\left(-\tfrac{v(r)}{r}\vec{e_r}\right)\\ &= \left(\tfrac{1}{r}\tfrac{d}{dr}(rv'(r)) - \tfrac{v(r)}{r^2}\right)\vec{e_\theta}\end{aligned} \tag{14}$$

Collecting the scalar coefficients of the three vectors $\vec{e_r}, \vec{e_\theta}, \vec{e_z}$, we obtain the following system of equations:

$$\boxed{\left\{\begin{array}{l} -\rho\frac{v^2}{r} = -\frac{\partial P}{\partial r}\\ 0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{\mu}{r}\frac{d}{dr}(rv'(r)) - \mu\frac{v(r)}{r^2}\\ 0 = -\rho g - \frac{\partial P}{\partial z}\end{array}\right.} \tag{15}$$

### 3.3 Integration of the Navier–Stokes equations

The third equation of the system is the same as in Cartesian coordinates and just expresses the hydrostatic dependence of the pressure field. Integrating the second equation (the one corresponding to the coefficients along $\vec{e_\theta}$) is enough to determine the velocity field. First of all, we notice that $\partial P/\partial\theta$ depends only on $r$:

$$\frac{\partial P}{\partial \theta} = \mu \frac{d}{dr}(rv'(r)) - \mu \frac{v(r)}{r}, \tag{16}$$

hence there exists two functions of $r$ only, call them $C$ and $D$, such that

$$P(r, \theta, z) = C(r)\theta - \rho g z + D(r). \tag{17}$$

However, since $\theta$ is defined up to a multiple of $2\pi$, the function $\theta \mapsto P(r, \theta, z)$ must be $2\pi$-periodic for all values of $r$ and $z$ between the two cylinders. The only way to satisfy this periodicity condition is to have $C(r) = 0$. Hence the pressure does not depend on the orthoradial angle:

$$\frac{\partial P}{\partial \theta} = 0, \tag{18}$$

and Eq. 16 become an equation in $v$ only:

$$\frac{d}{dr}(rv'(r)) - \frac{v(r)}{r} = 0. \tag{19}$$

We rewrite it as

$$v'' + \frac{v'}{r} - \frac{v}{r^2} = 0. \tag{20}$$

We integrate this equation once: there exists a constant $D$ such that

$$v' + \frac{v}{r} = D. \tag{21}$$

The function $r \mapsto Dr/2$ is a particular solution of this equation, and we can introduce another constant $E$ (and redefine the unknown constant $D$) such that

$$v(r) = \frac{E}{r} + Dr. \tag{22}$$

Since the radius of the inner cylinder is strictly positive, this expression is finite for all values of $r$ corresponding to points between the two cylinders, and we can determine the two constants using the two boundary conditions:

$$v(a) = 0, \quad v(b) = b\omega. \tag{23}$$

Hence

$$\begin{cases} 0 & = \frac{E}{a} + Da \\ b\omega & = \frac{E}{b} + Db. \end{cases} \tag{24}$$

Hence

$$D = \frac{b^2\omega}{b^2 - a^2}, \quad E = -\frac{a^2 b^2 \omega}{b^2 - a^2}. \tag{25}$$

from which we obtain the expression of the velocity field:

$$\boxed{\vec{v}(r, \theta, z) = \left(\frac{b^2\omega}{b^2 - a^2}\left(-\frac{a^2}{r} + r\right)\right)\vec{e_\theta}.} \tag{26}$$

and also the pressure by integration of the component of the Navier–Stokes equation along direction $\vec{e_r}$:

$$P(r, \theta, z) = -\rho g z + \rho \int_a^b \left(\frac{v^2(r)}{r}\right) dr + constant \tag{27}$$

# 4 Remark on incompressibility

# 5 Application: the Couette flow as a viscosimeter

## 5.1 Using the material law of Newtonian fluids

In order to maintain the movement of the external cylinder at a constant speed, the operator must exert a force to compensate the friction force exerted by the fluid on the external cylinder.

To compute this force, we have to use the expression of the stress tensor, which we only have in Cartesian coordinates.

$$\vec{v}(\vec{x}) = u_1 \vec{e_1} + u_1 \vec{e_2}, \tag{28}$$

where from the expression in cylindrical coordinates we have

$$u_1 = -\frac{x_2}{\sqrt{x_1^2 + x_2^2}} \left( \frac{E}{\sqrt{x_1^2 + x_2^2}} + D\sqrt{x_1^2 + x_2^2} \right) \tag{29}$$

$$u_2 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \left( \frac{E}{\sqrt{x_1^2 + x_2^2}} + D\sqrt{x_1^2 + x_2^2} \right) \tag{30}$$

We want to minimize the amount of computation required. By cylindrical symmetry, it is enough to compute the norm of the force at point of coordinates $(x_1 = b, x_2 = 0, z = 0)$, at which $r = b$ and $\vec{e_\theta} = \vec{e_2}$.

## 5.2 Numerical application

Let us watch a demonstration of a commercially available Couette viscosimeter. From our observations we can estimate $\omega$, $a$ and $b$.

The force is applied by transmitting gravity along a pulley system (masses are suspended at the end of a pulley to make the external cylinder rotate). Can we guess the applied mass if we know the viscosity of water?

In practice of course the mass is known and we want to measure the viscosity. For instance, let us assume the same mass is attached to the same device (i.e. the same force is applied), but the viscosimeter is filled with a different liquid. Let us observe the number of turns per unit of time, and infer the viscosity. Once the diameters of the coaxial cylinders are known, and the space between them has been filled with a liquid, the viscosity of the liquid can be measured using a scale (measure of applied force) and a watch (measure of the frequency of rotation $\omega$).