Applied Elasticity in Engineering

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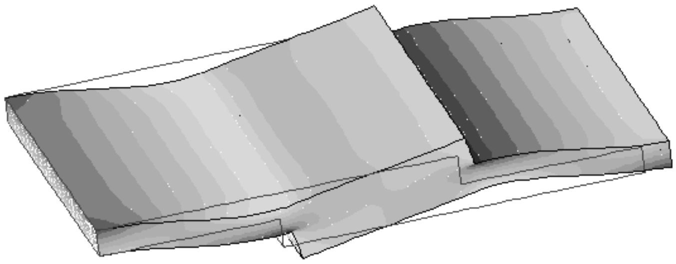

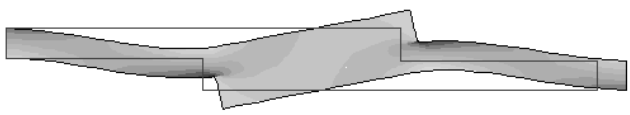

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Applied Elasticity in Engineering

Lecture notes - course 4A450

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Contents

Preface

These lecture notes present the theory of "Applied Elasticity". The first word of the title indicates that the theory can be *applied* to study, analyze and, hopefully, solve practical problems in engineering. It is obvious that we mean Mechanical Engineering, where the mechanical behavior of structures and materials is the subject of our studies. We do not consider the dynamical behavior, which is typically the subject of *dynamics*, so the loading of the material is static and vibrations are assumed to be of no importance.

The second word of the title of this lecture notes indicates that the material must always be elastic. Although there are materials, which stay elastic even when their deformation is very large – e.g. elastomers or rubber materials – this is not what we will look at. We confine ourselves to small deformations. When deformations in such materials become larger than a certain threshold, the elastic behavior is lost and permanent or plastic deformation will occur. We will not study plastic deformation in this course, but what we certainly have to do is to search for and formulate the limits of the elastic regime.

The theory in these notes is formulated with vectors and tensors, so the first chapter explains what we need to know of this mathematical language.

In the second chapter, much attention is devoted to the mathematical description of the deformation. The small deformation theory is derived as a special case of the general deformation theory. It is important to understand the assumptions and simplifications, which are introduced in this procedure.

Deformation is provoked by external loads, leading to stresses in the material. These stresses have to satisfy some general laws of physics: the balance laws for momentum and moment of momentum. The momentum balance appears to result in a vectorial partial differential equation, which has to be solved to determine the stresses. Only for very simple statically determinate problems, this can be done directly. Most problems in mechanics, however, are statically indeterminate and in that case the deformation has to be taken into account.

Deformation and stresses are obviously related by the properties of the material. The material behavior is modeled mathematically with stress-strain relations, also referred to as constitutive relations. For small elastic deformation, these relations are linear.

Combining balance laws and stress-strain relations leads to equations, from which the deformation can be solved. However, only for simple problems in terms of geometry and loading conditions. an analytical solution can be determined. For more practical problems we have to search for approximate solutions, which can be found by using numerical approximations.

Numerical solutions of mechanical problems are routinely determined with the Finite Element Method (FEM). In this course, we will use the commercial FEM package MSC.Marc. Making a model and observing the analysis results is done by the graphical interface MSC.Mentat. First we will analyze problems for which an analytical solution can be determined. The differences between analytical and numerical solutions, however small these may be, will help us to understand the FEM procedures. Obviously, the numerical method can be used for problems for which an analytical solution can not be determined.

Chapter 1

Vectors and tensors

In mechanics and other fields of physics, quantities are represented by vectors and tensors. Essential manipulations with these quantities will be summarized in this appendix. For quantitative calculations and programming, components of vectors and tensors are needed, which can be determined in a coordinate system with respect to a vector basis.

1.1 Vector

A vector represents a physical quantity which is characterized by its direction and its magnitude. The length of the vector represents the magnitude, while its direction is denoted with a unit vector along its axis, also called the working line. The zero vector is a special vector having zero length.

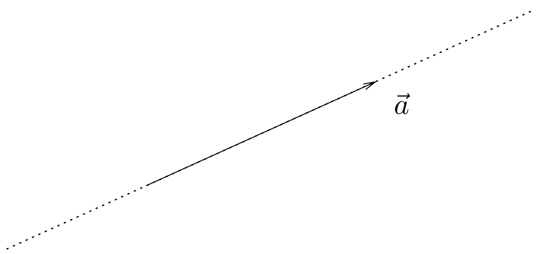


Fig. 1.1 : *A vector $\vec{a}$ and its working line*

$$\vec{a} = ||\vec{a}||\ \vec{e}$$

length	:	$\|\|\vec{a}\|\|$		
direction vector	:	$\vec{e}$	;	$\|\|\vec{e}\|\| = 1$
zero vector	:	$\vec{0}$		
unit vector	:	$\vec{e}$	;	$\|\|\vec{e}\|\| = 1$

1.1.1 Scalar multiplication

A vector can be multiplied with a scalar, which results in a new vector with the same axis. A negative scalar multiplier reverses the vector's direction.

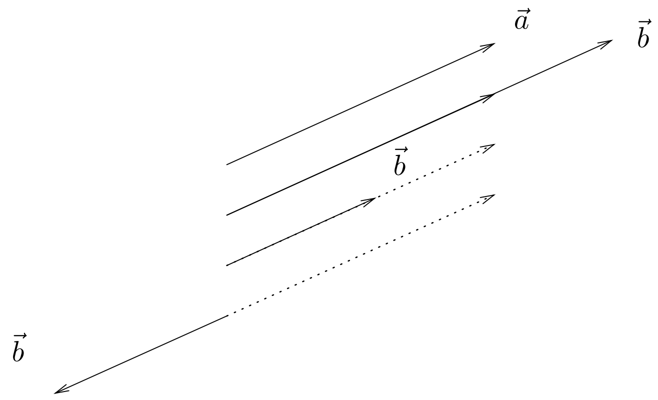


Fig. 1.2 : *Scalar multiplication of a vector* $\vec{a}$

$$\vec{b} = \alpha\vec{a}$$

1.1.2 Sum of two vectors

Adding two vectors results in a new vector, which is the diagonal of the parallelogram, spanned by the two original vectors.

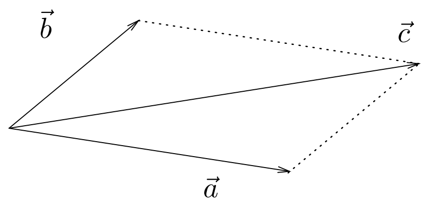


Fig. 1.3 : *Addition of two vectors*

$$\vec{c} = \vec{a} + \vec{b}$$

1.1.3 Scalar product

The scalar or inner product of two vectors is the product of their lengths and the cosine of the smallest angle between them. The result is a scalar, which explains its name. Because the product is generally denoted with a dot between the vectors, it is also called the dot product.

The scalar product is commutative and linear. According to the definition it is zero for two perpendicular vectors.

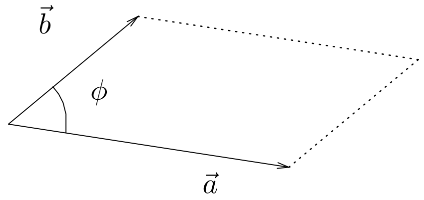


Fig. 1.4 : *Scalar product of two vectors* $\vec{a}$ *and* $\vec{b}$

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \; ||\vec{b}|| \; \cos(\phi)$$

$$\vec{a} \cdot \vec{a} = ||\vec{a}||^2 \geq 0 \qquad ; \qquad \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad ; \qquad \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

1.1.4 Vector product

The vector product of two vectors results in a new vector, who's axis is perpendicular to the plane of the two original vectors. Its direction is determined by the right-hand rule. Its length equals the area of the parallelogram, spanned by the original vectors.

Because the vector product is often denoted with a cross between the vectors, it is also referred to as the cross product. Instead of the cross other symbols are used however, e.g.:

$$\vec{a} \times \vec{b} \quad ; \quad \vec{a} * \vec{b}$$

The vector product is linear but not commutative.

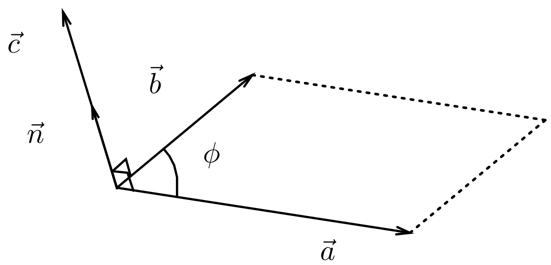


Fig. 1.5 : *Vector product of two vectors* $\vec{a}$ *and* $\vec{b}$

$$\vec{c} = \vec{a} * \vec{b} = \{||\vec{a}|| \; ||\vec{b}|| \; \sin(\phi)\}\vec{n}$$
$$= [\text{area parallelogram}] \; \vec{n}$$

$$\vec{b} * \vec{a} = -\vec{a} * \vec{b} \qquad ; \qquad \vec{a} * (\vec{b} * \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

1.1.5 Triple product

The triple product of three vectors is a combination of a vector product and a scalar product, where the first one has to be calculated first because otherwise we would have to take the vector product of a vector and a scalar, which is meaningless.

The triple product is a scalar, which is positive for a right-handed set of vectors and negative for a left-handed set. Its absolute value equals the volume of the parallelepiped, spanned by the three vectors. When the vectors are in one plane, the spanned volume and thus the triple product is zero. In that case the vectors are not independent.

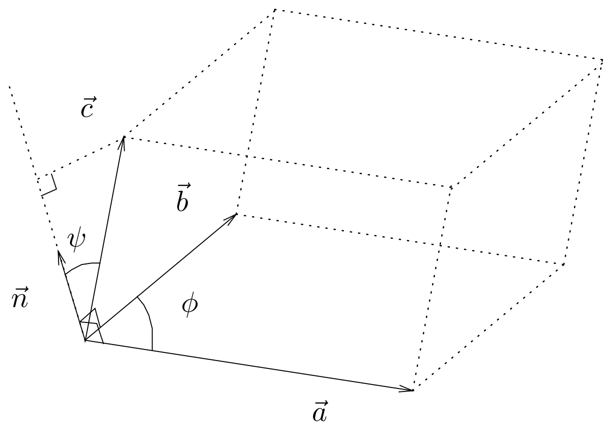


Fig. 1.6 : *Triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$*

$$\vec{a} * \vec{b} \cdot \vec{c} = \{||\vec{a}|| \; ||\vec{b}|| \; \sin(\phi)\}\{\vec{n} \cdot \vec{c}\}$$
$$= \{||\vec{a}|| \; ||\vec{b}|| \; \sin(\phi)\}\{||\vec{c}|| \cos(\psi)\}$$
$$= |\text{volume parallelepiped}|$$

$$\begin{array}{llll} > 0 & \rightarrow & \vec{a}, \vec{b}, \vec{c} & \text{right handed} \\ < 0 & \rightarrow & \vec{a}, \vec{b}, \vec{c} & \text{left handed} \\ = 0 & \rightarrow & \vec{a}, \vec{b}, \vec{c} & \text{dependent} \end{array}$$

1.1.6 Tensor product

The tensor product of two vectors represents a dyad, which is a linear vector transformation. A dyad is a special tensor – to be discussed later –, which explains the name of this product. Because it is often denoted without a symbol between the two vectors, it is also referred to as the open product.

The tensor product is not commutative. Swapping the vectors results in the conjugate or transposed or adjoint dyad. In the special case that it is commutative, the dyad is called symmetric.

A conjugate dyad is denoted with the index $(\ )^c$ or the index $(\ )^T$ (transpose). Both indices are used in these notes.

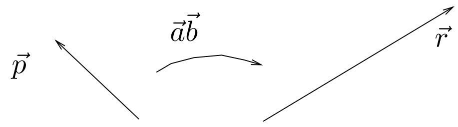


Fig. 1.7 : *A dyad is a linear vector transformation*

$$\vec{a}\vec{b} = \text{dyad} = \text{linear vector transformation}$$

$$\vec{a}\vec{b} \cdot \vec{p} = \vec{a}(\vec{b} \cdot \vec{p}) = \vec{r}$$
$$\vec{a}\vec{b} \cdot (\alpha\vec{p} + \beta\vec{q}) = \alpha\vec{a}\vec{b} \cdot \vec{p} + \beta\vec{a}\vec{b} \cdot \vec{q} = \alpha\vec{r} + \beta\vec{s}$$

conjugated dyad	$(\vec{a}\vec{b})^c = \vec{b}\vec{a} \neq \vec{a}\vec{b}$
symmetric dyad	$(\vec{a}\vec{b})^c = \vec{a}\vec{b}$

1.1.7 Vector basis

A vector basis in a three-dimensional space is a set of three vectors not in one plane. These vectors are referred to as independent. Each fourth vector can be expressed in the three base vectors.

When the vectors are mutually perpendicular, the basis is called orthogonal. If the basis consists of mutually perpendicular unit vectors, it is called orthonormal.

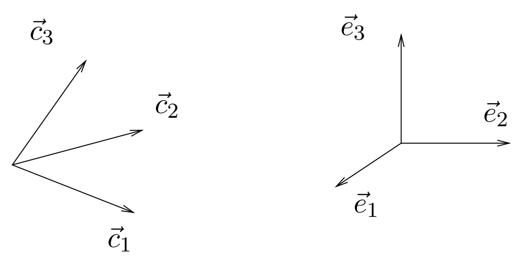


Fig. 1.8 : *A random and an orthonormal vector basis in three-dimensional space*

random basis	$\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$;	$\vec{c}_1 * \vec{c}_2 \cdot \vec{c}_3 \neq 0$
orthonormal basis	$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$	(δ_{ij} = Kronecker delta)
	$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \quad \rightarrow$	$\vec{e}_i \cdot \vec{e}_j = 0 \quad \vert \quad i \neq j \quad ; \quad \vec{e}_i \cdot \vec{e}_i = 1$
right-handed basis	$\vec{e}_1 * \vec{e}_2 = \vec{e}_3 \quad ;$	$\vec{e}_2 * \vec{e}_3 = \vec{e}_1 \quad ; \quad \vec{e}_3 * \vec{e}_1 = \vec{e}_2$

1.1.8 Matrix representation of a vector

In every point of a three-dimensional space three independent vectors exist. Here we assume that these base vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are orthonormal, i.e. orthogonal (= perpendicular) and having length 1. A fourth vector $\vec{a}$ can be written as a weighted sum of these base vectors. The coefficients are the components of $\vec{a}$ with relation to $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. The component a_i represents the length of the projection of the vector $\vec{a}$ on the line with direction $\vec{e}_i$.

We can denote this in several ways. In index notation a short version of the above mentioned summation is based on the Einstein summation convention. In column notation, (transposed) columns are used to store the components of $\vec{a}$ and the base vectors and the usual rules for the manipulation of columns apply.

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \sum_{i=1}^{3} a_i\vec{e}_i = a_i\vec{e}_i = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \underset{\sim}{a}^T \underset{\sim}{\vec{e}} = \underset{\sim}{\vec{e}}^T \underset{\sim}{a}$$

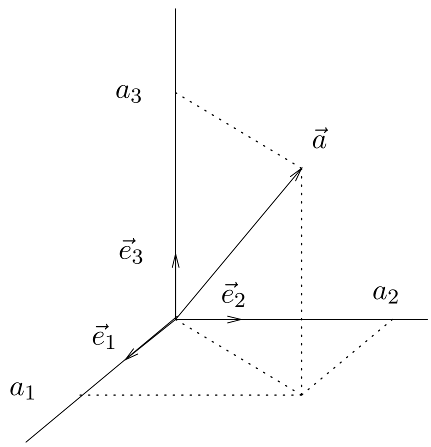


Fig. 1.9 : *A vector represented with components w.r.t. an orthonormal vector basis*

1.1.9 Components

The components of a vector $\vec{a}$ with respect to an orthonormal basis can be determined directly. All components, stored in column $\underset{\sim}{a}$, can then be calculated as the inner product of vector $\vec{a}$ and the column $\underset{\sim}{\vec{e}}$ containing the base vectors.

$$a_i = \vec{a} \cdot \vec{e}_i \qquad i = 1,2,3 \quad \rightarrow$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \vec{a} \cdot \vec{e}_1 \\ \vec{a} \cdot \vec{e}_2 \\ \vec{a} \cdot \vec{e}_3 \end{bmatrix} = \vec{a} \cdot \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \vec{a} \cdot \underset{\sim}{\vec{e}}$$

1.2 Coordinate systems

1.2.1 Cartesian coordinate system

A point in a Cartesian coordinate system is identified by three independent Cartesian coordinates, which measure distances along three perpendicular coordinate axes in a reference point, the origin.

In each point three coordinate axes exist which are parallel to the original coordinate axes. Base vectors are unit vectors tangential to the coordinate axes. They are orthogonal and independent of the Cartesian coordinates.

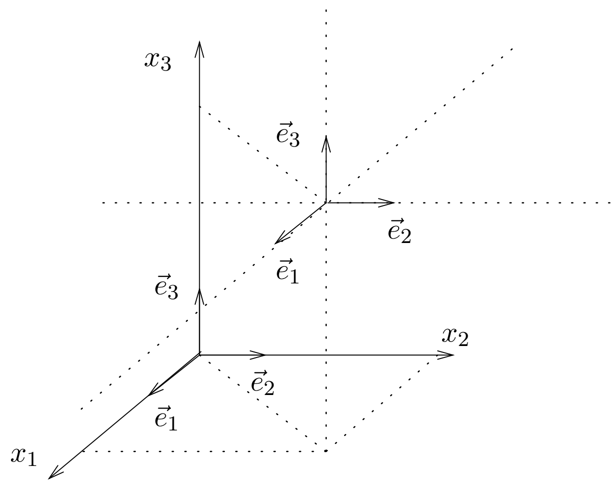


Fig. 1.10 : *Cartesian coordinate system*

Cartesian coordinates	:	(x_1, x_2, x_3)	or	(x, y, z)
Cartesian basis	:	$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$	or	$\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$

1.2.2 Cylindrical coordinate system

A point in a cylindrical coordinate system is identified by three independent cylindrical coordinates. Two of these measure a distance, respectively from (r) and along (z) a reference axis in a reference point, the origin. The third coordinate measures an angle (θ), rotating from a reference plane around the reference axis.

In each point three coordinate axes exist, two linear and one circular. Base vectors are unit vectors tangential to these coordinate axes. They are orthonormal and two of them depend on the angular coordinate.

The cylindrical coordinates can be transformed into Cartesian coordinates :

$$x_1 = r\cos(\theta) \quad ; \quad x_2 = r\sin(\theta) \quad ; \quad x_3 = z$$
$$r = \sqrt{x_1^2 + x_2^2} \quad ; \quad \theta = \arctan\left[\frac{x_2}{x_1}\right] \quad ; \quad z = x_3$$

The unit tangential vectors to the coordinate axes constitute an orthonormal vector base $\{\vec{e}_r, \vec{e}_t, \vec{e}_z\}$. The derivatives of these base vectors can be calculated.

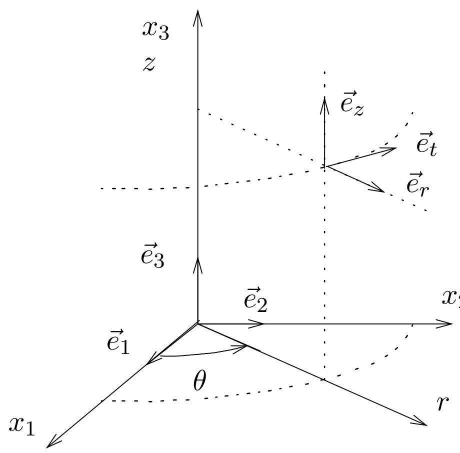


Fig. 1.11 : *Cylindrical coordinate system*

cylindrical coordinates	:	(r, θ, z)
cylindrical basis	:	$\{\vec{e}_r(\theta), \vec{e}_t(\theta), \vec{e}_z\}$

$$\vec{e}_r(\theta) = \cos(\theta)\vec{e}_1 + \sin(\theta)\vec{e}_2 \quad ; \quad \vec{e}_t(\theta) = -\sin(\theta)\vec{e}_1 + cos(\theta)\vec{e}_2 \quad ; \quad \vec{e}_z = \vec{e}_3$$

$$\frac{\partial \vec{e}_r}{\partial \theta} = -\sin(\theta)\vec{e}_1 + cos(\theta)\vec{e}_2 = \vec{e}_t \quad ; \quad \frac{\partial \vec{e}_t}{\partial \theta} = -\cos(\theta)\vec{e}_1 - \sin(\theta)\vec{e}_2 = -\vec{e}_r$$

1.2.3 Spherical coordinate system

A point in a spherical coordinate system is identified by three independent spherical coordinates. One measures a distance (r) from a reference point, the origin. The two other coordinates measure angles (θ and ϕ) w.r.t. two reference planes.

In each point three coordinate axes exist, one linear and two circular. Base vectors are unit vectors tangential to these coordinate axes. They are orthonormal and depend on the angular coordinates.

The spherical coordinates can be translated to Cartesian coordinates and vice versa :

$$
\begin{array}{lll}
x_1 = r\cos(\theta)\sin(\phi) & ; \quad x_2 = r\sin(\theta)\sin(\phi) & ; \quad x_3 = r\cos(\phi) \\
r = \sqrt{x_1^2 + x_2^2 + x_3^2} & ; \quad \phi = \arccos\left[\frac{x_3}{r}\right] & ; \quad \theta = \arctan\left[\frac{x_2}{x_1}\right]
\end{array}
$$

The unit tangential vectors to the coordinate axes constitute an orthonormal vector base $\{\vec{e}_r, \vec{e}_t, \vec{e}_\phi\}$. The derivatives of these base vectors can be calculated.

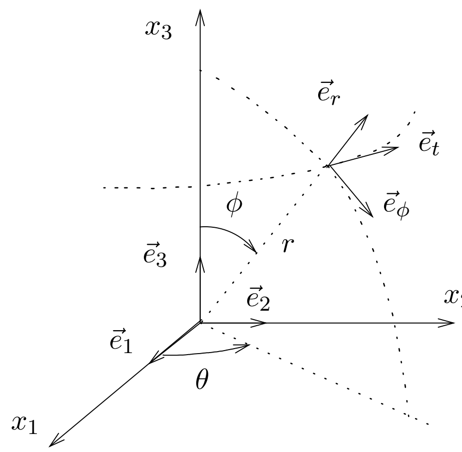


Fig. 1.12 : *Spherical coordinate system*

spherical coordinates	:	(r, θ, ϕ)
spherical basis	:	$\{\vec{e}_r(\theta,\phi), \vec{e}_t(\theta), \vec{e}_\phi(\theta,\phi)\}$

$$
\begin{aligned}
\vec{e}_r(\theta,\phi) &= \cos(\theta)\sin(\phi)\vec{e}_1 + \sin(\theta)\sin(\phi)\vec{e}_2 + \cos(\phi)\vec{e}_3 \\
\vec{e}_t(\theta) &= -\sin(\theta)\vec{e}_1 + cos(\theta)\vec{e}_2 \\
\vec{e}_\phi(\theta,\phi) &= \cos(\theta)\cos(\phi)\vec{e}_1 + \sin(\theta)\cos(\phi)\vec{e}_2 - \sin(\phi)\vec{e}_3
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \vec{e}_r}{\partial \theta} &= -\sin(\theta)\sin(\phi)\vec{e}_1 + \cos(\theta)\sin(\phi)\vec{e}_2 = \sin(\phi)\vec{e}_t \\
\frac{\partial \vec{e}_r}{\partial \phi} &= \cos(\theta)\cos(\phi)\vec{e}_1 + \sin(\theta)\cos(\phi)\vec{e}_2 - \sin(\phi)\vec{e}_3 = \vec{e}_\phi \\
\frac{d\vec{e}_t}{d\theta} &= -\cos(\theta)\vec{e}_1 - \sin(\theta)\vec{e}_2 = -\sin(\phi)\vec{e}_r - \cos(\phi)\vec{e}_\phi \\
\frac{\partial \vec{e}_\phi}{\partial \theta} &= -\sin(\theta)\cos(\phi)\vec{e}_1 + \cos(\theta)\cos(\phi)\vec{e}_2 = \cos(\phi)\vec{e}_t \\
\frac{\partial \vec{e}_\phi}{\partial \phi} &= -\cos(\theta)\sin(\phi)\vec{e}_1 - \sin(\theta)\sin(\phi)\vec{e}_2 - \cos(\phi)\vec{e}_3 = -\vec{e}_r
\end{aligned}
$$

1.2.4 Polar coordinates

In two dimensions the cylindrical coordinates are often referred to as polar coordinates.

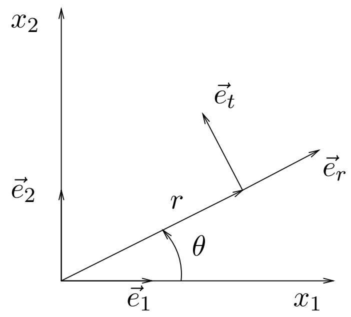


Fig. 1.13 : *Polar coordinates*

polar coordinates	:	(r, θ)
polar basis	:	$\{\vec{e}_r(\theta), \vec{e}_t(\theta)\}$

$$\vec{e}_r(\theta) = \cos(\theta)\vec{e}_1 + \sin(\theta)\vec{e}_2$$
$$\vec{e}_t(\theta) = \frac{d\vec{e}_r(\theta)}{d\theta} = -\sin(\theta)\vec{e}_1 + cos(\theta)\vec{e}_2 \qquad \rightarrow \qquad \frac{d\vec{e}_t(\theta)}{d\theta} = -\vec{e}_r(\theta)$$

1.3 Position vector

A point in a three-dimensional space can be identified with a position vector $\vec{x}$, originating from the fixed origin.

1.3.1 Position vector and Cartesian components

In a Cartesian coordinate system the components of this vector $\vec{x}$ w.r.t. the Cartesian basis are the Cartesian coordinates of the considered point.

The incremental position vector $d\vec{x}$ points from one point to a neighbor point and has its components w.r.t. the local Cartesian vector base.

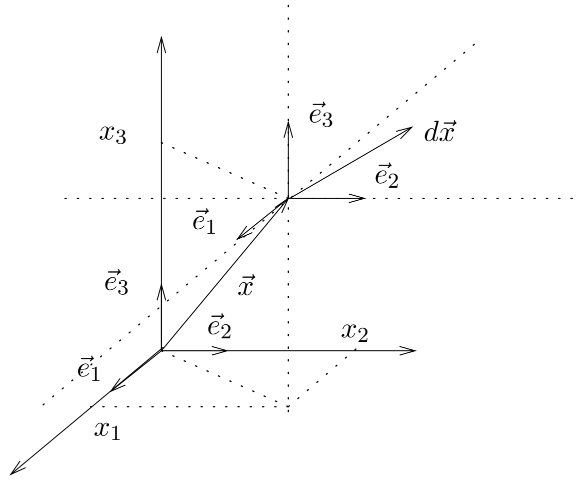


Fig. 1.14 : *Position vector*

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$
$$\vec{x} + d\vec{x} = (x_1 + dx_1)\vec{e}_1 + (x_2 + dx_2)\vec{e}_2 + (x_3 + dx_3)\vec{e}_3$$

incremental position vector	$d\vec{x} = dx_1\vec{e}_1 + dx_2\vec{e}_2 + dx_3\vec{e}_3$
components of $d\vec{x}$	$dx_1 = d\vec{x} \cdot \vec{e}_1 \quad ; \quad dx_2 = d\vec{x} \cdot \vec{e}_2 \quad ; \quad dx_3 = d\vec{x} \cdot \vec{e}_3$

1.3.2 Position vector and cylindrical components

In a cylindrical coordinate system the position vector $\vec{x}$ has two components.

The incremental position vector $d\vec{x}$ has three components w.r.t. the local cylindrical vector base.

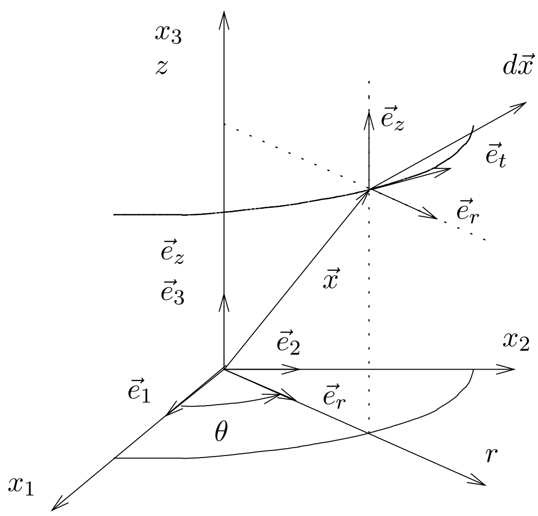


Fig. 1.15 : *Position vector*

$$
\begin{aligned}
\vec{x} &= r\vec{e}_r(\theta) + z\vec{e}_z \\
\vec{x} + d\vec{x} &= (r + dr)\vec{e}_r(\theta + d\theta) + (z + dz)\vec{e}_z \\
&= (r + dr)\left\{\vec{e}_r(\theta) + \frac{d\vec{e}_r}{d\theta}\,d\theta\right\} + (z + dz)\vec{e}_z \\
&= r\vec{e}_r(\theta) + z\vec{e}_z + r\vec{e}_t(\theta)d\theta + dr\vec{e}_r(\theta) + \vec{e}_t(\theta)drd\theta + dz\vec{e}_z
\end{aligned}
$$

incremental position vector $\quad d\vec{x} = dr\,\vec{e}_r(\theta) + r\,d\theta\,\vec{e}_t(\theta) + dz\,\vec{e}_z$

components of $d\vec{x}$ $\quad dr = d\vec{x}\cdot\vec{e}_r \quad ; \quad d\theta = \frac{1}{r}\,d\vec{x}\cdot\vec{e}_t \quad ; \quad dz = d\vec{x}\cdot\vec{e}_z$

1.3.3 Position vector and spherical components

In a spherical coordinate system the position vector $\vec{x}$ has only one component, which is its length.

The incremental position vector $d\vec{x}$ has three components w.r.t. the local spherical vector base.

$$
\begin{aligned}
\vec{x} &= r\vec{e}_r(\theta,\phi) \\
\vec{x} + d\vec{x} &= (r + dr)\vec{e}_r(\theta + d\theta, \phi + d\phi) \\
&= (r + dr)\left\{\vec{e}_r(\theta,\phi) + \frac{\partial\vec{e}_r}{\partial\theta}\,d\theta + \frac{\partial\vec{e}_r}{\partial\phi}\,d\phi\right\} \\
&= r\vec{e}_r(\theta,\phi) + r\sin(\phi)\vec{e}_t(\theta)d\theta + r\vec{e}_\phi(\theta,\phi)d\phi + dr\vec{e}_r(\theta,\phi)
\end{aligned}
$$

incremental position vector $\quad d\vec{x} = dr\,\vec{e}_r(\theta,\phi) + r\sin(\phi)\,d\theta\,\vec{e}_t(\theta) + r\,d\phi\,\vec{e}_\phi(\theta,\phi)$

components of $d\vec{x}$ $\quad dr = d\vec{x}\cdot\vec{e}_r \quad ; \quad d\theta = \frac{1}{r\sin(\phi)}\,d\vec{x}\cdot\vec{e}_t \quad ; \quad d\phi = \frac{1}{r}\,d\vec{x}\cdot\vec{e}_\phi$

1.4 Gradient operator

In mechanics (and physics in general) it is necessary to determine changes of scalars, vectors and tensors w.r.t. the spatial position. This means that derivatives w.r.t. the spatial coordinates have to be determined. An important operator, used to determine these spatial derivatives is the gradient operator.

1.4.1 Variation of a scalar function

Consider a scalar function f of the scalar variable x. The variation of the function value between two neighboring values of x can be expressed with a Taylor series expansion. If the variation is very small, this series can be linearized, which implies that only the first-order derivative of the function is taken into account.

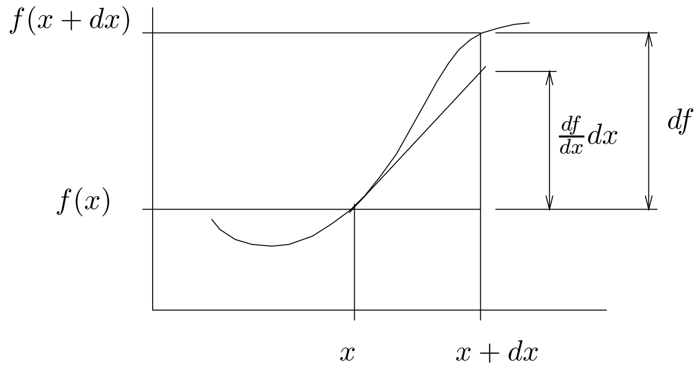


Fig. 1.16 : *Variation of a scalar function of one variable*

$$\begin{aligned} df &= f(x+dx) - f(x) \\ &= f(x) + \left.\frac{df}{dx}\right|_x dx + \tfrac{1}{2}\left.\frac{d^2f}{dx^2}\right|_x dx^2 + .. - f(x) \\ &\approx \left.\frac{df}{dx}\right|_x dx \end{aligned}$$

Consider a scalar function f of two independent variables x and y. The variation of the function value between two neighboring points can be expressed with a Taylor series expansion. If the variation is very small, this series can be linearized, which implies that only first-order derivatives of the function are taken into account.

$$\begin{aligned} df &= f(x+dx, y+dy) + \left.\frac{\partial f}{\partial x}\right|_{(x,y)} dx + \left.\frac{\partial f}{\partial y}\right|_{(x,y)} dy + .. - f(x,y) \\ &\approx \left.\frac{\partial f}{\partial x}\right|_{(x,y)} dx + \left.\frac{\partial f}{\partial y}\right|_{(x,y)} dy \end{aligned}$$

A function of three independent variables x, y and z can be differentiated likewise to give the variation.

$$df \approx \left.\frac{\partial f}{\partial x}\right|_{(x,y,z,)} dx + \left.\frac{\partial f}{\partial y}\right|_{(x,y,z,)} dy + \left.\frac{\partial f}{\partial z}\right|_{(x,y,z,)} dz +$$

Spatial variation of a Cartesian scalar function

Consider a scalar function of three Cartesian coordinates x, y and z. The variation of the function value between two neighboring points can be expressed with a linearized Taylor series expansion. The variation in the coordinates can be expressed in the incremental position vector between the two points. This leads to the gradient operator.

The gradient (or nabla or del) operator $\vec{\nabla}$ is not a vector, because it has no length or direction. The gradient of a scalar a is a vector : $\vec{\nabla} a$

$$
\begin{aligned}
da &= dx\frac{\partial a}{\partial x} + dy\frac{\partial a}{\partial y} + dz\frac{\partial a}{\partial z} = (d\vec{x}\cdot\vec{e}_x)\frac{\partial a}{\partial x} + (d\vec{x}\cdot\vec{e}_y)\frac{\partial a}{\partial y} + (d\vec{x}\cdot\vec{e}_z)\frac{\partial a}{\partial z} \\
&= d\vec{x}\cdot\left[\vec{e}_x\frac{\partial a}{\partial x} + \vec{e}_y\frac{\partial a}{\partial y} + \vec{e}_z\frac{\partial a}{\partial z}\right] = d\vec{x}\cdot(\vec{\nabla}a)
\end{aligned}
$$

gradient operator $$\vec{\nabla} = \left[\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right] = \underset{\sim}{\vec{e}}^T\underset{\sim}{\nabla} = \underset{\sim}{\nabla}^T\underset{\sim}{\vec{e}}$$

Spatial variation of a cylindrical scalar function

Consider a scalar function of three cylindrical coordinates r, θ and z. The variation of the function value between two neighboring points can be expressed with a linearized Taylor series expansion. The variation in the coordinates can be expressed in the incremental position vector between the two points. This leads to the gradient operator.

$$
\begin{aligned}
da &= dr\,\frac{\partial a}{\partial r} + d\theta\,\frac{\partial a}{\partial \theta} + dz\,\frac{\partial a}{\partial z} = (d\vec{x}\cdot\vec{e}_r)\,\frac{\partial a}{\partial r} + (\frac{1}{r}\,d\vec{x}\cdot\vec{e}_t)\,\frac{\partial a}{\partial \theta} + (d\vec{x}\cdot\vec{e}_z)\,\frac{\partial a}{\partial z} \\
&= d\vec{x}\cdot\left[\vec{e}_r\,\frac{\partial a}{\partial r} + \frac{1}{r}\,\vec{e}_t\,\frac{\partial a}{\partial \theta} + \vec{e}_z\,\frac{\partial a}{\partial z}\right] = d\vec{x}\cdot(\vec{\nabla}a)
\end{aligned}
$$

gradient operator $$\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \vec{e}_z\frac{\partial}{\partial z} = \underset{\sim}{\vec{e}}^T\underset{\sim}{\nabla} = \underset{\sim}{\nabla}^T\underset{\sim}{\vec{e}}$$

Spatial variation of a spherical scalar function

Consider a scalar function of three spherical coordinates r, θ and ϕ. The variation of the function value between two neighboring points can be expressed with a linearized Taylor series expansion. The variation in the coordinates can be expressed in the incremental position vector between the two points. This leads to the gradient operator.

$$
\begin{aligned}
da &= dr\frac{\partial a}{\partial r} + d\theta\frac{\partial a}{\partial \theta} + d\phi\frac{\partial a}{\partial \phi} = (d\vec{x}\cdot\vec{e}_r)\frac{\partial a}{\partial r} + (\frac{1}{r\sin(\phi)}\,d\vec{x}\cdot\vec{e}_t)\frac{\partial a}{\partial \theta} + (\frac{1}{r}\,d\vec{x}\cdot\vec{e}_\phi)\frac{\partial a}{\partial \phi} \\
&= d\vec{x}\cdot\left[\vec{e}_r\frac{\partial a}{\partial r} + \frac{1}{r\sin(\phi)}\,\vec{e}_t\frac{\partial a}{\partial \theta} + \frac{1}{r}\,\vec{e}_\phi\frac{\partial a}{\partial \phi}\right] = d\vec{x}\cdot(\vec{\nabla}a)
\end{aligned}
$$

gradient operator $$\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\frac{1}{r\sin(\phi)}\,\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r}\,\frac{\partial}{\partial \phi} = \underset{\sim}{\vec{e}}^T\underset{\sim}{\nabla} = \underset{\sim}{\nabla}^T\underset{\sim}{\vec{e}}$$

1.4.2 Spatial derivatives of a vector function

For a vector function $\vec{a}(\vec{x})$ the variation can be expressed as the inner product of the difference vector $d\vec{x}$ and the gradient of the vector $\vec{a}$. The latter entity is a dyad. The inner product of the gradient operator $\vec{\nabla}$ and $\vec{a}$ is called the divergence of $\vec{a}$. The outer product is referred to as the rotation or curl.

When cylindrical or spherical coordinates are used, the base vectors are (partly) functions of coordinates. Differentiation must than be done with care.

The gradient, divergence and rotation can be written in components w.r.t. a vector basis. The rather straightforward algebraic notation can be easily elaborated. However, the use of column/matrix notation results in shorter and more transparent expressions.

$$\mathsf{grad}(\vec{a}) = \vec{\nabla}\vec{a} \quad ; \quad \mathsf{div}(\vec{a}) = \vec{\nabla}\cdot\vec{a} \quad ; \quad \mathsf{rot}(\vec{a}) = \vec{\nabla} * \vec{a}$$

Cartesian components

The gradient of a vector $\vec{a}$ can be written in components w.r.t. the Cartesian vector basis $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$. The base vectors are independent of the coordinates, so only the components of the vector need to be differentiated. The divergence is the inner product of $\vec{\nabla}$ and $\vec{a}$ and thus results in a scalar value. The curl results in a vector.

$$\begin{aligned}
\vec{\nabla}\vec{a} &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)(a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \\
&= \vec{e}_x a_{x,x}\vec{e}_x + \vec{e}_x a_{y,x}\vec{e}_y + \vec{e}_x a_{z,x}\vec{e}_z + \vec{e}_y a_{x,y}\vec{e}_x + \\
&\quad \vec{e}_y a_{y,y}\vec{e}_y + \vec{e}_y a_{z,y}\vec{e}_z + \vec{e}_z a_{x,z}\vec{e}_x + \vec{e}_z a_{y,z}\vec{e}_y + \vec{e}_z a_{z,z}\vec{e}_z \\
&= \begin{bmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{bmatrix} = \underset{\sim}{\vec{e}}^T \left(\underset{\sim}{\nabla}\, \underset{\sim}{a}^T\right) \underset{\sim}{\vec{e}} \\
\vec{\nabla}\cdot\vec{a} &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)\cdot(a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \\
&= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = \mathsf{tr}\left(\underset{\sim}{\nabla}\, \underset{\sim}{a}^T\right) = \mathsf{tr}\left(\vec{\nabla}\vec{a}\right) \\
\vec{\nabla} * \vec{a} &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right) * (a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \\
&= \left\{a_{z,y} - a_{y,z}\right\}\vec{e}_x + \left\{a_{x,z} - a_{z,x}\right\}\vec{e}_y + \left\{a_{y,x} - a_{x,y}\right\}\vec{e}_z
\end{aligned}$$

Cylindrical components

The gradient of a vector $\vec{a}$ can be written in components w.r.t. the cylindrical vector basis $\{\vec{e}_r, \vec{e}_t, \vec{e}_z\}$. The base vectors $\vec{e}_r$ and $\vec{e}_t$ depend on the coordinate θ, so they have to be differentiated together with the components of the vector. The result is a 3×3 matrix.

$$\begin{aligned}
\vec{\nabla}\vec{a} &= \{\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_z\frac{\partial}{\partial z}\}\{a_r\vec{e}_r + a_t\vec{e}_t + a_z\vec{e}_z\} \\
&= \vec{e}_r a_{r,r}\vec{e}_r + \vec{e}_r a_{t,r}\vec{e}_t + \vec{e}_r a_{z,r}\vec{e}_z + \vec{e}_t\frac{1}{r}a_{r,t}\vec{e}_r + \vec{e}_t\frac{1}{r}a_{t,t}\vec{e}_t + \vec{e}_t\frac{1}{r}a_{z,t}\vec{e}_z + \vec{e}_t\frac{1}{r}a_r\vec{e}_t - \vec{e}_t\frac{1}{r}a_t\vec{e}_r \\
&\quad \vec{e}_z a_{r,z}\vec{e}_r + \vec{e}_z a_{t,z}\vec{e}_t + \vec{e}_z a_{z,z}\vec{e}_z
\end{aligned}$$

$$
\begin{aligned}
&= \underset{\sim}{\vec{e}}^T \left\{ \left(\underset{\sim}{\nabla}\underset{\sim}{a}^T\right) \underset{\sim}{\vec{e}} + \left[\begin{array}{c} 0 \\ \frac{1}{r}\,\vec{e}_t a_r - \frac{1}{r}\,\vec{e}_r a_t \\ 0 \end{array} \right] \} \right\} = \underset{\sim}{\vec{e}}^T \left\{ \left(\underset{\sim}{\nabla}\underset{\sim}{a}^T\right) \underset{\sim}{\vec{e}} + \left[\begin{array}{ccc} 0 & 0 & 0 \\ -\frac{1}{r}\,a_t & \frac{1}{r}\,a_r & 0 \\ 0 & 0 & 0 \end{array} \right] \underset{\sim}{\vec{e}} \right\} \\
&= \underset{\sim}{\vec{e}}^T (\underset{\sim}{\nabla}\underset{\sim}{a}^T)\underset{\sim}{\vec{e}} + \underset{\sim}{\vec{e}}^T \underline{h}\,\underset{\sim}{\vec{e}}
\end{aligned}
$$

$$
\vec{\nabla} \cdot \vec{a} = \mathsf{tr}(\underset{\sim}{\nabla}\underset{\sim}{a}^T) + \mathsf{tr}(\underline{h}) = a_{r,r} + \frac{1}{r}\,a_{t,t} + a_{z,z} + \frac{1}{r}\,a_r
$$

$$
\begin{aligned}
\vec{\nabla} * \vec{a} &= \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_t \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z}\right) * (a_r\vec{e}_r + a_t\vec{e}_t + a_z\vec{e}_z) \\
&= \vec{e}_r * \left\{ a_{r,r}\vec{e}_r + a_{t,r}\vec{e}_t + a_{z,r}\vec{e}_z \right\} + \vec{e}_t * \frac{1}{r}\left\{ a_{r,t}\vec{e}_r + a_r\vec{e}_t + a_{t,t}\vec{e}_t - a_t\vec{e}_r + a_{z,t}\vec{e}_z \right\} + \\
&\quad \vec{e}_z * \left\{ a_{r,z}\vec{e}_r + a_{t,z}\vec{e}_t + a_{z,z}\vec{e}_z \right\} \\
&= a_{t,r}\vec{e}_z - a_{z,r}\vec{e}_t + \frac{1}{r}\left\{ - a_{r,t}\vec{e}_z + a_t\vec{e}_z + a_{z,t}\vec{e}_r \right\} + a_{r,z}\vec{e}_t - a_{t,z}\vec{e}_r \\
&= \left[\frac{1}{r}a_{z,t} - a_{t,z}\right]\vec{e}_r + \left[a_{r,z} - a_{z,r}\right]\vec{e}_t + \left[a_{t,r} - \frac{1}{r}a_{r,t} + \frac{1}{r}a_t\right]\vec{e}_z
\end{aligned}
$$

Laplace operator

The Laplace operator appears in many equations. It is the inner product of the gradient operator with itself.

Laplace operator	$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$
Cartesian components	$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
cylindrical components	$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$
spherical components	$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \left(\frac{1}{r\sin(\phi)}\right)^2 \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2\tan(\phi)}\frac{\partial}{\partial \phi}$

1.5 2nd-order tensor

A scalar function f takes a scalar variable, e.g. p as input and produces another scalar variable, say q, as output, so we have :

$$
q = f(p) \qquad \text{or} \qquad p \xrightarrow{f} q
$$

Such a function is also called a projection.

Instead of input and output being a scalar, they can also be a vector. A tensor is the equivalent of a function f in this case. What makes tensors special is that they are linear functions, a very important property. A tensor is written here in bold face character. The tensors which are introduced first and will be used most of the time are second-order tensors. Each second-order tensor can be written as a summation of a number of dyads. Such a representation is not unique, in fact the number of variations is infinite. With this representation in mind we can accept that the tensor relates input and output vectors with an inner product :

$$\vec{q} = \boldsymbol{A} \cdot \vec{p} \qquad \text{or} \qquad \vec{p} \xrightarrow{\boldsymbol{A}} \vec{q}$$

tensor = linear projection	$\boldsymbol{A} \cdot (\alpha\vec{m} + \beta\vec{n}) = \alpha\boldsymbol{A} \cdot \vec{m} + \beta\boldsymbol{A} \cdot \vec{n}$
representation	$\boldsymbol{A} = \alpha_1\vec{a}_1\vec{b}_1 + \alpha_2\vec{a}_2\vec{b}_2 + \alpha_3\vec{a}_3\vec{b}_3 + ..$

1.5.1 Components of a tensor

As said before, a second-order tensor can be represented as a summation of dyads and this can be done in an infinite number of variations. When we write each vector of the dyadic products in components w.r.t. a three-dimensional vector basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ it is immediately clear that all possible representations result in the same unique sum of nine independent dyadic products of the base vectors. The coefficients of these dyads are the components of the tensor w.r.t. the vector basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

The components of the tensor $\boldsymbol{A}$ can be stored in a 3×3 matrix $\underline{A}$. The components can also be stored in a column. In that case the sequence of these columns must be always chosen the same. This latter *column notation* of a tensor is especially convenient for certain mathematical elaborations and also for the manipulations with fourth-order tensors, which will be introduced later.

$$\boldsymbol{A} = \alpha_1\vec{a}_1\vec{b}_1 + \alpha_2\vec{a}_2\vec{b}_2 + \alpha_3\vec{a}_3\vec{b}_3 + ..$$

each vector in components w.r.t. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ $\quad\rightarrow$

$$\begin{aligned}
\boldsymbol{A} &= \alpha_1(a_{11}\vec{e}_1 + a_{12}\vec{e}_2 + a_{13}\vec{e}_3)(b_{11}\vec{e}_1 + b_{12}\vec{e}_2 + b_{13}\vec{e}_3) + \\
&\quad\; \alpha_2(a_{21}\vec{e}_1 + a_{22}\vec{e}_2 + a_{23}\vec{e}_3)(b_{21}\vec{e}_1 + b_{22}\vec{e}_2 + b_{23}\vec{e}_3) + ... \\
&= A_{11}\vec{e}_1\vec{e}_1 + A_{12}\vec{e}_1\vec{e}_2 + A_{13}\vec{e}_1\vec{e}_3 + \\
&\quad\; A_{21}\vec{e}_2\vec{e}_1 + A_{22}\vec{e}_2\vec{e}_2 + A_{23}\vec{e}_2\vec{e}_3 + \\
&\quad\; A_{31}\vec{e}_3\vec{e}_1 + A_{32}\vec{e}_3\vec{e}_2 + A_{33}\vec{e}_3\vec{e}_3
\end{aligned}$$

matrix notation

$$\boldsymbol{A} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \underset{\sim}{\vec{e}}^T \underline{A}\, \underset{\sim}{\vec{e}}$$

column notation

$$\boldsymbol{A} \quad\rightarrow\quad \underline{A} \quad\rightarrow\quad \underset{\approx}{A} = \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{12} & A_{21} & A_{23} & A_{32} & A_{31} & A_{13} \end{bmatrix}^T$$

1.5.2 Spatial derivatives of a tensor function

For a tensor function $\boldsymbol{A}(\vec{x})$ the gradient, divergence and rotation or curl can be calculated. When cylindrical or spherical coordinates are used, the base vectors are (partly) functions of coordinates. Differentiation must than be done with care.

The gradient, divergence and rotation can be written in components w.r.t. a vector basis. The rather straightforward algebraic notation can be easily elaborated. However, the use of column/matrix notation results in shorter and more transparent expressions.

Only the divergence w.r.t. a cylindrical vector basis is elaborate here.

$$\mathsf{grad}(\boldsymbol{A}) = \vec{\nabla}\boldsymbol{A} \quad ; \quad \mathsf{div}(\boldsymbol{A}) = \vec{\nabla}\cdot\boldsymbol{A} \quad ; \quad \mathsf{rot}(\boldsymbol{A}) = \vec{\nabla} * \boldsymbol{A}$$

Divergence of a tensor in cylindrical components

The divergence of a second order tensor can be written in components w.r.t. a cylindrical coordinate system. Index notation is used as an intermediate in this derivation. The indices i, j, k take the "values" $1 = r, 2 = t(= \theta), 3 = z$.

$$\begin{aligned}
\vec{\nabla}\cdot\boldsymbol{A} &= \vec{e}_i\cdot\nabla_i(\vec{e}_j A_{jk}\vec{e}_k) \\
&= \vec{e}_i\cdot(\nabla_i\vec{e}_j)A_{jk}\vec{e}_k + \vec{e}_i\cdot\vec{e}_j(\nabla_i A_{jk})\vec{e}_k + \vec{e}_i\cdot\vec{e}_j A_{jk}(\nabla_i\vec{e}_k) \\
&= \vec{e}_i\cdot(\nabla_i\vec{e}_j)A_{jk}\vec{e}_k + \delta_{ij}(\nabla_i A_{jk})\vec{e}_k + \delta_{ij}A_{jk}(\nabla_i\vec{e}_k)
\end{aligned}$$

$$\nabla_i\vec{e}_j = \delta_{i2}\delta_{1j}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2j}\,\frac{1}{r}\,\vec{e}_r$$

$$\begin{aligned}
&= \vec{e}_i\cdot(\delta_{i2}\delta_{1j}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2j}\,\frac{1}{r}\,\vec{e}_r)A_{jk}\vec{e}_k + \delta_{ij}(\nabla_i A_{jk})\vec{e}_k + \delta_{ij}A_{jk}(\delta_{i2}\delta_{1k}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2k}\,\frac{1}{r}\,\vec{e}_r) \\
&= \vec{e}_i\cdot(\delta_{i2}\delta_{1j}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2j}\,\frac{1}{r}\,\vec{e}_r)A_{jk}\vec{e}_k + \delta_{ij}(\nabla_i A_{jk})\vec{e}_k + (\delta_{i2}\delta_{1k}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2k}\,\frac{1}{r}\,\vec{e}_r)A_{jk}\delta_{ij} \\
&= \vec{e}_t\cdot(\delta_{1j}\,\frac{1}{r}\,\vec{e}_t - \delta_{2j}\,\frac{1}{r}\,\vec{e}_r)A_{jk}\vec{e}_k + (\nabla_j A_{jk})\vec{e}_k + (\delta_{j2}\delta_{1k}\,\frac{1}{r}\,\vec{e}_t - \delta_{j2}\delta_{2k}\,\frac{1}{r}\,\vec{e}_r)A_{jk} \\
&= \delta_{1j}\,\frac{1}{r}\,A_{jk}\vec{e}_k + (\nabla_j A_{jk})\vec{e}_k + (\delta_{j2}\delta_{1k}\,\frac{1}{r}\,\vec{e}_t - \delta_{j2}\delta_{2k}\,\frac{1}{r}\,\vec{e}_r)A_{jk} \\
&= \frac{1}{r}A_{1k}\vec{e}_k + (\nabla_j A_{jk})\vec{e}_k + \frac{1}{r}\,(A_{21}\vec{e}_t - A_{22}\vec{e}_r) \\
&= (\frac{1}{r}A_{11} - \frac{1}{r}A_{22})\vec{e}_1 + (\frac{1}{r}A_{12} + \frac{1}{r}A_{21})\vec{e}_2 + \frac{1}{r}A_{13}\vec{e}_3 + (\nabla_j A_{jk})\vec{e}_k \\
&= g_k\vec{e}_k + \nabla_j A_{jk}\vec{e}_k \\
&= \underset{\sim}{g}^T\underset{\sim}{\vec{e}} + (\underset{\sim}{\nabla}^T\underline{A})\underset{\sim}{\vec{e}} \\
&= (\underset{\sim}{\nabla}^T\underline{A})\underset{\sim}{\vec{e}} + \underset{\sim}{g}^T\underset{\sim}{\vec{e}} \qquad \text{with} \qquad \underset{\sim}{g}^T = \frac{1}{r}\left[\ (A_{11} - A_{22}) \quad (A_{12} + A_{21}) \quad A_{33}\ \right]
\end{aligned}$$

1.5.3 Special tensors

Some second-order tensors are considered special with regard to their results and/or their representation.

The dyad is generally considered to be a special second-order tensor. The null or zero tensor projects each vector onto the null vector which has zero length and undefined direction. The unit tensor projects each vector onto itself. Conjugating (or transposing) a second-order tensor implies conjugating each of its dyads. We could also say that the front- and back-end of the tensor are interchanged.

Matrix representation of special tensors, e.g. the null tensor, the unity tensor and the conjugate of a tensor, result in obvious matrices.

dyad	:	$\vec{a}\vec{b}$		
null tensor	:	$\boldsymbol{O}$	$\rightarrow$	$\boldsymbol{O} \cdot \vec{p} = \vec{0}$
unit tensor	:	$\boldsymbol{I}$	$\rightarrow$	$\boldsymbol{I} \cdot \vec{p} = \vec{p}$
conjugated	:	$\boldsymbol{A}^c$	$\rightarrow$	$\boldsymbol{A}^c \cdot \vec{p} = \vec{p} \cdot \boldsymbol{A}$

null tensor $\rightarrow$ null matrix

$$\underline{O} = \underset{\sim}{\vec{e}} \cdot \boldsymbol{O} \cdot \underset{\sim}{\vec{e}}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

unity tensor $\rightarrow$ unity matrix

$$\underline{I} = \underset{\sim}{\vec{e}} \cdot \boldsymbol{I} \cdot \underset{\sim}{\vec{e}}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad \boldsymbol{I} = \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3 = \underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}}$$

conjugate tensor $\rightarrow$ transpose matrix

$$\underline{A} = \underset{\sim}{\vec{e}} \cdot \boldsymbol{A} \cdot \underset{\sim}{\vec{e}}^T \quad \rightarrow \quad \underline{A}^T = \underset{\sim}{\vec{e}} \cdot \boldsymbol{A}^c \cdot \underset{\sim}{\vec{e}}^T$$

1.5.4 Manipulations

We already saw that we can conjugate a second-order tensor, which is an obvious manipulation, taking in mind its representation as the sum of dyads. This also leads automatically to multiplication of a tensor with a scalar, summation and taking the inner product of two tensors. The result of all these basic manipulations is a new second-order tensor.

When we take the double inner product of two second-order tensors, the result is a scalar value, which is easily understood when both tensors are considered as the sum of dyads again.

scalar multiplication	$\boldsymbol{B} = \alpha \boldsymbol{A}$
summation	$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$
inner product	$\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$

double inner product $\qquad \boldsymbol{A} : \boldsymbol{B} = \boldsymbol{A}^c : \boldsymbol{B}^c = \text{scalar}$

$$\text{NB :} \qquad \boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A}$$
$$\boldsymbol{A}^2 = \boldsymbol{A} \cdot \boldsymbol{A} \qquad ; \qquad \boldsymbol{A}^3 = \boldsymbol{A} \cdot \boldsymbol{A} \cdot \boldsymbol{A} \qquad ; \qquad \text{etc.}$$

1.5.5 Scalar functions of a tensor

Scalar functions of a tensor exist, which play an important role in physics and mechanics. As their name indicates, the functions result in a scalar value. This value is independent of the matrix representation of the tensor. In practice, components w.r.t. a chosen vector basis are used to calculate this value. However, the resulting value is independent of the chosen base and therefore the functions are called invariants of the tensor. Besides the Euclidean norm, we introduce three other (fundamental) invariants of the second-order tensor.

1.5.6 Euclidean norm

The first function presented here is the Euclidean norm of a tensor, which can be seen as a kind of weight or length. A vector base is in fact not needed at all to calculate the Euclidean norm. Only the length of a vector has to be measured. The Euclidean norm has some properties that can be proved which is not done here. Besides the Euclidean norm other norms can be defined.

$$m = ||\boldsymbol{A}|| = \max_{\vec{e}} ||\boldsymbol{A} \cdot \vec{e}\,|| \qquad \forall \quad \vec{e} \quad \text{with} \quad ||\vec{e}|| = 1$$

properties

1. $||\boldsymbol{A}|| \geq 0$
2. $||\alpha \boldsymbol{A}|| = |\alpha| \; ||\boldsymbol{A}||$
3. $||\boldsymbol{A} \cdot \boldsymbol{B}|| \leq ||\boldsymbol{A}|| \; ||\boldsymbol{B}||$
4. $||\boldsymbol{A} + \boldsymbol{B}|| \leq ||\boldsymbol{A}|| + ||\boldsymbol{B}||$

1.5.7 1st invariant

The first invariant is also called the trace of the tensor. It is a linear function. Calculation of the trace is easily done using the matrix of the tensor w.r.t. an orthonormal vector basis.

$$\begin{aligned} J_1(\boldsymbol{A}) &= \text{tr}(\boldsymbol{A}) \\ &= \frac{1}{\vec{c}_1 * \vec{c}_2 \cdot \vec{c}_3} [\vec{c}_1 \cdot \boldsymbol{A} \cdot (\vec{c}_2 * \vec{c}_3) + \text{cycl.}] \end{aligned}$$

properties

1. $J_1(\boldsymbol{A}) = J_1(\boldsymbol{A}^c)$
2. $J_1(\boldsymbol{I}) = 3$
3. $J_1(\alpha \boldsymbol{A}) = \alpha J_1(\boldsymbol{A})$
4. $J_1(\boldsymbol{A} + \boldsymbol{B}) = J_1(\boldsymbol{A}) + J_1(\boldsymbol{B})$
5. $J_1(\boldsymbol{A} \cdot \boldsymbol{B}) = \boldsymbol{A} : \boldsymbol{B} \quad \rightarrow \quad J_1(\boldsymbol{A}) = \boldsymbol{A} : \boldsymbol{I}$

1.5.8 2nd invariant

The second invariant can be calculated as a function of the trace of the tensor and the trace of the tensor squared. The second invariant is a quadratic function of the tensor.

$$\begin{aligned} J_2(\boldsymbol{A}) &= \tfrac{1}{2}\{\mathsf{tr}^2(\boldsymbol{A}) - \mathsf{tr}(\boldsymbol{A}^2)\} \\ &= \frac{1}{\vec{c}_1 * \vec{c}_2 \cdot \vec{c}_3}[\vec{c}_1 \cdot (\boldsymbol{A} \cdot \vec{c}_2) * (\boldsymbol{A} \cdot \vec{c}_3) + \mathsf{cycl.}] \end{aligned}$$

properties

1. $J_2(\boldsymbol{A}) = J_2(\boldsymbol{A}^c)$
2. $J_2(\boldsymbol{I}) = 3$
3. $J_2(\alpha\boldsymbol{A}) = \alpha^2 J_2(\boldsymbol{A})$

1.5.9 3rd invariant

The third invariant is also called the determinant of the tensor. It is easily calculated from the matrix w.r.t. an orthonormal vector basis. The determinant is a third-order function of the tensor. It is an important value when it comes to check whether a tensor is regular or singular.

$$\begin{aligned} J_3(\boldsymbol{A}) &= \det(\boldsymbol{A}) \\ &= \frac{1}{\vec{c}_1 * \vec{c}_2 \cdot \vec{c}_3}[(\boldsymbol{A} \cdot \vec{c}_1) \cdot (\boldsymbol{A} \cdot \vec{c}_2) * (\boldsymbol{A} \cdot \vec{c}_3)] \end{aligned}$$

properties

1. $J_3(\boldsymbol{A}) = J_3(\boldsymbol{A}^c)$
2. $J_3(\boldsymbol{I}) = 1$
3. $J_3(\alpha\boldsymbol{A}) = \alpha^3 J_3(\boldsymbol{A})$
4. $J_3(\boldsymbol{A} \cdot \boldsymbol{B}) = J_3(\boldsymbol{A}) J_3(\boldsymbol{B})$

1.5.10 Invariants w.r.t. an orthonormal basis

From the matrix of a tensor w.r.t. an orthonormal basis, the three invariants can be calculated straightforwardly.

$$\boldsymbol{A} \quad \rightarrow \quad \underline{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$
\begin{aligned}
J_1(\boldsymbol{A}) &= \mathsf{tr}(\boldsymbol{A}) = \mathsf{tr}(\underline{A}) \\
&= A_{11} + A_{22} + A_{33} \\[2ex]
J_2(\boldsymbol{A}) &= \tfrac{1}{2}\{\mathsf{tr}^2(\boldsymbol{A}) - \mathsf{tr}(\boldsymbol{A}^2)\} \\[2ex]
J_3(\boldsymbol{A}) &= \det \boldsymbol{A} = \det(\underline{A}) \\
&= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{21}A_{32}A_{13} \\
&\qquad - (A_{13}A_{22}A_{31} + A_{12}A_{21}A_{33} + A_{23}A_{32}A_{11})
\end{aligned}
$$

1.5.11 Regular $\sim$ singular tensor

When a second-order tensor is regular, its determinant is not zero. If the inner product of a regular tensor with a vector results in the null vector, it must be so that the former vector is also a null vector. Considering the matrix of the tensor, this implies that its rows and columns are independent.

A second-order tensor is singular, when its determinant equals zero. In that case the inner product of the tensor with a vector, being not the null vector, **may** result in the null vector. The rows and columns of the matrix of the singular tensor are dependent.

$$
\begin{array}{lclcl}
\det(\boldsymbol{A}) \neq 0 & \leftrightarrow & \boldsymbol{A} \quad \textbf{regular} & \leftrightarrow & [\boldsymbol{A} \cdot \vec{a} = \vec{0} \leftrightarrow \vec{a} = \vec{0}] \\
\det(\boldsymbol{A}) = 0 & \leftrightarrow & \boldsymbol{A} \quad \textbf{singular} & \leftrightarrow & [\boldsymbol{A} \cdot \vec{a} = \vec{0} \quad : \quad \vec{a} \neq \vec{0}]
\end{array}
$$

1.5.12 Eigenvalues and eigenvectors

Taking the inner product of a tensor with one of its eigenvectors results in a vector with the same direction – better : working line – as the eigenvector, but not necessarily the same length. It is standard procedure that an eigenvector is taken to be a unit vector. The length of the new vector is the eigenvalue associated with the eigenvector.

$$
\boldsymbol{A} \cdot \vec{n} = \lambda\, \vec{n} \qquad \text{with} \qquad \vec{n} \neq \vec{0}
$$

From its definition we can derive an equation from which the eigenvalues and the eigenvectors can be determined. The coefficient tensor of this equation must be singular, as eigenvectors are never the null vector. Demanding its determinant to be zero results in a third-order equation, the characteristic equation, from which the eigenvalues can be solved.

$$
\begin{array}{l}
\boldsymbol{A} \cdot \vec{n} = \lambda \vec{n} \quad \rightarrow \quad \boldsymbol{A} \cdot \vec{n} - \lambda \vec{n} = \vec{0} \quad \rightarrow \quad \boldsymbol{A} \cdot \vec{n} - \lambda \boldsymbol{I} \cdot \vec{n} = \vec{0} \quad \rightarrow \\
(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \vec{n} = \vec{0} \quad \text{with} \quad \vec{n} \neq \vec{0} \quad \rightarrow \\
\boldsymbol{A} - \lambda \boldsymbol{I} \quad \text{singular} \quad \rightarrow \quad \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \quad \rightarrow \\
\det(\underline{A} - \lambda \underline{I}) = 0 \quad \rightarrow \quad \text{characteristic equation} \\
\text{characteristic equation :} \quad \text{3 roots} \quad : \quad \lambda_1, \lambda_2, \lambda_3
\end{array}
$$

After determining the eigenvalues, the associated eigenvectors can be determined from the original equation.

eigenvector to λ_i, $i \in \{1,2,3\}$:	$(\boldsymbol{A} - \lambda_i \boldsymbol{I}) \cdot \vec{n}_i = \vec{0}$ or	$(\underline{A} - \lambda_i \underline{I})\underset{\sim}{n}_i = \underset{\sim}{0}$

dependent set of equations	$\rightarrow$	only ratio $n_1 : n_2 : n_3$ can be calculated
components n_1, n_2, n_3 calculation	$\rightarrow$	extra equation necessary
normalize eigenvectors	$\rightarrow$	$\lVert\vec{n}_i\rVert = 1 \quad \rightarrow \quad n_1^2 + n_2^2 + n_3^2 = 1$

It can be shown that the three eigenvectors are orthonormal when all three eigenvalues have different values. When two eigenvalues are equal, the two associated eigenvectors can be chosen perpendicular to each other, being both already perpendicular to the third eigenvector. With all eigenvalues equal, each set of three orthonormal vectors are principal directions. The tensor is then called 'isotropic'.

1.5.13 Relations between invariants

The three principal invariants of a tensor are related through the Cayley-Hamilton theorem. The lemma of Cayley-Hamilton states that every second-order tensor obeys its own characteristic equation. This rule can be used to reduce tensor equations to maximum second-order. The invariants of the inverse of a non-singular tensor are related. Any function of the principal invariants of a tensor is invariant as well.

Cayley-Hamilton theorem $\qquad \boldsymbol{A}^3 - J_1(\boldsymbol{A})\boldsymbol{A}^2 + J_2(\boldsymbol{A})\boldsymbol{A} - J_3(\boldsymbol{A})\boldsymbol{I} = \boldsymbol{O}$

relation between invariants of $\boldsymbol{A}^{-1}$

$$J_1(\boldsymbol{A}^{-1}) = \frac{J_2(\boldsymbol{A})}{J_3(\boldsymbol{A})} \quad ; \quad J_2(\boldsymbol{A}^{-1}) = \frac{J_1(\boldsymbol{A})}{J_3(\boldsymbol{A})} \quad ; \quad J_3(\boldsymbol{A}^{-1}) = \frac{1}{J_3(\boldsymbol{A})}$$

1.6 Special tensors

Physical phenomena and properties are commonly characterized by tensorial variables. In derivations the *inverse* of a tensor is frequently needed and can be calculated uniquely when the tensor is regular. In continuum mechanics the *deviatoric* and *hydrostatic* part of a tensor are often used.

Tensors may have specific properties due to the nature of physical phenomena and quantities. Many tensors are for instance *symmetric*, leading to special features concerning eigenvalues and eigenvectors. Rotation (rate) is associated with *skew-symmetric* and *orthogonal* tensors.

inverse tensor	$\boldsymbol{A}^{-1} \quad \rightarrow \quad \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$
deviatoric part of a tensor	$\boldsymbol{A}^d = \boldsymbol{A} - \frac{1}{3}\mathrm{tr}(\boldsymbol{A})\boldsymbol{I}$
symmetric tensor	$\boldsymbol{A}^c = \boldsymbol{A}$
skew-symmetric tensor	$\boldsymbol{A}^c = -\boldsymbol{A}$
positive definite tensor	$\vec{a} \cdot \boldsymbol{A} \cdot \vec{a} > 0 \quad \forall \quad \vec{a} \neq \vec{0}$
orthogonal tensor	$(\boldsymbol{A} \cdot \vec{a}) \cdot (\boldsymbol{A} \cdot \vec{b}) = \vec{a} \cdot \vec{b} \quad \forall \quad \vec{a}, \vec{b}$
adjugated tensor	$(\boldsymbol{A} \cdot \vec{a}) * (\boldsymbol{A} \cdot \vec{b}) = \boldsymbol{A}^a \cdot (\vec{a} * \vec{b}) \quad \forall \quad \vec{a}, \vec{b}$

1.6.1 Inverse tensor

The inverse $\boldsymbol{A}^{-1}$ of a tensor $\boldsymbol{A}$ only exists if $\boldsymbol{A}$ is regular, i.e. if $\det(\boldsymbol{A}) \neq 0$. Inversion is applied to solve $\vec{x}$ from the equation $\boldsymbol{A} \cdot \vec{x} = \vec{y}$ giving $\vec{x} = \boldsymbol{A}^{-1} \cdot \vec{y}$.

The inverse of a tensor product $\boldsymbol{A} \cdot \boldsymbol{B}$ equals $\boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1}$, so the sequence of the tensors is reversed.

The matrix $\underline{A}^{-1}$ of tensor $\boldsymbol{A}^{-1}$ is the inverse of the matrix $\underline{A}$ of $\boldsymbol{A}$. Calculation of $\underline{A}^{-1}$ can be done with various algorithms.

$$\det(\boldsymbol{A}) \neq 0 \quad \leftrightarrow \quad \exists! \quad \boldsymbol{A}^{-1} \quad | \quad \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$$

property

$$\left.\begin{array}{l} (\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \vec{a} = \vec{b} \quad \rightarrow \quad \vec{a} = (\boldsymbol{A} \cdot \boldsymbol{B})^{-1} \cdot \vec{b} \\ (\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \vec{a} = \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \vec{a}) = \vec{b} \quad \rightarrow \\ \qquad \boldsymbol{B} \cdot \vec{a} = \boldsymbol{A}^{-1} \cdot \vec{b} \quad \rightarrow \quad \vec{a} = \boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1} \cdot \vec{b} \end{array}\right\} \quad \rightarrow$$

$$(\boldsymbol{A} \cdot \boldsymbol{B})^{-1} = \boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1}$$

components $\quad$ (minor(A_{ij}) = determinant of sub-matrix of A_{ij})

$$A^{-1}_{ji} = \frac{1}{\det(\underline{A})} (-1)^{i+j} \; \mathrm{minor}(A_{ij})$$

1.6.2 Deviatoric part of a tensor

Each tensor can be written as the sum of a deviatoric and a hydrostatic part. In mechanics this decomposition is often applied because both parts reflect a special aspect of deformation or stress state.

$$\boldsymbol{A}^d = \boldsymbol{A} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{A})\boldsymbol{I} \quad ; \quad \tfrac{1}{3}\mathrm{tr}(\boldsymbol{A})\boldsymbol{I} = \boldsymbol{A}^h = \text{hydrostatic or spherical part}$$

properties

1. $(\boldsymbol{A} + \boldsymbol{B})^d = \boldsymbol{A}^d + \boldsymbol{B}^d$
2. $\mathrm{tr}(\boldsymbol{A}^d) = 0$
3. eigenvalues (μ_i) and eigenvectors ($\vec{m}_i$)

$$\begin{array}{lll}
\det(\boldsymbol{A}^d - \mu \boldsymbol{I}) = 0 & \rightarrow & \\
\det(\boldsymbol{A} - \{\frac{1}{3}\mathsf{tr}(\boldsymbol{A}) + \mu\}\boldsymbol{I}) = 0 & \rightarrow & \mu = \lambda - \frac{1}{3}\mathsf{tr}(\boldsymbol{A}) \\
(\boldsymbol{A}^d - \mu \boldsymbol{I}) \cdot \vec{m} = \vec{0} & \rightarrow & \\
(\boldsymbol{A} - \{\frac{1}{3}\mathsf{tr}(\boldsymbol{A}) + \mu\}\boldsymbol{I}) \cdot \vec{m} = \vec{0} & \rightarrow & \\
(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \vec{m} = \vec{0} & \rightarrow & \vec{m} = \vec{n}
\end{array}$$

1.6.3 Symmetric tensor

A second order tensor is the sum of dyads. When each dyad is written in reversed order, the conjugate tensor $\boldsymbol{A}^c$ results. The tensor is symmetric when each dyad in its sum is symmetric.

A very convenient property of a symmetric tensor is that all eigenvalues and associated eigenvectors are real. The eigenvectors are or can be chosen to be orthonormal. They can be used as an orthonormal vector base. Writing $\boldsymbol{A}$ in components w.r.t. this basis results in the spectral representation of the tensor. The matrix $\underline{A}$ is a diagonal matrix with the eigenvalues on the diagonal.

Scalar functions of a tensor $\boldsymbol{A}$ can be calculated using the spectral representation, considering the fact that the eigenvectors are not changed.

$$\boldsymbol{A}^c = \boldsymbol{A}$$

properties

$$\left.\begin{array}{lllll}
1. & \text{eigenvalues and eigenvectors are real} & & & \\
2. & \lambda_i \ \ \text{different} & \rightarrow & \vec{n}_i & \perp \\
3. & \lambda_i \ \ \text{not different} & \rightarrow & \vec{n}_i & \text{chosen} \perp
\end{array}\right\} \rightarrow$$

eigenvectors span orthonormal basis $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$

proof of $\perp$:

$$\sigma_i \vec{n}_i = \boldsymbol{\sigma} \cdot \vec{n}_i \quad \rightarrow \quad \vec{n}_i = \frac{1}{\sigma_i} \boldsymbol{\sigma} \cdot \vec{n}_i \quad \rightarrow$$
$$\vec{n}_i \cdot \vec{n}_j = \frac{1}{\sigma_i} \vec{n}_j \cdot \boldsymbol{\sigma} \cdot \vec{n}_i = \frac{1}{\sigma_j} \vec{n}_i \cdot \boldsymbol{\sigma} \cdot \vec{n}_j \quad \rightarrow \quad \vec{n}_i \cdot \boldsymbol{\sigma} \cdot \vec{n}_j = 0 \quad \rightarrow \quad \vec{n}_i \cdot \vec{n}_j = 0$$

spectral representation of $\boldsymbol{A}$

$$\begin{aligned}
\boldsymbol{A} &= \boldsymbol{A} \cdot \boldsymbol{I} = \boldsymbol{A} \cdot (\vec{n}_1 \vec{n}_1 + \vec{n}_2 \vec{n}_2 + \vec{n}_3 \vec{n}_3) \\
&= \lambda_1 \vec{n}_1 \vec{n}_1 + \lambda_2 \vec{n}_2 \vec{n}_2 + \lambda_3 \vec{n}_3 \vec{n}_3
\end{aligned}$$

functions

$$\begin{aligned}
\boldsymbol{A}^{-1} &= \frac{1}{\lambda_1} \vec{n}_1 \vec{n}_1 + \frac{1}{\lambda_2} \vec{n}_2 \vec{n}_2 + \frac{1}{\lambda_3} \vec{n}_3 \vec{n}_3 + \\
\sqrt{\boldsymbol{A}} &= \sqrt{\lambda_1} \vec{n}_1 \vec{n}_1 + \sqrt{\lambda_2} \vec{n}_2 \vec{n}_2 + \sqrt{\lambda_3} \vec{n}_3 \vec{n}_3 \\
\ln \boldsymbol{A} &= \ln \lambda_1 \vec{n}_1 \vec{n}_1 + \ln \lambda_2 \vec{n}_2 \vec{n}_2 + \ln \lambda_3 \vec{n}_3 \vec{n}_3 \\
\sin \boldsymbol{A} &= \sin(\lambda_1) \vec{n}_1 \vec{n}_1 + \sin(\lambda_2) \vec{n}_2 \vec{n}_2 + \sin(\lambda_3) \vec{n}_3 \vec{n}_3 \\
J_1(\boldsymbol{A}) &= \mathsf{tr}(\boldsymbol{A}) = \lambda_1 + \lambda_2 + \lambda_3 \\
J_2(\boldsymbol{A}) &= \tfrac{1}{2}\{\mathsf{tr}^2(\boldsymbol{A}) - \mathsf{tr}(\boldsymbol{A} \cdot \boldsymbol{A})\} = \tfrac{1}{2}\{(\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\} \\
J_2(\boldsymbol{A}) &= \det(\boldsymbol{A}) = \lambda_1 \lambda_2 \lambda_3
\end{aligned}$$

1.6.4 Skew-symmetric tensor

The conjugate of a skew-symmetric tensor is the negative of the tensor.

The double dot product of a skew-symmetric and a symmetric tensor is zero. Because the unity tensor is also a symmetric tensor, the trace of a skew-symmetric tensor must be zero.

A skew-symmetric tensor has one unique *axial vector*.

$$\boldsymbol{A}^c = -\,\boldsymbol{A}$$

properties

$$\begin{array}{ll}
1. & \boldsymbol{A} : \boldsymbol{B} = \mathsf{tr}(\boldsymbol{A} \cdot \boldsymbol{B}) = \mathsf{tr}(\boldsymbol{A}^c \cdot \boldsymbol{B}^c) = \boldsymbol{A}^c : \boldsymbol{B}^c \\
 & \left.\begin{array}{l} \boldsymbol{A}^c = -\boldsymbol{A} \quad \rightarrow \quad \boldsymbol{A} : \boldsymbol{B} = -\boldsymbol{A} : \boldsymbol{B}^c \\ \boldsymbol{B}^c = \boldsymbol{B} \quad \rightarrow \quad \boldsymbol{A} : \boldsymbol{B} = -\boldsymbol{A} : \boldsymbol{B} \end{array}\right\} \quad \rightarrow \quad \boldsymbol{A} : \boldsymbol{B} = 0 \\
2. & \boldsymbol{B} = \boldsymbol{I} \quad \rightarrow \quad \mathsf{tr}(\boldsymbol{A}) = \boldsymbol{A} : \boldsymbol{I} = 0 \\
3. & \boldsymbol{A} \cdot \vec{q} = \vec{p} \quad \rightarrow \quad \vec{q} \cdot \boldsymbol{A} \cdot \vec{q} = \vec{q} \cdot \boldsymbol{A}^c \cdot \vec{q} = -\,\vec{q} \cdot \boldsymbol{A} \cdot \vec{q} \quad \rightarrow \\
 & \vec{q} \cdot \boldsymbol{A} \cdot \vec{q} = 0 \quad \rightarrow \quad \vec{q} \cdot \vec{p} = 0 \quad \rightarrow \quad \vec{q} \perp \vec{p} \quad \rightarrow \\
 & \exists! \quad \vec{\omega} \quad \text{such that} \quad \boldsymbol{A} \cdot \vec{q} = \vec{p} = \vec{\omega} * \vec{q}
\end{array}$$

The components of the axial vector $\vec{\omega}$ associated with the skew-symmetric tensor $\boldsymbol{A}$ can be expressed in the components of $\boldsymbol{A}$. This involves the solution of a system of three equations.

$$\boldsymbol{A} \cdot \vec{q} = \underset{\sim}{\vec{e}}^T \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underset{\sim}{\vec{e}}^T \begin{bmatrix} A_{11}q_1 + A_{12}q_2 + A_{13}q_3 \\ A_{21}q_1 + A_{22}q_2 + A_{23}q_3 \\ A_{31}q_1 + A_{32}q_2 + A_{33}q_3 \end{bmatrix}$$

$$\begin{aligned}
\vec{\omega} * \vec{q} &= (\omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3) * (q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3) \\
&= \omega_1 q_2(\vec{e}_3) + \omega_1 q_3(-\,\vec{e}_2) + \omega_2 q_1(-\,\vec{e}_3) + \omega_2 q_3(\vec{e}_1) + \\
&\quad\ \omega_3 q_1(\vec{e}_2) + \omega_3 q_2(-\,\vec{e}_1) \\
&= \underset{\sim}{\vec{e}}^T \begin{bmatrix} \omega_2 q_3 - \omega_3 q_2 \\ \omega_3 q_1 - \omega_1 q_3 \\ \omega_1 q_2 - \omega_2 q_1 \end{bmatrix} \qquad \rightarrow \qquad \underline{A} = \begin{bmatrix} 0 & -\,\omega_3 & \omega_2 \\ \omega_3 & 0 & -\,\omega_1 \\ -\,\omega_2 & \omega_1 & 0 \end{bmatrix}
\end{aligned}$$

1.6.5 Positive definite tensor

The diagonal matrix components of a positive definite tensor must all be positive numbers. A positive definite tensor cannot be skew-symmetric. When it is symmetric, all eigenvalues must be positive. In that case the tensor is automatically regular, because its inverse exists.

$$\vec{a} \cdot \boldsymbol{A} \cdot \vec{a} > 0 \qquad \forall \quad \vec{a} \neq \vec{0}$$

properties

$$
\begin{array}{ll}
1. & \boldsymbol{A} \text{ cannot be skew-symmetric, because :} \\
 & \left.\begin{array}{l} \vec{a} \cdot \boldsymbol{A} \cdot \vec{a} = \vec{a} \cdot \boldsymbol{A}^c \cdot \vec{a} \quad \rightarrow \\ \vec{a} \cdot (\boldsymbol{A} - \boldsymbol{A}^c) \cdot \vec{a} = 0 \\ \boldsymbol{A} \quad \text{skew-symm.} \quad \rightarrow \quad \boldsymbol{A}^c = -\boldsymbol{A} \end{array}\right\} \quad \rightarrow \quad \vec{a} \cdot \boldsymbol{A} \cdot \vec{a} = 0 \quad \forall \quad \vec{a} \\
2. & \boldsymbol{A} = \boldsymbol{A}^c \quad \rightarrow \quad \vec{n}_i \cdot \boldsymbol{A} \cdot \vec{n}_i = \lambda_i > 0 \quad \rightarrow \\
 & \text{all eigenvalues positive} \quad \rightarrow \quad \text{regular}
\end{array}
$$

1.6.6 Orthogonal tensor

When an orthogonal tensor is used to transform a vector, the length of that vector remains the same.

The inverse of an orthogonal tensor equals the conjugate of the tensor. This implies that the columns of its matrix are orthonormal, which also applies to the rows. This means that an orthogonal tensor is either a rotation tensor or a mirror tensor.

The determinant of $\boldsymbol{A}$ has either the value $+1$, in which case $\boldsymbol{A}$ is a rotation tensor, or -1, when $\boldsymbol{A}$ is a mirror tensor.

$$(\boldsymbol{A} \cdot \vec{a}) \cdot (\boldsymbol{A} \cdot \vec{b}) = \vec{a} \cdot \vec{b} \qquad \forall \quad \vec{a}, \vec{b}$$

properties

$$
\begin{array}{ll}
1. & (\boldsymbol{A} \cdot \vec{v}) \cdot (\boldsymbol{A} \cdot \vec{v}) = \vec{v} \cdot \vec{v} \quad \rightarrow \quad ||\boldsymbol{A} \cdot \vec{v}|| = ||\vec{v}|| \\
2. & \vec{a} \cdot \boldsymbol{A}^c \cdot \boldsymbol{A} \cdot \vec{b} = \vec{a} \cdot \vec{b} \quad \rightarrow \quad \boldsymbol{A} \cdot \boldsymbol{A}^c = \boldsymbol{I} \quad \rightarrow \quad \boldsymbol{A}^c = \boldsymbol{A}^{-1} \\
3. & \det(\boldsymbol{A} \cdot \boldsymbol{A}^c) = \det(\boldsymbol{A})^2 = \det(\boldsymbol{I}) = 1 \quad \rightarrow \\
 & \det(\boldsymbol{A}) = \pm 1 \quad \rightarrow \quad \boldsymbol{A} \quad \text{regular}
\end{array}
$$

Rotation of a vector base

A rotation tensor $\boldsymbol{Q}$ can be used to rotate an orthonormal vector basis $\underset{\sim}{\vec{m}}$ to $\underset{\sim}{\vec{n}}$. It can be shown that the matrix $\underline{Q}^{(n)}$ of $\boldsymbol{Q}$ w.r.t. $\underset{\sim}{\vec{n}}$ is the same as the matrix $\underline{Q}^{(m)}$ w.r.t. $\underset{\sim}{\vec{m}}$.

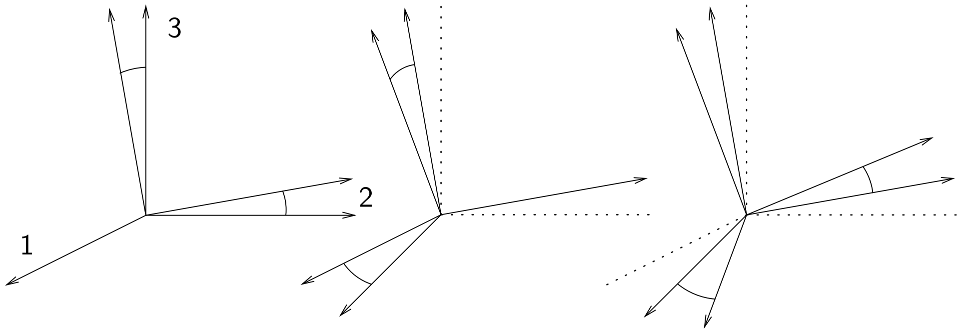
The column with the rotated base vectors $\underset{\sim}{\vec{n}}$ can be expressed in the column with the initial base vectors $\underset{\sim}{\vec{m}}$ as : $\underset{\sim}{\vec{n}} = \underline{Q}^T \underset{\sim}{\vec{m}}$, so using the transpose of the rotation matrix $\underline{Q}$.

$$
\left.\begin{array}{l} \vec{n}_1 = \boldsymbol{Q} \cdot \vec{m}_1 \\ \vec{n}_2 = \boldsymbol{Q} \cdot \vec{m}_2 \\ \vec{n}_3 = \boldsymbol{Q} \cdot \vec{m}_3 \end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l} \vec{n}_1 \vec{m}_1 = \boldsymbol{Q} \cdot \vec{m}_1 \vec{m}_1 \\ \vec{n}_2 \vec{m}_2 = \boldsymbol{Q} \cdot \vec{m}_2 \vec{m}_2 \\ \vec{n}_3 \vec{m}_3 = \boldsymbol{Q} \cdot \vec{m}_3 \vec{m}_3 \end{array}\right\} \quad \rightarrow \quad \boldsymbol{Q} = \underset{\sim}{\vec{n}}^T \underset{\sim}{\vec{m}}
$$

$$
\left.\begin{array}{l} \underline{Q}^{(n)} = \underset{\sim}{\vec{n}} \cdot \boldsymbol{Q} \cdot \underset{\sim}{\vec{n}}^T = (\underset{\sim}{\vec{n}} \cdot \underset{\sim}{\vec{n}}^T) \underset{\sim}{\vec{m}} \cdot \underset{\sim}{\vec{n}}^T = \underset{\sim}{\vec{m}} \cdot \underset{\sim}{\vec{n}}^T \\ \underline{Q}^{(m)} = \underset{\sim}{\vec{m}} \cdot \boldsymbol{Q} \cdot \underset{\sim}{\vec{m}}^T = \underset{\sim}{\vec{m}} \cdot \underset{\sim}{\vec{n}}^T (\underset{\sim}{\vec{m}} \cdot \underset{\sim}{\vec{m}}^T) = \underset{\sim}{\vec{m}} \cdot \underset{\sim}{\vec{n}}^T \end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l} \underline{Q}^{(n)} = \underline{Q}^{(m)} = \underline{Q} \\ \underset{\sim}{\vec{m}} = \underline{Q} \, \underset{\sim}{\vec{n}} \quad \rightarrow \quad \underset{\sim}{\vec{n}} = \underline{Q}^T \underset{\sim}{\vec{m}} \end{array}\right\}
$$

We consider a rotation of the vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ to $\{\vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3\}$, which is the result of three subsequent rotations : 1) rotation about the 1-axis, 2) rotation about the new 2-axis and 3) rotation about the new 3-axis.

For each individual rotation the rotation matrix can be determined.



$$\left.\begin{aligned} \vec{\varepsilon}_1^{\,(1)} &= \vec{e}_1 \\ \vec{\varepsilon}_2^{\,(1)} &= c^{(1)}\vec{e}_2 + s^{(1)}\vec{e}_3 \\ \vec{\varepsilon}_3^{\,(1)} &= -s^{(1)}\vec{e}_2 + c^{(1)}\vec{e}_3 \end{aligned}\right\} \qquad \underline{Q}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^{(1)} & -s^{(1)} \\ 0 & s^{(1)} & c^{(1)} \end{bmatrix}$$

$$\left.\begin{aligned} \vec{\varepsilon}_1^{\,(2)} &= c^{(2)}\vec{\varepsilon}_1^{\,(1)} - s^{(2)}\vec{\varepsilon}_3^{\,(1)} \\ \vec{\varepsilon}_2^{\,(2)} &= \vec{\varepsilon}_2^{\,(1)} \\ \vec{\varepsilon}_3^{\,(2)} &= s^{(2)}\vec{\varepsilon}_1^{\,(1)} + c^{(2)}\vec{\varepsilon}_3^{\,(1)} \end{aligned}\right\} \qquad \underline{Q}_2 = \begin{bmatrix} c^{(2)} & 0 & s^{(2)} \\ 0 & 1 & 0 \\ -s^{(2)} & 0 & c^{(2)} \end{bmatrix}$$

$$\left.\begin{aligned} \vec{\varepsilon}_1^{\,(3)} &= c^{(3)}\vec{\varepsilon}_1^{\,(2)} + s^{(3)}\vec{\varepsilon}_2^{\,(2)} \\ \vec{\varepsilon}_2^{\,(3)} &= -s^{(3)}\vec{\varepsilon}_1^{\,2)} + c^{(3)}\vec{\varepsilon}_2^{\,(2)} \\ \vec{\varepsilon}_3^{\,(3)} &= \vec{\varepsilon}_3^{\,(2)} \end{aligned}\right\} \qquad \underline{Q}_3 = \begin{bmatrix} c^{(3)} & -s^{(3)} & 0 \\ s^{(3)} & c^{(3)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The total rotation matrix $\underline{Q}$ is the product of the individual rotation matrices.

$$\left.\begin{aligned} \underset{\sim}{\vec{\varepsilon}}^{\,(1)} &= \underline{Q}_1^{\,T}\underset{\sim}{\vec{e}} \\ \underset{\sim}{\vec{\varepsilon}}^{\,(2)} &= \underline{Q}_2^{\,T}\underset{\sim}{\vec{\varepsilon}}^{\,(1)} \\ \underset{\sim}{\vec{\varepsilon}}^{\,(3)} &= \underline{Q}_3^{\,T}\underset{\sim}{\vec{\varepsilon}}^{\,(2)} = \underset{\sim}{\vec{\varepsilon}} \end{aligned}\right\} \rightarrow \quad \begin{aligned} &\underset{\sim}{\vec{\varepsilon}} = \underline{Q}_3^{\,T}\underline{Q}_2^{\,T}\underline{Q}_1^{\,T}\underset{\sim}{\vec{e}} = \underline{Q}^T\underset{\sim}{\vec{e}} \\ &\underset{\sim}{\vec{e}} = \underline{Q}\,\underset{\sim}{\vec{\varepsilon}} \end{aligned}$$

$$\underline{Q} = \begin{bmatrix} c^{(2)}c^{(3)} & -c^{(2)}s^{(3)} & s^{(2)} \\ c^{(1)}s^{(3)} + s^{(1)}s^{(2)}c^{(3)} & c^{(1)}c^{(3)} - s^{(1)}s^{(2)}s^{(3)} & -s^{(1)}c^{(2)} \\ s^{(1)}s^{(3)} - c^{(1)}s^{(2)}c^{(3)} & s^{(1)}c^{(3)} + c^{(1)}s^{(2)}s^{(3)} & c^{(1)}c^{(2)} \end{bmatrix}$$

A tensor $\boldsymbol{A}$ with matrix $\underline{A}$ w.r.t. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ has a matrix $\underline{A}^*$ w.r.t. basis $\{\vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3\}$. The matrix $\underline{A}^*$ can be calculated from $\underline{A}$ by multiplication with $\underline{Q}$, the rotation matrix w.r.t. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

The column $\underset{\approx}{A}$ with the 9 components of $\boldsymbol{A}$ can be transformed to $\underset{\approx}{A}^*$ by multiplication with the 9x9 transformation matrix $\underline{T}$: $\underset{\approx}{A}^* = \underline{T}\,\underset{\approx}{A}$. When $\boldsymbol{A}$ is symmetric, the transformation matrix $\underline{T}$ is 6x6. Note that $\underline{T}$ is **not** the representation of a tensor.

The matrix $\underline{T}$ is not orthogonal, but its inverse can be calculated easily by reversing the rotation angles : $\underline{T}^{-1} = \underline{T}(-\alpha_1, -\alpha_2, -\alpha_3)$. The use of the transformation matrix $\underline{T}$ is described in detail in appendix B.

$$
\begin{aligned}
\boldsymbol{A} &= \underset{\sim}{\vec{e}}^{\,T}\, \underline{A}\, \underset{\sim}{\vec{e}} = \underset{\sim}{\vec{\varepsilon}}^{\,T}\, \underline{A}^*\, \underset{\sim}{\vec{\varepsilon}} \quad \rightarrow \\
\underline{A}^* &= \underset{\sim}{\vec{\varepsilon}} \cdot \underset{\sim}{\vec{e}}^{\,T}\, \underline{A}\, \underset{\sim}{\vec{e}} \cdot \underset{\sim}{\vec{\varepsilon}}^{\,T} = \underline{Q}^T\, \underline{A}\, \underline{Q} \\
\underset{\approx}{A}^* &= \underline{T}\, \underset{\approx}{A}
\end{aligned}
$$

1.6.7 Adjugated tensor

The definition of the adjugate tensor resembles that of the orthogonal tensor, only now the scalar product is replaced by a vector product.

$$
(\boldsymbol{A} \cdot \vec{a}) * (\boldsymbol{A} \cdot \vec{b}) = \boldsymbol{A}^a \cdot (\vec{a} * \vec{b}) \quad \forall \quad \vec{a}, \vec{b}
$$

property $\qquad \boldsymbol{A}^c \cdot \boldsymbol{A}^a = \det(\boldsymbol{A}) \boldsymbol{I}$

1.7 Fourth-order tensor

Transformation of second-order tensors are done by means of a fourth-order tensor. A second-order tensor is mapped onto a different second-order tensor by means of the double inner product with a fourth-order tensor. This mapping is linear.

A fourth-order tensor can be written as a finite sum of quadrades, being open products of four vectors. When quadrades of three base vectors in three-dimensional space are used, the number of independent terms is 81, which means that the fourth-order tensor has 81 components. In index notation this can be written very short. Use of matrix notation requires the use of a $3 \times 3 \times 3 \times 3$ matrix.

$$
{}^4\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{C}
$$

tensor = linear transformation $\qquad {}^4\boldsymbol{A} : (\alpha \boldsymbol{M} + \beta \boldsymbol{N}) = \alpha\, {}^4\boldsymbol{A} : \boldsymbol{M} + \beta\, {}^4\boldsymbol{A} : \boldsymbol{N}$

representation $\qquad {}^4\boldsymbol{A} = \alpha_1 \vec{a}_1 \vec{b}_1 \vec{c}_1 \vec{d}_1 + \alpha_2 \vec{a}_2 \vec{b}_2 \vec{c}_2 \vec{d}_2 + \alpha_3 \vec{a}_3 \vec{b}_3 \vec{c}_3 \vec{d}_3 + ..$

components $\qquad {}^4\boldsymbol{A} = \vec{e}_i \vec{e}_j A_{ijkl} \vec{e}_k \vec{e}_l$

1.7.1 Conjugated fourth-order tensor

Different types of conjugated tensors are associated with a fourth-order tensor. This also implies that there are different types of symmetries involved.

A left or right symmetric fourth-order tensor has 54 independent components. A tensor which is left and right symmetric has 36 independent components. A middle symmetric tensor has 45 independent components. A total symmetric tensor is left, right and middle symmetric and has 21 independent components.

fourth-order tensor	:	${}^4\boldsymbol{A} = \vec{a}\,\vec{b}\,\vec{c}\,\vec{d}$
total conjugate	:	${}^4\boldsymbol{A}^c = \vec{d}\,\vec{c}\,\vec{b}\,\vec{a}$
right conjugate	:	${}^4\boldsymbol{A}^{rc} = \vec{a}\,\vec{b}\,\vec{d}\,\vec{c}$
left conjugate	:	${}^4\boldsymbol{A}^{lc} = \vec{b}\,\vec{a}\,\vec{c}\,\vec{d}$
middle conjugate	:	${}^4\boldsymbol{A}^{mc} = \vec{a}\,\vec{c}\,\vec{b}\,\vec{d}$

symmetries

left	${}^4\boldsymbol{A} = {}^4\boldsymbol{A}^{lc}$	; $\boldsymbol{B} : {}^4\boldsymbol{A} = \boldsymbol{B}^c : {}^4\boldsymbol{A}$	$\forall \quad \boldsymbol{B}$
right	${}^4\boldsymbol{A} = {}^4\boldsymbol{A}^{rc}$	; ${}^4\boldsymbol{A} : \boldsymbol{B} = {}^4\boldsymbol{A} : \boldsymbol{B}^c$	$\forall \quad \boldsymbol{B}$
middle	${}^4\boldsymbol{A} = {}^4\boldsymbol{A}^{mc}$		
total	${}^4\boldsymbol{A} = {}^4\boldsymbol{A}^{c}$	; $\boldsymbol{B} : {}^4\boldsymbol{A} : \boldsymbol{C} = \boldsymbol{C}^c : {}^4\boldsymbol{A} : \boldsymbol{B}^c$	$\forall \quad \boldsymbol{B}, \boldsymbol{C}$

1.7.2 Fourth-order unit tensor

The fourth-order unit tensor maps each second-order tensor onto itself. The symmetric fourth-order unit tensor, which is total symmetric, maps a second-order tensor on its symmetric part.

$$ {}^4\boldsymbol{I} : \boldsymbol{B} = \boldsymbol{B} \qquad \forall \quad \boldsymbol{B} $$

components

$$ \begin{aligned} {}^4\boldsymbol{I} &= \vec{e}_1\vec{e}_1\vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_1\vec{e}_1\vec{e}_2 + \vec{e}_3\vec{e}_1\vec{e}_1\vec{e}_3 + \vec{e}_1\vec{e}_2\vec{e}_2\vec{e}_1 + \ldots \\ &= \vec{e}_i\vec{e}_j\vec{e}_j\vec{e}_i = \vec{e}_i\vec{e}_j\delta_{il}\delta_{jk}\vec{e}_k\vec{e}_l \end{aligned} $$

${}^4\boldsymbol{I}$ **not** left- or right symmetric

$$ \begin{aligned} &{}^4\boldsymbol{I} : \boldsymbol{B} = \boldsymbol{B} \neq \boldsymbol{B}^c = {}^4\boldsymbol{I} : \boldsymbol{B}^c \\ &\boldsymbol{B} : {}^4\boldsymbol{I} = \boldsymbol{B} \neq \boldsymbol{B}^c = \boldsymbol{B}^c : {}^4\boldsymbol{I} \end{aligned} $$

symmetric fourth-order tensor

$$ {}^4\boldsymbol{I}^s = \tfrac{1}{2}\left({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}\right) = \tfrac{1}{2}\,\vec{e}_i\vec{e}_j(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})\vec{e}_k\vec{e}_l $$

1.7.3 Products

Inner and double inner products of fourth-order tensors with fourth- and second-order tensors, result in new fourth-order or second-order tensors. Calculating such products requires that some rules have to be followed.

$$ {}^4\boldsymbol{A} \cdot \boldsymbol{B} = {}^4\boldsymbol{C} \qquad \rightarrow \qquad A_{ijkm}B_{ml} = C_{ijkl} $$

$$ \begin{aligned} &{}^4\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{C} \qquad \rightarrow \qquad A_{ijkl}B_{lk} = C_{ij} \\ &{}^4\boldsymbol{A} : \boldsymbol{B} \neq \boldsymbol{B} : {}^4\boldsymbol{A} \end{aligned} $$

$$
\begin{array}{lcl}
{}^4\boldsymbol{A} : {}^4\boldsymbol{B} = {}^4\boldsymbol{C} & \rightarrow & A_{ijmn}B_{nmkl} = C_{ijkl} \\
{}^4\boldsymbol{A} : {}^4\boldsymbol{B} \neq {}^4\boldsymbol{B} : {}^4\boldsymbol{A} & &
\end{array}
$$

rules

$$
\begin{array}{l}
{}^4\boldsymbol{A} : (\boldsymbol{B} \cdot \boldsymbol{C}) = ({}^4\boldsymbol{A} \cdot \boldsymbol{B}) : \boldsymbol{C} \\
\boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{B}^c \cdot \boldsymbol{A}^c = {}^4\boldsymbol{I}^s : (\boldsymbol{A} \cdot \boldsymbol{B}) = ({}^4\boldsymbol{I}^s \cdot \boldsymbol{A}) : \boldsymbol{B}
\end{array}
$$

Chapter 2

Kinematics

The motion and deformation of a three-dimensional continuum is studied in *continuum mechanics*. A continuum is an ideal material body, where the neighborhood of a material point is assumed to be dense and fully occupied with other material points. The real microstructure of the material (molecules, crystals, particles, ...) is not considered. The deformation is also continuous, which implies that the neighborhood of a material point always consists of the same collection of material points.

Kinematics describes the transformation of a material body from its undeformed to its deformed state without paying attention to the cause of deformation. In the mathematical formulation of kinematics a Lagrangian or an Eulerian approach can be chosen. (It is also possible to follow a so-called Arbitrary-Lagrange-Euler approach.)

The undeformed state is indicated as the state at time t_0 and the deformed state as the state at the current time t. When the deformation process is time- or rate-independent, the time variable must be considered to be a fictitious time, only used to indicate subsequent moments in the deformation process.

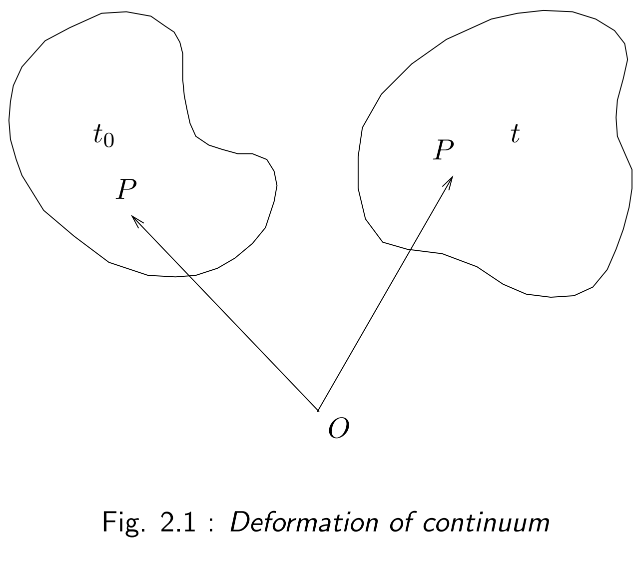


Fig. 2.1 : *Deformation of continuum*

2.1 Identification of points

Describing the deformation of a material body cannot be done without a proper identification of the individual material points.

2.1.1 Material coordinates

Each point of the material can be identified by or labeled with material coordinates. In a three-dimensional space three coordinates $\{\xi_1, \xi_2, \xi_3\}$ are needed and sufficient to identify a point uniquely. The material coordinates of a material point do never change. They can be stored in a column $\underset{\sim}{\xi} \; : \; \underset{\sim}{\xi}^T = [\; \xi_1 \quad \xi_2 \quad \xi_3 \;]$.

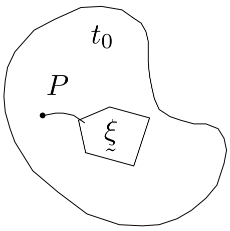


Fig. 2.2 : *Material coordinates*

2.1.2 Position vectors

A point of the material can also be identified with its position in space. Two position vectors can be chosen for this purpose : the position vector in the undeformed state, $\vec{x}_0$, or the position vector in the current, deformed state, $\vec{x}$. Both position vectors can be considered to be a function of the material coordinates $\underset{\sim}{\xi}$.

Each point is always identified with one position vector. One spatial position is always occupied by one material point. For a continuum the position vector is a continuous differentiable function.

Using a vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, components of the position vectors can be determined and stored in columns.

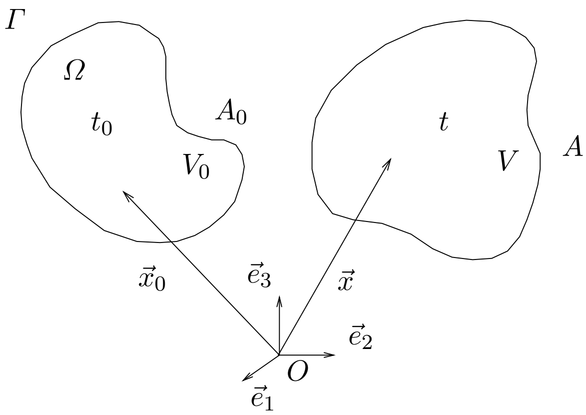


Fig. 2.3 : *Position vector*

undeformed configuration (t_0)	$\vec{x}_0 = \vec{\chi}(\underset{\sim}{\xi}, t_0) = x_{01}\vec{e}_1 + x_{02}\vec{e}_2 + x_{03}\vec{e}_3$
deformed configuration (t)	$\vec{x} = \vec{\chi}(\underset{\sim}{\xi}, t) = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$

2.1.3 Lagrange - Euler

When a *Lagrangian formulation* is used to describe state transformation, all variables are determined in material points which are identified in the undeformed state with their *initial* position vector $\vec{x}_0$. When an *Eulerian formulation* is used, all variables are determined in material points which are identified in the deformed state with their *current* position vector $\vec{x}$. For a scalar quantity a, this can be formally written with a function $\mathcal{A}_L$ or $\mathcal{A}_E$, respectively.

$$\begin{array}{lll} \text{Lagrange} & : & a = \mathcal{A}_L(\vec{x}_0, t) \\ \text{Euler} & : & a = \mathcal{A}_E(\vec{x}, t) \end{array}$$

The difference da of a scalar quantity a in two adjacent points P and Q can be calculated in both the Lagrangian and the Eulerian framework.

$$\begin{array}{lll} \text{Lagrange} & : & da = a_Q - a_P = \mathcal{A}_L(\vec{x}_0 + d\vec{x}_0, t) - \mathcal{A}_L(\vec{x}_0, t) = d\vec{x}_0 \cdot \left. (\vec{\nabla}_0 a)\right|_t \\ \text{Euler} & : & da = a_Q - a_P = \mathcal{A}_E(\vec{x} + d\vec{x}, t) - \mathcal{A}_E(\vec{x}, t) = d\vec{x} \cdot \left. (\vec{\nabla} a)\right|_t \end{array}$$

This leads to the definition of two gradient operators, $\vec{\nabla}_0$ and $\vec{\nabla}$, respectively.

$$\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3}$$
$$\vec{\nabla}_0 = \vec{e}_1 \frac{\partial}{\partial x_{01}} + \vec{e}_2 \frac{\partial}{\partial x_{02}} + \vec{e}_3 \frac{\partial}{\partial x_{03}}$$

For a vectorial quantity $\vec{a}$, the spatial difference $d\vec{a}$ in two adjacent points, can also be calculated, using either $\vec{\nabla}_0$ or $\vec{\nabla}$. For the position vectors, the gradients result in the unity tensor $\boldsymbol{I}$.

$$\vec{\nabla}\vec{x} = \boldsymbol{I} \qquad ; \qquad \vec{\nabla}_0\vec{x}_0 = \boldsymbol{I}$$

2.2 Deformation

Upon deformation, a material point changes position from $\vec{x}_0$ to $\vec{x}$. This is denoted with a displacement vector $\vec{u}$. In three-dimensional space this vector has three components : u_1, u_2 and u_3.

The deformation of the material can be described by the displacement vector of all the material points. This, however, is not a feasible procedure. Instead, we consider the deformation of an infinitesimal material volume in each point, which can be described with a *deformation tensor.*

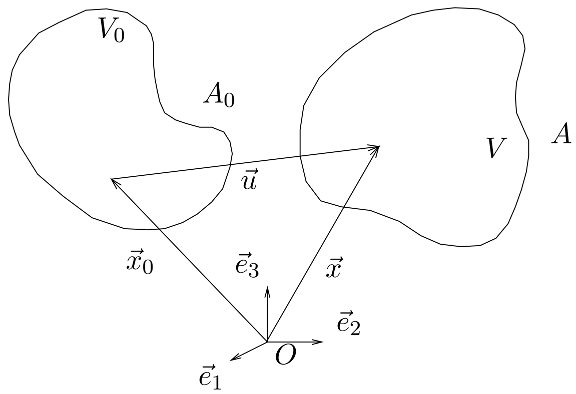


Fig. 2.4 : *Deformation of a continuum*

$$\vec{u} = \vec{x} - \vec{x}_0 = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$$

2.2.1 Deformation tensor

To introduce the deformation tensor, we first consider the deformation of an infinitesimal material line element, between two adjacent material points. The vector between these points in the undeformed state is $d\vec{x}_0$. Deformation results in a transformation of this vector to $d\vec{x}$, which can be denoted with a tensor, the deformation tensor $\boldsymbol{F}$. Using the gradient operator with respect to the undeformed state, the deformation tensor can be written as a gradient, which explains its much used name : *deformation gradient tensor.*

$$\begin{aligned} d\vec{x} &= \boldsymbol{F} \cdot d\vec{x}_0 \\ &= \vec{X}(\vec{x}_0 + d\vec{x}_0, \boldsymbol{t}) - \vec{X}(\vec{x}_0, \boldsymbol{t}) = d\vec{x}_0 \cdot \left(\vec{\nabla}_0\vec{x}\right) \\ &= \left(\vec{\nabla}_0\vec{x}\right)^c \cdot d\vec{x}_0 = \boldsymbol{F} \cdot d\vec{x}_0 \end{aligned}$$

$$\boxed{\boldsymbol{F} = \left(\vec{\nabla}_0\vec{x}\right)^c = \left[\left(\vec{\nabla}_0\vec{x}_0\right)^c + \left(\vec{\nabla}_0\vec{u}\right)^c\right] = \boldsymbol{I} + \left(\vec{\nabla}_0\vec{u}\right)^c}$$

In the undeformed configuration, an infinitesimal material volume is uniquely defined by three material line elements or material vectors $d\vec{x}_{01}$, $d\vec{x}_{02}$ and $d\vec{x}_{03}$. Using the deformation tensor $\boldsymbol{F}$, these vectors are transformed to the deformed state to become $d\vec{x}_1$, $d\vec{x}_2$ and $d\vec{x}_3$. These vectors span the deformed volume element, containing the same material points as in the initial volume element. It is thus obvious that $\boldsymbol{F}$ describes the transformation of the material.

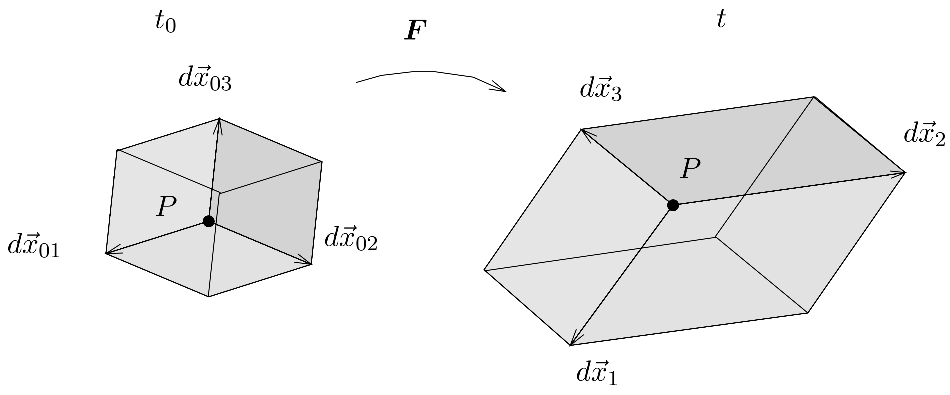


Fig. 2.5 : *Deformation tensor*

$$d\vec{x}_1 = \boldsymbol{F} \cdot d\vec{x}_{01} \quad ; \quad d\vec{x}_2 = \boldsymbol{F} \cdot d\vec{x}_{02} \quad ; \quad d\vec{x}_3 = \boldsymbol{F} \cdot d\vec{x}_{03}$$

Volume change

The three vectors which span the material element, can be combined in a triple product. The resulting scalar value is positive when the vectors are right-handed and represents the volume of the material element. In the undeformed state this volume is dV_0 and after deformation the volume is dV. Using the deformation tensor $\boldsymbol{F}$ and the definition of the determinant (third invariant) of a second-order tensor, the relation between dV and dV_0 can be derived.

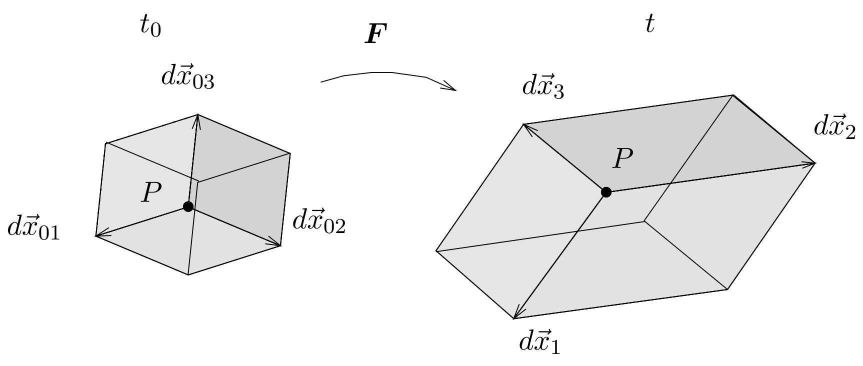


Fig. 2.6 : *Volume change*

undeformed configuration $\qquad dV_0 = d\vec{x}_{01} * d\vec{x}_{02} \cdot d\vec{x}_{03}$

current configuration

$$\begin{aligned} dV &= d\vec{x}_1 * d\vec{x}_2 \cdot d\vec{x}_3 \\ &= (\boldsymbol{F} \cdot d\vec{x}_{01}) * (\boldsymbol{F} \cdot d\vec{x}_{02}) \cdot (\boldsymbol{F} \cdot d\vec{x}_{03}) \\ &= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02} \cdot d\vec{x}_{03}) \\ &= \det(\boldsymbol{F}) dV_0 \end{aligned}$$

volume change factor $\boxed{J = \det(\boldsymbol{F}) = \dfrac{dV}{dV_0}}$

Area change

The vector product of two vectors along two material line elements represents a vector, the length of which equals the area of the parallelogram spanned by the vectors. Using the deformation tensor $\boldsymbol{F}$, the change of area during deformation can be calculated.

$$\begin{aligned}
dA\,\vec{n} &= d\vec{x}_1 * d\vec{x}_2 = (\boldsymbol{F}\cdot d\vec{x}_{01}) * (\boldsymbol{F}\cdot d\vec{x}_{02}) \\
dA\,\vec{n}\cdot(\boldsymbol{F}\cdot d\vec{x}_{03}) &= (\boldsymbol{F}\cdot d\vec{x}_{01}) * (\boldsymbol{F}\cdot d\vec{x}_{02})\cdot(\boldsymbol{F}\cdot d\vec{x}_{03}) \\
&= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot d\vec{x}_{03} \qquad \forall \quad d\vec{x}_{03} \quad \rightarrow \\
dA\,\vec{n}\cdot\boldsymbol{F} &= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02}) \\
dA\,\vec{n} &= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot\boldsymbol{F}^{-1} \\
&= \det(\boldsymbol{F})dA_0\,\vec{n}_0\cdot\boldsymbol{F}^{-1} \\
&= dA_0\,\vec{n}_0\cdot\,\big(\boldsymbol{F}^{-1}\det(\boldsymbol{F})\big)
\end{aligned}$$

Inverse deformation

The determinant of the deformation tensor, being the quotient of two volumes, is always a positive number. This implies that the deformation tensor is regular and that the inverse $\boldsymbol{F}^{-1}$ exists. It represents the transformation of the deformed state to the undeformed state. The gradient operators $\vec{\nabla}$ and $\vec{\nabla}_0$ are related by the (inverse) deformation tensor.

$$\begin{array}{ll}
\det(\boldsymbol{F}) = J > 0 \quad \rightarrow \quad \boldsymbol{F} \quad \text{regular} \quad \rightarrow \quad \text{inverse} \quad \boldsymbol{F}^{-1} \\
\text{inverse deformation :} \qquad d\vec{x}_0 = \boldsymbol{F}^{-c} d\vec{x}
\end{array}$$

$$\boldsymbol{I} = \boldsymbol{F}^{-c}\cdot\boldsymbol{F}^{c} \quad \rightarrow \quad \vec{\nabla}\vec{x} = \boldsymbol{F}^{-c}\cdot(\vec{\nabla}_0\vec{x}) \quad \rightarrow \quad \boxed{\vec{\nabla} = \boldsymbol{F}^{-c}\cdot\vec{\nabla}_0}$$

Homogeneous deformation

The deformation tensor describes the deformation of an infinitesimal material volume, initially located at position $\vec{x}_0$. The deformation tensor is generally a function of the position $\vec{x}_0$.

When $\boldsymbol{F}$ is not a function of position $\vec{x}_0$, the deformation is referred to as being *homogeneous*. In that case, each infinitesimal material volume shows the same deformation. The current position vector $\vec{x}$ can be related to the initial position vector $\vec{x}_0$ and an unknown rigid body translation $\vec{t}$.

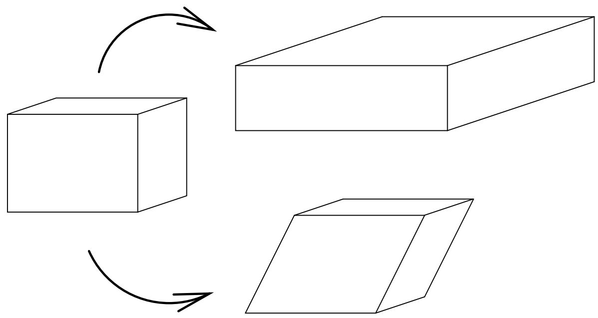


Fig. 2.7 : *Homogeneous deformation*

$$\vec{\nabla}_0 \vec{x} = \boldsymbol{F}^c = \text{uniform tensor} \quad \rightarrow \quad \vec{x} = (\vec{x}_0 \cdot \boldsymbol{F}^c) + \vec{t} = \boldsymbol{F} \cdot \vec{x}_0 + \vec{t}$$

2.2.2 Elongation and shear

During deformation a material line element $d\vec{x}_0$ is transformed to the line element $d\vec{x}$. The *elongation factor* or *stretch ratio* λ of the line element, is defined as the ratio of its length after and before deformation. The elongation factor can be expressed in $\boldsymbol{F}$ and $\vec{e}_0$, the unity direction vector of $d\vec{x}_0$. It follows that the elongation is calculated from the product $\boldsymbol{F}^c \cdot \boldsymbol{F}$, which is known as the right Cauchy-Green stretch tensor $\boldsymbol{C}$.

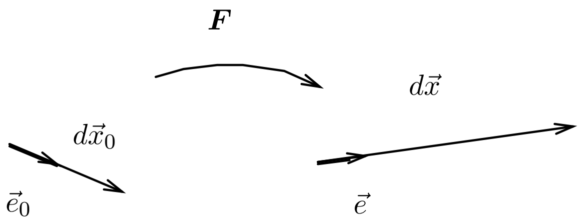


Fig. 2.8 : *Elongation of material line element*

$$\begin{aligned}\lambda(\vec{e}_0) &= \frac{||d\vec{x}||}{||d\vec{x}_0||} = \frac{\sqrt{d\vec{x} \cdot d\vec{x}}}{\sqrt{d\vec{x}_0 \cdot d\vec{x}_0}} = \frac{\sqrt{d\vec{x}_0 \cdot \boldsymbol{F}^c \cdot \boldsymbol{F} \cdot d\vec{x}_0}}{\sqrt{d\vec{x}_0 \cdot d\vec{x}_0}} = \frac{||d\vec{x}_0||}{||d\vec{x}_0||} \sqrt{\vec{e}_0 \cdot \boldsymbol{F}^c \cdot \boldsymbol{F} \cdot \vec{e}_0} \\ &= \sqrt{\vec{e}_0 \cdot \boldsymbol{F}^c \cdot \boldsymbol{F} \cdot \vec{e}_0} = \sqrt{\vec{e}_0 \cdot \boldsymbol{C} \cdot \vec{e}_0}\end{aligned}$$

We consider two material vectors in the undeformed state, $d\vec{x}_{01}$ and $d\vec{x}_{02}$, which are perpendicular. The shear deformation γ is defined as the cosine of θ, the angle between the two material vectors in the deformed state. The shear deformation can be expressed in $\boldsymbol{F}$ and $\vec{e}_{01}$ and $\vec{e}_{02}$, the unit direction vectors of $d\vec{x}_{01}$ and $d\vec{x}_{02}$. Again the shear is calculated from the right Cauchy-Green stretch tensor $\boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F}$.

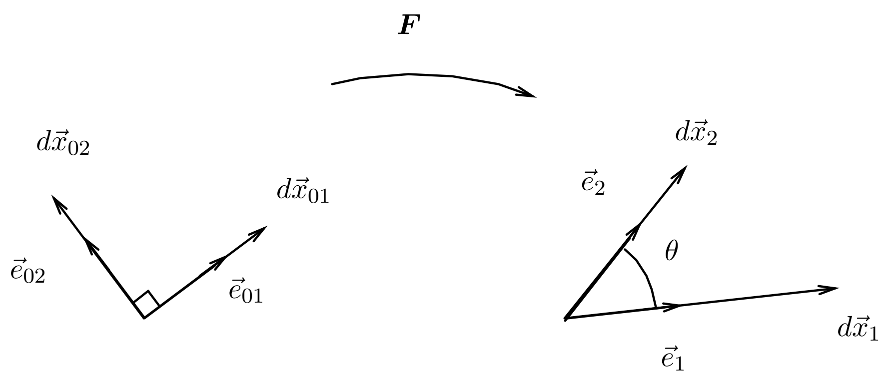


Fig. 2.9 : *Shear of two material line elements*

$$\gamma(\vec{e}_{01}, \vec{e}_{02}) = \sin(\frac{\pi}{2} - \theta) = \cos(\theta) = \frac{d\vec{x}_1 \cdot d\vec{x}_2}{||d\vec{x}_1||||d\vec{x}_2||} = \frac{\vec{e}_{01} \cdot \boldsymbol{F}^c \cdot \boldsymbol{F} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})} = \frac{\vec{e}_{01} \cdot \boldsymbol{C} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})}$$

2.2.3 Strains

The elongation of a material line element is completely described by the stretch ratio λ. When there is no deformation, we have $\lambda = 1$. It is often convenient to describe the elongation with a so-called elongational strain, which is zero when there is no deformation. A strain ε is defined as a function of λ, which has to satisfy certain requirements. Much used strain definitions are the linear, the logarithmic, the Green-Lagrange and the Euler-Almansi strain. One of the requirements of a strain definition is that it must linearize toward the linear strain, which is illustrated in the figure below.

linear	$\varepsilon_l = \lambda - 1$	logarithmic	$\varepsilon_{ln} = \ln(\lambda)$
Green-Lagrange	$\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$	Euler-Almansi	$\varepsilon_{ea} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$

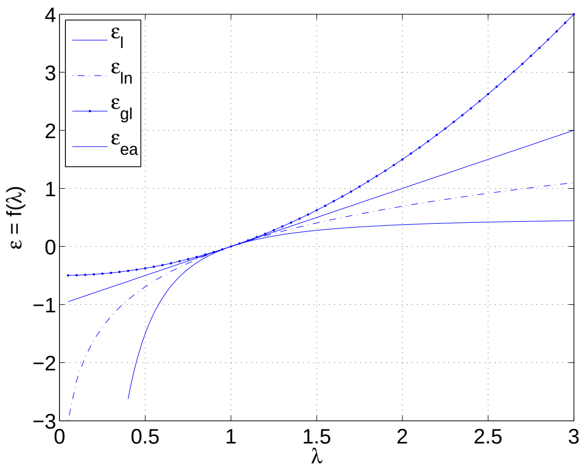


Fig. 2.10 : *Strain definitions*

2.2.4 Strain tensor

The Green-Lagrange strain of a line element with a known direction $\vec{e}_0$ in the undeformed state, can be calculated straightforwardly from the so-called Green-Lagrange strain tensor $\boldsymbol{E}$. Also the shear γ can be expressed in this tensor. For other strain definitions, different strain tensors are used, which are not descussed here.

$$\tfrac{1}{2}\left\{\lambda^2(\vec{e}_0) - 1\right\} = \vec{e}_0 \cdot \left[\tfrac{1}{2}\left(\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}\right)\right] \cdot \vec{e}_0 = \vec{e}_0 \cdot \boldsymbol{E} \cdot \vec{e}_0$$

$$\gamma(\vec{e}_{01}, \vec{e}_{02}) = \frac{\vec{e}_{01} \cdot [\,\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}\,] \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})} = \left(\frac{2}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})}\right) \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{02}$$

Green-Lagrange strain tensor : $\boxed{\boldsymbol{E} = \tfrac{1}{2}\left(\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}\right)}$

2.3 Principal directions of deformation

In each point P there is exactly one orthogonal material volume, which will not show any shear during deformation from t_0 to t. Rigid rotation may occur, although this is not shown in the figure.

The directions $\{1, 2, 3\}$ of the sides of the initial orthogonal volume are called *principal directions* of deformation and associated with them are the three *principal elongation factors* λ_1, λ_2 and λ_3. For this material volume the three principal elongation factors characterize the deformation uniquely. Be aware of the fact that the principal directions change when the deformation proceeds. They are a function of the time t.

The relative volume change J is the product of the three principal elongation factors. For incompressible material there is no volume change, so the above product will have value one.

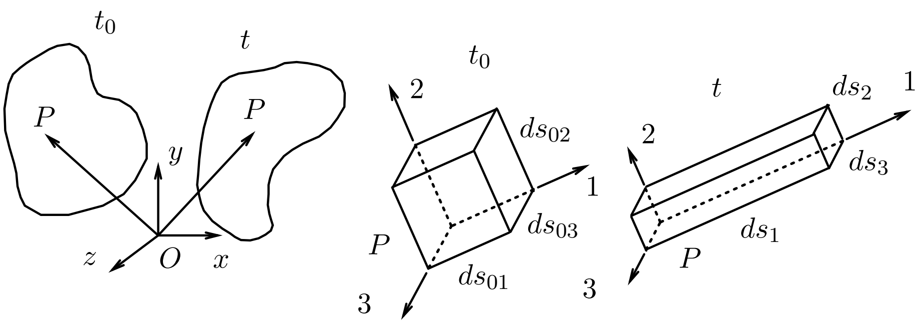


Fig. 2.11 : *Deformation of material cube with sides in principal directions*

$$\lambda_1 = \frac{ds_1}{ds_{01}} \quad ; \quad \lambda_2 = \frac{ds_2}{ds_{02}} \quad ; \quad \lambda_3 = \frac{ds_3}{ds_{03}} \quad ; \quad \gamma_{12} = \gamma_{23} = \gamma_{31} = 0$$

$$J = \frac{dV}{dV_0} = \frac{ds_1 ds_2 ds_3}{ds_{01} ds_{02} ds_{03}} = \lambda_1 \lambda_2 \lambda_3$$

2.4 Linear deformation

In linear elasticity theory deformations are very small. All kind of relations from general continuum mechanics theory may be linearized, resulting for instance in the linear strain tensor $\boldsymbol{\varepsilon}$, which is then fully expressed in the gradient of the displacement. The deformations are in fact so small that the geometry of the material body in the deformed state approximately equals that of the undeformed state.

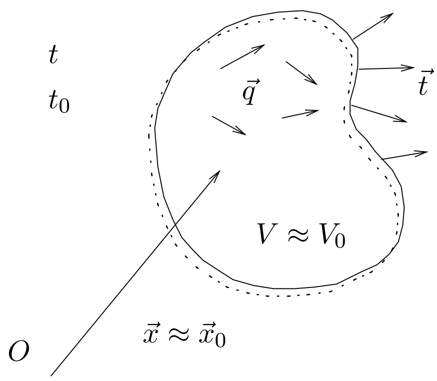


Fig. 2.12 : *Small deformation*

$$
\begin{aligned}
&\boldsymbol{F} = \boldsymbol{I} + (\vec{\nabla}_0 \vec{u})^c \quad \rightarrow \quad (\vec{\nabla}_0 \vec{u})^c = \boldsymbol{F} - \boldsymbol{I} \approx \boldsymbol{O} \quad \rightarrow \\
&\boldsymbol{E} = \tfrac{1}{2} \left\{ (\vec{\nabla}_0 \vec{u})^c + (\vec{\nabla}_0 \vec{u}) + (\vec{\nabla}_0 \vec{u})^c \cdot (\vec{\nabla}_0 \vec{u}) \right\} \\
&\quad \approx \tfrac{1}{2} \left\{ \vec{\nabla}_0 \vec{u})^c + (\vec{\nabla}_0 \vec{u}) \right\} \approx \tfrac{1}{2} \left\{ \vec{\nabla} \vec{u})^c + (\vec{\nabla} \vec{u}) \right\} = \boldsymbol{\varepsilon}
\end{aligned}
$$

Not only straining and shearing must be small to allow the use of linear strains, also the rigid body rotation must be small. This is immediately clear, when we consider the rigid rotation of a material line element PQ around the fixed point P. The x- and y-displacement of point Q, u and v respectively, are expressed in the rotation angle ϕ and the length of the line element dx_0. The nonlinear Green-Lagrange strain is always zero. The linear strain, however, is only zero for very small rotations.

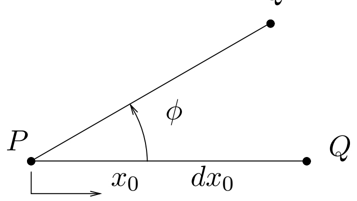


Fig. 2.13 : *Rigid rotation of a line element*

$$u = [\cos(\phi) - 1]\, dx_0 \quad ; \quad v = [\sin(\phi)]\, dx_0$$
$$\frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \quad ; \quad \frac{\partial v}{\partial x_0} = \sin(\phi)$$

Green-Lagrange strain
$$\begin{aligned} \varepsilon_{gl} &= \frac{\partial u}{\partial x_0} + \tfrac{1}{2}\left(\frac{\partial u}{\partial x_0}\right)^2 + \tfrac{1}{2}\left(\frac{\partial v}{\partial x_0}\right)^2 \\ &= \cos(\phi) - 1 + \tfrac{1}{2}\left[\cos(\phi) - 1\right]^2 + \tfrac{1}{2}\sin^2(\phi) = 0 \end{aligned}$$

linear strain
$$\varepsilon_l = \frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \neq 0$$
$$\text{small rotation} \quad \rightarrow \quad \varepsilon_l \approx 0$$

Elongational strain, shear strain volumetric strain

For small deformations and rotations the elongational and shear strain can be linearized and expressed in the linear strain tensor $\boldsymbol{\varepsilon}$. The volume change ratio J can be expressed in linear strain components and also linearized.

elongational strain
$$\begin{aligned} \tfrac{1}{2}\left(\lambda^2(\vec{e}_{01}) - 1\right) &= \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{01} \\ &\downarrow \\ \lambda(\vec{e}_{01}) - 1 &= \vec{e}_{01} \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_{01} \end{aligned}$$

shear strain
$$\begin{aligned} \gamma(\vec{e}_{01}, \vec{e}_{02}) = \sin\left(\tfrac{\pi}{2} - \theta\right) &= \left(\frac{2}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})}\right) \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{02} \\ &\downarrow \\ \tfrac{\pi}{2} - \theta &= 2\,\vec{e}_{01} \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_{02} \end{aligned}$$

volume change
$$\begin{aligned} J = \frac{dV}{dV_0} = \frac{ds_1 ds_2 ds_3}{ds_{01} ds_{02} ds_{03}} &= \lambda_1\lambda_2\lambda_3 = (\varepsilon_1 + 1)(\varepsilon_2 + 1)(\varepsilon_2 + 1) \\ &\downarrow \\ J &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + 1 = \mathrm{tr}(\boldsymbol{\varepsilon}) + 1 \end{aligned}$$

volume strain
$$J - 1 = \mathrm{tr}(\boldsymbol{\varepsilon})$$

2.4.1 Linear strain matrix

With respect to an orthogonal basis, the linear strain tensor can be written in components, resulting in the linear strain matrix.

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \quad \text{with} \quad \left\{ \begin{array}{l} \varepsilon_{21} = \varepsilon_{12} \\ \varepsilon_{32} = \varepsilon_{23} \\ \varepsilon_{31} = \varepsilon_{13} \end{array} \right.$$

2.4.2 Cartesian components

The linear strain components w.r.t. a Cartesian coordinate system are easily derived using the component expressions for the gradient operator and the displacement vector. For derivatives a short notation is used : $(\ )_{i,j} = \dfrac{\partial(\ )_i}{\partial x_j}$.

gradient operator $\qquad \vec{\nabla} = \vec{e}_x \dfrac{\partial}{\partial x} + \vec{e}_y \dfrac{\partial}{\partial y} + \vec{e}_z \dfrac{\partial}{\partial z}$

displacement vector $\qquad \vec{u} = u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z$

linear strain tensor $\qquad \boldsymbol{\varepsilon} = \frac{1}{2}\left\{(\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u})\right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2u_{x,x} & u_{x,y} + u_{y,x} & u_{x,z} + u_{z,x} \\ u_{y,x} + u_{x,y} & 2u_{y,y} & u_{y,z} + u_{z,y} \\ u_{z,x} + u_{x,z} & u_{z,y} + u_{y,z} & 2u_{z,z} \end{bmatrix}$$

Compatibility conditions

The six independent strain components are related to only three displacement components. Therefore the strain components cannot be independent. Six relations can be derived, which are referred to as the compatibility conditions.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \qquad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial z}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \qquad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial y^2} = \frac{\partial^2 \varepsilon_{yx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial x}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \qquad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial z^2} = \frac{\partial^2 \varepsilon_{zy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial y}$$

2.4.3 Cylindrical components

The linear strain components w.r.t. a cylindrical coordinate system are derived straightforwardly using the component expressions for the gradient operator and the displacement vector.

gradient operator $\qquad \vec{\nabla} = \vec{e}_r \dfrac{\partial}{\partial r} + \vec{e}_t \dfrac{1}{r}\dfrac{\partial}{\partial \theta} + \vec{e}_z \dfrac{\partial}{\partial z}$

displacement vector $\qquad \vec{u} = u_r \vec{e}_r(\theta) + u_t \vec{e}_t(\theta) + u_z \vec{e}_z$

linear strain tensor $\qquad \boldsymbol{\varepsilon} = \frac{1}{2}\left\{(\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u})\right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{rt} & \varepsilon_{rz} \\ \varepsilon_{tr} & \varepsilon_{tt} & \varepsilon_{tz} \\ \varepsilon_{zr} & \varepsilon_{zt} & \varepsilon_{zz} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ u_{z,r} + u_{r,z} & \frac{1}{r}u_{z,t} + u_{t,z} & 2u_{z,z} \end{bmatrix}$$

Compatibility condition

In the $r\theta$-plane there is one compatibility relation.

$$\frac{1}{r^2}\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \varepsilon_{tt}}{\partial r^2} - \frac{2}{r}\frac{\partial^2 \varepsilon_{rt}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \varepsilon_{tt}}{\partial r} - \frac{2}{r^2}\frac{\partial \varepsilon_{rt}}{\partial \theta} = 0$$

2.4.4 Principal strains and directions

Because the linear strain tensor is symmetric, it has three real-valued eigenvalues $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and associated eigenvectors $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. The eigenvectors are normalized to have unit length and they are mutually perpendicular, so they constitute an orthonormal vector base. The strain matrix w.r.t. this vector base is diagonal.

The eigenvalues are referred to as the *principal strains* and the eigenvectors as the *principal strain directions*. They are equivalent to the *principal directions* of deformatieon. Line elements along these directions in the undeformed state t_0 do not show any shear during deformation towards the current state t.

spectral form	$\boldsymbol{\varepsilon} = \varepsilon_1 \vec{n}_1 \vec{n}_1 + \varepsilon_2 \vec{n}_2 \vec{n}_2 + \varepsilon_3 \vec{n}_3 \vec{n}_3$
principal strain matrix	$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$

2.5 Special deformations

2.5.1 Planar deformation

It often happens that (part of) a structure is loaded in one plane. Moreover the load is often such that no bending out of that plane takes place. The resulting deformation is referred to as being planar.

Here it is assumed that the plane of deformation is the x_1x_2-plane. Note that in this planar deformation there still can be displacement perpendicular to the plane of deformation, which results in change of thickness.

The *in-plane* displacement components u_1 and u_2 are only a function of x_1 and x_2. The *out-of-plane* displacement u_3 may be a function of x_3 as well.

$$u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = u_3(x_1, x_2, x_3)$$

2.5.2 Plane strain

When the boundary conditions and the material behavior are such that displacement of material points are only in the x_1x_2-plane, the deformation is referred to as *plane strain* in the x_1x_2-plane. Only three relevant strain components remain.

$$
\begin{aligned}
&u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = 0 \\
&\varepsilon_{33} = 0 \quad ; \quad \gamma_{13} = \gamma_{23} = 0
\end{aligned}
$$

2.5.3 Axi-symmetric deformation

Many man-made and natural structures have an axi-symmetric geometry, which means that their shape and volume can be constructed by virtually rotating a cross section around the axis of revolution. Points are indicated with cylindrical coordinates $\{r, \theta, z\}$. When material properties and loading are also independent of the coordinate θ, the deformation and resulting stresses will be also independent of θ. With the *additional* assumption that no rotation around the z-axis takes place ($u_t = 0$), all state variables can be studied in one half of the cross section through the z-axis.

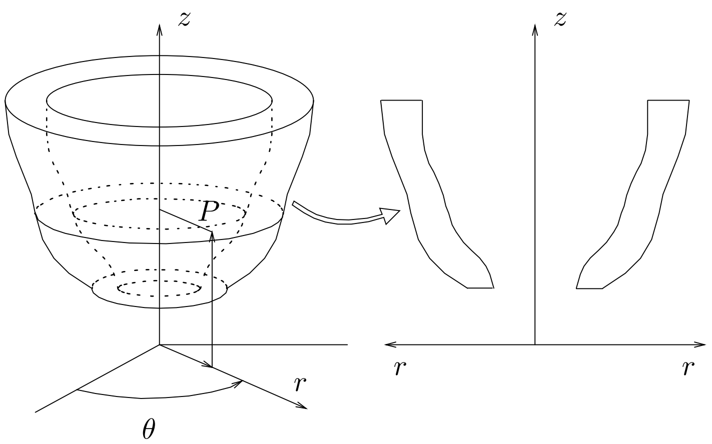


Fig. 2.14 : *Axi-symmetric deformation*

$$
\begin{aligned}
\frac{\partial}{\partial \theta} = 0 \qquad &\rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_t(r,z)\vec{e}_t(\theta) + u_z(r,z)\vec{e}_z \\
\frac{\partial}{\partial \theta} = 0 \text{ and } u_t = 0 \qquad &\rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_z(r,z)\vec{e}_z
\end{aligned}
$$

The strain-displacement relations are simplified considerably.

$$
\underset{\sim}{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & -\frac{1}{r}(u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ -\frac{1}{r}(u_t) + u_{t,r} & 2\frac{1}{r}(u_r) & u_{t,z} \\ u_{z,r} + u_{r,z} & u_{t,z} & 2u_{z,z} \end{bmatrix}
$$

$$\underline{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & 0 & u_{r,z} + u_{z,r} \\ 0 & 2\frac{1}{r}(u_r) & 0 \\ u_{z,r} + u_{r,z} & 0 & 2u_{z,z} \end{bmatrix} \quad \text{when} \quad u_t = 0$$

Axi-symmetric plane strain

When boundary conditions and material behavior are such that displacement of material points are only in the $r\theta$-plane, the deformation is referred to as *plane strain* in the $r\theta$-plane.

plane strain deformation

$$\left. \begin{array}{l} u_r = u_r(r, \theta) \\ u_t = u_t(r, \theta) \\ u_z = 0 \end{array} \right\} \quad \rightarrow \quad \varepsilon_{zz} = \gamma_{rz} = \gamma_{tz} = 0$$

linear strain matrix

$$\underline{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & u_{t,r} - \frac{1}{r}(u_t) & 0 \\ u_{t,r} - \frac{1}{r}(u_t) & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

plane strain deformation with $u_t = 0$

$$\left. \begin{array}{l} u_r = u_r(r) \\ u_z = 0 \end{array} \right\} \quad \rightarrow \quad \underline{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & 0 & 0 \\ 0 & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.6 Examples

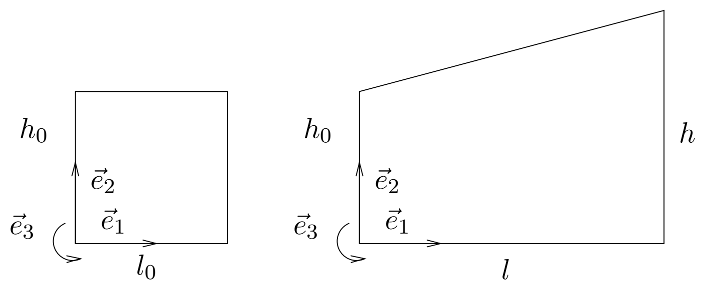
Inhomogeneous deformation

A rectangular block of material is deformed, as shown in the figure. The basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is orthonormal. The position vector of an arbitrary material point in undeformed and deformed state, respectively is :

$$\vec{x}_0 = x_{01}\vec{e}_1 + x_{02}\vec{e}_2 + x_{03}\vec{e}_3 \quad ; \quad \vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_{03}\vec{e}_3$$

There is no deformation in $\vec{e}_3$-direction. Deformation in the 12-plane is such that straight lines remain straight during deformation.

The deformation tensor can be calculated from the relation between the coordinates of the material point in undeformed and deformed state.

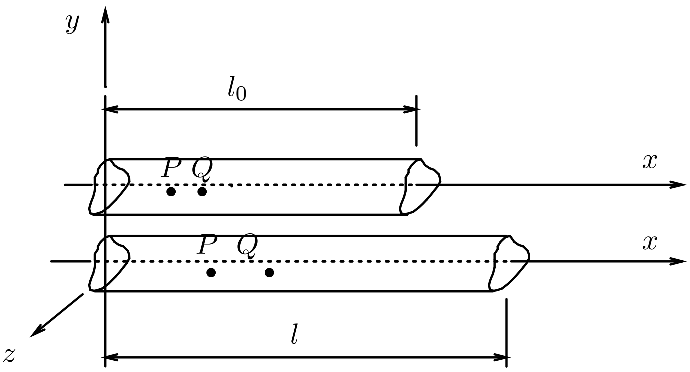


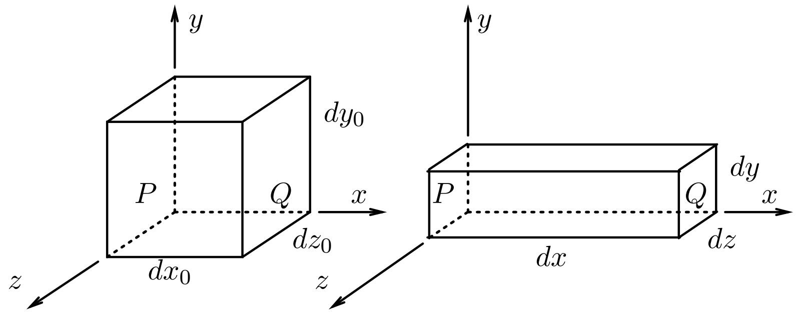
$$x_1 = \frac{l}{l_0} x_{01} \quad ; \quad x_2 = x_{02} + \frac{h - h_0}{h_0 l_0} x_{01} x_{02} \quad ; \quad x_3 = x_{03}$$

$$\begin{aligned}
\boldsymbol{F}^c &= \left(\vec{\nabla}_0 \vec{x}\right) \\
&= \left(\vec{e}_{01}\frac{\partial}{\partial x_{01}} + \vec{e}_{02}\frac{\partial}{\partial x_{02}} + \vec{e}_{03}\frac{\partial}{\partial x_{03}}\right)(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3) \\
&= \left(\vec{e}_{01}\frac{\partial}{\partial x_{01}} + \vec{e}_{02}\frac{\partial}{\partial x_{02}} + \vec{e}_{03}\frac{\partial}{\partial x_{03}}\right) \\
&\qquad \left[\left(\frac{l}{l_0} x_{01}\right)\vec{e}_1 + \left(x_{02} + \frac{h - h_0}{h_0 l_0} x_{01} x_{02}\right)\vec{e}_2 + (x_{03})\,\vec{e}_3\right] \\
&= \left(\frac{l}{l_0}\right)\vec{e}_{01}\vec{e}_1 + \left(\frac{h - h_0}{h_0 l_0} x_{02}\right)\vec{e}_{01}\vec{e}_2 + \left(1 + \frac{h - h_0}{h_0 l_0} x_{01}\right)\vec{e}_{02}\vec{e}_2 + \vec{e}_{03}\vec{e}_3
\end{aligned}$$

Strain $\sim$ displacement

The strain-displacement relations for the elongation of line elements can be derived by considering the elongational deformation of an infinitesimal cube of material e.g. in a tensile test.

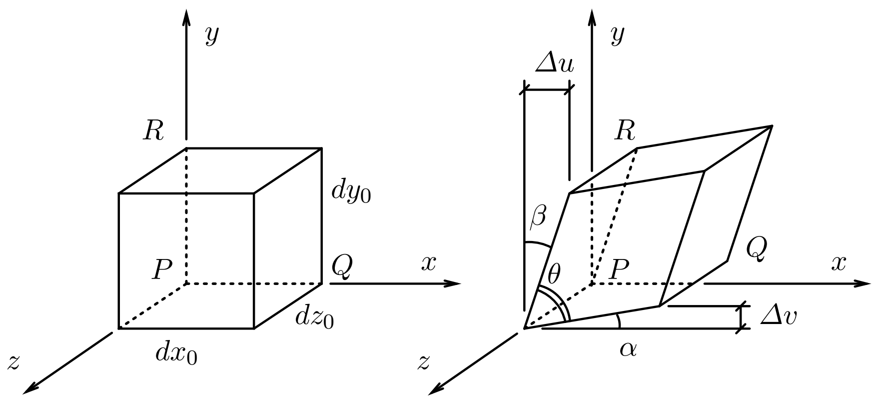




$$
\begin{aligned}
\varepsilon_{xx} &= \lambda_{xx} - 1 = \frac{dx}{dx_0} - 1 = \frac{dx - dx_0}{dx_0} = \frac{u_Q - u_P}{dx_0} \\
&= \frac{u(x_0 + dx_0) - u(x_0)}{dx_0} = \frac{\partial u}{\partial x_0} = \frac{\partial u}{\partial x} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} \qquad ; \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}
\end{aligned}
$$

Strain $\sim$ displacement

The strain-displacement relations for the shear of two line elements can be derived by considering the shear deformation of an infinitesimal cube of material e.g. in a torsion test.

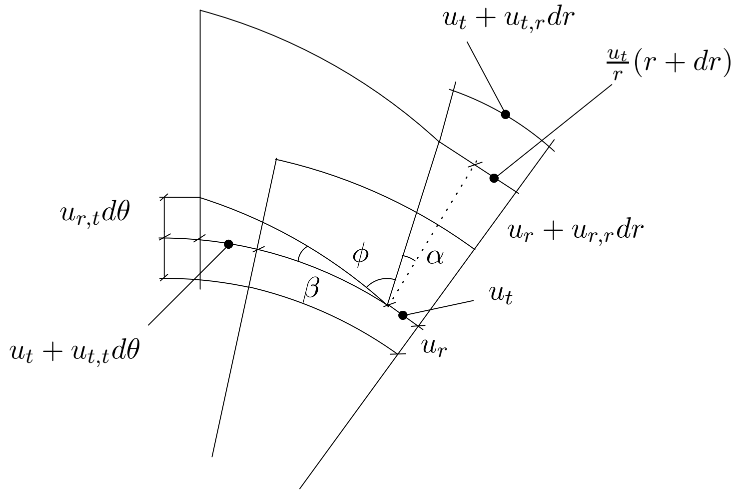


$$
\begin{aligned}
\gamma_{xy} &= \tfrac{\pi}{2} - \theta_{xy} = \alpha + \beta \approx \sin(\alpha) + \sin(\beta) \\
&= \frac{\Delta v}{dx_0} + \frac{\Delta u}{dy_0} = \frac{v_Q - v_P}{dx_0} + \frac{u_R - u_P}{dy_0} \\
&= \frac{v(x_0 + dx_0) - v(x_0)}{dx_0} + \frac{u(y_0 + dy_0) - u(y_0)}{dy_0} \\
&= \frac{\partial v}{\partial x_0} + \frac{\partial u}{\partial y_0} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad ; \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}
\end{aligned}
$$

Strain ~ displacement

Strain-displacement relations can be derived geometrically in the cylindrical coordinate system, as we did in the Cartesian coordinate system.

We consider the deformation of an infinitesimal part in the $r\theta$-plane and determine the elongational and shear strain components. The dimensions of the material volume in undeformed state are $dr \times rd\theta \times dz$.



$$
\begin{aligned}
\varepsilon_{rr} &= \frac{u_{r,r}dr}{dr} = u_{r,r} \\
\varepsilon_{tt} &= \frac{(r + u_r)d\theta - rd\theta}{rd\theta} + \frac{(u_t + u_{t,t}d\theta) - u_t}{rd\theta} = \frac{u_r}{r} + \frac{1}{r}\,u_{t,t} \\
\gamma_{rt} &= \frac{\pi}{2} - \phi = \alpha + \beta = \left(u_{t,r} - \frac{u_t}{r}\right) + \left(\frac{1}{r}\,u_{r,t}\right)
\end{aligned}
$$

Strain gages

Strain gages are used to measure strains on the surface of a thin walled pressure vessel. Three gages are glued on the surface, the second perpendicular to the first one and the third at an angle of 45^o between those two. Measured strains have values $\varepsilon_{g1}, \varepsilon_{g2}$ and ε_{g3}.

The linear strain tensor is written in components w.r.t. the Cartesian coordinate system with its x-axis along the first strain gage. The components $\varepsilon_{xx}, \varepsilon_{xy}$ and ε_{yy} have to be determined from the measured values.

To do this, we use the expression which gives us the strain in a specific direction, indicated by the unit vector $\vec{n}$.

$$\varepsilon_n = \vec{n} \cdot \boldsymbol{\varepsilon} \cdot \vec{n}$$

Because we have three different directions, where the strain is known, we can write this equation three times.

$$\varepsilon_{g1} = \vec{n}_{g1} \cdot \boldsymbol{\varepsilon} \cdot \vec{n}_{g1} = \underset{\sim}{n}_{g1}^T \underline{\varepsilon}\, \underset{\sim}{n}_{g1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon_{xx}$$

$$\varepsilon_{g2} = \vec{n}_{g2} \cdot \boldsymbol{\varepsilon} \cdot \vec{n}_{g2} = \underset{\sim}{n}_{g2}^T \underline{\varepsilon}\, \underset{\sim}{n}_{g2} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \varepsilon_{yy}$$

$$\varepsilon_{g3} = \vec{n}_{g3} \cdot \boldsymbol{\varepsilon} \cdot \vec{n}_{g3} = \underset{\sim}{n}_{g3}^T \underline{\varepsilon}\, \underset{\sim}{n}_{g3} = \tfrac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tfrac{1}{2}(\varepsilon_{xx} + 2\varepsilon_{xy} + \varepsilon_{yy})$$

The first two equations immediately lead to values for ε_{xx} and ε_{yy} and the remaining unknown, ε_{xy} can be solved from the last equation.

$$\left.\begin{aligned} \varepsilon_{xx} &= \varepsilon_{g1} \\ \varepsilon_{yy} &= \varepsilon_{g2} \\ \varepsilon_{xy} &= 2\varepsilon_{g3} - \varepsilon_{xx} - \varepsilon_{yy} \\ &= 2\varepsilon_{g3} - \varepsilon_{g1} - \varepsilon_{g2} \end{aligned}\right\} \quad \rightarrow \quad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_{g1} & 2\varepsilon_{g3} - \varepsilon_{g1} - \varepsilon_{g2} \\ 2\varepsilon_{g3} - \varepsilon_{g1} - \varepsilon_{g2} & \varepsilon_{g2} \end{bmatrix}$$

The three gages can be oriented at various angles with respect to each other and with respect to the coordinate system. However, the three strain components can always be solved from a set of three independent equations.

Chapter 3

Stresses

Kinematics describes the motion and deformation of a set of material points, considered here to be a continuous body. The cause of this deformation is not considered in kinematics. Motion and deformation may have various causes, which are collectively considered here to be external forces and moments.

Deformation of the material – not its motion alone – results in internal stresses. It is very important to calculate them accurately, because they may cause irreversible structural changes and even unallowable damage of the material.

3.1 Stress vector

Consider a material body in the deformed state, with edge and volume forces. The body is divided in two parts, where the cutting plane passes through the material point P. An edge load is introduced on both sides of the cutting plane to prevent separation of the two parts. In two associated points (= coinciding before the cut was made) in the cutting plane of both parts, these loads are of opposite sign, but have equal absolute value.

The resulting force on an area ΔA of the cutting plane in point P is $\Delta\vec{k}$. The resulting force per unit of area is the ratio of $\Delta\vec{k}$ and ΔA. The *stress vector* $\vec{p}$ in point P is defined as the limit value of this ratio for $\Delta A \rightarrow 0$. So, obviously, the stress vector is associated to both poin P and the cutting plane through this point.

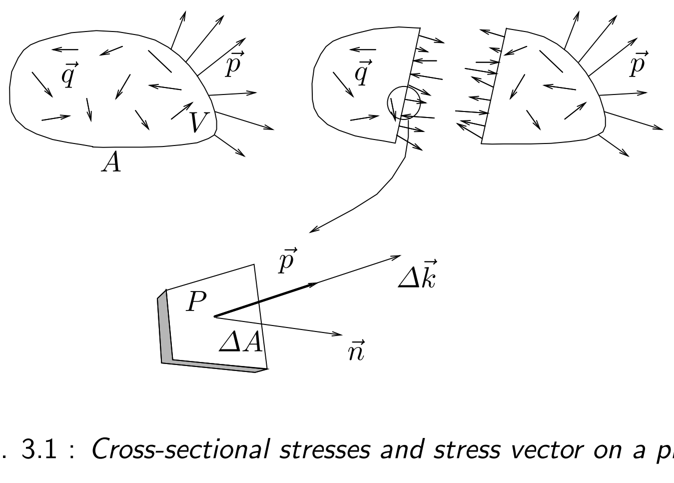


Fig. 3.1 : *Cross-sectional stresses and stress vector on a plane*

$$\boxed{\vec{p} = \lim_{\Delta A \to 0} \frac{\Delta \vec{k}}{\Delta A}}$$

3.1.1 Normal stress and shear stress

The stress vector $\vec{p}$ can be written as the sum of two other vectors. The first is the *normal stress vector* $\vec{p}_n$ in the direction of the unity normal vector $\vec{n}$ on ΔA. The second vector is in the plane and is called the *shear stress vector* $\vec{p}_s$.

The length of the normal stress vector is the *normal stress* p_n and the length of the shear stress vector is the *shear stress* p_s.

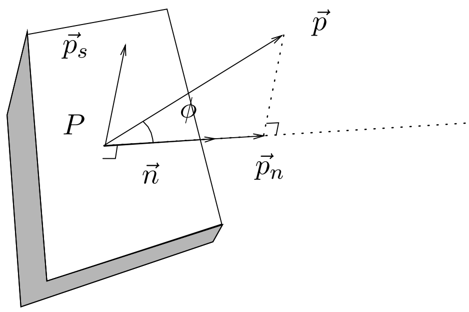


Fig. 3.2 : *Decomposition of stress vector in normal and shear stress*

normal stress	:	$p_n = \vec{p} \cdot \vec{n}$
tensile stress	:	positive ($\phi < \frac{\pi}{2}$)
compression stress	:	negative ($\phi > \frac{\pi}{2}$)
normal stress vector	:	$\vec{p}_n = p_n \vec{n}$
shear stress vector	:	$\vec{p}_s = \vec{p} - \vec{p}_n$
shear stress	:	$p_s = \|\|\vec{p}_s\|\| = \sqrt{\|\|\vec{p}\|\|^2 - p_n^2}$

3.2 Cauchy stress tensor

The stress vector can be calculated, using the *stress tensor* $\boldsymbol{\sigma}$, which represents the stress state in point P. The plane is identified by its unity normal vector $\vec{n}$. The stress vector is calculated according to Cauchy's theorem, which states that in each material point such a stress tensor must uniquely exist. ($\exists!$: there exists only one.)

Theorem of Cauchy : $\exists!$ tensor $\boldsymbol{\sigma}$ such that : $\boxed{\vec{p} = \boldsymbol{\sigma} \cdot \vec{n}}$

3.2.1 Cauchy stress matrix

With respect to an orthogonal basis, the Cauchy stress tensor $\boldsymbol{\sigma}$ can be written in components, resulting in the Cauchy stress matrix $\underline{\sigma}$, which stores the components of the Cauchy stress tensor w.r.t. an orthonormal vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. The components of the Cauchy stress matrix are components of stress vectors on the planes with unit normal vectors in the coordinate directions.

With our definition, the first index of a stress component indicates the direction of the stress vector and the second index indicates the normal of the plane where it is loaded. As an example, the stress vector on the plane with $\vec{n} = \vec{e}_1$ is considered.

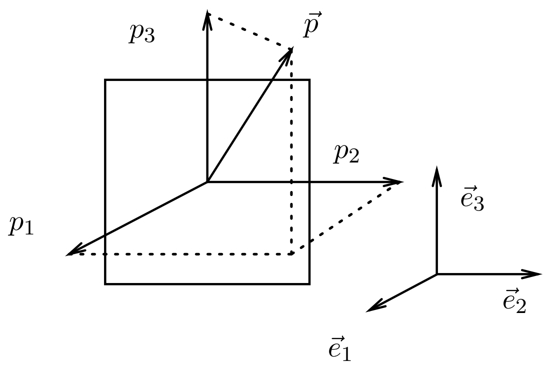
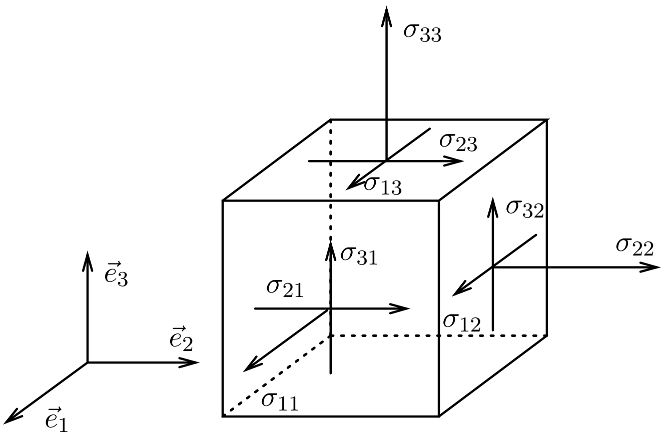


Fig. 3.3 : *Components of stress vector on a plane*

$$\vec{p} = \boldsymbol{\sigma} \cdot \vec{e}_1 \quad \rightarrow \quad \underset{\sim}{p} = \underset{\sim}{\vec{e}} \cdot \vec{p} = \underset{\sim}{\vec{e}} \cdot \boldsymbol{\sigma} \cdot \vec{e}_1 = \underset{\sim}{\vec{e}} \cdot (\underset{\sim}{\vec{e}}^T \underline{\sigma}\, \underset{\sim}{\vec{e}}) \cdot \vec{e}_1 = \underline{\sigma}\, \underset{\sim}{\vec{e}} \cdot \vec{e}_1$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}$$

The components of the Cauchy stress matrix can be represented as normal and shear stresses on the side planes of a stress cube.

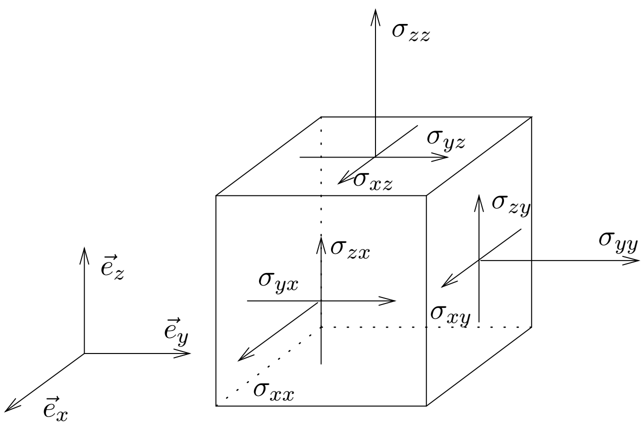


$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Fig. 3.4 : *Stress cube*

3.2.2 Cartesian components

In the Cartesian coordinate system the stress cube sides are parallel to the Cartesian coordinate axes. Stress components are indicated with the indices x, y and z.

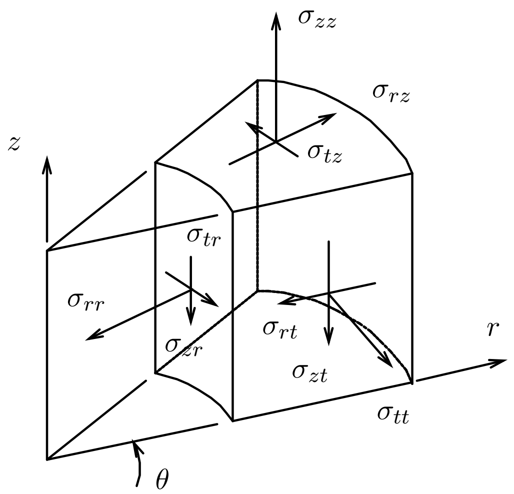


$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Fig. 3.5 : *Cartesian stress cube*

3.2.3 Cylindrical components

In the cylindrical coordinate system the stress 'cube' sides are parallel to the cylindrical coordinate axes. Stress components are indicated with the indices r, t and z.



$$\underline{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{rt} & \sigma_{rz} \\ \sigma_{tr} & \sigma_{tt} & \sigma_{tz} \\ \sigma_{zr} & \sigma_{zt} & \sigma_{zz} \end{bmatrix}$$

Fig. 3.6 : *Cylindrical stress "cube"*

3.3 Principal stresses and directions

It will be shown later that the stress tensor is symmetric. This means that it has three real-valued eigenvalues $\{\sigma_1, \sigma_2, \sigma_3\}$ and associated eigenvectors $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. The eigenvectors are normalized to have unit length and they are mutually perpendicular, so they constitute an orthonormal vector base. The stress matrix w.r.t. this vector base is diagonal.

The eigenvalues are referred to as the principal stresses and the eigenvectors as the principal stress directions. The stress cube with the normal principal stresses is referred to as the principal stress cube.

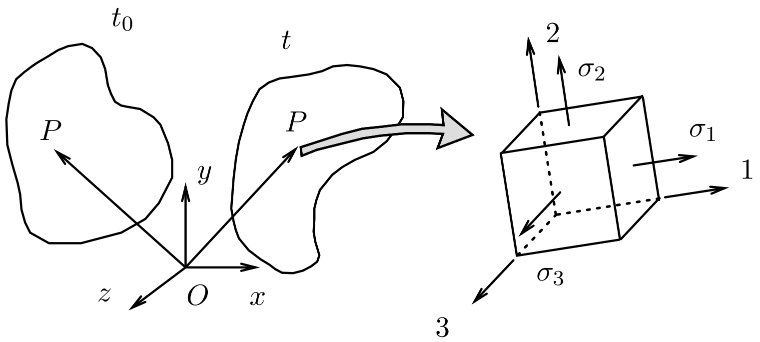


Fig. 3.7 : *Principal stress cube with principal stresses*

spectral form $\qquad \boldsymbol{\sigma} = \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3$

principal stress matrix $\qquad \underline{\sigma}_P = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

Stress transformation

We consider the two-dimensional plane with principal stress directions coinciding with the unity vectors $\vec{e}_1$ and $\vec{e}_2$. The principal stresses are σ_1 and σ_2. On a plane which is rotated anti-clockwise from $\vec{e}_1$ over an angle $\alpha < \frac{\pi}{2}$ the stress vector $\vec{p}$ and its normal and shear components can be calculated. They are indicated as σ_α and τ_α respectively.

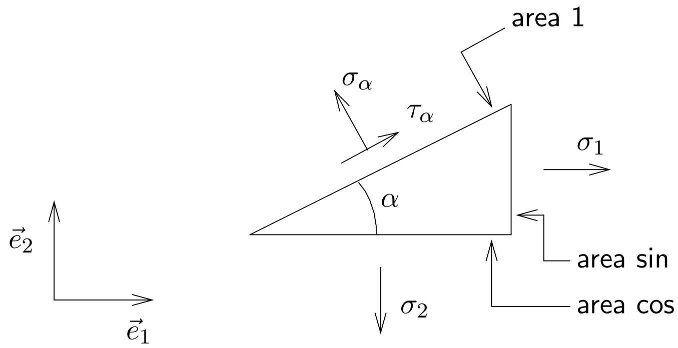


Fig. 3.8 : *Normal and shear stress on a plane*

$$
\begin{aligned}
&\boldsymbol{\sigma} = \sigma_1 \vec{e}_1 \vec{e}_1 + \sigma_2 \vec{e}_2 \vec{e}_2 \\
&\vec{n} = -\sin(\alpha)\vec{e}_1 + \cos(\alpha)\vec{e}_2 \\
&\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} = -\sigma_1 \sin(\alpha)\vec{e}_1 + \sigma_2 \cos(\alpha)\vec{e}_2 \\
&\sigma_\alpha = \sigma_1 \sin^2(\alpha) + \sigma_2 \cos^2(\alpha) \\
&\tau_\alpha = (\sigma_2 - \sigma_1)\sin(\alpha)\cos(\alpha)
\end{aligned}
$$

Mohr's circles of stress

From the relations for the normal and shear stress on a plane in between two principal stress planes, a relation between these two stresses and the principal stresses can be derived. The resulting relation is the equation of a circle in the $\sigma_\alpha\tau_\alpha$-plane, referred to as Mohr's circle for stress. The radius of the circle is $\frac{1}{2}(\sigma_1 - \sigma_2)$. The coordinates of its center are $\{\frac{1}{2}(\sigma_1 + \sigma_2), 0\}$.

Stresses on a plane, which is rotated over α w.r.t. a principal stress plane, can be found in the circle by rotation over 2α.

$$
\begin{aligned}
\sigma_\alpha &= \sigma_1 \sin^2(\alpha) + \sigma_2 \cos^2(\alpha) \\
&= \sigma_1(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\alpha)) + \sigma_2(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\alpha)) \\
&= \tfrac{1}{2}(\sigma_1 + \sigma_2) - \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos(2\alpha) \quad \rightarrow
\end{aligned}
$$

$$
\left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2 \cos^2(2\alpha)
$$

$$
\tau_\alpha = (\sigma_2 - \sigma_1)\sin(\alpha)\cos(\alpha) = \tfrac{1}{2}(\sigma_2 - \sigma_1)\sin(2\alpha) \quad \rightarrow
$$

$$
\tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_2 - \sigma_1)\right\}^2 \sin^2(2\alpha)
$$

add equations $\quad \rightarrow \quad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 + \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2$

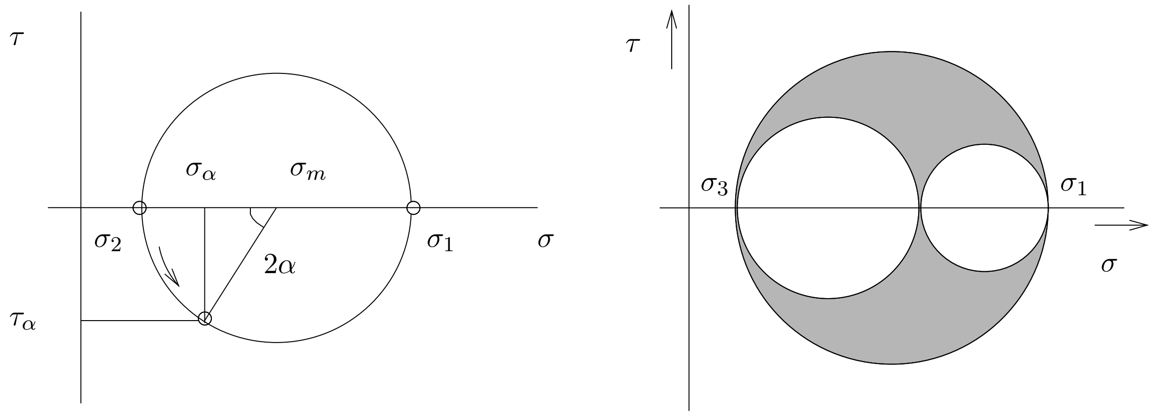


Fig. 3.9 : *Mohr's circles*

Because there are three principal stresses and principal stress planes, there are also three stress circles. It can be proven that each stress state is located on one of the circles or in the shaded area.

3.4 Special stress states

Some special stress states are illustrated here. Stress components are considered in the Cartesian coordinate system.

3.4.1 Uni-axial stress

An unidirectional stress state is what we have in a tensile bar or truss. The axial load N in a cross-section (area A in the deformed state) is the integral of the axial stress σ over A. For homogeneous material the stress is uniform in the cross-section and is called the *true* or *Cauchy stress*. When it is assumed to be uniform in the cross-section, it is the ratio of N and A. The *engineering stress* is the ratio of N and the initial cross-sectional area A_0, which makes calculation easy, because A does not have to be known. For small deformations it is obvious that $A \approx A_0$ and thus that $\sigma \approx \sigma_n$.

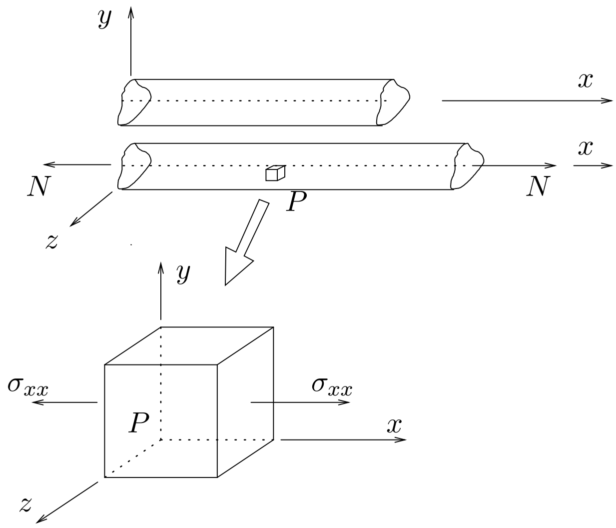
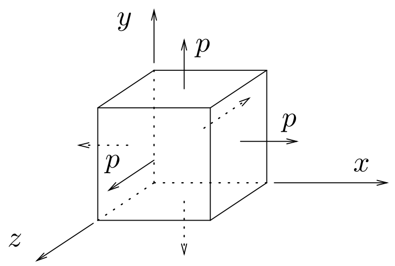


Fig. 3.10 : *Stresses on a small material volume in a tensile bar*

true or Cauchy stress	$\sigma = \dfrac{N}{A} = \sigma_{xx} \quad \rightarrow \quad \boxed{\boldsymbol{\sigma} = \sigma_{xx}\vec{e}_x\vec{e}_x}$
engineering stress	$\sigma_n = \dfrac{N}{A_0}$

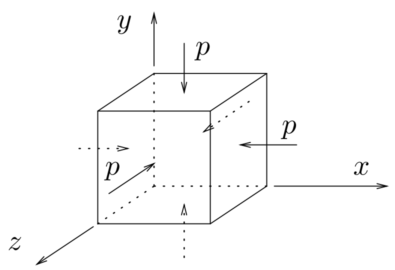
3.4.2 Hydrostatic stress

A hydrostatic loading of the material body results in a hydrostatic stress state in each material point P. This can again be indicated by stresses (either tensile or compressive) on a stress cube. The three stress variables, with the same value, are normal to the faces of the stress cube.



$\sigma_{xx} = p$
$\sigma_{yy} = p$
$\sigma_{zz} = p$

$$\boxed{\boldsymbol{\sigma} = p(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z)}$$



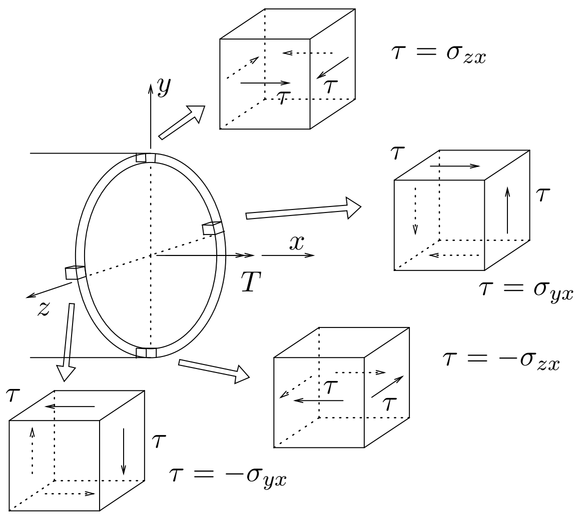
$\sigma_{xx} = -p$
$\sigma_{yy} = -p$
$\sigma_{zz} = -p$

$$\boxed{\boldsymbol{\sigma} = -p(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z)}$$

Fig. 3.11 : *Stresses on a material volume under hydrostatic loading*

3.4.3 Shear stress

The axial torsion of a thin-walled tube (radius R, wall thickness t) is the result of an axial torsional moment (torque) T. This load causes a shear stress τ in the cross-sectional wall. Although this shear stress has the same value in each point of the cross-section, the stress cube looks differently in each point because of the circumferential direction of τ.



$$\tau = \frac{T}{2\pi R^2 t}$$

Fig. 3.12 : *Stresses on a small material volume in the wall of a tube under shear loading*

$$\boxed{\boldsymbol{\sigma} = \tau(\vec{e}_i\vec{e}_j + \vec{e}_j\vec{e}_i) \text{ with } i \neq j}$$

3.4.4 Plane stress

When stresses on a plane perpendicular to the 3-direction are zero, the stress state is referred to as *plane stress* w.r.t. the 12-plane. Only three stress components are relevant in this case.

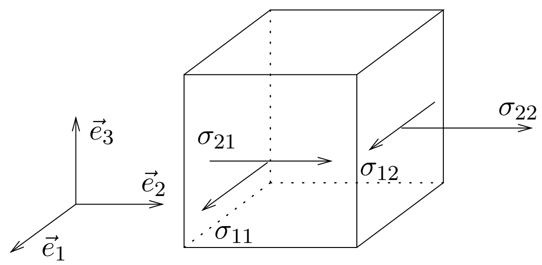


Fig. 3.13 : *Stress cube for plane stress in 12-plane*

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \quad \rightarrow \quad \boxed{\boldsymbol{\sigma} \cdot \vec{e}_3 = \vec{0}} \quad \rightarrow \quad \text{relevant stresses :} \quad \sigma_{11}, \sigma_{22}, \sigma_{12}$$

3.5 Resulting force on arbitrary material volume

A material body with volume V and surface area A is loaded with a volume load $\vec{q}$ per unit of mass and by a surface load $\vec{p}$ per unit of area. Taking a random part of the continuum with volume $\bar{V}$ and edge $\bar{A}$, the resulting force can be written as an integral over the volume, using Gauss' theorem. The load $\rho\vec{q}$ is a volume load per unit of volume, where ρ is the density of the material.

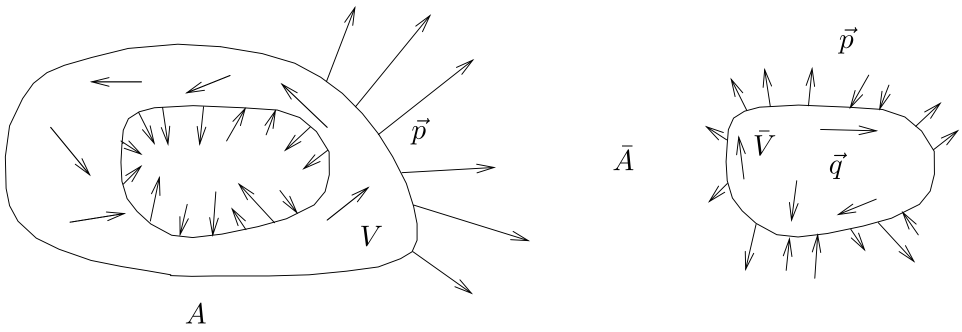


Fig. 3.14 : *Forces on a random section of a material body*

resulting force on $\bar{V}$ $$\vec{K} = \int_{\bar{V}} \rho\vec{q}\, dV + \int_{\bar{A}} \vec{p}\, dA = \int_{\bar{V}} \rho\vec{q}\, dV + \int_{\bar{A}} \vec{n} \cdot \boldsymbol{\sigma}^c \, dA$$

Gauss' theorem $\rightarrow$ $$\boxed{\vec{K} = \int_{\bar{V}} \left(\rho\vec{q} + \vec{\nabla} \cdot \boldsymbol{\sigma}^c\right) dV}$$

3.6 Resulting moment on arbitrary material volume

The resulting moment about a fixed point of the forces working in volume and edge points of a random part of the continuum body can be calculated by integration.

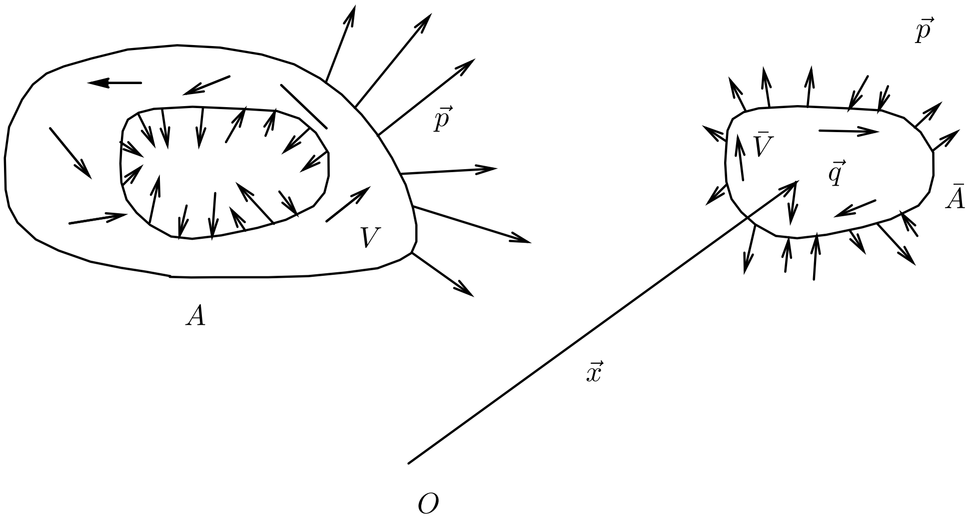


Fig. 3.15 : *Moments of forces on a random section of a material body*

resulting moment about O

$$\boxed{\vec{M}_O = \int\limits_{\bar{V}} \vec{x} * \rho\vec{q}\, dV + \int\limits_{\bar{A}} \vec{x} * \vec{p}\, dA}$$

3.7 Example

Principal stresses and stress directions

The stress state in a material point P is characterized by the stress tensor $\boldsymbol{\sigma}$, which is given in comonents with respect to an orthonormal basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$:

$$\boldsymbol{\sigma} = 10\vec{e}_1\vec{e}_1 + 6(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + 10\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3$$

The principal stresses are the eigenvalues of the tensor, which can be calculated as follows :

$$\begin{array}{l}
\det \begin{bmatrix} 10-\sigma & 6 & 0 \\ 6 & 10-\sigma & 0 \\ 0 & 0 & 1-\sigma \end{bmatrix} = 0 \quad \rightarrow \\
(10-\sigma)^2(1-\sigma) - 36(1-\sigma) = 0 \\
(1-\sigma)\{(10-\sigma)^2 - 36\} = 0 \\
(1-\sigma)(16-\sigma)(4-\sigma) = 0 \quad \rightarrow \\
\sigma_1 = 16 \quad ; \quad \sigma_2 = 4 \quad ; \quad \sigma_3 = 1
\end{array}$$

The eigenvectors are the principal stress directions.

$$\begin{array}{l}
\sigma_1 = 16 \quad \rightarrow \quad \begin{bmatrix} -6 & 6 & 0 \\ 6 & -6 & 0 \\ 0 & 0 & -15 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \\
\left.\begin{array}{l} \alpha_1 = \alpha_2 \quad ; \quad \alpha_3 = 0 \\ \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \end{array}\right\} \quad \rightarrow \quad \vec{n}_1 = \tfrac{1}{2}\sqrt{2}\vec{e}_1 + \tfrac{1}{2}\sqrt{2}\vec{e}_2 \\
\text{idem :} \qquad \vec{n}_2 = -\tfrac{1}{2}\sqrt{2}\vec{e}_1 + \tfrac{1}{2}\sqrt{2}\vec{e}_2 \quad ; \quad \vec{n}_3 = \vec{e}_3
\end{array}$$

The average stress can also be calculated

$$\sigma_m = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) = 7$$

The deviatoric stress is

$$\begin{aligned}
\boldsymbol{\sigma}^d &= \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} \\
&= \{10\vec{e}_1\vec{e}_1 + 6(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + 10\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3\} - 7\boldsymbol{I} \\
&= 3\vec{e}_1\vec{e}_1 + 6(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + 3\vec{e}_2\vec{e}_2 - 6\vec{e}_3\vec{e}_3
\end{aligned}$$

Chapter 4

Balance or conservation laws

In every physical process, so also during deformation of continuum bodies, some general accepted physical laws have to be obeyed : the conservation laws. During deformation the total *mass* has to be preserved and also the total *momentum* and *moment of momentum*. Because we do not consider dissipation and thermal effects, we will not discuss the conservation law for total energy.

4.1 Mass balance

The mass of each finite, randomly chosen volume of material points in the continuum body must remain the same during the deformation process. Because we consider here a finite volume, this is the so-called *global* version of the mass conservation law.

From the requirement that this global law must hold for every randomly chosen volume, the *local version* of the conservation law can be derived. This derivation uses an integral transformation, where the integral over the volume $\bar{V}$ in the deformed state is transformed into an integral over the volume $\bar{V}_0$ in the undeformed state. From the requirement that the resulting integral equation has to be satisfied for each volume $\bar{V}_0$, the local version of the mass balance results.

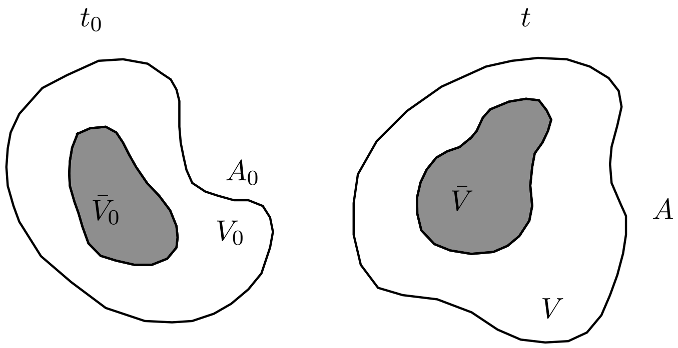


Fig. 4.1 : *Random volume in undeformed and deformed state*

$$\int_{\bar{V}} \rho \, dV = \int_{\bar{V}_0} \rho_0 \, dV_0 \qquad \forall \quad \bar{V} \qquad \rightarrow \qquad \int_{\bar{V}_0} (\rho J - \rho_0) \, dV_0 = 0 \qquad \forall \quad \bar{V}_0 \qquad \rightarrow$$

$$\rho J = \rho_0 \qquad\qquad \forall \quad \vec{x} \in V(t)$$

The local version, which is also referred to as the *continuity equation*, can also be derived directly by considering the mass dM of the infinitesimal volume dV of material points.

The time derivative of the mass conservation law is also used frequently. Because we focus attention on the same material particles, a so-called *material time derivative* is used, which is indicated as $(\dot{\ })$.

$$\begin{array}{ll} dM = dM_0 \quad \rightarrow \quad \rho dV = \rho_0 dV_0 \quad \rightarrow \\ \rho J = \rho_0 \qquad \forall \quad \vec{x} \in V(t) \quad \rightarrow \\ \dot{\rho} J + \rho \dot{J} = 0 \end{array}$$

4.2 Balance of momentum

According to the balance of momentum law, a point mass m which has a velocity $\vec{v}$, will change its momentum $\vec{i} = m\vec{v}$ under the action of a force $\vec{K}$. Analogously, the total force working on a randomly chosen volume of material points equals the change of the total momentum of the material points inside the volume. In the balance law, again a material time derivative is used, because we consider the same material points. The total force can be written as a volume integral of volume forces and the divergence of the stress tensor.

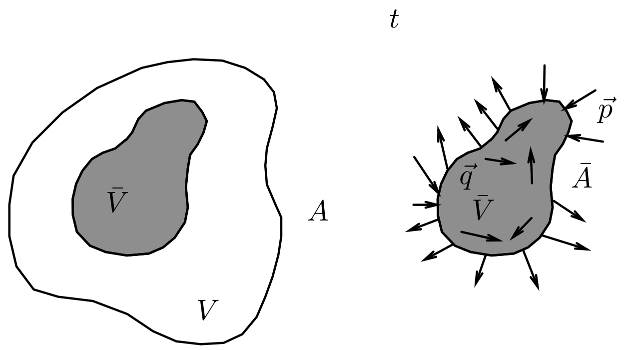


Fig. 4.2 : *Forces on random section of a material body*

$$\begin{aligned} \vec{K} &= \frac{D\vec{i}}{Dt} = \frac{D}{Dt} \int_{\bar{V}} \rho \vec{v} \, dV \qquad\qquad \forall \quad \bar{V} \quad \rightarrow \\ &= \frac{D}{Dt} \int_{\bar{V}_0} \rho \vec{v} J \, dV_0 = \int_{\bar{V}_0} \frac{D}{Dt} (\rho \vec{v} J) \, dV_0 \qquad\qquad \forall \quad \bar{V}_0 \\ &= \int_{\bar{V}_0} \left(\dot{\rho} \vec{v} J + \rho \dot{\vec{v}} J + \rho \vec{v} \dot{J} \right) dV_0 \qquad\qquad \forall \quad \bar{V}_0 \end{aligned}$$

$$\text{mass balance} \quad : \quad \dot{\rho} J + \rho \dot{J} = 0 \quad \rightarrow$$

$$= \int_{\bar{V}_0} \rho \dot{\vec{v}} J \, dV_0 = \int_{\bar{V}} \rho \dot{\vec{v}} \, dV \qquad \forall \quad \bar{V}$$

$$\int_{\bar{V}} \left(\rho \vec{q} + \vec{\nabla} \cdot \boldsymbol{\sigma}^c \right) dV = \int_{\bar{V}} \rho \dot{\vec{v}} \, dV \qquad \forall \quad \bar{V}$$

From the requirement that the global balance law must hold for every randomly chosen volume of material points, the local version of the balance of momentum can be derived, which must hold in every material point. In the derivation an integral transformation is used.

The local balance of momentum law is also called the *equation of motion.* For a stationary process, where the material velocity $\vec{v}$ in a fixed spatial point does not change, the equation is simplified. For a static process, where there is no acceleration of masses, the *equilibrium equation* results.

$$\begin{array}{ll}
\text{local version : equation of motion} & \vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \rho \dot{\vec{v}} = \rho \dfrac{\delta \vec{v}}{\delta t} + \rho \vec{v} \cdot \left(\vec{\nabla} \vec{v} \right) \quad \forall \; \vec{x} \in V(t) \\
\text{stationary} \quad \left(\dfrac{\delta \vec{v}}{\delta t} = 0 \right) & \vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \rho \vec{v} \cdot \left(\vec{\nabla} \vec{v} \right) \\
\text{static : equilibrium equation} & \boxed{\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \vec{0}}
\end{array}$$

4.2.1 Cartesian components

The equilibrium equation can be written in components w.r.t. a Cartesian vector basis. This results in three partial differential equations, one for each coordinate direction.

$$\begin{aligned}
\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x &= 0 \\
\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y &= 0 \\
\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z &= 0
\end{aligned}$$

4.2.2 Cylindrical components

Writing tensor and vectors in components w.r.t. a cylindrical vector basis is more elaborative because the cylindrical base vectors $\vec{e}_r$ and $\vec{e}_t$ are a function of the coordinate θ, so they have to be differentiated, when expanding the divergence term.

$$\begin{aligned}
\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r &= 0 \\
\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t &= 0 \\
\sigma_{zr,r} + \frac{1}{r}\sigma_{zt,t} + \frac{1}{r}\sigma_{zr} + \sigma_{zz,z} + \rho q_z &= 0
\end{aligned}$$

4.3 Balance of moment of momentum

The balance of moment of momentum states that the total moment about a fixed point of all forces working on a randomly chosen volume of material points ($\vec{M}_O$), equals the change of the total moment of momentum of the material points inside the volume, taken w.r.t. the same fixed point ($\vec{L}_O$).

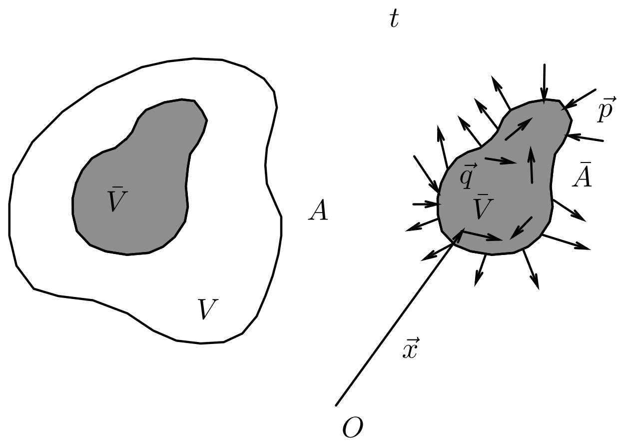


Fig. 4.3 : *Moment of forces on a random section of a material body*

$$\vec{M}_O = \frac{D\vec{L}_O}{Dt} = \frac{D}{Dt}\int\limits_{\bar{V}} \vec{x} * \rho\vec{v}\, dV \qquad \forall \quad \bar{V}$$

$$= \frac{D}{Dt}\int\limits_{\bar{V}_0} \vec{x} * \rho\vec{v}J\, dV_0 = \int\limits_{\bar{V}_0} \frac{D}{Dt}\left(\vec{x} * \rho\vec{v}J\right)\, dV_0 \qquad \forall \quad \bar{V}_0$$

$$= \int\limits_{\bar{V}_0} \left(\dot{\vec{x}} * \rho\vec{v}J + \vec{x} * \dot{\rho}\vec{v}J + \vec{x} * \rho\dot{\vec{v}}J + \vec{x} * \rho\vec{v}\dot{J}\right) dV_0 \qquad \forall \quad \bar{V}_0$$

$$\left.\begin{array}{ll} \text{mass balance} \quad : & \dot{\rho}J + \rho\dot{J} = 0 \\ \dot{\vec{x}} * \vec{v} = \vec{v} * \vec{v} = \vec{0} & \end{array}\right\} \quad \rightarrow$$

$$= \int\limits_{\bar{V}_0} \vec{x} * \rho\dot{\vec{v}}J\, dV_0 = \int\limits_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \qquad \forall \quad \bar{V}$$

$$\int\limits_{\bar{V}} \vec{x} * \rho\vec{q}\, dV + \int\limits_{\bar{A}} \vec{x} * \vec{p}\, dA = \int\limits_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \qquad \forall \quad \bar{V}$$

To derive a local version, the integral over the area $\bar{A}$ has to be transformed to an integral over the enclosed volume $\bar{V}$. In this derivation, the Levi-Civita tensor ${}^3\boldsymbol{\epsilon}$ is used, which is defined such that

$$\vec{a} * \vec{b} = {}^3\boldsymbol{\epsilon} : \vec{a}\,\vec{b}$$

holds for all vectors $\vec{a}$ and $\vec{b}$.

$$\int_{\bar{A}} \vec{x} * \vec{p}\, dA = \int_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x}\,\vec{p})\, dA = \int_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x}\,\boldsymbol{\sigma} \cdot \vec{n})\, dA = \int_{\bar{A}} \vec{n} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x}\boldsymbol{\sigma})\}^c\, dA$$

$$= \int_{\bar{V}} \vec{\nabla} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x}\boldsymbol{\sigma})\}^c\, dV \qquad \text{with} \quad {}^3\boldsymbol{\epsilon}^c = -{}^3\boldsymbol{\epsilon} \quad \rightarrow$$

$$= -\int_{\bar{V}} \vec{\nabla} \cdot \{(\vec{x}\boldsymbol{\sigma})^c : {}^3\boldsymbol{\epsilon}^c\}\, dV = -\int_{\bar{V}} \vec{\nabla} \cdot \{(\boldsymbol{\sigma}^c\vec{x}) : {}^3\boldsymbol{\epsilon}^c\}\, dV$$

$$= -\int_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} \cdot (\vec{\nabla}\vec{x}) : {}^3\boldsymbol{\epsilon}^c\right] dV$$

$$= -\int_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} : {}^3\boldsymbol{\epsilon}^c\right] dV = \int_{\bar{V}} \left[{}^3\boldsymbol{\epsilon} : \vec{x}(\vec{\nabla} \cdot \boldsymbol{\sigma}^c) + {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\right] dV$$

$$= \int_{\bar{V}} \left[{}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c + \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\right] dV$$

Substitution in the global version and using the local balance of momentum, leads to the local version of the balance of moment of momentum.

$$\int_{\bar{V}} \vec{x} * \rho\vec{q}\, dV + \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV + \int_{\bar{V}} \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\, dV = \int_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \quad \forall \quad \bar{V} \quad \rightarrow$$

$$\int_{\bar{V}} \vec{x} * \left[\rho\vec{q} + (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) - \rho\dot{\vec{v}}\right] dV + \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow$$

$$\int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow \quad {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c = \vec{0} \quad \forall \quad \vec{x} \in \bar{V}$$

Because the components of ${}^3\boldsymbol{\epsilon}$ equal 1 for permutation $\{i,j,k\} = \{123, 312, 231\}$, -1 for $\{i,j,k\} = \{321, 132, 213\}$, and 0 if indices are repeated, it can be derived that

$$\begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the local version states that the Cauchy stress tensor is symmetric.

$$\boxed{\boldsymbol{\sigma}^c = \boldsymbol{\sigma}} \qquad\qquad \forall \quad \vec{x} \in V(t)$$

4.3.1 Cartesian and cylindrical components

With respect to a Cartesian or cylindrical basis the symmetry of the stress tensor results in three equations.

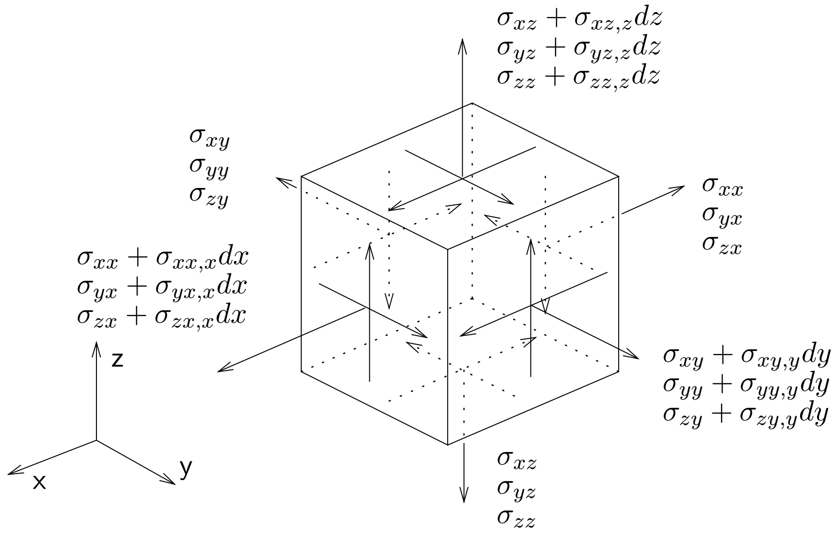
$$\underline{\sigma} = \underline{\sigma}^T \qquad \rightarrow$$

Cartesian : $\sigma_{xy} = \sigma_{yx}$; $\sigma_{yz} = \sigma_{zy}$; $\sigma_{zx} = \sigma_{xz}$
cylindrical : $\sigma_{rt} = \sigma_{tr}$; $\sigma_{tz} = \sigma_{zt}$; $\sigma_{zr} = \sigma_{rz}$

4.4 Examples

Equilibrium of forces : Cartesian

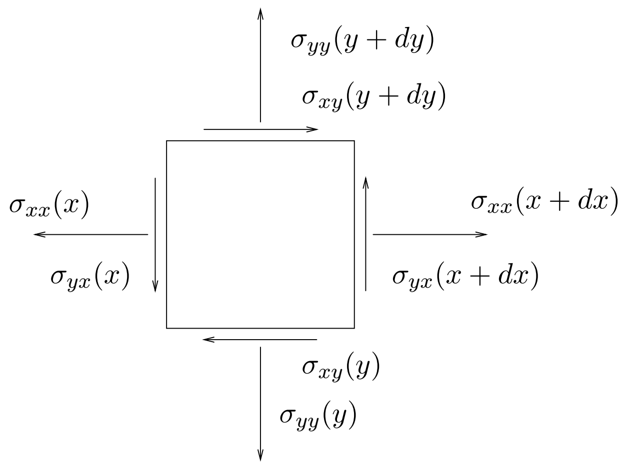
The equilibrium equations in the three coordinate directions can be derived by considering the force equilibrium of the Cartesian stress cube.



$$(\sigma_{xx} + \sigma_{xx,x}dx)dydz + (\sigma_{xy} + \sigma_{xy,y}dy)dxdz + (\sigma_{xz} + \sigma_{xz,z}dz)dxdy - (\sigma_{xx})dydz - (\sigma_{xy})dxdz - (\sigma_{xz})dxdy + \rho q_x dxdydz = 0$$

Equilibrium of moments : Cartesian

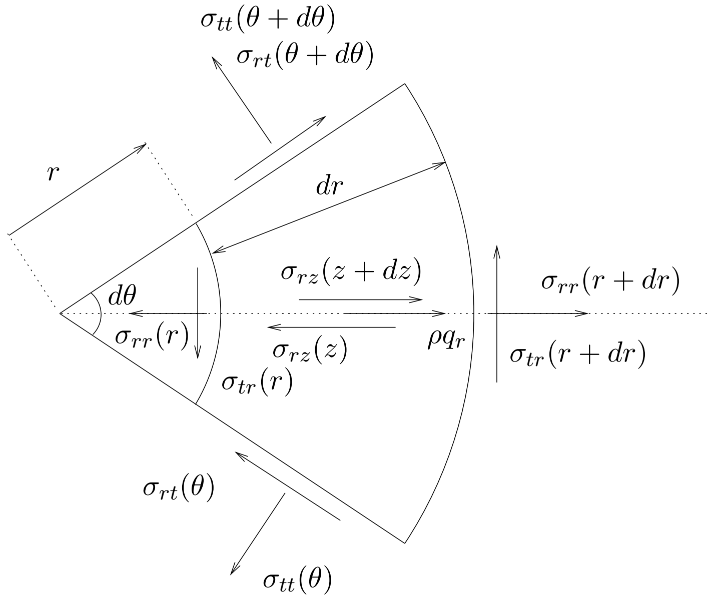
The forces, working on the Cartesian stress cube, have a moment w.r.t. a certain point in space. The sum of all the moments must be zero. We consider the moments of forces in the xy-plane w.r.t. the z-axis through the center of the cube. Anti-clockwise moments are positive.



$$\begin{aligned}
&\sigma_{yx} dydz\,\tfrac{1}{2}dx + \sigma_{yx} dydz\,\tfrac{1}{2}dx + \sigma_{yx,x} dxdydz\tfrac{1}{2}dx \\
&\quad - \sigma_{xy} dxdz\,\tfrac{1}{2}dy - \sigma_{xy} dxdz\,\tfrac{1}{2}dy - \sigma_{xy,x} dxdydz\,\tfrac{1}{2}dy = 0 \\
&\sigma_{yx} - \sigma_{xy} = 0 \qquad \rightarrow \quad \sigma_{yx} = \sigma_{xy}
\end{aligned}$$

Equilibrium of forces : cylindrical

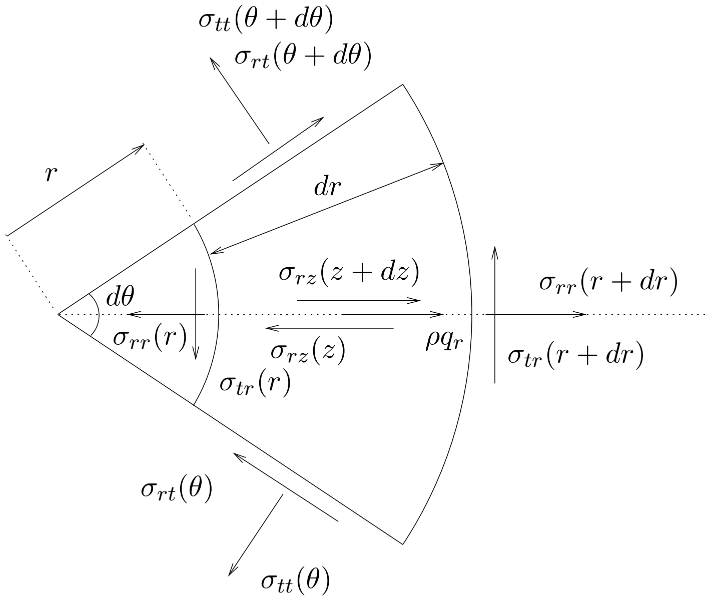
The equilibrium equations in the three coordinate directions can be derived by considering the force equilibrium of the cylindrical stress 'cube'. Here only the equilibrium in r-direction is considered. The stress components are a function of the three cylindrical coordinates r, θ and z, but only the relevant (changing) ones are indicated.



$$\begin{aligned}
&-\sigma_{rr}(r) r d\theta dz - \sigma_{rz}(z) r dr d\theta - \sigma_{rt}(\theta) dr dz - \sigma_{tt}(\theta) dr \tfrac{1}{2} d\theta dz \\
&\quad + \sigma_{rr}(r + dr)(r + dr) d\theta dz + \sigma_{rz}(z + dz) r dr d\theta \\
&\quad + \sigma_{rt}(\theta + d\theta) dr dz - \sigma_{tt}(\theta + d\theta) dr \tfrac{1}{2} d\theta dz + \rho q_r r dr d\theta dz = 0 \\
&\sigma_{rr,r} r dr d\theta dz + \sigma_{rr} dr d\theta dz + \sigma_{rz,z} r dr d\theta dz + \sigma_{rt,t} dr d\theta dz \\
&\quad - \sigma_{tt}(\theta) dr d\theta dz + \rho q_r dr d\theta dz = 0 \\
&\qquad \sigma_{rr,r} + \frac{1}{r}\sigma_{rr} + \sigma_{rz,z} + \frac{1}{r}\sigma_{rt,t} - \frac{1}{r}\sigma_{tt} + \rho q_r = 0
\end{aligned}$$

Equilibrium of moments : cylindrical

The forces, working on the cylindrical stress cube, have a moment w.r.t. a certain point in space. The sum of all the moments mus be zero. We consider the moments of forces in the $r\theta$-plane w.r.t. the z-axis through the center of the cube.



$$
\begin{aligned}
&\sigma_{tr}(r) r d\theta dz\, \tfrac{1}{2} dr + \sigma_{tr}(r+dr)(r+dr) d\theta dz\, \tfrac{1}{2} dr \\
&\quad - \sigma_{rt}(\theta) dr dz\, \tfrac{1}{2} r d\theta - \sigma_{rt}(\theta + d\theta) dr dz\, \tfrac{1}{2} r d\theta = 0 \\
&\sigma_{tr} r dr d\theta dz - \sigma_{rt} r dr d\theta dz = 0 \qquad \rightarrow \qquad \sigma_{tr} = \sigma_{rt}
\end{aligned}
$$

Chapter 5

Linear elastic material

For linear elastic material behavior the stress tensor $\boldsymbol{\sigma}$ is related to the linear strain tensor $\boldsymbol{\varepsilon}$ by the constant fourth-order stiffness tensor ${}^4\boldsymbol{C}$:

$$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}$$

The relevant components of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ w.r.t. an orthonormal vector basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are stored in columns $\underset{\approx}{\sigma}$ and $\underset{\approx}{\varepsilon}$. Note that we use double "waves" to indicate that the columns contain components of a second-order tensor.

$$\begin{aligned}
\underset{\approx}{\sigma}^T &= [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{21}\ \sigma_{23}\ \sigma_{32}\ \sigma_{31}\ \sigma_{13}] \\
\underset{\approx}{\varepsilon}^T &= [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \varepsilon_{12}\ \varepsilon_{21}\ \varepsilon_{23}\ \varepsilon_{32}\ \varepsilon_{31}\ \varepsilon_{13}]
\end{aligned}$$

The relation between these columns is given by the 9×9 matrix $\underline{\underline{C}}$, which stores the components of ${}^4\boldsymbol{C}$ and is referred to as the material stiffness matrix. Note again the use of double underscore to indicate that the matrix contains components of a fourth-order tensor.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\
C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\
C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\
C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\
C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\
C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\
C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\
C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\
C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331}
\end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

The stored energy per unit of volume is :

$$W = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} = \left[\tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}\right]^c = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C}^c : \boldsymbol{\varepsilon}$$

which implies that ${}^4\boldsymbol{C}$ is total-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^c$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^T$.

As the stress tensor is symmetric, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^c$, the tensor ${}^4\boldsymbol{C}$ must be left-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^{lc}$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^{LT}$. As also the strain tensor is symmetric, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^c$, the constitutive relation can be written with a 6×6 stiffness matrix.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & [C_{1121}+C_{1112}] & [C_{1132}+C_{1123}] & [C_{1113}+C_{1131}] \\ C_{2211} & C_{2222} & C_{2233} & [C_{2221}+C_{2212}] & [C_{2232}+C_{2223}] & [C_{2213}+C_{2231}] \\ C_{3311} & C_{3322} & C_{3333} & [C_{3321}+C_{3312}] & [C_{3332}+C_{3323}] & [C_{3313}+C_{3331}] \\ C_{1211} & C_{1222} & C_{1233} & [C_{1221}+C_{1212}] & [C_{1232}+C_{1223}] & [C_{1213}+C_{1231}] \\ C_{2311} & C_{2322} & C_{2333} & [C_{2321}+C_{2312}] & [C_{2332}+C_{2323}] & [C_{2313}+C_{2331}] \\ C_{3111} & C_{3122} & C_{3133} & [C_{3121}+C_{3112}] & [C_{3132}+C_{3123}] & [C_{3113}+C_{3131}] \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

The components of $\underline{\underline{C}}$ must be determined experimentally, by prescribing strains and measuring stresses and vice versa. It is clear that only the summation of the components in the 4th, 5th and 6th columns can be determined and for that reason, it is assumed that the stiffness tensor is right-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^{rc}$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^{RT}$.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

To restore the symmetry of the stiffness matrix, the factor 2 in the last three columns is swapped to the column with the strain components. The shear components are replaced by the shear strains : $2\varepsilon_{ij} = \gamma_{ij}$. This leads to a symmetric stiffness matrix $\underline{\underline{C}}$ with 21 independent components.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

5.1 Material symmetry

Almost all materials have some material symmetry, originating from the micro structure, which implies that the number of independent material parameters is reduced. The following names refer to increasing material symmetry and thus to decreasing number of material parameters :

monoclinic → orthotropic → quadratic → transversal isotropic → cubic → isotropic

5.1.1 Monoclinic

In each material point of a monoclinic material there is one symmetry plane, which we take here to be the 12-plane. Strain components w.r.t. two vector bases $\underset{\sim}{\vec{e}} = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]^T$ and $\underset{\sim}{\vec{e}}^* = [\vec{e}_1 \ \vec{e}_2 \ -\vec{e}_3]^T$ must result in the same stresses. It can be proved that all components of the stiffness matrix, with an odd total of the index 3, must be zero. This implies :

$$C_{2311} = C_{2322} = C_{2333} = C_{2321} = C_{3111} = C_{3122} = C_{3133} = C_{3121} = 0$$

A monoclinic material is characterized by 13 material parameters. In the figure the directions with equal properties are indicated with an equal number of lines.

Monoclinic symmetry is found in e.g. gypsum ($CaSO_4 2H_2O$).

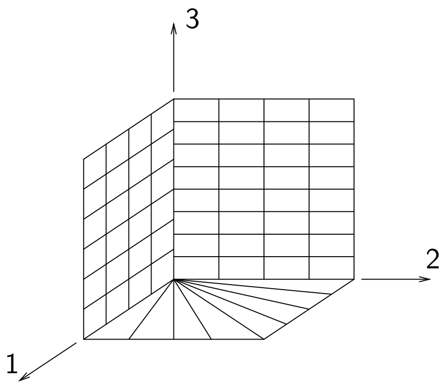


Fig. 5.1 : *One symmetry plane for monoclinic material symmetry*

$$\underline{\underline{C}} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & 0 & 0 \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2323} & C_{2331} \\ 0 & 0 & 0 & 0 & C_{3123} & C_{3131} \end{bmatrix}$$

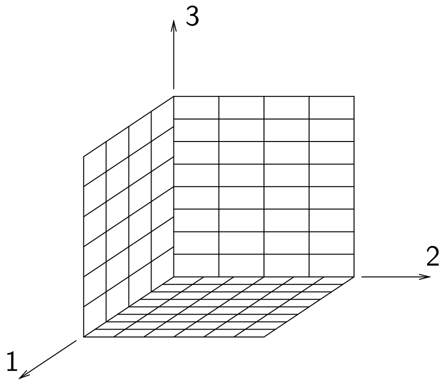
5.1.2 Orthotropic

In a point of an orthotropic material there are three symmetry planes which are perpendicular. We choose them here to coincide with the Cartesian coordinate planes. In addition to the implications for monoclinic symmetry, we can add the requirements

$$C_{1112} = C_{2212} = C_{3312} = C_{3123} = 0$$

An orthotropic material is characterized by 9 material parameters. In the stiffness matrix, they are now indicated as A, B, C, Q, R, S, K, L and M.

Orthotropic symmetry is found in orthorhombic crystals (e.g. cementite, Fe_3C) and in composites with fibers in three perpendicular directions.



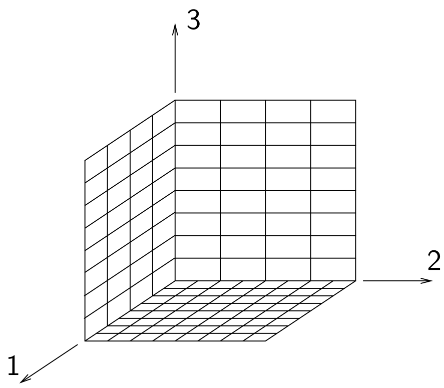
$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix}$$

Fig. 5.2 : *Three symmetry planes for orthotropic material symmetry*

5.1.3 Quadratic

If in an orthotropic material the properties in two of the three symmetry planes are the same, the material is referred to as quadratic. Here we assume the behavior to be identical in the $\vec{e}_1$- and the $\vec{e}_2$-directions, however there is no isotropy in the 12-plane. This implies : $A = B$, $S = R$ and $M = L$. Only 6 material parameters are needed to describe the mechanical material behavior.

Quadratic symmetry is found in tetragonal crystals e.g. TiO_2 and white tin $Sn\beta$.



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

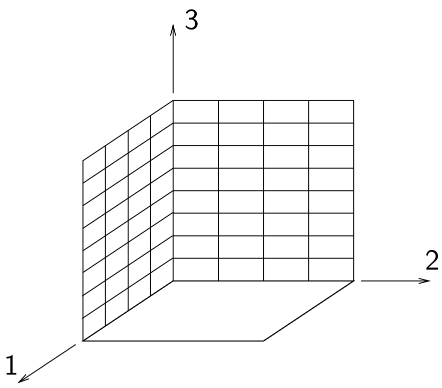
Fig. 5.3 : *Quadratic material*

5.1.4 Transversal isotropic

When the material behavior in the 12-plane is isotropic, an additional relation between parameters can be deduced. To do this, we consider a pure shear deformation in the 12-plane, where a shear stress τ leads to a shear γ. The principal stress and strain directions coincide due to the isotropic behavior in the plane. In the principal directions the relation between principal stresses and strains follow from the material stiffness matrix.

$$
\begin{array}{l}
\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \quad \rightarrow \quad \det(\underline{\sigma} - \sigma \underline{I}) = 0 \quad \rightarrow \quad \begin{array}{l} \sigma_1 = \tau \\ \sigma_2 = -\tau \end{array} \\
\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} \quad \rightarrow \quad \det(\underline{\varepsilon} - \varepsilon \underline{I}) = 0 \quad \rightarrow \quad \begin{array}{l} \varepsilon_1 = \frac{1}{2}\gamma \\ \varepsilon_2 = -\frac{1}{2}\gamma \end{array} \\
\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} A & Q \\ Q & A \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \rightarrow \quad \begin{array}{l} \sigma_1 = A\varepsilon_1 + Q\varepsilon_2 = \tau = K\gamma \\ \sigma_2 = Q\varepsilon_1 + A\varepsilon_2 = -\tau = -K\gamma \end{array} \quad \rightarrow \\
\left. \begin{array}{l} (A - Q)(\varepsilon_1 - \varepsilon_2) = 2K\gamma \\ \varepsilon_1 = \frac{1}{2}\gamma \quad ; \quad \varepsilon_1 = -\frac{1}{2}\gamma \end{array} \right\} \quad \rightarrow \quad \boxed{K = \frac{1}{2}(A - Q)}
\end{array}
$$

Examples of transversal isotropy are found in hexagonal crystals (CHP, Zn, Mg, Ti) and honeycomb composites. The material behavior of these materials can be described with 5 material parameters.



$$
\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}
$$

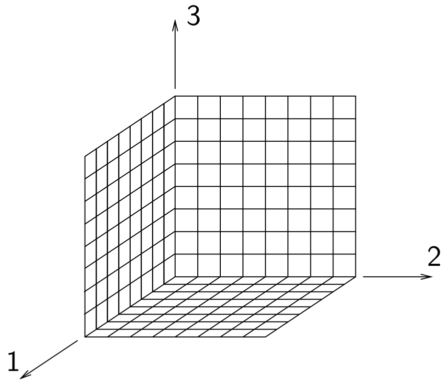
with $K = \frac{1}{2}(A - Q)$

Fig. 5.4 : *Transversal material*

5.1.5 Cubic

In the three perpendicular material directions the material properties are the same. In the symmetry planes there is no isotropic behavior. Only 3 material parameters remain.

Examples of cubic symmetry are found in BCC and FCC crystals (e.g. in Ag, Cu, Au, Fe, NaCl).



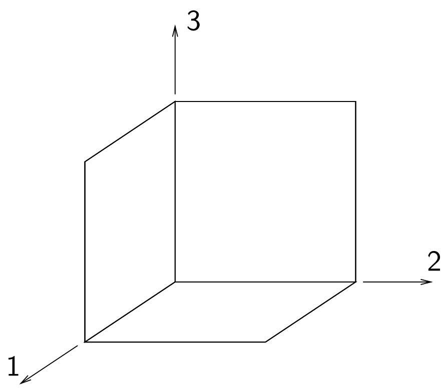
$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

Fig. 5.5 : *Cubic material*

5.1.6 Isotropic

In all three directions the properties are the same and in each plane the properties are isotropic. Only 2 material parameters remain.

Isotropic material behavior is found for materials having a microstructure, which is sufficiently randomly oriented and distributed on a very small scale. This applies to metals with a randomly oriented polycrystalline structure, ceramics with a random granular structure and composites with random fiber/particle orientation.



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

with $\quad L = \frac{1}{2}(A - Q)$

Fig. 5.6 : *Isotropic material*

5.2 Engineering parameters

In engineering practice the linear elastic material behavior is characterized by Young's moduli, shear moduli and Poisson ratios. They have to be measured in tensile and shear experiments. In this section these parameters are introduced for an isotropic material.

For orthotropic and transversal isotropic material, the stiffness and compliance matrices, expressed in engineering parameters, can be found in appendix A.

5.2.1 Isotropic

For isotropic materials the material properties are the same in each direction. The mechanical behavior is characterized by two independent material parameters : Young's modulus E, which characterizes the tensile stiffness and Poisson's ratio ν, which determines the contraction. The shear modulus G describes the shear behavior and is not independent but related to E and ν. To express the material constants A, Q and L in the parameters E, ν and G, two simple tests are considered : a tensile test along the 1-axis and a shear test in the 13-plane.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A-Q)$$

In a tensile test the contraction strain ε_d and the axial stress σ are related to the axial strain ε. The expressions for A, Q and L result after some simple mathematics.

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon & \varepsilon_d & \varepsilon_d & 0 & 0 & 0 \end{bmatrix} \quad ; \quad \underset{\approx}{\sigma}^T = \begin{bmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left. \begin{array}{l} \sigma = A\varepsilon + 2Q\varepsilon_d \\ 0 = Q\varepsilon + (A+Q)\varepsilon_d \quad \rightarrow \quad \varepsilon_d = -\dfrac{Q}{A+Q}\varepsilon = -\nu\varepsilon \end{array} \right\} \rightarrow$$

$$\sigma = A\varepsilon - 2Q\nu\varepsilon = (A - 2Q\nu)\varepsilon = E\varepsilon$$

$$\left. \begin{array}{l} Q(1-\nu) = A\nu \\ A - 2Q\nu = E \end{array} \right\} \rightarrow \quad \boxed{A = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}} \quad \rightarrow$$

$$\boxed{Q = \frac{\nu E}{(1+\nu)(1-2\nu)}} \quad ; \quad \boxed{L = \tfrac{1}{2}(A-Q) = \frac{E}{2(1+\nu)}}$$

When we analyze a shear test, the relation between the shear strain γ and the shear stress τ is given by the shear modulus G. For isotropic material G is a function of E and ν. For non-isotropic materials, the shear moduli are independent parameters.

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \quad ; \quad \underset{\approx}{\sigma}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \tau \end{bmatrix}$$

$$\tau = L\gamma = \frac{E}{2(1+\nu)}\gamma = G\gamma$$

For an isotropic material, a hydrostatic stress will only result in volume change. The relation between the volume strain and the hydrostatic stress is given by the bulk modulus K, which is a function of E and ν.

$$
\begin{aligned}
J - 1 &= \lambda_{11}\lambda_{22}\lambda_{33} \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \\
&= \frac{1-2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{K}\,\tfrac{1}{3}\mathrm{tr}(\underline{\sigma})
\end{aligned}
$$

Besides Young's modulus, shear modulus, bulk modulus and Poisson ratio in some formulations the so-called Lamé coefficients λ and μ are used, where $\mu = G$ and λ is a function of E and ν. The next tables list the relations between all these parameters.

	E,ν	λ,G	K,G	E,G	E,K
E	E	$\frac{(2G+3\lambda)G}{\lambda+G}$	$\frac{9KG}{3K+G}$	E	E
ν	ν	$\frac{\lambda}{2(\lambda+G)}$	$\frac{3K-2G}{2(3K+G)}$	$\frac{E-2G}{2G}$	$\frac{3K-E}{6K}$
G	$\frac{E}{2(1+\nu)}$	G	G	G	$\frac{3KE}{9K-E}$
K	$\frac{E}{3(1-2\nu)}$	$\frac{3\lambda+2G}{3}$	K	$\frac{EG}{3(3G-E)}$	K
λ	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	λ	$\frac{3K-2G}{3}$	$\frac{G(E-2G)}{3G-E}$	$\frac{3K(3K-E)}{9K-E}$

	E,λ	G,ν	λ,ν	λK	K,ν
E	E	$2G(1+\nu)$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$3K(1-2\nu)$
ν	$\frac{-E-\lambda+\sqrt{(E+\lambda)^2+8\lambda^2}}{4\lambda}$	ν	ν	$\frac{\lambda}{3K-\lambda}$	ν
G	$\frac{-3\lambda+E+\sqrt{(3\lambda-E)^2+8\lambda E}}{4}$	G	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{3(K-\lambda)}{2}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$
K	$\frac{E-3\lambda+\sqrt{(E-3\lambda)^2-12\lambda E}}{6}$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$\frac{\lambda(1+\nu)}{3\nu}$	K	K
λ	λ	$\frac{2G\nu}{1-2\nu}$	λ	λ	$\frac{3K\nu}{1+\nu}$

5.3 Isotropic material tensors

Isotropic linear elastic material behavior is characterized by only two independent material constants, for which we can choose Young's modulus E and Poisson's ratio ν. The isotropic material law can be written in tensorial form, where $\boldsymbol{\sigma}$ is related to $\boldsymbol{\varepsilon}$ with a fourth-order material stiffness tensor ${}^4\boldsymbol{C}$.

In column/matrix notation the strain components are related to the stress components by a 6×6 compliance matrix. Inversion leads to the 6×6 stiffness matrix, which relates strain components to stress components. It should be noted that shear strains are denoted as ε_{ij} and not as γ_{ij}, as was done before.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} \quad \rightarrow \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\, \underset{\approx}{\sigma}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \frac{E}{(1+\nu)} \begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} \quad \rightarrow \quad \underset{\approx}{\sigma} = \underline{\underline{C}}\, \underset{\approx}{\varepsilon}$$

The stiffness matrix is written as the sum of two matrices, which can then be written in tensorial form.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \left[\frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{E}{(1+\nu)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Tensorial notation

The first matrix is the matrix representation of the fourth-order tensor $\boldsymbol{II}$. The second matrix is the representation of the symmetric fourth-order tensor ${}^4\boldsymbol{I}^s$. The resulting fourth-order material stiffness tensor ${}^4\boldsymbol{C}$ contains two material constants c_0 and c_1. It is observed that $c_0 = \lambda$ and $c_1 = 2\mu$, where λ and μ are the Lamé coefficients introduced earlier.

$$\boldsymbol{\sigma} = \left[c_0 \boldsymbol{II} + c_1\, {}^4\boldsymbol{I}^s \right] : \boldsymbol{\varepsilon} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}$$

with $\quad {}^4\boldsymbol{I}^s = \frac{1}{2}\left({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc} \right)$

and $\quad c_0 = \dfrac{E\nu}{(1+\nu)(1-2\nu)} \quad ; \quad c_1 = \dfrac{E}{1+\nu}$

Hydrostatic and deviatoric strain/stress

The strain and stress tensors can both be written as the sum of an hydrostatic - $(.)^h$ - and a deviatoric - $(.)^d$ - part. Doing so, the stress-strain relation can be easily inverted.

$$\begin{aligned}
\boldsymbol{\sigma} &= {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} = \left[c_0 \boldsymbol{I}\boldsymbol{I} + c_1\, {}^4\boldsymbol{I}^s\right] : \boldsymbol{\varepsilon} \\
&= c_0 \mathsf{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1\boldsymbol{\varepsilon} = c_0 \mathsf{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1 \left\{\boldsymbol{\varepsilon}^d + \tfrac{1}{3}\mathsf{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}\right\} \\
&= (c_0 + \tfrac{1}{3}c_1)\mathsf{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1\boldsymbol{\varepsilon}^d = (3c_0 + c_1)\tfrac{1}{3}\mathsf{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1\boldsymbol{\varepsilon}^d = (3c_0 + c_1)\boldsymbol{\varepsilon}^h + c_1\boldsymbol{\varepsilon}^d \\
&= \boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d \quad \rightarrow
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^d \\
&= \frac{1}{3c_0 + c_1}\,\boldsymbol{\sigma}^h + \frac{1}{c_1}\,\boldsymbol{\sigma}^d = \frac{1}{3c_0 + c_1}\,\tfrac{1}{3}\mathsf{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1}{c_1}\,\left\{\boldsymbol{\sigma} - \tfrac{1}{3}\mathsf{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right\} \\
&= -\,\frac{c_0}{(3c_0 + c_1)c_1}\,\mathsf{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1}{c_1}\,\boldsymbol{\sigma} = \left[-\,\frac{c_0}{(3c_0 + c_1)c_1}\boldsymbol{I}\boldsymbol{I} + \frac{1}{c_1}\,{}^4\boldsymbol{I}^s\right] : \boldsymbol{\sigma} \\
&= {}^4\boldsymbol{S} : \boldsymbol{\sigma}
\end{aligned}$$

5.4 Planar deformation

In many cases the state of strain or stress is planar. Both for plane strain and for plane stress, only strains and stresses in a plane are related by the material law. Here we assume that this plane is the 12-plane. For plane strain we than have $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, and for plane stress $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$. The material law for these planar situations can be derived from the three-dimensional stress-strain relations. In the following sections the result is shown for orthotropic material. For cases with more material symmetry, the planar stress-strain relations can be simplified accordingly. The corresponding stiffness and compliance matrices can be found in appendix A.

The planar stress-strain laws can be derived either from the stiffness matrix $\underline{\underline{C}}$ or from the compliance matrix $\underline{\underline{S}}$.

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \quad \rightarrow \quad \underline{\underline{S}} = \underline{\underline{C}}^{-1} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix}$$

5.4.1 Plane strain and plane stress

For a plane strain state with $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, the stress σ_{33} can be expressed in the planar strains ε_{11} and ε_{22}. The material stiffness matrix $\underline{\underline{C}}_{\varepsilon}$ can be extracted directly from $\underline{\underline{C}}$. The material compliance matrix $\underline{\underline{S}}_{\varepsilon}$ has to be derived by inversion.

$$\sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22} = -\frac{r}{c}\sigma_{11} - \frac{s}{c}\sigma_{22}$$

$$\underline{\underline{C}}_{\varepsilon} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix}$$

$$\underline{\underline{S}}_{\varepsilon} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} = \underline{\underline{C}}^{-1} = \frac{1}{Q^2 - BA} \begin{bmatrix} -B & Q & 0 \\ Q & -A & 0 \\ 0 & & \dfrac{Q^2 - BA}{K} \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} ac - r^2 & qc - rs & 0 \\ qc - rs & bc - s^2 & 0 \\ 0 & 0 & kc \end{bmatrix}$$

For the plane stress state, with $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$, the two-dimensional material law can be easily derived from the three-dimensional compliance matrix $\underline{\underline{S}}_{\varepsilon}$. The strain ε_{33} can be directly expressed in σ_{11} and σ_{22}. The material stiffness matrix has to be derived by inversion.

$$\varepsilon_{33} = r\sigma_{11} + s\sigma_{22} = -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22}$$

$$\underline{\underline{S}}_{\sigma} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix}$$

$$\underline{\underline{C}}_{\sigma} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} = \underline{\underline{S}}_{\sigma}^{-1} = \frac{1}{q^2 - ba} \begin{bmatrix} -b & q & 0 \\ q & -a & 0 \\ 0 & & \dfrac{q^2 - ba}{k} \end{bmatrix}$$

$$= \frac{1}{C} \begin{bmatrix} AC - R^2 & QC - RS & 0 \\ QC - RS & BC - S^2 & 0 \\ 0 & 0 & KC \end{bmatrix}$$

In general we can write the stiffness and compliance matrix for planar deformation as a 3×3 matrix with components, which are specified for plane strain ($p = \varepsilon$) or plane stress ($p = \sigma$).

$$\underline{\underline{C}}_{p} = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_{p} = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix}$$

5.5 Thermo-elasticity

A temperature change ΔT of an unrestrained material invokes deformation. The total strain results from both mechanical and thermal effects and when deformations are small the total strain $\boldsymbol{\varepsilon}$ can be written as the sum of mechanical strains $\boldsymbol{\varepsilon}_m$ and thermal strains $\boldsymbol{\varepsilon}_T$. The thermal strains are related to the temperature change ΔT by the coefficient of thermal expansion tensor $\boldsymbol{A}$.

The stresses in terms of strains are derived by inversion of the compliance matrix $\underline{S}$.

For thermally isotropic materials only the linear coefficient of thermal expansion α is relevant.

Anisotropic

$$\begin{aligned}
\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_T = {}^4\boldsymbol{S} : \boldsymbol{\sigma} + \boldsymbol{A}\Delta T & &\rightarrow & \underset{\approx}{\varepsilon} &= \underset{\approx}{\varepsilon}_m + \underset{\approx}{\varepsilon}_T = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \underline{\underline{A}}\,\Delta T \\
\boldsymbol{\sigma} &= {}^4\boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{A}\Delta T) & &\rightarrow & \underset{\approx}{\sigma} &= \underline{\underline{C}}\left(\underset{\approx}{\varepsilon} - \underline{\underline{A}}\,\Delta T\right)
\end{aligned}$$

Isotropic

$$\begin{aligned}
\boldsymbol{\varepsilon} &= {}^4\boldsymbol{S} : \boldsymbol{\sigma} + \alpha\,\Delta T\boldsymbol{I} & &\rightarrow & \underset{\approx}{\varepsilon} &= \underline{\underline{S}}\,\underset{\approx}{\sigma} + \alpha\,\Delta T\,\underset{\approx}{I} \\
\boldsymbol{\sigma} &= {}^4\boldsymbol{C} : (\boldsymbol{\varepsilon} - \alpha\Delta T\boldsymbol{I}) & &\rightarrow & \underset{\approx}{\sigma} &= \underline{\underline{C}}(\underset{\approx}{\varepsilon} - \alpha\,\Delta T\,\underset{\approx}{I})
\end{aligned}$$

For orthotropic material, this can be written in full matrix notation.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} A+Q+R \\ Q+B+S \\ R+S+C \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

5.5.1 Plane strain/stress

When the material is in plane strain or plane stress state, only three strains and stresses are relevant, here indicated with indices 1 and 2. The thermo-elastic stress-strain law can than be expressed in the elasticity parameters and the coefficient of thermal expansion. The relation is presented for orthotropic material. It is assumed that the thermal expansion is isotropic.

Plane stress

$$\begin{aligned}\varepsilon_{33} &= -\frac{R}{C}\,\varepsilon_{11} - \frac{S}{C}\,\varepsilon_{22} + \frac{1}{C}\,(R+S+C)\,\alpha\,\Delta T \qquad &\text{(from } \underline{\underline{C}}\text{)}\\ &= r\sigma_{11} + s\sigma_{22} + \alpha\Delta T &\text{(from } \underline{\underline{S}}\text{)}\end{aligned}$$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix} a & q & 0\\ q & b & 0\\ 0 & 0 & k\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} + \alpha\Delta T\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0\\ Q_\sigma & B_\sigma & 0\\ 0 & 0 & K\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} - \alpha\Delta T\begin{bmatrix}A_\sigma + Q_\sigma\\ B_\sigma + Q_\sigma\\ 0\end{bmatrix}$$

Plane strain

$$\begin{aligned}\sigma_{33} &= R\varepsilon_{11} + S\varepsilon_{22} - \alpha(R+S+C)\,\Delta T \qquad &\text{(from } \underline{\underline{C}}\text{)}\\ &= -\frac{r}{c}\,\sigma_{11} - \frac{s}{c}\,\sigma_{22} - \frac{\alpha}{c}\,\Delta T &\text{(from } \underline{\underline{S}}\text{)}\end{aligned}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} A & Q & 0\\ Q & B & 0\\ 0 & 0 & K\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} - \alpha\Delta T\begin{bmatrix}A + Q + R\\ B + Q + S\\ 0\end{bmatrix}$$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0\\ q_\varepsilon & b_\varepsilon & 0\\ 0 & 0 & k\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} + \alpha\Delta T\begin{bmatrix}1 + q_\varepsilon S + a_\varepsilon R\\ 1 + q_\varepsilon R + b_\varepsilon S\\ 0\end{bmatrix}$$

planar general

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} A_p & Q_p & 0\\ Q_p & B_p & 0\\ 0 & 0 & K\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} - \alpha\Delta T\begin{bmatrix}\Theta_{p1}\\ \Theta_{p2}\\ 0\end{bmatrix}$$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix} a_p & q_p & 0\\ q_p & b_p & 0\\ 0 & 0 & k\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} + \alpha\Delta T\begin{bmatrix}\theta_{p1}\\ \theta_{p2}\\ 0\end{bmatrix}$$

Chapter 6

Elastic limit criteria

Loading of a material body causes deformation of the structure and, consequently, strains and stresses in the material. When either strains or stresses (or both combined) become too large, the material will be damaged, which means that irreversible microstructural changes will result. The structural and/or functional requirements of the structure or product will be hampered, which is referred to as failure.

Their are several failure modes, listed in the table below, each of them associated with a failure mechanism. In the following we will only consider plastic yielding. When the stress state exceeds the yield limit, the material behavior will not be elastic any longer. Irreversible microstructural changes (crystallographic slip in metals) will cause permanent (= plastic) deformation.

failure mode	**mechanism**
plastic yielding	crystallographic slip (metals)
brittle fracture	(sudden) breakage of bonds
progressive damage	micro-cracks $\rightarrow$ growth $\rightarrow$ coalescence
fatigue	damage/fracture under cyclic loading
dynamic failure	vibration $\rightarrow$ resonance
thermal failure	creep / melting
elastic instabilities	buckling $\rightarrow$ plastic deformation

6.1 Yield function

In a one-dimensional stress state (tensile test), yielding will occur when the absolute value of the stress σ reaches the initial yield stress σ_{y0}. This can be tested with a yield criterion, where a yield function f is used. When $f < 0$ the material behaves elastically and when $f = 0$ yielding occurs. Values $f > 0$ cannot be reached.

$$f(\sigma) = \sigma^2 - \sigma_{y0}^2 = 0 \quad \rightarrow \quad g(\sigma) = \sigma^2 = \sigma_{y0}^2 = g_t = \text{limit in tensile test}$$

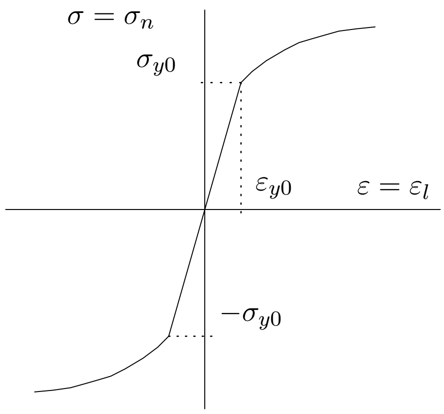


Fig. 6.1 : *Tensile curve with initial yield stress*

In a three-dimensional stress space, the yield criterion represents a yield surface. For elastic behavior ($f < 0$) the stress state is located inside the yield surface and for $f = 0$, the stress state is on the yield surface. Because $f > 0$ cannot be realized, stress states outside the yield surface can not exist. For isotropic material behavior, the yield function can be expressed in the principal stresses σ_1, σ_2 and σ_3. It can be visualized as a yield surface in the three-dimensional principal stress space.

$$\begin{array}{llll} f(\boldsymbol{\sigma}) = 0 & \rightarrow & g(\boldsymbol{\sigma}) = g_t & : \text{ yield surface in 6D stress space} \\ f(\sigma_1, \sigma_2, \sigma_3) = 0 & \rightarrow & g(\sigma_1, \sigma_2, \sigma_3) = g_t & : \text{ yield surface in 3D principal stress space} \end{array}$$

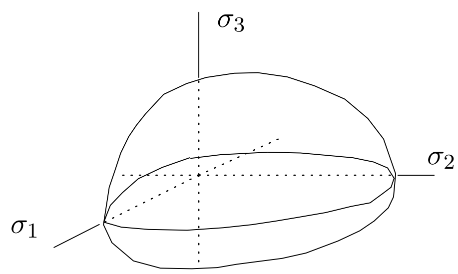


Fig. 6.2 : *Yield surface in three-dimensional principal stress space*

6.2 Principal stress space

The three-dimensional stress space is associated with a material point and has three axes, one for each principal stress value in that point. In the origin of the three-dimensional principal stress space, where $\sigma_1 = \sigma_2 = \sigma_3 = 0$, three orthonormal vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ constitute a vector base. The stress state in the material point is characterized by the principal stresses and thus by a point in stress space with "coordinates" σ_1, σ_2 and σ_3. This point can also be identified with a vector $\vec{\sigma}$, having components σ_1, σ_2 and σ_3 with respect to the vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

The hydrostatic axis, where $\sigma_1 = \sigma_2 = \sigma_3$ can be identified with a unit vector $\vec{e}_p$. Perpendicular to $\vec{e}_p$ in the $\vec{e}_1\vec{e}_p$-plane a unit vector $\vec{e}_q$ can be defined. Subsequently the unit vector $\vec{e}_r$ is defined perpendicular to the $\vec{e}_p\vec{e}_q$-plane.

The vectors $\vec{e}_q$ and $\vec{e}_r$ span the so-called Π-plane perpendicular to the hydrostatic axis. Vectors $\vec{e}_p$, $\vec{e}_q$ and $\vec{e}_r$ constitute a orthonormal vector base. A random unit vector $\vec{e}_t(\phi)$ in the Π-plane can be expressed in $\vec{e}_q$ and $\vec{e}_r$.

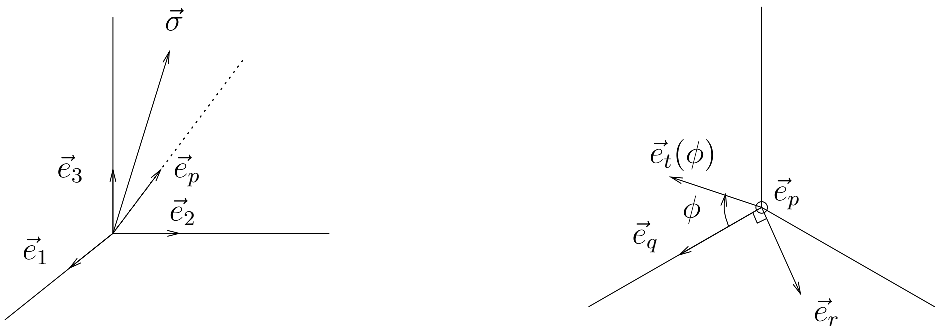


Fig. 6.3 : *Principal stress space*

hydrostatic axis $\quad \vec{e}_p = \frac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3)$ with $||\vec{e}_p|| = 1$

plane perpendicular to hydrostatic axis

$$
\begin{aligned}
\vec{e}_q^{\,*} &= \vec{e}_1 - (\vec{e}_p \cdot \vec{e}_1)\vec{e}_p = \vec{e}_1 - \tfrac{1}{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) = \tfrac{1}{3}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) \\
\vec{e}_q &= \tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) \\
\vec{e}_r &= \vec{e}_p * \vec{e}_q = \tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) * \tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) = \tfrac{1}{2}\sqrt{2}(\vec{e}_2 - \vec{e}_3)
\end{aligned}
$$

vector in Π-plane $\quad \vec{e}_t(\phi) = \cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r$

A stress state can be represented by a vector in the principal stress space. This vector can be written as the sum of a vector along the hydrostatic axis and a vector in the Π-plane. These vectors are referred to as the hydrostatic and the deviatoric part of the stress vector.

$$
\begin{aligned}
\vec{\sigma} &= \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3 = \vec{\sigma}^h + \vec{\sigma}^d \\
\vec{\sigma}^h &= (\vec{\sigma}\cdot\vec{e}_p)\vec{e}_p = \sigma^h\vec{e}_p = \tfrac{1}{3}\sqrt{3}(\sigma_1+\sigma_2+\sigma_3)\vec{e}_p = \sqrt{3}\sigma_m\vec{e}_p \\
\sigma^h &= \tfrac{1}{3}\sqrt{3}(\sigma_1+\sigma_2+\sigma_3) \\
\vec{\sigma}^d &= \vec{\sigma} - (\vec{\sigma}\cdot\vec{e}_p)\vec{e}_p \\
&= \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3 - \tfrac{1}{3}\sqrt{3}(\sigma_1+\sigma_2+\sigma_3)\tfrac{1}{3}\sqrt{3}(\vec{e}_1+\vec{e}_2+\vec{e}_3) \\
&= \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3 - \tfrac{1}{3}(\sigma_1\vec{e}_1 + \sigma_2\vec{e}_1 + \sigma_3\vec{e}_1 + \sigma_1\vec{e}_2 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_2 + \sigma_1\vec{e}_3 + \sigma_2\vec{e}_3 + \sigma_3\vec{e}_3) \\
&= \tfrac{1}{3}\{(2\sigma_1-\sigma_2-\sigma_3)\vec{e}_1 + (-\sigma_1+2\sigma_2-\sigma_3)\vec{e}_2 + (-\sigma_1-\sigma_2+2\sigma_3)\vec{e}_3\} \\
\sigma^d &= ||\vec{\sigma}^d|| = \sqrt{\vec{\sigma}^d\cdot\vec{\sigma}^d} \\
&= \tfrac{1}{3}\sqrt{(2\sigma_1-\sigma_2-\sigma_3)^2 + (-\sigma_1+2\sigma_2-\sigma_3)^2 + (-\sigma_1-\sigma_2+2\sigma_3)^2} \\
&= \sqrt{\tfrac{2}{3}(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1)} \\
&= \sqrt{\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d}
\end{aligned}
$$

Because the stress vector in the principal stress space can also be written as the sum of three vectors along the base vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, the principal stresses can be expressed in σ^h and σ^d.

$$
\begin{aligned}
\vec{\sigma} &= \vec{\sigma}^h + \vec{\sigma}^d = \sigma^h\vec{e}_p + \sigma^d\vec{e}_t(\phi) \\
&= \sigma^h\vec{e}_p + \sigma^d\{\cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r\} \\
&= \sigma^h\tfrac{1}{3}\sqrt{3}(\vec{e}_1+\vec{e}_2+\vec{e}_3) + \sigma^d\{\cos(\phi)\tfrac{1}{6}\sqrt{6}(2\vec{e}_1-\vec{e}_2-\vec{e}_3) - \sin(\phi)\tfrac{1}{2}\sqrt{2}(\vec{e}_2-\vec{e}_3)\} \\
&= \{\tfrac{1}{3}\sqrt{3}\,\sigma^h + \tfrac{1}{3}\sqrt{6}\,\sigma^d\cos(\phi)\}\vec{e}_1 + \\
&\quad \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d\cos(\phi) - \tfrac{1}{2}\sqrt{2}\,\sigma^d\sin(\phi)\}\vec{e}_2 + \\
&\quad \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d\cos(\phi) + \tfrac{1}{2}\sqrt{2}\,\sigma^d\sin(\phi)\}\vec{e}_3 \\
&= \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3
\end{aligned}
$$

6.3 Yield criteria

In the following sections, various yield criteria are presented. Each of them starts from a hypothesis, stating when the material will yield. Such a hypothesis is based on experimental observation and is valid for a specific (class of) material(s).

The yield criteria can be visualized in several stress spaces:

- the two-dimensional (σ_1, σ_2)-space for plane stress states with $\sigma_3 = 0$,
- the three-dimensional $(\sigma_1, \sigma_2, \sigma_3)$-space,
- the Π-plane and
- the $\sigma\tau$-plane, where Mohr's circles are used.

6.3.1 Maximum stress/strain

The maximum stress/strain criterion states that yielding occurs when one of the stress/strain components exceeds a limit value.

This criterion is used for orthotropic materials.

maximum stress	$(\sigma_{11} = s_X) \vee (\sigma_{22} = s_Y) \vee (\sigma_{12} = s_S) \vee \cdots$
maximum strain	$(\varepsilon_{11} = e_X) \vee (\varepsilon_{22} = e_Y) \vee (\sigma_{12} = e_S) \vee \cdots$

6.3.2 Maximum principal stress (Rankine)

The maximum principal stress (or Rankine) criterion states that

yielding occurs when the maximum principal stress reaches a limit value.

The absolute value is used to arrive at the same elasticity limit in tension and compression.

The Rankine criterion is used for brittle materials like cast iron. At failure these materials show *cleavage fracture*.

$$\boxed{\sigma_{max} = max(|\sigma_i| \;;\; i = 1,2,3) = \sigma_{max_t} = \sigma_{y0}}$$

The figure shows the yield surface in the principal stress space for a plane stress state with $\sigma_3 = 0$.

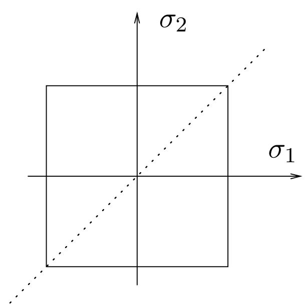


Fig. 6.4 : *Rankine yield surface in two-dimensional principal stress space*

In the three-dimensional stress space the yield surface is a cube with side-length $2\sigma_{y0}$.

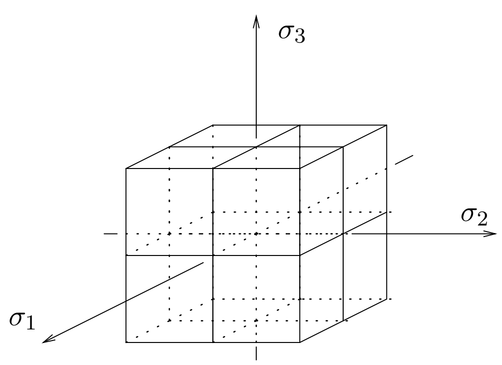


Fig. 6.5 : *Rankine yield surface in three-dimensional principal stress space*

In the (σ, τ)-space the Rankine criterion is visualized by to limits, which can not be exceeded by the absolute maximum of the principal stress.

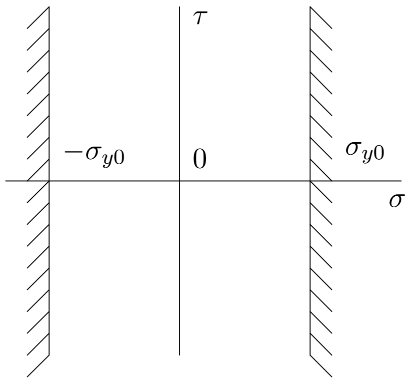


Fig. 6.6 : *Rankin yield limits in* (σ, τ)*-space*

6.3.3 Maximum principal strain (Saint Venant)

The maximum principal strain (or Saint Venant) criterion states that

yielding occurs when the maximum principal strain reaches a limit value.

From a tensile experiment this limit value appears to be the ratio of uni-axial yield stress and Young's modulus.

For $\sigma_1 > \sigma_2 > \sigma_3$, the maximum principal strain can be calculated from Hooke's law and its limit value can be expressed in the initial yield value σ_{yo} and Young's modulus E.

$$\varepsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 = \frac{\sigma_{y0}}{E} \quad \rightarrow \quad \sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma_{y0}$$

For other sequences of the principal stresses, relations are similar and can be used to construct the yield curve/surface in 2D/3D principal stress space.

$$\boxed{\varepsilon_{max} = max(|\varepsilon_i| \ ; \ i = 1, 2, 3) = \varepsilon_{max_t} = \frac{\sigma_{y0}}{E}}$$

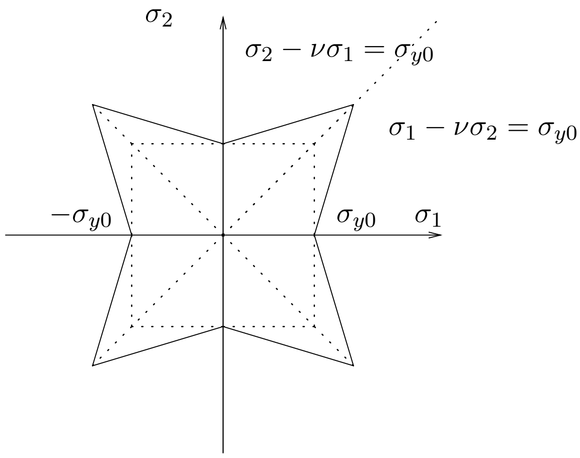


Fig. 6.7 : *Saint-Venant's yield curve in two-dimensional principal stress space*

6.3.4 Tresca

The Tresca criterion (Tresca, Coulomb, Mohr, Guest (1864)) states that

yielding occurs when the maximum shear stress reaches a limit value.

In a tensile test the limit value for the shear stress appears to be half the uni-axial yield stress.

$$\tau_{max} = \frac{1}{2}\left(\sigma_{max} - \sigma_{min}\right) = \tau_{max_t} = \frac{1}{2}\sigma_{y0} \quad \rightarrow \quad \boxed{\bar{\sigma}_{TR} = \sigma_{max} - \sigma_{min} = \sigma_{y0}}$$

Using Mohr's circles, it is easily seen how the maximum shear stress can be expressed in the maximum and minimum principal stresses.

For the plane stress case ($\sigma_3 = 0$) the yield curve in the $\sigma_1\sigma_2$-plane can be constructed using Mohr's circles. When both principal stresses are positive numbers, the yielding occurs when the largest reaches the one-dimensional yield stress σ_{y0}. When σ_1 is positive (= tensile stress), compression in the perpendicular direction, so a negative σ_2, implies that σ_1 must decrease to remain at the yield limit. Using Mohr's circles, this can easily be observed.

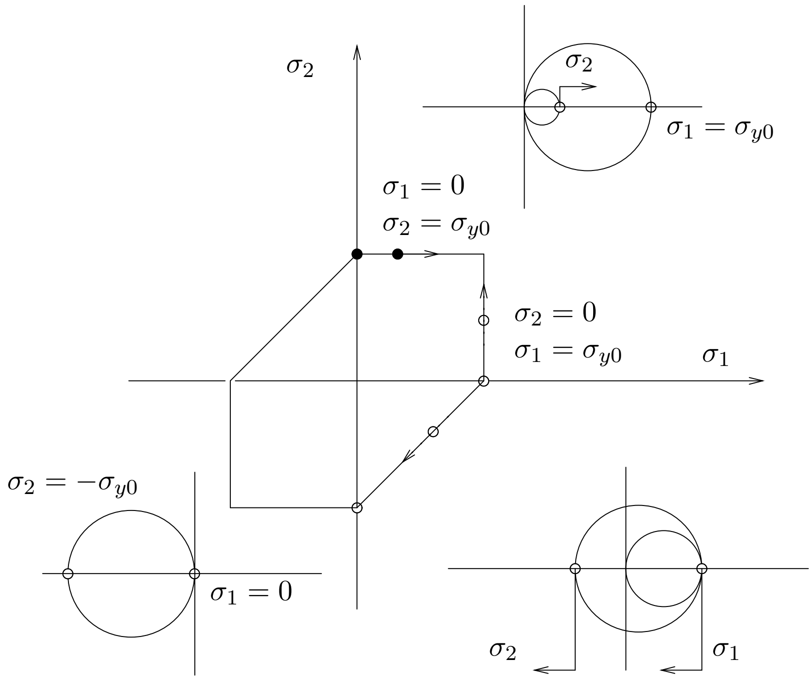


Fig. 6.8 : *Tresca yield curve in two-dimensional principal stress space*

$$
\begin{array}{llll}
\sigma_1 \geq 0 & ; & \sigma_2 \geq 0 & \tau_{max} = \sigma_1 | \sigma_2 = \frac{1}{2}\sigma_{y0} \\
\sigma_1 \geq 0 & ; & \sigma_2 < 0 & \tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sigma_{y0}
\end{array}
$$

Adding an extra hydrostatic stress state implies a translation in the three-dimensional principal stress space

$$\{\sigma_1, \sigma_2, \sigma_3\} \quad \rightarrow \quad \{\sigma_1 + c, \sigma_2 + c, \sigma_2 + c\}$$

i.e. a translation parallel to the hydrostatic axis where $\sigma_1 = \sigma_2 = \sigma_3$. This will never result in yielding or more plastic deformation, so the yield surface is a cylinder with its axis coinciding with (or parallel to) the hydrostatic axis.

In the Π-plane, the Tresca criterion is a regular 6-sided polygonal.

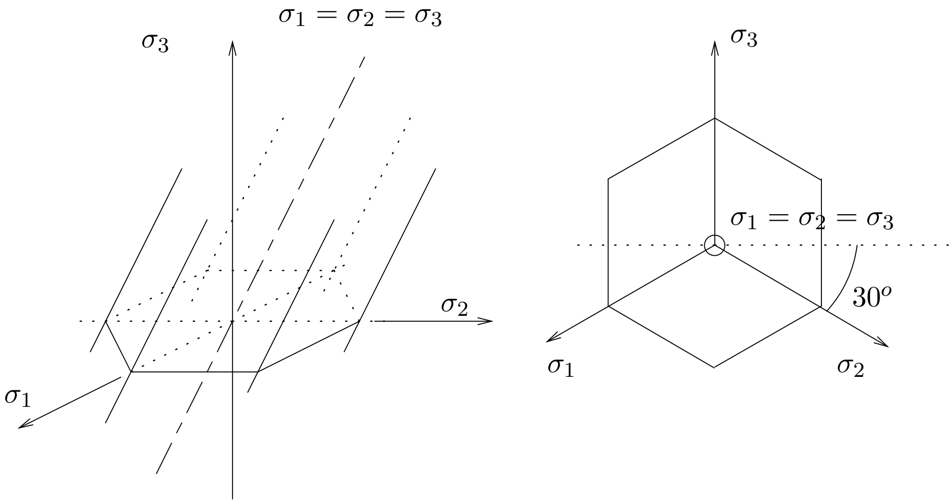


Fig. 6.9 : *Tresca yield surface in three-dimensional principal stress space and the Π-plane*

In the $\sigma\tau$-plane the Tresca yield criterion can be visualized with Mohr's circles.

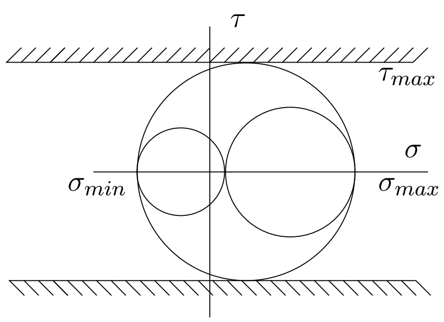


Fig. 6.10 : *Mohr's circles and Tresca yield limits in (σ, τ)-space*

6.3.5 Von Mises

According to the Von Mises elastic limit criterion (Von Mises, Hubert, Hencky (1918)),

yielding occurs when the specific shape deformation elastic energy reaches a critical value.

The specific *shape deformation energy* is also referred to as *distortional energy* or *deviatoric energy* or *shear strain energy.* It can be derived by splitting up the total specific elastic energy W into a hydrostatic part W^h and a deviatoric part W^d.

The deviatoric W^d can be expressed in $\boldsymbol{\sigma}^d$ and the hydrostatic W^h can be expressed in the mean stress $\sigma_m = \frac{1}{3}\text{tr}(\boldsymbol{\sigma})$.

$$
\begin{aligned}
W &= \tfrac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\sigma} : {}^4\boldsymbol{S} : \boldsymbol{\sigma} = \tfrac{1}{2}\left(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d\right) : {}^4\boldsymbol{S} : \left(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d\right) \\
&\qquad\qquad \boldsymbol{\sigma}^h = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} \\
&\qquad\qquad \boldsymbol{\sigma}^d = \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} \quad \rightarrow \quad \boldsymbol{I} : \boldsymbol{\sigma}^d = \mathrm{tr}(\boldsymbol{\sigma}^d) = 0 \\
&\qquad\qquad {}^4\boldsymbol{S} = -\frac{\nu}{E}\boldsymbol{I}\boldsymbol{I} + \frac{1+\nu}{E}\,{}^4\boldsymbol{I}^s \\
&= \tfrac{1}{2}\left(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d\right) : \left[-\frac{\nu}{E}\,\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \boldsymbol{O} + \frac{1+\nu}{3E}\,\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1+\nu}{E}\,\boldsymbol{\sigma}^d\right] \\
&= \tfrac{1}{2}\left(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d\right) : \left[\frac{1-2\nu}{3E}\,\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1+\nu}{E}\,\boldsymbol{\sigma}^d\right] = \tfrac{1}{2}\left[\frac{1-2\nu}{3E}\,\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1+\nu}{E}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d\right] \\
&= \frac{1}{18K}\,\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = W^h + W^d
\end{aligned}
$$

The deviatoric part is sometimes expressed in the second invariant J_2 of the deviatoric stress tensor. This shape deformation energy W^d can be expressed in the principal stresses. For the tensile test the shape deformation energy W_t^d can be expressed in the yield stress σ_{y0}. The Von Mises yield criterion $W^d = W_t^d$ can than be written as $\bar{\sigma}_{VM} = \sigma_{y0}$, where $\bar{\sigma}_{VM}$ is the equivalent or effective Von Mises stress, a function of all principal stresses.

The equivalent Von Mises stress $\bar{\sigma}_{VM}$ is sometimes replaced by the octahedral shear stress $\tau_{oct} = \frac{1}{3}\sqrt{2}\bar{\sigma}_{VM}$.

$$
\begin{aligned}
W^d &= \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{4G}\,2J_2 \\
&= \frac{1}{4G}\,\left\{\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right\} : \left\{\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right\} = \frac{1}{4G}\left[\boldsymbol{\sigma} : \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma})\right] \\
&= \frac{1}{4G}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \tfrac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2\right] \\
&= \frac{1}{4G}\,\tfrac{1}{3}\,\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]
\end{aligned}
$$

$$
W_t^d = \frac{1}{4G}\,\tfrac{2}{3}\,\sigma_t^2 = \frac{1}{4G}\,\tfrac{2}{3}\,\sigma_{y0}^2
$$

$$
W^d = W_t^d \quad \rightarrow
$$

$$
1) \quad \tfrac{1}{2}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\} = \sigma_{y0}^2 \quad \rightarrow
$$

$$
\boxed{\bar{\sigma}_{VM} = \sqrt{\tfrac{1}{2}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}} = \sigma_{y0}}
$$

$$
2) \quad \tfrac{1}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \tfrac{1}{3}\sigma_{y0}^2 \quad \rightarrow \quad \boxed{\bar{\sigma}_{VM} = \sqrt{\tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} = \sqrt{3J_2} = \sigma_{y0}}
$$

The Von Mises yield criterion can be expressed in Cartesian stress components.

$$\begin{aligned}
\tfrac{2}{3}\,\sigma_{VM}^2 &= \mathsf{tr}(\underline{\sigma}^d\underline{\sigma}^d) \quad \mathsf{with}\ \underline{\sigma}^d = \underline{\sigma} - \tfrac{1}{3}\mathsf{tr}(\underline{\sigma})\underline{I} \\
&= \left(\tfrac{2}{3}\sigma_{xx} - \tfrac{1}{3}\sigma_{yy} - \tfrac{1}{3}\sigma_{zz}\right)^2 + \sigma_{xy}^2 + \sigma_{xz}^2 + \\
&\quad \left(\tfrac{2}{3}\sigma_{yy} - \tfrac{1}{3}\sigma_{zz} - \tfrac{1}{3}\sigma_{xx}\right)^2 + \sigma_{yz}^2 + \sigma_{yx}^2 + \\
&\quad \left(\tfrac{2}{3}\sigma_{zz} - \tfrac{1}{3}\sigma_{xx} - \tfrac{1}{3}\sigma_{yy}\right)^2 + \sigma_{zx}^2 + \sigma_{zy}^2 \\
&= \tfrac{2}{3}\left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2\right) - \tfrac{2}{3}\left(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}\right) + 2\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right)
\end{aligned}$$

For plane stress ($\sigma_3 = 0$), the yield curve is an ellipse in the $\sigma_1\sigma_2$-plane. The length of the principal axes of the ellipse is $\sqrt{2}\sigma_{y0}$ and $\sqrt{\frac{1}{3}}\sigma_{y0}$.

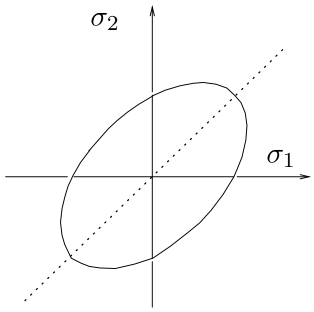


Fig. 6.11 : *Von Mises yield curve in two-dimensional principal stress space*

The three-dimensional Von Mises yield criterion is the equation of a cylindrical surface in three-dimensional principal stress space. Because hydrostatic stress does not influence yielding, the axis of the cylinder coincides with the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$.

In the Π-plane, the Von Mises criterion is a circle with radius $\sqrt{\frac{2}{3}}\sigma_{y0}$.

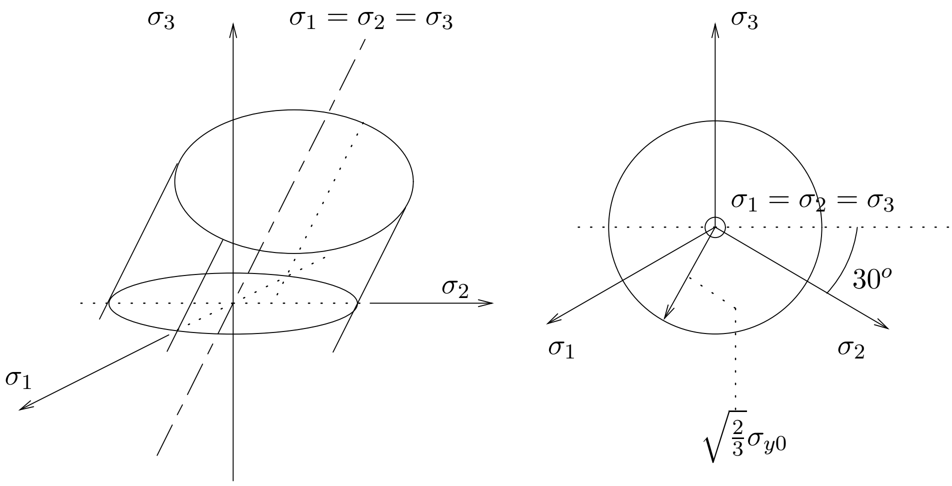


Fig. 6.12 : *Von Mises yield surface in three-dimensional principal stress space and the Π-plane*

6.3.6 Beltrami-Haigh

According to the elastic limit criterion of Beltrami-Haigh,

yielding occurs when the total specific elastic energy W reaches a critical value.

$$
\begin{aligned}
W &= \frac{1}{18K}\,\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d \\
&= \frac{1}{18K}\,(\sigma_1+\sigma_2+\sigma_3)^2 + \frac{1}{4G}\left\{\sigma_1^2+\sigma_2^2+\sigma_3^2 - \tfrac{1}{3}\,(\sigma_1+\sigma_2+\sigma_3)^2\right\} \\
&= \left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1+\sigma_2+\sigma_3)^2 + \frac{1}{4G}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) \\
W_t &= \left(\frac{1}{18K} - \frac{1}{12G}\right)\sigma^2 + \frac{1}{4G}\,\sigma^2 = \frac{1}{2E}\,\sigma^2 = \frac{1}{2E}\,\sigma_{y0}^2
\end{aligned}
$$

$$
\boxed{2E\left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1+\sigma_2+\sigma_3)^2 + \frac{2E}{4G}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) = \sigma_{y0}^2}
$$

The yield criterion contains elastic material parameters and thus depends on the elastic properties of the material. In three-dimensional principal stress space the yield surface is an ellipsoid. The longer axis coincides with (or is parallel to) the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$.

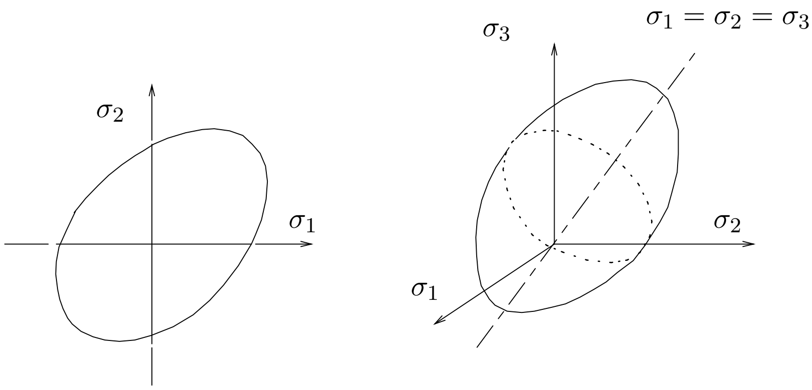


Fig. 6.13 : *Beltrami-Haigh yield curve and surface in principal stress space*

6.3.7 Mohr-Coulomb

A prominent difference in behavior under tensile and compression loading is seen in much materials, e.g. concrete, sand, soil and ceramics. In a tensile test such a material may have a yield stress σ_{ut} and in compression a yield stress σ_{uc} with $\sigma_{uc} > \sigma_{ut}$. The Mohr-Coulomb yield criterion states that

yielding occurs when the shear stress reaches a limit value.

For a plane stress state with $\sigma_3 = 0$ the yield contour in the $\sigma_1\sigma_2$-plane can be constructed in the same way as has been done for the Tresca criterion.

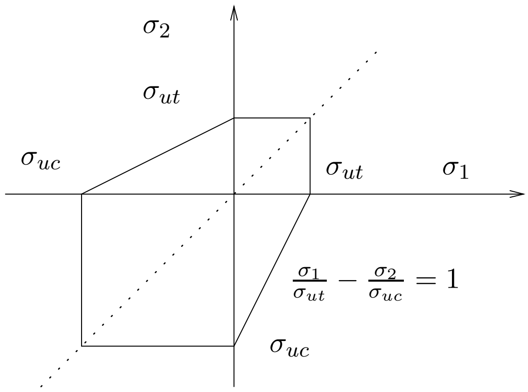


Fig. 6.14 : *Mohr-Coulomb yield curve in two-dimensional principal stress space*

The yield surface in the three-dimensional principal stress space is a cone with axis along the hydrostatic axis.

The intersection with the plane $\sigma_3 = 0$ gives the yield contour for plane stress.

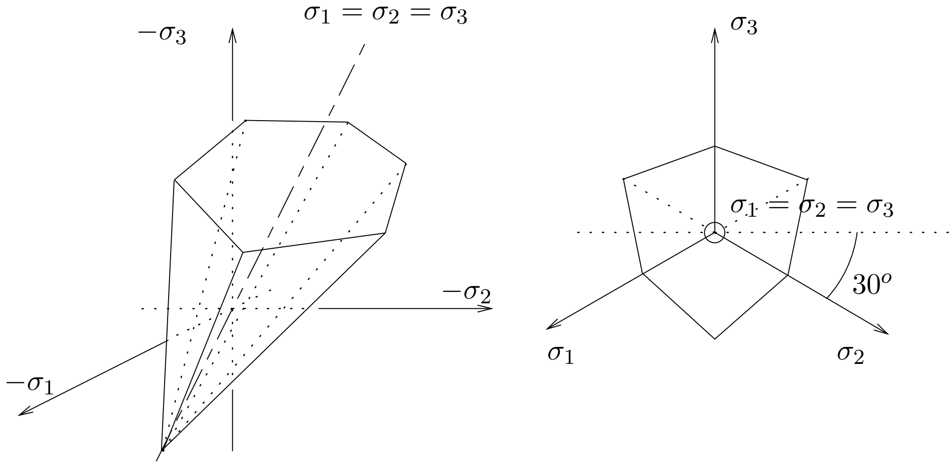


Fig. 6.15 : *Mohr-Coulomb yield surface in three-dimensional principal stress space and the Π-plane*

6.3.8 Drucker-Prager

For materials with internal friction and maximum adhesion, yielding can be described by the Drucker-Prager yield criterion. It relates to the Mohr-Coulomb criterion in the same way as the Von Mises criterion relates to the Tresca criterion.

For a plane stress state with $\sigma_3 = 0$ the Drucker-Prager yield contour in the $\sigma_1\sigma_2$-plane is a shifted ellipse.

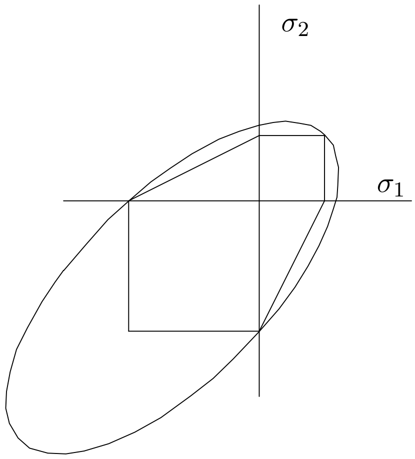


Fig. 6.16 : *Drucker-Prager yield curve in two-dimensional principal stress space*

In three-dimensional principal stress space the Drucker-Prager yield surface is a cone with circular cross-section.

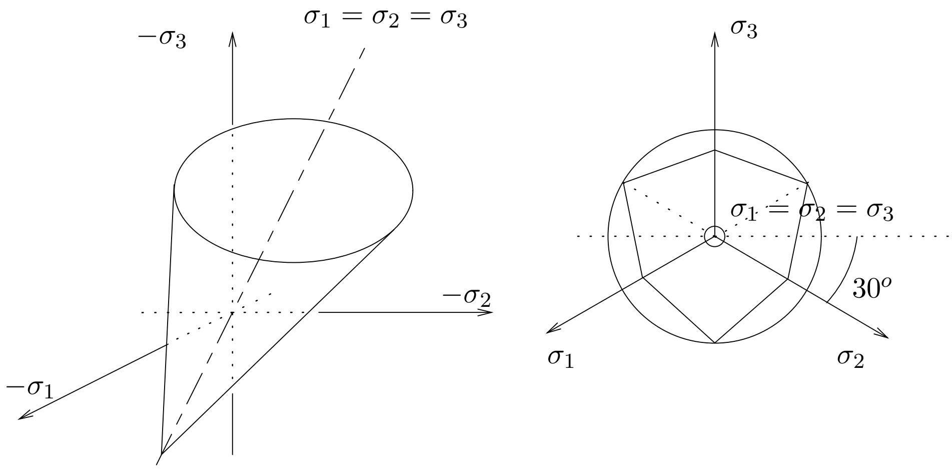


Fig. 6.17 : *Drucker-Prager yield surface in three-dimensional principal stress space and the Π-plane*

6.3.9 Other yield criteria

There are many more yield criteria, which are used for specific materials and loading conditions. The criteria of Hill, Hoffman and Tsai-Wu are used for orthotropic materials. In these criteria, there is a distinction between tensile and compressive stresses and their respective limit values.

6.4 Examples

Equivalent Von Mises stress

The stress state in a point is represented by the next Cauchy stress tensor :

$$\boldsymbol{\sigma} = 3\sigma\vec{e}_1\vec{e}_1 - \sigma\vec{e}_2\vec{e}_2 - 2\sigma\vec{e}_3\vec{e}_3 + \sigma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)$$

The Cauchy stress matrix is

$$\underline{\sigma} = \begin{bmatrix} 3\sigma & \sigma & 0 \\ \sigma & -\sigma & 0 \\ 0 & 0 & -2\sigma \end{bmatrix}$$

The Von Mises equivalent stress is defined as

$$\bar{\sigma}_{VM} = \sqrt{\tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} = \sqrt{\tfrac{3}{2}\text{tr}(\boldsymbol{\sigma}^d \cdot \boldsymbol{\sigma}^d)} = \sqrt{\tfrac{3}{2}\text{tr}(\underline{\sigma}^d\underline{\sigma}^d)}$$

The trace of the matrix product is calculated first, using the average stress $\sigma_m = \frac{1}{3}\text{tr}(\underline{\sigma})$.

$$\begin{aligned}\text{tr}(\underline{\sigma}^d\underline{\sigma}^d) &= \text{tr}([\underline{\sigma} - \sigma_m\underline{I}]\,[\underline{\sigma} - \sigma_m\underline{I}]) = \text{tr}(\underline{\sigma}\,\underline{\sigma} - 2\sigma_m\underline{I} + \sigma_m^2\underline{I}) = \text{tr}(\underline{\sigma}\,\underline{\sigma}) - 6\sigma_m + 3\sigma_m^2 \\ &= \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{13}^2 - \tfrac{2}{3}\sigma_{11} - \tfrac{2}{3}\sigma_{22} - \tfrac{2}{3}\sigma_{33} + \\ &\quad \tfrac{1}{3}\sigma_{11}^2 + \tfrac{1}{3}\sigma_{22}^2 + \tfrac{1}{3}\sigma_{33}^2 + \tfrac{2}{3}\sigma_{11}\sigma_{22} + \tfrac{2}{3}\sigma_{22}\sigma_{33} + \tfrac{2}{3}\sigma_{33}\sigma_{11}\end{aligned}$$

Substitution of the given values for the stress components leads to

$$\text{tr}(\underline{\sigma}^d\underline{\sigma}^d) = 16\sigma^2 \quad \rightarrow \quad \bar{\sigma}_{VM}^2 = 24\sigma^2 \quad \rightarrow \quad \bar{\sigma}_{VM} = 2\sqrt{6}\,\sigma$$

Equivalent Von Mises and Tresca stresses

The Cauchy stress matrix for a stress state is

$$\underline{\sigma} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

with all component values positive.
The Tresca yield criterion states that yielding will occur when the maximum shear stress reaches a limit value, which is determined in a tensile experiment. The equivalent Tresca stress is two times this maximum shear stress.

$$\bar{\sigma}_{TR} = 2\tau_{max} = \sigma_{max} - \sigma_{min}$$

The limit value is the one-dimensional yield stress σ_{y0}. To calulate $\bar{\sigma}_{TR}$, we need the prinipal stresses, which can be determined by requiring the matrix $\underline{\sigma} - s\underline{I}$ to be singular.

$$\det(\underline{\sigma} - s\underline{I}) = \det \begin{bmatrix} \sigma - s & \tau & 0 \\ \tau & \sigma - s & 0 \\ 0 & 0 & \sigma - s \end{bmatrix} = 0 \quad \rightarrow$$

$$(\sigma - s)^3 - \tau^2(\sigma - s) = 0 \quad \rightarrow \quad (\sigma - s)\{(\sigma - s)^2 - \tau^2\} = 0 \quad \rightarrow$$

$$(\sigma - s)(\sigma - s + \tau)(\sigma - s - \tau) = 0 \quad \rightarrow$$

$$\sigma_1 = \sigma_{max} = \sigma + \tau \quad ; \quad \sigma_2 = \sigma \quad ; \quad \sigma_3 = \sigma_{min} = \sigma - \tau$$

The equivalent Tresca stress is

$$\bar{\sigma}_{TR} = 2\tau$$

so yielding according to Tresca will occur when

$$\tau = \tfrac{1}{2}\,\sigma_{y0}$$

The equivalent Von Mises stress is expressed in the principal stresses :

$$\bar{\sigma}_{VM} = \sqrt{\tfrac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}}$$

and can be calculated by substitution,

$$\bar{\sigma}_{VM} = \sqrt{3\tau^2} = \sqrt{3}\tau$$

Yielding according to Von Mises will occur when the equivalent stress reaches a limit value, the one-dimensional yield stress σ_{y0}, which results in

$$\tau = \tfrac{1}{3}\sqrt{3}\,\sigma_{y0}$$

Chapter 7

Governing equations

In this chapter we will recall the equations, which have to be solved to determine the deformation of a three-dimensional linear elastic material body under the influence of an external load. The equations will be written in component notation w.r.t. a Cartesian and a cylindrical vector base and simplified for plane strain, plane stress and axi-symmetry. The material behavior is assumed to be isotropic.

7.1 Vector/tensor equations

The deformed (current) state is determined by 12 state variables : 3 displacement components and 9 stress components. These unknown quantities must be solved from 12 equations : 6 equilibrium equations and 6 constitutive equations.

With proper boundary (and initial) conditions the equations can be solved, which, for practical problems, must generally be done numerically. The compatibility equations are generally satisfied for the chosen strain-displacement relation. In some solution approaches they are used instead of the equilibrium equations.

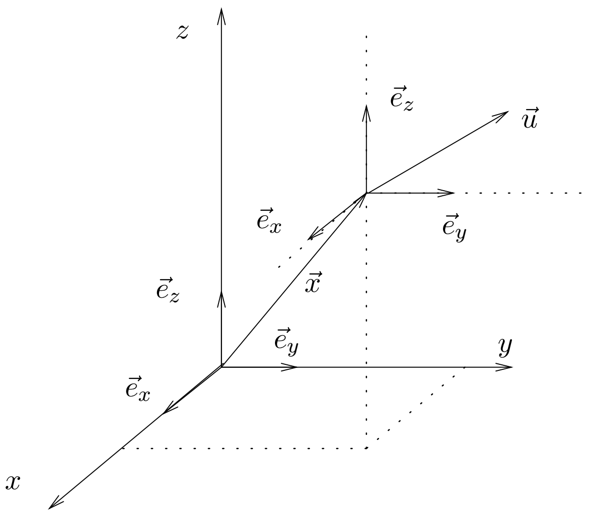
gradient operator	:	$\vec{\nabla} = \underset{\sim}{\nabla}^T \underset{\sim}{\vec{e}}$
position	:	$\vec{x} = \underset{\sim}{x}^T \underset{\sim}{\vec{e}}$
displacement	:	$\vec{u} = \underset{\sim}{u}^T \underset{\sim}{\vec{e}}$
strain tensor	:	$\boldsymbol{\varepsilon} = \frac{1}{2}\left\{\left(\vec{\nabla}\vec{u}\right)^c + \left(\vec{\nabla}\vec{u}\right)\right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$
compatibility	:	$\nabla^2 \{\mathsf{tr}(\boldsymbol{\varepsilon})\} - \vec{\nabla}\cdot(\vec{\nabla}\cdot\boldsymbol{\varepsilon})^c = 0$
stress tensor	:	$\boldsymbol{\sigma} = \underset{\sim}{\vec{e}}^T \underline{\sigma}\, \underset{\sim}{\vec{e}}$
eq.of motion	:	$\vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \rho\ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^c$
material law	:	$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \quad ; \quad \boldsymbol{\varepsilon} = {}^4\boldsymbol{C}^{-1} : \boldsymbol{\sigma} = {}^4\boldsymbol{S} : \boldsymbol{\sigma}$

7.2 Three-dimensional scalar equations

The vectors and tensors can be written in components with respect to a three-dimensional vector basis. For various problems in mechanics, it will be suitable to choose either a Cartesian coordinate system or a cylindrical coordinate system.

7.2.1 Cartesian components

The governing equations are written in components w.r.t. a Cartesian vector base. In the column with strain components, we use ε instead of γ. Therefore, the shear related parameters in $\underline{\underline{C}}$ and $\underline{\underline{S}}$ are multiplied with or divided by 2, respectively.



$$\vec{x} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$
$$\underset{\sim}{x}^T = \left[\begin{array}{ccc} x & y & z \end{array} \right]$$
$$\underset{\sim}{\nabla}^T = \left[\begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array} \right]$$

Fig. 7.1 : *Cartesian components of position vector and gradient operator*

$$\underset{\sim}{u}^T = \left[\begin{array}{ccc} u_x & u_y & u_z \end{array} \right] \qquad \rightarrow \qquad \underset{\approx}{\varepsilon}^T = \left[\begin{array}{cccccc} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \varepsilon_{xy} & \varepsilon_{yz} & \varepsilon_{zx} \end{array} \right]$$

$$\underline{\varepsilon} = \frac{1}{2} \left[\begin{array}{ccc} 2u_{x,x} & u_{x,y} + u_{y,x} & u_{x,z} + u_{z,x} \\ u_{y,x} + u_{x,y} & 2u_{y,y} & u_{y,z} + u_{z,y} \\ u_{z,x} + u_{x,z} & u_{z,y} + u_{y,z} & 2u_{z,z} \end{array} \right]$$

$$2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$
$$\varepsilon_{xx,yz} + \varepsilon_{yz,xx} - \varepsilon_{zx,xy} - \varepsilon_{xy,xz} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$

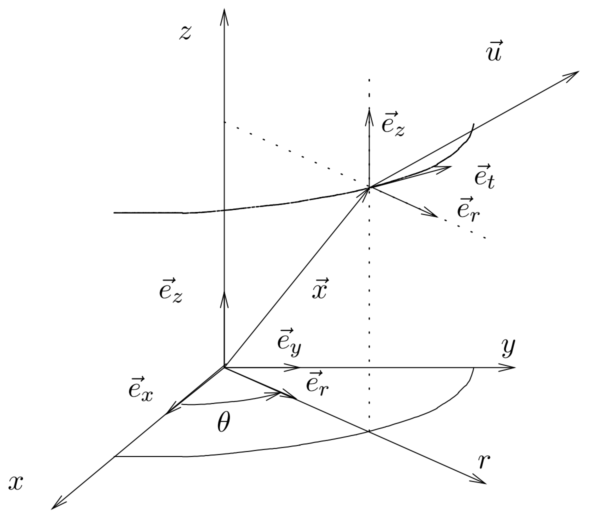
$$\underset{\approx}{\sigma}^T = \left[\begin{array}{cccccc} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{array} \right]$$

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x = \rho \ddot{u}_x \qquad (\sigma_{xy} = \sigma_{yx})$$
$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y = \rho \ddot{u}_y \qquad (\sigma_{yz} = \sigma_{zy})$$
$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z = \rho \ddot{u}_z \qquad (\sigma_{zx} = \sigma_{xz})$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon} \qquad ; \qquad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}$$

7.2.2 Cylindrical components

The governing equations are written in components w.r.t. a cylindrical vector base.



$$\vec{x} = r\vec{e}_r(\theta) + z\vec{e}_z$$
$$\underset{\sim}{x}^T = \left[\begin{array}{ccc} r & \theta & z \end{array}\right]$$

$$x = r\cos(\theta) \;;\; y = r\sin(\theta)$$

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_t \qquad \frac{d\vec{e}_t}{d\theta} = -\vec{e}_r$$

$$\underset{\sim}{\nabla}^T = \left[\begin{array}{ccc} \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \end{array}\right]$$

Fig. 7.2 : *Cylindrical components of position vector and gradient operator*

$$\underset{\sim}{u}^T = \left[\begin{array}{ccc} u_r & u_t & u_z \end{array}\right] \qquad \rightarrow \qquad \underset{\approx}{\varepsilon}^T = \left[\begin{array}{cccccc} \varepsilon_{rr} & \varepsilon_{tt} & \varepsilon_{zz} & \varepsilon_{rt} & \varepsilon_{tz} & \varepsilon_{zr} \end{array}\right]$$

$$\underline{\varepsilon} = \frac{1}{2}\left[\begin{array}{ccc} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ u_{z,r} + u_{r,z} & \frac{1}{r}u_{z,t} + u_{t,z} & 2u_{z,z} \end{array}\right]$$

$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0 \quad \rightarrow$ cyc. 2x
$\varepsilon_{rr,tz} + \varepsilon_{tz,rr} - \varepsilon_{zr,rt} - \varepsilon_{rt,rz} = 0 \quad \rightarrow$ cyc. 2x

$$\underset{\approx}{\sigma}^T = \left[\begin{array}{cccccc} \sigma_{rr} & \sigma_{tt} & \sigma_{zz} & \sigma_{rt} & \sigma_{tz} & \sigma_{zr} \end{array}\right]$$

$$\sigma_{rr,r} + \tfrac{1}{r}\,\sigma_{rt,t} + \tfrac{1}{r}\,(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = \rho \ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$
$$\sigma_{tr,r} + \tfrac{1}{r}\,\sigma_{tt,t} + \tfrac{1}{r}\,(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t = \rho \ddot{u}_t \qquad (\sigma_{tz} = \sigma_{zt})$$
$$\sigma_{zr,r} + \tfrac{1}{r}\,\sigma_{zt,t} + \tfrac{1}{r}\,\sigma_{zr} + \sigma_{zz,z} + \rho q_z = \rho \ddot{u}_z \qquad (\sigma_{zr} = \sigma_{rz})$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon} \qquad ; \qquad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}$$

7.3 Material law

When deformations are small, every material will show linear elastic behavior. For orthotropic material there are 9 independent material constants. When there is more material symmetry, this number decreases. Finally, isotropic material can be characterized with only two material constants.

Be aware that we use now the strain components ε_{ij} and not the shear components γ_{ij}. In an earlier chapter, the parameters for orthotropic, transversally isotropic and isotropic material were rewritten in terms of engineering parameters: Young's moduli and Poisson's ratio's.

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2K & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L & 0 \\ 0 & 0 & 0 & 0 & 0 & 2M \end{bmatrix} \quad \rightarrow \quad \underline{\underline{S}} = \underline{\underline{C}}^{-1} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}k & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}l & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}m \end{bmatrix}$$

quadratic	$B = A \ ; \ S = R \ ; \ M = L;$
transversal isotropic	$B = A \ ; \ S = R \ ; \ M = L \ ; \ K = \frac{1}{2}(A - Q)$
cubic	$C = B = A \ ; \ S = R = Q \ ; \ M = L = K \neq \frac{1}{2}(A - Q)$
isotropic	$C = B = A \ ; \ S = R = Q \ ; \ M = L = K = \frac{1}{2}(A - Q)$

7.4 Planar deformation

In many applications the loading and deformation is in one plane. The result is that the material body is in a state of plane strain or plane stress. The governing equations can than be simplified considerably.

7.4.1 Cartesian

In a plane strain situation, deformation in one direction – here the z-direction – is suppressed. In a plane stress situation, stresses on one plane – here the plane with normal in z-direction – are zero.

Eliminating σ_{zz} for plane strain and ε_{zz} for plane stress leads to a simplified Hooke's law. Also the equilibrium equation in the z-direction is automatically satisfied and has become obsolete.

$$\left. \begin{array}{lcl} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \end{array} \right\} \qquad \left\{ \begin{array}{l} u_x = u_x(x, y) \\ u_y = u_y(x, y) \end{array} \right.$$

$$\underset{\sim}{\varepsilon}^T = \left[\ \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{xy} \ \right] = \left[\ u_{x,x} \quad u_{y,y} \quad \frac{1}{2}(u_{x,y} + u_{y,x}) \ \right]$$

$$2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0$$

$$\underset{\sim}{\sigma}^T = \left[\begin{array}{ccc} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{array}\right]$$

$$\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \qquad (\sigma_{xy} = \sigma_{yx})$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y$$

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{bmatrix}$$

7.4.2 Cylindrical

In a plane strain situation, deformation in one direction – here the z-direction – is suppressed. In a plane stress situation, stresses on one plane – here the plane with normal in z-direction – are zero.

Eliminating σ_{zz} for plane strain and ε_{zz} for plane stress leads to a simplified Hooke's law. Also the equilibrium equation in the z-direction is automatically satisfied and has become obsolete.

$$\left.\begin{array}{lll} \textsf{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \textsf{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta) \end{array}\right.$$

$$\underset{\sim}{\varepsilon}^T = \left[\begin{array}{ccc} \varepsilon_{rr} & \varepsilon_{tt} & \varepsilon_{rt} \end{array}\right] = \left[\begin{array}{ccc} u_{r,r} & \frac{1}{r}(u_r + u_{t,t}) & \frac{1}{2}\left(\frac{1}{r}(u_{r,t} - u_t) + u_{t,r}\right) \end{array}\right]$$

$$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0$$

$$\underset{\sim}{\sigma}^T = \left[\begin{array}{ccc} \sigma_{rr} & \sigma_{tt} & \sigma_{rt} \end{array}\right]$$

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$

$$\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \rho q_t = \rho \ddot{u}_t$$

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{bmatrix}$$

7.4.3 Cylindrical : axi-symmetric + $u_t = 0$

When geometry and boundary conditions are such that we have $\dfrac{\partial(\ )}{\partial\theta} = (\ )_t = 0$ the situation is referred to as being axi-symmetric.

In many cases boundary conditions are such that there is no displacement of material points in tangential direction ($u_t = 0$). In that case we have $\varepsilon_{rt} = 0 \rightarrow \sigma_{rt} = 0$

$$\left.\begin{array}{lll} \textsf{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \textsf{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r) \\ u_t = 0 \end{array}\right.$$

$$
\begin{aligned}
&\underset{\sim}{\varepsilon}^T = \left[\begin{array}{cc} \varepsilon_{rr} & \varepsilon_{tt} \end{array} \right] = \left[\begin{array}{cc} u_{r,r} & \frac{1}{r}(u_r) \end{array} \right] \\
&\varepsilon_{rr} = u_{r,r} = (r\varepsilon_{tt})_{,r} = \varepsilon_{tt} + r\varepsilon_{tt,r} \\
\\
&\underset{\sim}{\sigma}^T = \left[\begin{array}{cc} \sigma_{rr} & \sigma_{tt} \end{array} \right] \\
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \\
&\underline{\underline{C}}_p = \left[\begin{array}{cc} A_p & Q_p \\ Q_p & B_p \end{array} \right] \qquad ; \qquad \underline{\underline{S}}_p = \left[\begin{array}{cc} a_p & q_p \\ q_p & b_p \end{array} \right]
\end{aligned}
$$

7.5 Inconsistency plane stress

Although for plane stress the out-of-plane shear stresses must be zero, they are not, when calculated afterwards from the strains. This inconsistency is inherent to the plane stress assumption. Deviations must be small to render the assumption of plane stress valid.

$$
\begin{aligned}
\sigma_{xz} &= 2K\varepsilon_{xz} = 2Ku_{z,x} \neq 0 \\
\sigma_{yz} &= 2K\varepsilon_{yz} = 2Ku_{z,y} \neq 0
\end{aligned}
$$

Chapter 8

Analytical solution strategies

8.1 Governing equations for unknowns

The deformation of a three-dimensional continuum in three-dimensional space is described by the displacement vector $\vec{u}$ of each material point. Due to the deformation, stresses arise and the stress state is characterized by the stress tensor $\boldsymbol{\sigma}$. For static problems, this tensor has to satisfy the equilibrium equations. Solving stresses from these equations is generally not possible and additional equations are needed, which relate stresses to deformation. These constitutive equations, which describe the material behavior, relate the stress tensor $\boldsymbol{\sigma}$ to the strain tensor $\boldsymbol{\varepsilon}$, which is a function of the displacement gradient tensor $(\vec{\nabla}\vec{u})$. Components of this strain tensor cannot be independent and are related by the compatibility equations.

unknown variables

displacements	$\vec{u} = \vec{u}(\vec{x})$	
deformation tensor	$\boldsymbol{F} = \left(\vec{\nabla}_0 \vec{x}\right)^C$	
stresses	$\boldsymbol{\sigma}$	

equations

compatibility	$\nabla^2 \{\mathsf{tr}(\boldsymbol{\varepsilon})\} - \vec{\nabla} \cdot (\vec{\nabla} \cdot \boldsymbol{\varepsilon})^c = 0$	
equilibrium	$\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho\vec{q} = \rho\ddot{\vec{u}}$;	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
material law	$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{F})$	

8.2 Boundary conditions

Some of the governing equations are partial differential equations, where differentiation is done w.r.t. the spatial coordinates. These differential equations can only be solved when proper boundary conditions are specified. In each boundary point of the material body, either the displacement or the load must be prescribed. It is also possible to specify a relation between displacement and load in such a point.

When the acceleration of the material points cannot be neglected, the equilibrium equation becomes the equation of motion, with $\rho\ddot{\vec{u}}$ as its right-hand term. In that case a solution can only be determined when proper initial conditions are prescribed, i.e. initial displacement, velocity or acceleration. In this section we will assume $\ddot{\vec{u}} = \vec{0}$.

$$\boxed{\begin{array}{l} \vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho\vec{q} = \vec{0} \qquad \forall \quad \vec{x} \in V \\ \vec{u} = \vec{u}_p \qquad \forall \quad \vec{x} \in A_u \\ \vec{p} = \vec{n} \cdot \boldsymbol{\sigma} = \vec{p}_p \qquad \forall \quad \vec{x} \in A_p \end{array}}$$

8.2.1 Saint-Venant's principle

The so-called *Saint-Venant principle* states that, if a load on a structure is replaced by a statically equivalent load, the resulting strains and stresses in the structure will only be altered near the regions where the load is applied. With this principle in mind, the real boundary conditions can often be modeled in a simplified way. Concentrated forces can for instance be replaced by distributed loads, and vice versa. Stresses and strains will only differ significantly in the neighborhood of the boundary, where the load is applied.

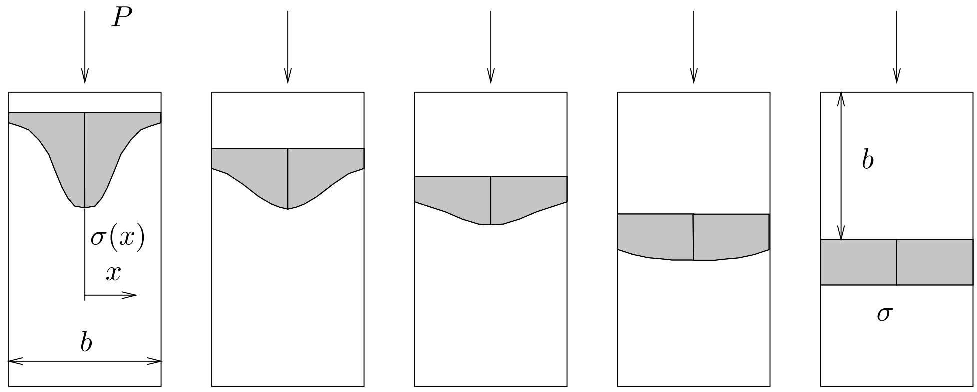


Fig. 8.1 : *Saint-Venant principle*

$$P = \int_A \sigma(x)\, dA = \sigma\, A \qquad ; \qquad A = b * t$$

8.2.2 Superposition

Under the assumption of small deformations and linear elastic material behavior, the governing equations, which must be solved to determine deformation and stresses (= solution S)

are linear. When boundary conditions (fixations and loads (L)), which are needed for the solution, are also linear, the total problem is linear and the principle of superposition holds.

The principle of superposition states that the solution S for a given combined load $L = L_1 + L_2$ is the sum of the solution S_1 for load L_1 and the solution S_2 for L_2, so : $S = S_1 + S_2$.

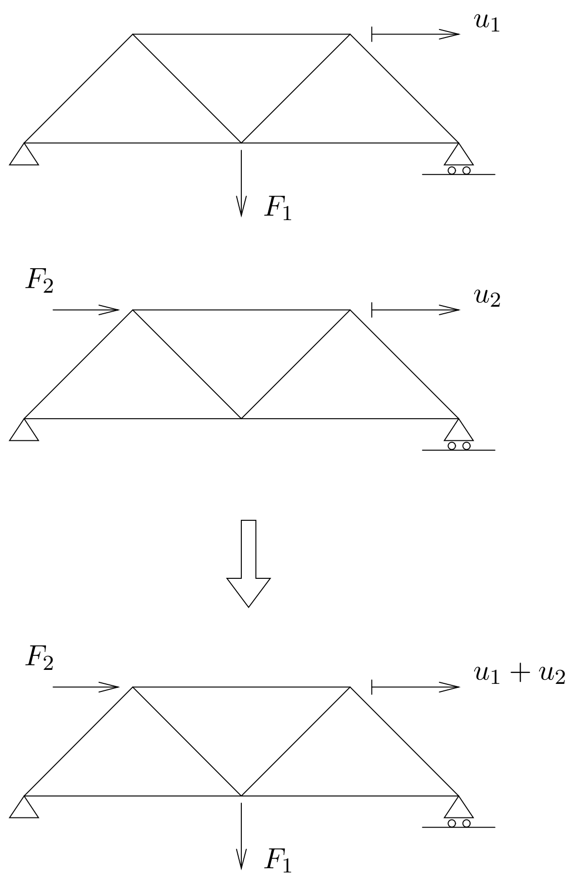


Fig. 8.2 : *Principle of superposition*

8.3 Solution : displacement method

In the *displacement method* the constitutive relation for the stress tensor is substituted in the force equilibrium equation.

Subsequently the strain tensor is replaced by its definition in terms of the displacement gradient. This results in a differential equation in the displacement $\vec{u}$, which can be solved when proper boundary conditions are specified.

In a Cartesian coordinate system the vector/tensor formulation can be replaced by index notation. It is elaborated here for the case of linear elasticity theory.

$$\left.\begin{array}{l}\vec{\nabla}\cdot\boldsymbol{\sigma}^c+\rho\vec{q}=\vec{0}\\ \boldsymbol{\sigma}={}^4\boldsymbol{C}:\boldsymbol{\varepsilon}\end{array}\right\}\quad\rightarrow\quad\left.\begin{array}{l}\vec{\nabla}\cdot\left({}^4\boldsymbol{C}:\boldsymbol{\varepsilon}\right)^c+\rho\vec{q}=\vec{0}\\ \boldsymbol{\varepsilon}=\frac{1}{2}\left\{\left(\vec{\nabla}\vec{u}\right)^c+\left(\vec{\nabla}\vec{u}\right)\right\}\end{array}\right\}\quad\rightarrow$$

$$\vec{\nabla}\cdot\left\{{}^4\boldsymbol{C}:\left(\vec{\nabla}\vec{u}\right)\right\}^c+\rho\vec{q}=\vec{0}\quad\rightarrow\quad\vec{u}\quad\rightarrow\quad\boldsymbol{\varepsilon}\quad\rightarrow\quad\boldsymbol{\sigma}$$

Cartesian index notation

$$\left.\begin{array}{l}\sigma_{ij,j}+\rho q_i=0_i\\ \sigma_{ij}=C_{ijlk}\varepsilon_{lk}\end{array}\right\}\quad\rightarrow\quad\left.\begin{array}{l}C_{ijkl}\varepsilon_{lk,j}+\rho q_i=0_i\\ \varepsilon_{lk}=\frac{1}{2}\left(u_{l,k}+u_{k,l}\right)\end{array}\right\}\quad\rightarrow$$

$$C_{ijkl}u_{l,kj}+\rho q_i=0_i\quad\rightarrow\quad u_i\quad\rightarrow\quad\varepsilon_{ij}\quad\rightarrow\quad\sigma_{ij}$$

8.3.1 Navier equations

The displacement method is elaborated for planar deformation in a Cartesian coordinate system. Linear deformation and linear elastic material behavior is assumed. Elimination and substitution results in two partial differential equations for the two displacement components. For the sake of simplicity, we do not consider thermal loading here.

$$\left.\begin{array}{l}\sigma_{xx,x}+\sigma_{xy,y}+\rho q_x=\rho\ddot{u}_x\qquad;\qquad\sigma_{yx,x}+\sigma_{yy,y}+\rho q_y=\rho\ddot{u}_y\\[1em] \sigma_{xx}=A_p\varepsilon_{xx}+Q_p\varepsilon_{yy}\\ \sigma_{yy}=Q_p\varepsilon_{xx}+B_p\varepsilon_{yy}\\ \sigma_{xy}=2K\varepsilon_{xy}\end{array}\right\}$$

$$\left.\begin{array}{l}A_p\varepsilon_{xx,x}+Q_p\varepsilon_{yy,x}+2K\varepsilon_{xy,y}+\rho q_x=\rho\ddot{u}_x\\ 2K\varepsilon_{xy,x}+Q_p\varepsilon_{xx,y}+B_p\varepsilon_{yy,y}+\rho q_y=\rho\ddot{u}_y\end{array}\right\}$$

$$\left.\begin{array}{l}A_pu_{x,xx}+Q_pu_{y,yx}+K(u_{x,yy}+u_{y,xy})+\rho q_x=\rho\ddot{u}_x\\ K(u_{x,yx}+u_{y,xx})+Q_pu_{x,xy}+B_pu_{y,yy}+\rho q_y=\rho\ddot{u}_y\end{array}\right\}$$

$$\left.\begin{array}{l}A_pu_{x,xx}+Ku_{x,yy}+(Q_p+K)u_{y,yx}+\rho q_x=\rho\ddot{u}_x\\ Ku_{y,xx}+B_pu_{y,yy}+(Q_p+K)u_{x,xy}+\rho q_y=\rho\ddot{u}_y\end{array}\right\}$$

8.3.2 Axi-symmetric with $u_t = 0$

Many engineering problems present a rotational symmetry w.r.t. an axis. They are *axi-symmetric*. In many cases the tangential displacement is zero : $u_t = 0$. This implies that there are no shear strains and stresses.

displacements	$u_r = u_r(r) \quad ; \quad u_z = u_z(r,z)$
strains	$\varepsilon_{rr} = u_{r,r}\ ;\ \varepsilon_{tt} = \frac{1}{r}u_r\ ;\ \varepsilon_{zz} = u_{z,z}$
stresses	$\sigma_{tz} = 0 \quad ; \quad \sigma_{rz} \approx 0 \quad ; \quad \sigma_{tr} = 0$

$$\text{eq. of motion} \qquad \sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r$$

The radial and tangential stresses are related to the radial and tangential strains by the planar material law. Material parameters are indicated as A_p, B_p and Q_p and can later be specified for a certain material and for plane strain or plane stress. With the strain-displacement relations the equation of motion can be transformed into a differential equation for the radial displacement u_r

$$\left.\begin{array}{l} \sigma_{rr} = A_p \varepsilon_{rr} + Q_p \varepsilon_{tt} - \Theta_{p1} \alpha \Delta T \\ \sigma_{tt} = Q_p \varepsilon_{rr} + B_p \varepsilon_{tt} - \Theta_{p2} \alpha \Delta T \end{array}\right\} \quad \rightarrow \quad \text{eq. of motion} \quad \rightarrow$$

$$\boxed{u_{r,rr} + \frac{1}{r}\, u_{r,r} - \zeta^2 \, \frac{1}{r^2}\, u_r = f(r)}$$

$$\text{with} \qquad \zeta = \sqrt{\frac{B_p}{A_p}}$$

$$\text{and} \qquad f(r) = \frac{\rho}{A_p}\,(\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p}\,\alpha(\Delta T)_{,r} + \frac{\Theta_{p1} - \Theta_{p2}}{A_p}\frac{1}{r}\,\alpha\Delta T$$

8.4 Solution : stress method

In the *stress method*, the constitutive relation for the strain tensor is substituted in the compatibility equation, resulting in a partial differential equation for the stress tensor. This equation and the equilibrium equations constitute a set of coupled equations from which the stress tensor has to be solved.

For planar problems, this can be elaborated and results in the Beltrami-Mitchell equation for the stress components. It is again assmed that deformations are small and the material behavior is linearly elastic.

Solution of the stress equation(s) is done by introducing the so-called *Airy stress function.*

8.4.1 Beltrami-Mitchell equation

The compatibility equation for planar deformation can be expressed in stress components, resulting in the Beltrami-Mitchell equation.

$$\left.\begin{array}{l} \varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\varepsilon_{xy,xy} \\ \\ \varepsilon_{xx} = a_p \sigma_{xx} + q_p \sigma_{yy} \\ \varepsilon_{yy} = q_p \sigma_{xx} + b_p \sigma_{yy} \\ \varepsilon_{xy} = \frac{1}{2} k \sigma_{xy} \end{array}\right\} \quad \rightarrow \quad \left\{\begin{array}{l} k\sigma_{xy,xy} = \\ \qquad a_p \sigma_{xx,yy} + q_p \sigma_{yy,yy} + \\ \qquad q_p \sigma_{xx,xx} + b_p \sigma_{yy,xx} \end{array}\right.$$

equilibrium

$$\left.\begin{array}{lll}\sigma_{xx,x}+\sigma_{xy,y}=-\rho q_x & \rightarrow & \sigma_{xy,xy}+\sigma_{xx,xx}=-\rho q_{x,x}\\ \sigma_{yx,x}+\sigma_{yy,y}=-\rho q_y & \rightarrow & \sigma_{xy,xy}+\sigma_{yy,yy}=-\rho q_{y,y}\end{array}\right\} \rightarrow$$
$$2\sigma_{xy,xy}+\sigma_{xx,xx}+\sigma_{yy,yy}=-\rho q_{x,x}-\rho q_{y,y}$$

Beltrami-Mitchell equation

$$(k+2q_p)\left(\sigma_{xx,xx}+\sigma_{yy,xx}\right)+2a_p\sigma_{xx,yy}+2b_p\sigma_{yy,yy}=-k\rho(q_{x,x}+q_{y,y})$$

8.4.2 Beltrami-Mitchell equation for thermal loading

With thermal strains, the compatibility equation for planar deformation can again be expressed in stress components. Combination with the equilibrium equations results in the Belrami-Mitchell equation for thermal loading.

$$\left.\begin{array}{l}\varepsilon_{xx,yy}+\varepsilon_{yy,xx}=2\varepsilon_{xy,xy}\\ \\ \varepsilon_{xx}=a_p\sigma_{xx}+q_p\sigma_{yy}+\alpha\Delta T\\ \varepsilon_{yy}=q_p\sigma_{xx}+b_p\sigma_{yy}+\alpha\Delta T\\ \varepsilon_{xy}=\frac{1}{2}k\sigma_{xy}\end{array}\right\} \rightarrow \left\{\begin{array}{l}k\sigma_{xy,xy}=\\ \qquad a_p\sigma_{xx,yy}+q_p\sigma_{yy,yy}+\\ \qquad q_p\sigma_{xx,xx}+b_p\sigma_{yy,xx}\\ \qquad \alpha(\Delta T)_{,xx}+\alpha(\Delta T)_{,yy}\end{array}\right.$$

equilibrium $\qquad 2\sigma_{xy,xy}+\sigma_{xx,xx}+\sigma_{yy,yy}=-\rho q_{x,x}-\rho q_{y,y}$

Beltrami-Mitchell equation for thermal loading

$$(k+2q_p)\sigma_{xx,xx}+(k+2q_p)\sigma_{yy,yy}+2a_p\sigma_{xx,yy}+2b_p\sigma_{yy,xx}=-k\rho(q_{x,x}+q_{y,y})-2\alpha\left\{(\Delta T)_{xx}+(\Delta T)_{yy}\right\}$$

8.4.3 Airy stress function method

In the stress function method an Airy stress function ψ is introduced and the stress tensor is related to it in such a way that the tensor obeys the equilibrium equations

$$\vec{\nabla}\cdot\boldsymbol{\sigma}^c=\vec{0}$$

Using Hooke's law, the strain tensor can be expressed in the Airy function. Substitution of this $\boldsymbol{\varepsilon}(\psi)$ relation in one of the compatibility equations results in a partial differential equation for the Airy function, which can be solved with the proper boundary conditions.

In a Cartesian coordinate system the vector/tensor formulation can be replaced by index notation.

The material compliance tensor is : $\qquad {}^4\boldsymbol{S}=-\dfrac{\nu}{E}\boldsymbol{I}\boldsymbol{I}+\dfrac{1+\nu}{E}\,{}^4\boldsymbol{I}^s$

$$\left.\begin{array}{l}\text{Airy stress function : } \psi(\vec{x}) \\ \boldsymbol{\sigma} = -\,\vec{\nabla}(\vec{\nabla}\psi) + \left(\nabla^2\psi\right)\boldsymbol{I} \\ \boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \boldsymbol{\sigma}\end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l}\boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \left[-\vec{\nabla}(\vec{\nabla}\psi) + (\nabla^2\psi)\boldsymbol{I}\right] \\ \nabla^2\left(\mathrm{tr}(\boldsymbol{\varepsilon})\right) - \vec{\nabla}\cdot\left(\vec{\nabla}\cdot\boldsymbol{\varepsilon}\right)^c = 0\end{array}\right\}$$

$$\nabla^2(\nabla^2\psi) = \nabla^4\psi = 0 \quad \rightarrow \quad \psi \quad \rightarrow \quad \boldsymbol{\sigma} \quad \rightarrow \quad \boldsymbol{\varepsilon}$$

Cartesian index notation

$$\left.\begin{array}{l}\text{Airy stress function : } \psi(x_i) \\ \sigma_{ij} = -\psi_{,ij} + \delta_{ij}\psi_{,kk} \\ \varepsilon_{ij} = S_{ijlk}\,\sigma_{kl}\end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l}\varepsilon_{ij} = S_{ijlk}\,(-\,\psi_{,kl} + \delta_{kl}\psi_{,mm}) \\ \varepsilon_{ii,jj} - \varepsilon_{ij,ij} = 0\end{array}\right\}$$

$$\psi_{,iijj} = 0 \quad \rightarrow \quad \psi \quad \rightarrow \quad \sigma_{ij} \quad \rightarrow \quad \varepsilon_{ij}$$

Planar, Cartesian

The stress function method is elaborated for planar deformation in a Cartesian coordinate system.

$$\left.\begin{array}{l}\sigma_{xx} = -\psi_{,xx} + \delta_{xx}(\psi_{,xx} + \psi_{,yy}) = \psi_{,yy} \\ \sigma_{yy} = -\psi_{,yy} + \delta_{yy}(\psi_{,xx} + \psi_{,yy}) = \psi_{,xx} \\ \sigma_{xy} = -\psi_{,xy} \\ \\ \varepsilon_{xx} = a_p\sigma_{xx} + q_p\varepsilon_{yy} \\ \varepsilon_{yy} = q_p\sigma_{xx} + b_p\sigma_{yy} \\ \varepsilon_{xy} = \frac{1}{2}k\sigma_{xy}\end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l}\varepsilon_{xx} = a_p\psi_{,yy} + q_p\psi_{,xx} \\ \varepsilon_{yy} = q_p\psi_{,yy} + b_p\psi_{,xx} \\ \varepsilon_{xy} = -\frac{1}{2}k\psi_{,xy} \\ \\ \varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\varepsilon_{xy,xy}\end{array}\right\}$$

$$b_p\psi_{,xxxx} + a_p\psi_{,yyyy} + (2q_p + k)\psi_{,xxyy} = 0$$

isotropic $\qquad a_p = b_p = \dfrac{1}{E} \quad ; \quad q_p = \dfrac{-\nu}{E} \quad ; \quad k = \dfrac{2(1+\nu)}{E} \quad \rightarrow$

bi-harmonic equation $\qquad \psi_{,xxxx} + \psi_{,yyyy} + 2\psi_{,xxyy} = 0$

Planar, cylindrical

In a cylindrical coordinate system, the bi-harmonic equation can be derived by transformation.

gradient and Laplace operator

$$\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \quad \rightarrow \text{2D} \rightarrow$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

bi-harmonic equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right)=0$$

stress components

$$\begin{aligned}\sigma_{rr}&=\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \qquad ; \qquad \sigma_{tt}=\frac{\partial^2\psi}{\partial r^2}\\ \sigma_{rt}&=\frac{1}{r^2}\frac{\partial\psi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\end{aligned}$$

8.5 Weighted residual formulation for 3D deformation

For an approximation, the equilibrium equation is not satisfied exactly in each material point. The error can be "smeared out" over the material volume, using a *weighting function* $\vec{w}(\vec{x})$.

equilibrium equation	$\vec{\nabla}\cdot\boldsymbol{\sigma}^c+\rho\vec{q}=\vec{0} \qquad \forall\ \vec{x}\in V$
approximation $\rightarrow$ residual	$\vec{\nabla}\cdot\boldsymbol{\sigma}^{c^*}+\rho\vec{q}=\vec{\Delta}(\vec{x})\neq\vec{0} \qquad \forall\ \vec{x}\in V$
weighted error is "smeared out"	$\int\limits_V \vec{w}(\vec{x})\cdot\vec{\Delta}(\vec{x})\,dV$

When the weighted residual integral is satisfied for each allowable weighting function $\vec{w}$, the equilibrium equation is satisfied in each point of the material.

$$\int\limits_V \vec{w}\cdot\left[\vec{\nabla}\cdot\boldsymbol{\sigma}^c+\rho\vec{q}\right]dV=0 \qquad \forall\quad \vec{w}(\vec{x}) \quad\leftrightarrow\quad \vec{\nabla}\cdot\boldsymbol{\sigma}^c+\rho\vec{q}=\vec{0} \qquad \forall\ \vec{x}\in V$$

In the weighted residual integral, one term contains the divergence of the stress tensor. This means that the integral can only be evaluated, when the derivatives of the stresses are continuous over the domain of integration. This requirement can be relaxed by applying partial integration to the term with the stress divergence. The result is the so-called weak formulation of the weighted residual integral.

Gauss theorem is used to transfer the volume integral with the term $\vec{\nabla}.(\ )$ to a surface integral. Also $\vec{p}=\boldsymbol{\sigma}\cdot\vec{n}=\vec{n}\cdot\boldsymbol{\sigma}^c$ and $\boldsymbol{\sigma}=\boldsymbol{\sigma}^c$ is used.

$$\left.\begin{aligned}&\int\limits_V \vec{w}\cdot\left[\vec{\nabla}\cdot\boldsymbol{\sigma}^c+\rho\vec{q}\right]dV=0\\ &\vec{\nabla}\cdot(\boldsymbol{\sigma}^c\cdot\vec{w})=(\vec{\nabla}\vec{w})^c:\boldsymbol{\sigma}^c+\vec{w}\cdot(\vec{\nabla}\cdot\boldsymbol{\sigma}^c)\end{aligned}\right\}\quad\rightarrow$$

$$\int_V \left[\vec{\nabla} \cdot (\boldsymbol{\sigma}^c \cdot \vec{w}) - (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma}^c + \vec{w} \cdot \rho\vec{q} \right] dV = 0 \qquad \forall\, \vec{w}$$

$$\int_V (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma}^c \, dV = \int_V \vec{w} \cdot \rho\vec{q}\, dV + \int_A \vec{n} \cdot \boldsymbol{\sigma}^c \cdot \vec{w}\, dA \qquad \forall\, \vec{w}$$

$$\int_V (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma}\, dV = \int_V \vec{w} \cdot \rho\vec{q}\, dV + \int_A \vec{w} \cdot \vec{p}\, dA \qquad \forall\, \vec{w}$$

8.5.1 Weighted residual formulation for linear deformation

When deformation and rotations are small, the deformation is geometrically linear. The deformed state is almost equal to the undeformed state.

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^c : \boldsymbol{\sigma}\, dV_0 = \int_{V_0} \vec{w} \cdot \rho\vec{q}\, dV_0 + \int_{A_0} \vec{w} \cdot \vec{p}\, dA_0 \qquad \forall\, \vec{w}$$

The material behavior is described by Hooke's law, which can be substituted in the weighted residual integral, according to the displacement solution method.

$$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} = {}^4\boldsymbol{C} : \tfrac{1}{2} \left\{ (\vec{\nabla}_0\vec{u}) + (\vec{\nabla}_0\vec{u})^c \right\} = {}^4\boldsymbol{C} : (\vec{\nabla}_0\vec{u})$$

The weighted residual integral is now completely expressed in the displacement $\vec{u}$. Approximate solutions can be determined with the finite element method.

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^c : {}^4\boldsymbol{C} : (\vec{\nabla}_0\vec{u})\, dV_0 = \int_{V_0} \vec{w} \cdot \rho\vec{q}\, dV_0 + \int_{A_0} \vec{w} \cdot \vec{p}\, dA_0 \qquad \forall\, \vec{w}$$

8.6 Finite element method for 3D deformation

Discretisation

The integral over the volume V is written as a sum of integrals over smaller volumes, which collectively constitute the whole volume. Such a small volume V^e is called an element. Subdividing the volume implies that also the surface with area A is subdivided in element surfaces (faces) with area A^e.

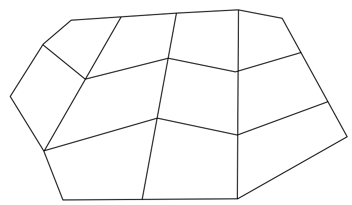


Fig. 8.3 : *Finite element discretisation*

$$\sum_e \int\limits_{V^e} (\vec{\nabla}\vec{w})^c : {}^4\boldsymbol{C} : (\vec{\nabla}\vec{u})\, dV^e = \sum_e \int\limits_{V^e} \vec{w} \cdot \rho\vec{q}\, dV^e + \sum_{e_A} \int\limits_{A^e} \vec{w} \cdot \vec{p}\, dA^e \qquad \forall\ \vec{w}$$

Isoparametric elements

Each point of a three-dimensional element can be identified with three local coordinates $\{\xi_1, \xi_2, \xi_3\}$. In two dimensions we need two and in one dimension only one local coordinate.

The real geometry of the element can be considered to be the result of a deformation from the original cubic, square or line element with (side) length 2. The deformation can be described with a deformation matrix, which is called the *Jacobian matrix* $\underline{J}$. The determinant of this matrix relates two infinitesimal volumes, areas or lengths of both element representations.

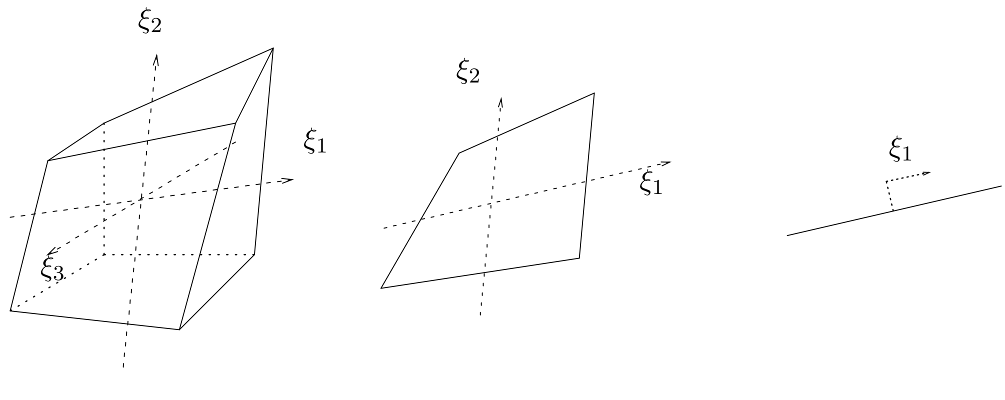


Fig. 8.4 : *Isoparametric elements*

isoparametric (local) coordinates	$(\xi_1, \xi_2, \xi_3) \quad ; \quad -1 \leq \xi_i \leq 1 \quad i = 1, 2, 3$
Jacobian matrix	$\underline{J} = \left(\nabla_{\underset{\sim}{\xi}} \underset{\sim}{x}^T\right)^T \quad ; \quad dV^e = \det(\underline{J})\, d\xi_1 d\xi_2 d\xi_3$

Interpolation

The value of the unknown quantity – here the displacement vector $\vec{u}$ – in an arbitrary point of the element, can be interpolated between the values of that quantity in certain fixed points of the element : the *element nodes. Interpolation functions* ψ are a function of the isoparametric coordinates.

The components of the vector $\vec{u}$ are stored in a column $\underset{\sim}{u}$. The nodal displacement components are stored in the column $\underset{\sim}{u}^e$. The position $\vec{x}$ of a point within the element is interpolated between the nodal point positions, the components of which are stored in the column $\underset{\sim}{x}^e$. Generally, that the interpolations for position and displacement are chosen to be the same.

Besides $\vec{u}$ and $\vec{x}$, the weighting function $\vec{w}$ also needs to be interpolated between nodal values. When this interpolation is the same as that for the displacement, the so-called *Galerkin* procedure is followed, which is generally the case for simple elements, considered here.

We consider the vector function $\vec{a}$ to be interpolated, where *nep* is the number of element nodes. Components a_i of $\vec{a}$ w.r.t. a global vector base, can then also be interpolated.

$$
\begin{aligned}
\vec{a} &= \psi^1 \vec{a}^1 + \psi^2 \vec{a}^2 + \cdots + \psi^{nep} \vec{a}^{nep} = \underset{\sim}{\psi}^T \underset{\sim}{\vec{a}}^e \quad \rightarrow \\
a_i &= \psi^1 a_i^1 + \psi^2 a_i^2 + \cdots + \psi^{nep} a_i^{nep} = \sum_{\alpha=1}^{nep} \psi^\alpha a_i^\alpha = \underset{\sim}{\psi}^T \underset{\sim}{a}_i^e \qquad \rightarrow \qquad \underset{\sim}{a} = \underline{\Psi}\, \underset{\sim}{a}^e
\end{aligned}
$$

The gradient of the vector function $\vec{a}$ also has to be elaborated. The gradient is referred to as the second-order tensor $\boldsymbol{L}^c$, which can be written in components w.r.t. a vector basis. The components are stored in a column $\underset{\approx}{L}$. This column can be written as the product of the so-called *B-matrix*, which contains the derivatives of the interpolation functions, and the column with nodal components of $\vec{a}$.

$$
\boldsymbol{L}^c = \left(\vec{\nabla} \vec{a} \right) \quad \rightarrow \quad \underline{L}^T = \underset{\sim}{\nabla} \underset{\sim}{a}^T + \underline{h} \quad \rightarrow \quad \underset{\approx}{L}_t = \underline{B}\, \underset{\sim}{a}^e
$$

Weighted residual integral

Interpolations for both the displacement and the weighting function and their respective derivatives are substituted in the weighted residual integrals of each element.

$$
f_i^e = \int_{V^e} (\vec{\nabla} \vec{w})^c : {}^4\boldsymbol{C} : (\vec{\nabla} \vec{u})\, dV^e \quad ; \quad f_e^e = \int_{V^e} \vec{w} \cdot \rho \vec{q}\, dV^e + \int_{A^e} \vec{w} \cdot \vec{p}\, dA^e
$$

$$
f_i^e = \underset{\sim}{w}^{eT} \left[\int_{V^e} \underline{B}^T \underline{\underline{C}}\, \underline{B}\, dV^e \right] \underset{\sim}{u}^e \quad ; \quad f_e^e = \underset{\sim}{w}^{eT} \left[\int_{V^e} \underline{\Psi}^T \rho \underset{\sim}{q}\, dV^e \right] + \underset{\sim}{w}^{eT} \left[\int_{A^e} \underline{\Psi}^T \underset{\sim}{p}\, dA^e \right]
$$

The volume integral in f_i^e is the element stiffness matrix $\underline{K}^e$. The integrals in f_e^e represent the external load and are summarized in the column $\underset{\sim}{f}_e^e$.

$$
f_i^e = \underset{\sim}{w}^{eT} \underline{K}^e\, \underset{\sim}{u}^e \qquad ; \qquad f_e^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}_e^e
$$

Integration

Calculating the element stiffness matrix $\underline{K}^e$ and the external loads $\underset{\sim}{f}{}_e^e$ implies the evaluation of an integral over the element volume V^e and the element surface A^e. This integration is done numerically, using a fixed set of *nip* Gauss-points, which have s specific location in the element. The value of the integrand is calculated in each Gauss-point and multiplied with a Gauss-point-specific weighting factor c^{ip} and added.

$$\int\limits_{V^e} g(x_1, x_2, x_3)\, dV^e = \int\limits_{\xi_1=-1}^{1} \int\limits_{\xi_2=-1}^{1} \int\limits_{\xi_3=-1}^{1} f(\xi_1, \xi_2, \xi_3)\, d\xi_1 d\xi_2 d\xi_3 = \sum_{ip=1}^{nip} c^{ip}\, f(\xi_1^{ip}, \xi_2^{ip}, \xi_3^{ip})$$

Assembling

The weighted residual contribution of all elements have to be collected into the total weighted residual integral. This means that all elements are connected or assembled. This assembling is an administrative procedure. All the element matrices and columns are placed at appropriate locations into the structural or global stiffness matrix $\underline{K}$ and the load column $\underset{\sim}{f}{}_e$.

Because the resulting equation has to be satisfied for all $\underset{\sim}{w}$, the nodal displacements $\underset{\sim}{u}$ have to satisfy a set of equations.

$$\begin{aligned}
&\sum_e \underset{\sim}{w}^{eT} \underline{K}^e\, \underset{\sim}{u}^e = \sum_e \underset{\sim}{w}^{eT} \underset{\sim}{f}{}_e^e \quad && \rightarrow \\
&\underset{\sim}{w}^T \underline{K}\, \underset{\sim}{u} = \underset{\sim}{w}^T \underset{\sim}{f}{}_e \qquad \forall\ \underset{\sim}{w} && \rightarrow \\
&\underline{K}\, \underset{\sim}{u} = \underset{\sim}{f}{}_e &&
\end{aligned}$$

Boundary conditions

The initial governing equations were differential equations, which obviously need boundary conditions to arrive at a unique solution. The boundary conditions are prescribed displacements or forces in certain material points. After finite element discretisation, displacements and forces can be applied in nodal points.

The set of nodal equations $\underline{K}\underset{\sim}{u} = \underset{\sim}{f}{}_e$ cannot be solved yet, because the structural stiffness matrix $\underline{K}$ is singular and cannot be inverted. First some essential boundary conditions must be applied, which prevent the rigid body motion of the material and renders the equations solvable.

Chapter 9

Analytical solutions

In the following sections we present various problems, which have an analytical solution. The equations are presented and the solution is given without extensive derivations. Many problems involve the calculation of integration constants from boundary conditions. For such problems these integration constants can be found in appendix E. Examples with numerical values for parameters, are presented. More examples can be found in the above-mentioned appendix.

9.1 Cartesian, planar

For planar problems in a Cartesian coordinate system, two partial differential equations for the displacement components u_x and u_y, the so-called Navier equations, have to be solved, using specific boundary conditions. Only for very simply cases, this can be done analytically. For practical problems, approximate solutions have to be determined with numerical solution procedures. The Navier equations have been derived in section 8.3.1 and are repeated below for the static case, where no material acceleration is considered. The material parameters A_p, B_p, Q_p and K have to be specified for plane stress or plane strain and for the material model concerned (see section 5.4.1 and appendix A.

$$\boxed{\begin{aligned} &A_p u_{x,xx} + K u_{x,yy} + (Q_p + K) u_{y,yx} + \rho q_x = 0 \\ &K u_{y,xx} + B_p u_{y,yy} + (Q_p + K) u_{x,xy} + \rho q_y = 0 \end{aligned}}$$

9.1.1 Tensile test

When a square plate (length a) of homogeneous material is loaded uniaxially by a uniform tensile edge load p, this load constitutes an equilibrium system, i.e. the stresses satisfy the equilibrium equations : $\sigma_{xx} = p$ and $\sigma_{yy} = \sigma_{xy} = \sigma_{zz} = 0$. The deformation can be calculated directly from Hooke's law.

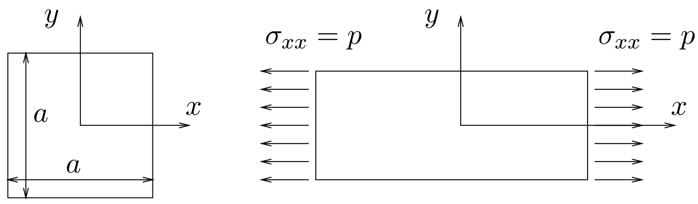


Fig. 9.1 : *Uniaxial tensile test*

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} = \frac{p}{E} \quad \rightarrow \quad u_x = \frac{p}{E}x + c \quad ; \quad u_x(x=0) = 0 \rightarrow c = 0$$
$$u_x = \frac{p}{E}x \quad \rightarrow \quad u_x(x=a) = \frac{p}{E}a$$
$$\varepsilon_{yy} = -\nu\varepsilon_{xx} = -\nu\frac{p}{E} \quad \rightarrow \quad u_y = -\nu\frac{p}{E}y + c \quad ; \quad u_y(y=0) = 0 \quad \rightarrow \quad c = 0$$
$$u_y = -\nu\frac{p}{E}y \quad ; \quad u_y(y=a/2) = -\nu\frac{p}{E}\frac{a}{2}$$

9.1.2 Orthotropic plate

A square plate is loaded in its plane so that a plane stress state can be assumed with $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. The plate material is a "matrix" in which long fibers are embedded, which have all the same orientation along the direction indicated as 1 in the "material" $1,2$-coordinate system. Both matrix and fibers are linearly elastic. The volume fraction of the fibers is V. The angle between the 1-direction and the x-axis is α. In the $1,2$-coordinate system the material behavior for plane stress is given by the orthotropic material law, which is found in appendix A.

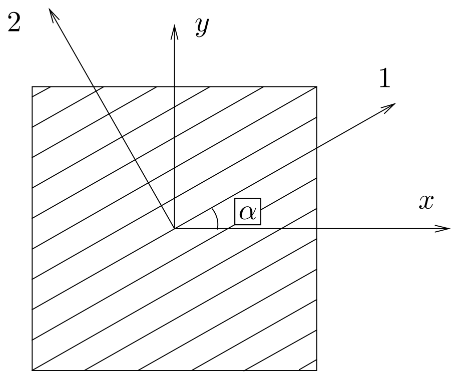


Fig. 9.2 : *Orthotropic plate*

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{1-\nu_{12}\nu_{21}} \begin{bmatrix} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & (1-\nu_{12}\nu_{21})G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \quad \rightarrow \quad \underset{\approx}{\sigma}^* = \underline{\underline{C}}^* \underset{\approx}{\varepsilon}^*$$

In appendix B the transformation of matrix components due to a rotation of the coordinate axes is described for the three-dimensional case. For planar deformation, the anticlockwise rotation is only about the global z-axis. For stress and strain components, stored in columns $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$, respectively, the transformation is described by transformation matrices $\underline{T}_\sigma$ for stress and $\underline{T}_\varepsilon$ for strain. The components of these matrices are cosine (c) and sine (s) functions of the rotation angle α, which is positive for an anti-clockwise rotation about the z-axis.

$$\underset{\sim}{\sigma}^* = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{12} \end{bmatrix}^T \qquad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}^T \qquad \underset{\sim}{\sigma}^* = \underline{T}_\sigma \, \underset{\sim}{\sigma}$$

$$\underset{\sim}{\varepsilon}^* = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \gamma_{12} \end{bmatrix}^T \qquad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix}^T \qquad \underset{\sim}{\varepsilon}^* = \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon}$$

$$\underline{T}_\sigma = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \qquad \underline{T}_\varepsilon = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix}$$

$$\underline{T}_\sigma^{-1} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \qquad \underline{T}_\varepsilon^{-1} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{bmatrix}$$

The properties in the material coordinate system are known. The stress-strain relations in the global coordinate system can than be calculated.

$$\underset{\sim}{\sigma}^* = \underline{\underline{C}}^* \, \underset{\sim}{\varepsilon}^* \quad \rightarrow \quad \underline{T}_\sigma \underset{\sim}{\sigma} = \underline{\underline{C}}^* \, \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon} \quad \rightarrow \quad \underset{\sim}{\sigma} = \underline{T}_\sigma^{-1} \, \underline{\underline{C}}^* \, \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon} = \underline{\underline{C}} \, \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon}^* = \underline{\underline{S}}^* \, \underset{\sim}{\sigma}^* \quad \rightarrow \quad \underline{T}_\varepsilon \underset{\sim}{\varepsilon} = \underline{\underline{S}}^* \, \underline{T}_\sigma \, \underset{\sim}{\sigma} \quad \rightarrow \quad \underset{\sim}{\varepsilon} = \underline{T}_\varepsilon^{-1} \, \underline{\underline{S}}^* \, \underline{T}_\sigma \, \underset{\sim}{\sigma} = \underline{\underline{S}} \, \underset{\sim}{\sigma}$$

Example : stiffness of an orthotropic plate

The material parameters in the material $1,2$-coordinate system are known from experiments :

$$E_1 = 100 \text{ N/mm}^2 \; ; \; E_2 = 20 \text{ N/mm}^2 \; ; \; G_{12} = 50 \text{ N/mm}^2 \; ; \; \nu_{12} = 0.4$$

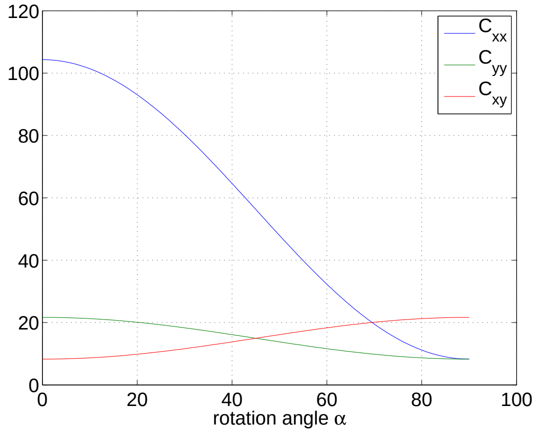
Due to symmety of the stiffness matrix (and of course the compliance matrix), the second Poisson ratio can be calculated :

$$\nu_{12} E_2 = \nu_{21} E_1 \quad \rightarrow \quad \nu_{21} = \nu_{12} \frac{E_2}{E_1} = (0.4) * (20/100) = 0.08$$

The stiffness matrix in the material coordinate system can than be calculated. When the material coordinate system is rotated over $\alpha = 20^o$ anti-clockwise w.r.t. the global x-axis, the transformation matrices can be generated and used to calculate the stiffness matrix w.r.t. the global axes.

$$\underline{\underline{C}}^* = \begin{bmatrix} 103.3058 & 8.2645 & 0 \\ 8.2645 & 20.6612 & 0 \\ 0 & 0 & 50.0000 \end{bmatrix} \quad ; \quad \underline{\underline{C}} = \begin{bmatrix} 93.0711 & 9.8316 & -31.8180 \\ 19.4992 & 20.0940 & 31.8180 \\ 29.9029 & -3.3414 & 39.0683 \end{bmatrix}$$

We can also concentrate on components of $\underline{\underline{C}}$ and investigate how they changes, when the rotation angle α varies within a certain range. The next plot shows C_{xx}, C_{yy} and C_{xy} as a function of α.



9.2 Axi-symmetric, planar, $u_t = 0$

The differential equation for the radial displacement u_r is derived in chapter 8 by substitution of the stress-strain relation (material law) and the strain-displacement relation in the equilibrium equation w.r.t. the radial direction. It is repeated here for orthotropic material behavior with isotropic thermal expansion. Material parameters A_p and Q_p have to be specified for plane stress and plane strain and can be found in appendix A.

$$\boxed{u_{r,rr} + \frac{1}{r}\,u_{r,r} - \zeta^2\,\frac{1}{r^2}\,u_r = f(r)}$$

$$\text{with} \qquad \zeta = \sqrt{\frac{B_p}{A_p}}$$

$$\text{and} \qquad f(r) = \frac{\rho}{A_p}\,(\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p}\,\alpha(\Delta T)_{,r} + \frac{\Theta_{p1} - \Theta_{p2}}{A_p}\,\frac{1}{r}\,\alpha\Delta T$$

A general solution for the differential equation can be determined as the addition of the homogeneous solution $\hat{u}_r$ and the particulate solution $\bar{u}_r$, which depends on the specific loading $f(r)$. From the general solution the radial and tangential strains can be calculated according to their definitions.

$$\begin{aligned}
&\hat{u}_r = r^\lambda \quad\rightarrow\quad \hat{u}_{r,r} = \lambda\,r^{\lambda-1} \quad\rightarrow\quad \hat{u}_{r,rr} = \lambda(\lambda-1)\,r^{\lambda-2} \quad\rightarrow \\
&\left[\lambda(\lambda-1) + \lambda - \zeta^2\right] r^{\lambda-2} = 0 \quad\rightarrow \\
&\lambda^2 = \zeta^2 \quad\rightarrow\quad \lambda = \pm\zeta \quad\rightarrow\quad \hat{u}_r = c_1\,r^\zeta + c_2 r^{-\zeta} \\
&u_r = c_1\,r^\zeta + c_2 r^{-\zeta} + \bar{u}_r
\end{aligned}$$

The general solution for radial displacement, strains and stresses is presented here.

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} + \bar{u}_r$

$$\varepsilon_{rr} = c_1 \zeta r^{\zeta-1} - c_2 \zeta r^{-\zeta-1} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1} + \frac{\bar{u}_r}{r}$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} + A_p \bar{u}_{r,r} + Q_p \frac{\bar{u}_r}{r} - \Theta_{p1}\alpha\Delta T$$

$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} + Q_p \bar{u}_{r,r} + B_p \frac{\bar{u}_r}{r} - \Theta_{p2}\alpha\Delta T$$

For isotropic material the relations can be simplified, as in that case we have $A_p = B_p$ and thus $\zeta = 1$.

general solution $\quad u_r = c_1 r + \frac{c_2}{r} + \bar{u}_r$

$$\varepsilon_{rr} = c_1 - c_2 r^{-2} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2} + \frac{\bar{u}_r}{r}$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} + A_p \bar{u}_{r,r} + Q_p \frac{\bar{u}_r}{r} - \Theta_{p1}\alpha\Delta T$$

$$\sigma_{tt} = (Q_p + A_p)c_1 - (Q_p - A_p)\frac{c_2}{r^2} + Q_p \bar{u}_{r,r} + A_p \frac{\bar{u}_r}{r} - \Theta_{p1}\alpha\Delta T$$

When there is no right-hand loading term $f(r)$ in the differential equation, the particulate part $\bar{u}_r$ will be zero. Then, for isotropic material, the radial and tangential strains are uniform, i.e. no function of the radius r. For a state of plane stress, the axial strain is calculated as a weighted summation of the in-plane strains, so also ε_{zz} will be uniform (see section 5.4.1). The thicknesss of the axi-symmetric object will remain uniform. For non-isotropic material behavior this is not the case, however.

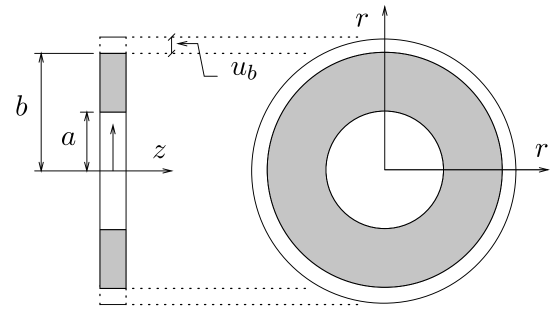
Loading and boundary conditions

In the following subsections, different geometries and loading conditions will be considered. The external load determines the right-hand side $f(r)$ of the differential equation and as a consequence the particulate part $\bar{u}_r$ of the general solution. Boundary conditions must be used subsequently to determine the integration constants c_1 and c_2. Finally the parameters A_p, B_p and Q_p must be chosen in accordance with the material behavior and specified for plane stress or plane strain (see appendix A).

The algebra, which is involved with these calculations, is not very difficult, but rather cumbersome. In appendix E a number of examples is presented. When numerical values are provided, displacements, strains and stresses can be calculated and plotted with a Matlab program, which is available on the website of this course. Based on the input, it selects the proper formulas for the calculation. Instructions for its use can be found in the program source file. The figures in the next subsections are made with this program.

9.2.1 Prescribed edge displacement

The outer edge of a disc with a central hole is given a prescribed displacement $u(r = b) = u_b$. The inner edge is stress-free. With these boundary conditions, the integration constants in the general solution can be determined. They can be found in appendix E.



$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$u_r(r = b) = u_b$$
$$\sigma_{rr}(r = a) = 0$$
$$c_1, c_2 \; : \; \text{App. E}$$

Fig. 9.3 : *Edge displacement of circular disc*

For the parameter values listed below, the radial displacement u_r and the stresses are calculated and plotted as a function of the radius r.

| $u_b = 0.01$ m | $a = 0.25$ m | $b = 0.5$ m | $h = 0.05$ m | $E = 250$ GPa | $\nu = 0.33$ |

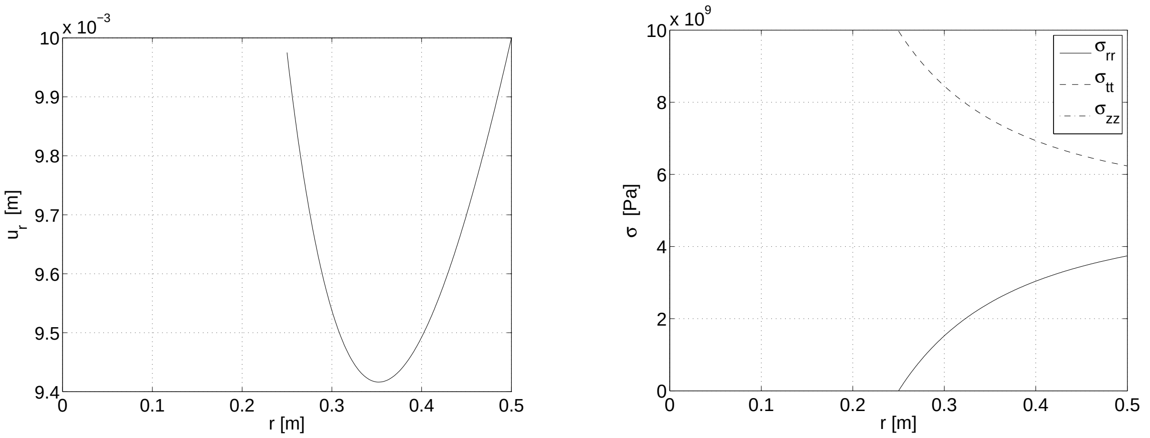
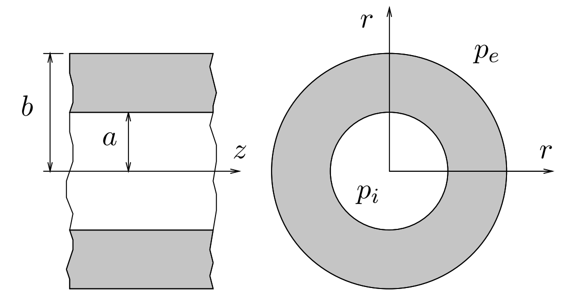


Fig. 9.4 : *Displacement and stresses for plane stress* ($\sigma_{zz} = 0$)

9.2.2 Edge load

A cylinder has inner radius $r = a$ and outer radius $r = b$. It is loaded with an internal (p_i) and/or an external (p_e) pressure.

The general solution to the equilibrium equation has two integration constants, which have to be determined from boundary conditions. In appendix E they are determined for the case that an open cylinder is subjected to an internal pressure p_i and an external pressure p_e. For plane stress ($\sigma_{zz} = 0$) the cylinder is free to deform in axial direction. The solution was first derived by Lamé in 1833 and therefore this solution is referred to as Lamé's equations. When these integration constants for isotropic material are substituted in the stress solution, it appears that the stresses are independent of the material parameters. This implies that radial and tangential stresses are the same for plane stress and plane strain. For the plane strain case, the axial stress σ_{zz} can be calculated directly from the radial and tangential stresses.



$$
\begin{aligned}
& f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0 \\
& \sigma_{rr}(r = a) = -p_i \\
& \sigma_{rr}(r = b) = -p_e \\
& c_1, c_2 \ : \ \text{App. E}
\end{aligned}
$$

Fig. 9.5 : *Cross-section of a thick-walled circular cylinder*

$$
\sigma_{rr} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} - \frac{a^2 b^2 (p_i - p_e)}{b^2 - a^2} \frac{1}{r^2} \quad ; \quad \sigma_{tt} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} + \frac{a^2 b^2 (p_i - p_e)}{b^2 - a^2} \frac{1}{r^2}
$$

The Tresca and Von Mises limit criteria for a pressurized cylinder can be calculated according to their definitions (see chapter 6).

$$
\begin{aligned}
\sigma_{TR} &= 2\tau_{max} = \max \left[\, |\sigma_{rr} - \sigma_{tt}|, |\sigma_{tt} - \sigma_{zz}|, |\sigma_{zz} - \sigma_{rr}| \,\right] \\
\sigma_{VM} &= \sqrt{\tfrac{1}{2} \left\{ (\sigma_{rr} - \sigma_{tt})^2 + (\sigma_{tt} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2 \right\}}
\end{aligned}
$$

An open cylinder is analyzed with the parameters from the table below. Stresses are plotted as a function of the radius.

$p_i = 100$ MPa	$a = 0.25$ m	$b = 0.5$ m	$h = 0.5$ m	$E = 250$ GPa	$\nu = 0.33$

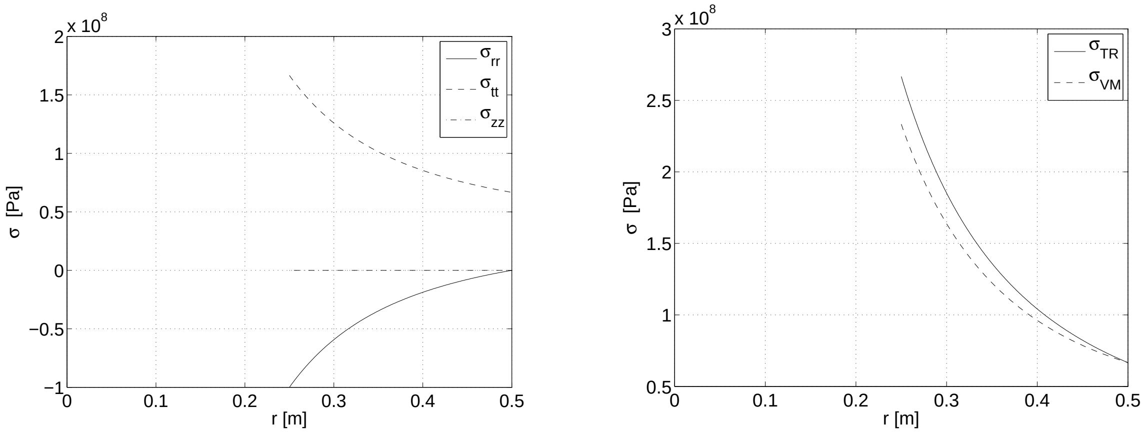


Fig. 9.6 : *Stresses in a thick-walled pressurized cylinder for plane stress ($\sigma_{zz} = 0$)*

That the inner material is under much higher tangential stress than the outer material, can be derived by reasoning, when we only consider an internal pressure. This pressure will result in enlargement of the diameter for each value of r, but it will also compress the material and result in reduction of the wall thickness. The inner diameter will thus increase more than the outer diameter – which is also calculated and plotted in the figure below – and the tangential stress will be much higher at the inner edge.

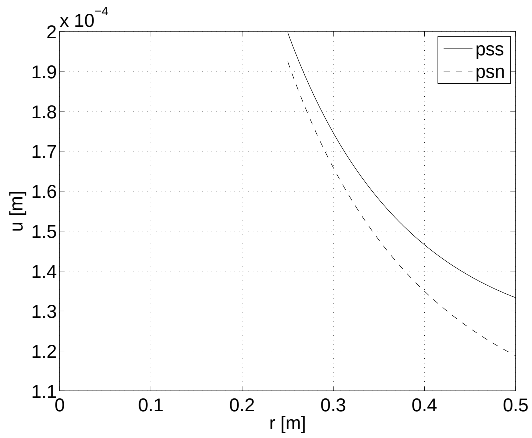


Fig. 9.7 : *Radial displacement in a thick-walled pressurized cylinder for plane stress (pss) and plane strain (psn)*

Closed cylinder

A closed cylinder is loaded in axial direction by the internal and the external pressure. This load leads to an axial stress σ_{zz}, which is uniform over the wall thickness. It can be determined from axial equilibrium and can be considered as an The radial and tangential stress are not influenced by this axial load.

The resulting radial displacement due to the contraction caused by the axial load, u_{ra}, can be calculated from Hooke's law.

$$\text{axial equilibrium} \qquad \sigma_{zz} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} \quad \rightarrow \quad u_{ra} = \varepsilon_{tta} r = -\frac{\nu}{E}\sigma_{zz} r$$

9.2.3 Shrink-fit compound pressurized cylinder

A compound cylinder is assembled of two individual cylinders. Before assembling the outer radius of the inner cylinder b_i is larger than the inner radius of the outer cylinder a_o. The difference between the two radii is the *shrinking allowance* $b_i - a_o$. By applying a pressure at the outer surface of the inner cylinder and at the inner surface of the outer cylinder, a clearance between the two cylinders is created and the cylinders can be assembled. After assembly the pressure is released, the clearance is eliminated and the two cylinders are fitted together.

The cylinders can also be assembled by heating up the outer cylinder to ΔT, due to which it will expand. The radial displacement of the inner radius has to be larger than the shrinking allowance. With α being the coefficient of thermal expansion, this means :

$$\varepsilon_{tt_{\Delta T}}(r = a_o) = u_{r_o}(r = a_o)/a_o = \alpha\Delta T \rightarrow u_{r_o}(r = a_o) = a_o\alpha\Delta T > b_i - a_o$$

After assembly the outer cylinder is cooled down again and the two cylinders are fitted together.

Residual stresses will remain in both cylinders. At the interface between the two cylinders the radial stress is the contact pressure, indicated as p_c. The stresses in both cylinders, loaded with this contact pressure, can be calculated with the Lame's equations.

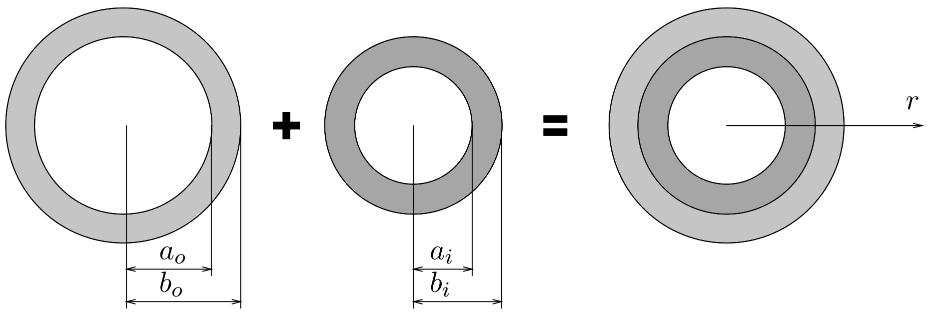


Fig. 9.8 : *Shrink-fit assemblage of circular cylinders*

$$\sigma_{rr_i} = \frac{-p_c b_i^2}{b_i^2 - a_i^2} + \frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2)r^2} \quad ; \quad \sigma_{tt_i} = \frac{-p_c b_i^2}{b_i^2 - a_i^2} - \frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2)r^2}$$

$$\sigma_{rr_o} = \frac{p_c a_o^2}{b_o^2 - a_o^2} - \frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2)r^2} \quad ; \quad \sigma_{tt_o} = \frac{p_c a_o^2}{b_o^2 - a_o^2} + \frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2)r^2}$$

The radial displacement can also be calculated for both cylinders. For plane stress, these relations are shown below. They can be derived with subsequential reference to the pages a26 and a6. The inner and outer radius of the compound cylinder is then known. The *radial interference* δ is the difference the displacemnts at the interface, which is located at radius r_c.

$$u_{r_i}(r = a_i) = -\frac{2}{E}\frac{p_c a_i b_i^2}{b_i^2 - a_i^2} \quad ; \quad u_{r_o}(r = b_o) = \frac{2}{E}\frac{p_c a_o^2 b_o}{b_o^2 - a_o^2}$$

$$u_{r_o}(r = a_o) = \frac{1-\nu}{E}\frac{p_c a_o^2}{b_o^2 - a_o^2}a_o + \frac{1+\nu}{E}\frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2)}\frac{1}{a_o}$$

$$u_{r_i}(r = b_i) = -\frac{1-\nu}{E}\frac{p_c b_i^2}{b_i^2 - a_i^2}b_i - \frac{1+\nu}{E}\frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2)}\frac{1}{b_i}$$

$$r_i = a_i + u_i(r = a_i) \quad ; \quad r_o = b_o + u_o(r = b_o)$$

The location of the contact interface is indicated as r_c. The contact pressure p_c can be solved from the relation for r_c.

$$r_c = b_i + u_{r_i}(r = b_i) = a_o + u_{r_o}(r = a_o) \quad \rightarrow$$

$$p_c = \frac{E(b_i - a_o)(b_o^2 - a_o^2)(b_i^2 - a_i^2)}{a_o(b_i^2 - a_i^2)\{(b_0^2 + a_o^2) + \nu(b_o^2 - a_o^2)\} + b_i(b_o^2 - a_o^2)\{(b_i^2 + a_i^2) - \nu(b_i^2 - a_i^2)\}}$$

For the parameter values listed below, the radial and tangential stresses are calculated and plotted as a function of the radius r.

$a_i = 0.4$ m	$b_i = 0.7$ m	$a_o = 0.699$ m	$b_o = 1$ m	$E = 200$ GPa	$\nu = 0.3$

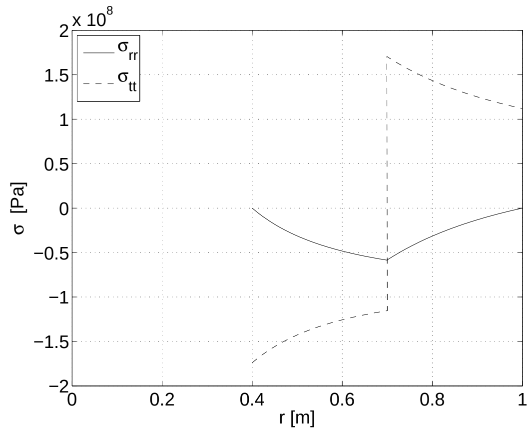


Fig. 9.9 : *Residual stresses in shrink-fit assemblage of two cylinders*

Shrink-fit and service state

After assembly, the compound cylinder is loaded with an internal pressure p_i. The residual stresses from the shrink-fit stage and the stresses due to the internal pressure can be added, based on the superposition principle. The resulting stresses are lower than the stresses due to internal pressure. Compound cylinders can carry large pressures more efficiently.

9.2.4 Circular hole in infinite medium

When a circular hole is located in an infinite medium, we can derive the stresses from Lame's equations by taking $b \rightarrow \infty$. For the general case this leads to limit values of the integration constants c_1 and c_2. These can then be substituted in the general solutions for displacement and stresses.

Pressurized hole in infinite medium

For a pressurized hole in an infinite medium the external pressure p_e is zero. In that case the absolute values of radial and tangential stresses are equal. The radial displacement can also be calculated.

$$b \rightarrow \infty \quad ; \quad p_i = p \quad ; \quad p_e = 0 \qquad \rightarrow \qquad \sigma_{rr} = -\frac{pa^2}{r^2} \quad ; \quad \sigma_{tt} = \frac{pa^2}{r^2}$$

For a plane stress state and with parameter values listed below, the radial displacement and the stresses are calculated and plotted as a function of the radius. Note that we take a large but finite value of for b.

| p_i = 100 MPa | a = 0.2 m | b = 20 m | h = 0.5 m | E = 200 GPa | ν = 0.3 |

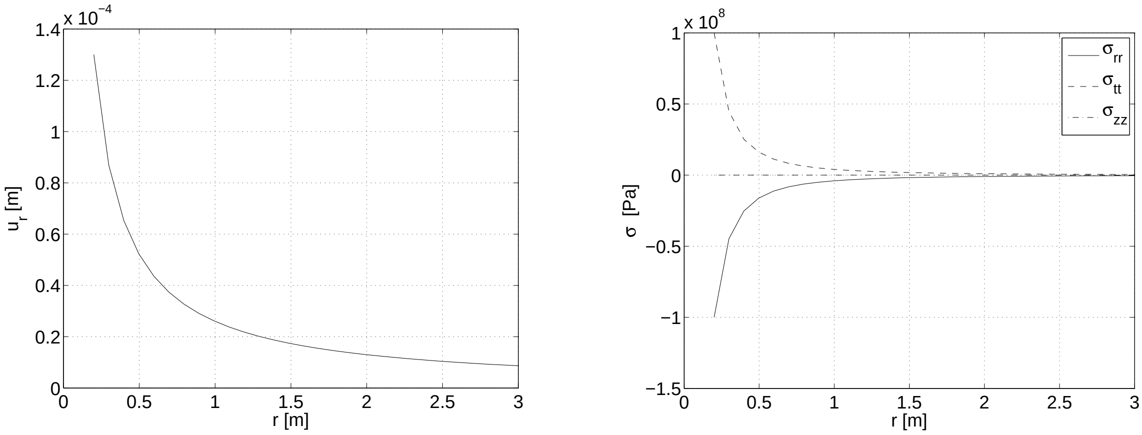


Fig. 9.10 : *Displacement and stresses in a pressurized circular hole in an infinite medium*

Stress-free hole in bi-axially loaded infinite medium

We consider the case of a stress-free hole of radius a in an infinite medium, which is bi-axially loaded at infinity by a uniform load T, equal in x- and y-direction. Because the load is applied at boundaries which are at infinite distance from the hole center, the bi-axial load is equivalent to an externally applied radial edge load $p_e = -T$.

Radial and tangential stresses are different in this case. The tangential stress is maximum for $r = a$ and equals $2T$.

$$b \rightarrow \infty \quad ; \quad p_i = 0 \quad ; \quad p_e = -T \qquad \rightarrow \qquad \sigma_{rr} = T\left(1 - \frac{a^2}{r^2}\right) \quad ; \quad \sigma_{tt} = T\left(1 + \frac{a^2}{r^2}\right)$$

stress concentration factor $\qquad K_t = \dfrac{\sigma_{max}}{T} = \dfrac{\sigma_{tt}(r=a)}{T} = \dfrac{2T}{T} = 2$

For a plane stress state and with parameter values listed below, the radial displacement and the stresses are calculated and plotted as a function of the radius.

| p_e = - 100 MPa | a = 0.2 m | b = 20 m | h = 0.5 m | E = 200 GPa | ν = 0.3 |

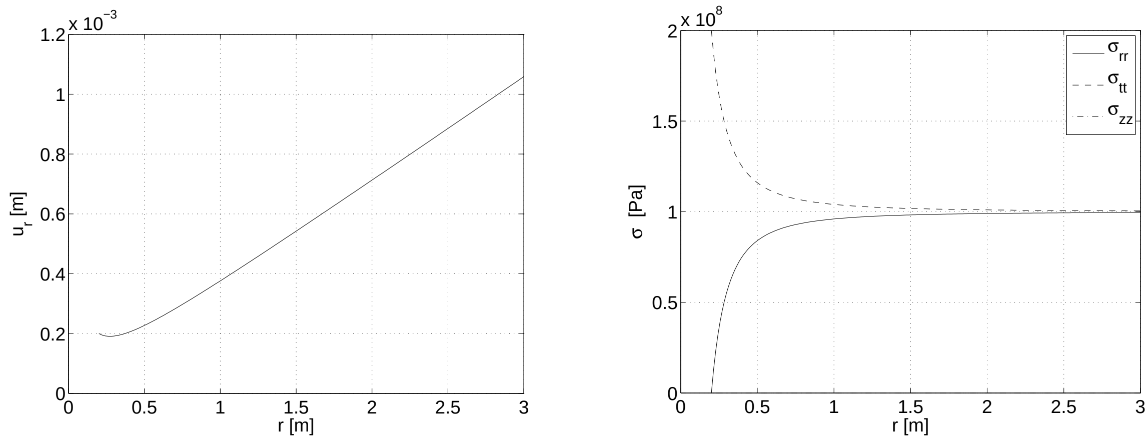


Fig. 9.11 : *Displacement and stresses in a pressurized circular hole in an infinite medium*

9.2.5 Centrifugal load

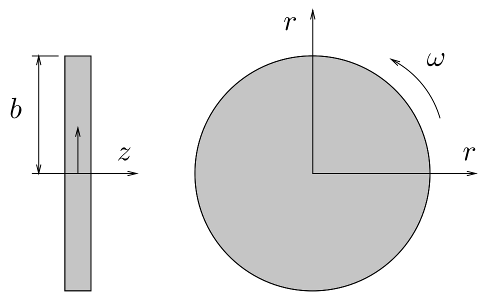
A circular disc, made of isotropic material, rotates with angular velocity ω [rad/s]. The outer radius of the disc is taken to be b. The disc may have a central circular hole with radius a. When $a = 0$ there is no hole and the disc is called "solid". Boundary conditions for a disc with a central hole are rather different than those for a "solid" disc, which results in different solutions for radial displacement and stresses. The external load $f(r)$ is the result of the radial acceleration of the material (see appendix C).

$$\text{external load} \qquad \ddot{u}_r = -\omega^2 r \quad \rightarrow \quad f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$

For orthotropic and isotropic material, the general solution for the radial displacement and the radial and tangential stresses can be calculated.

Solid disc

In a disc without a central hole (solid disc) there are material points at radius $r = 0$. To prevent infinite displacements for $r \rightarrow 0$ the second integration constant c_2 must be zero. At the outer edge the radial stress σ_{rr} must be zero, because this edge is unloaded. With these boundary conditions the integration constants in the general solution can be calculated (see appendix E).



$$u_r(r = 0) \neq \infty$$
$$\sigma_{rr}(r = b) = 0$$
$$c_1, c_2 \;:\; \text{App. E}$$

Fig. 9.12 : *A rotating solid disc*

For a plane stress state and with the listed parameter values, the stresses are calculated and plotted as a function of the radius.

| $\omega = 6$ c/s | $a = 0$ m | $b = 0.5$ m | $t = 0.05$ m | $\rho = 7500$ kg/m^3 | $E = 200$ GPa | $\nu = 0.3$ |

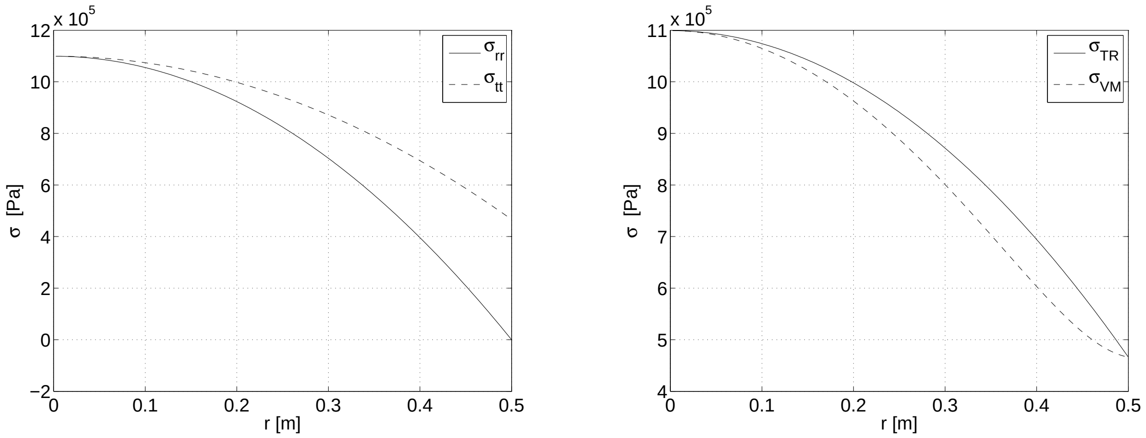


Fig. 9.13 : *Stresses in a rotating solid disc in plane stress*

In a rotating solid disc the radial and tangential stresses are equal in the center of the disc. They both decrease with increasing radius, where of course the radial stress reduces to zero at the outer radius. The equivalent Tresca and Von Mises stresses are not very different and also decrease with increasing radius. In the example the disc is assumed to be in a state of plane stress.

When the same disc is fixed between two rigid plates, a plane strain state must be modelled. In that case the axial stress is not zero. As can be seen in the plots below, the axial stress influences the Tresca and Von Mises equivalent stresses.

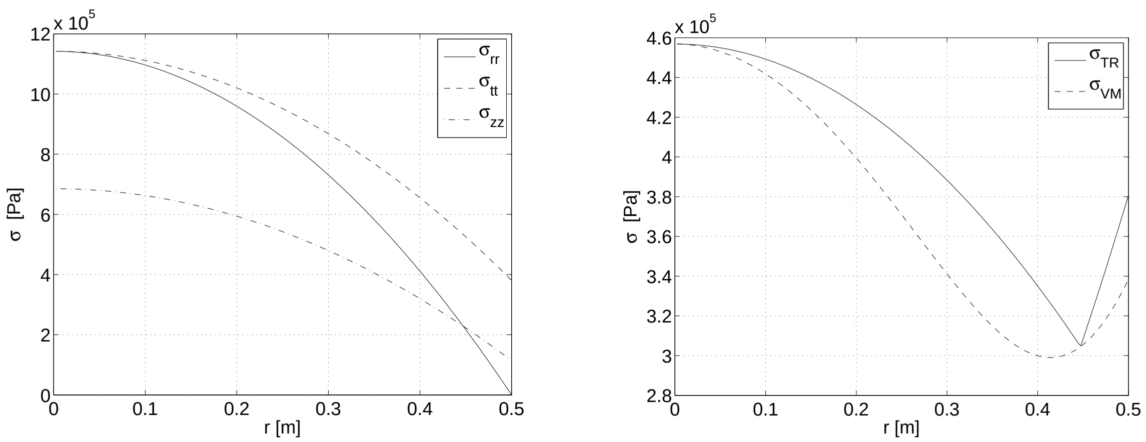
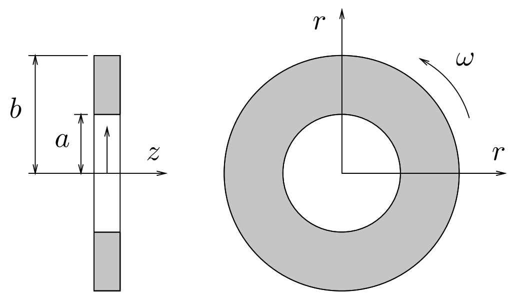


Fig. 9.14 : *Stresses in a rotating solid disc in plane strain*

Disc with central hole

When the disc has a central circular hole, the radial stress at the inner edge and at the outer edge must both be zero, which provides two equations to solve the two integration constants.



$$\sigma_{rr}(r = a) = 0$$
$$\sigma_{rr}(r = b) = 0$$
$$c_1, c_2 \ : \ \text{App. E}$$

Fig. 9.15 : *A rotating disc with central hole*

For a plane stress state and with parameter values listed below, the radial displacement and the stresses are calculated and plotted as a function of the radius.

$$\mid \omega = 6 \text{ c/s} \mid a = 0.2 \text{ m} \mid b = 0.5 \text{ m} \mid t = 0.05 \text{ m} \mid \rho = 7500 \text{ kg/m}^3 \mid E = 200 \text{ GPa} \mid \nu = 0.3 \mid$$

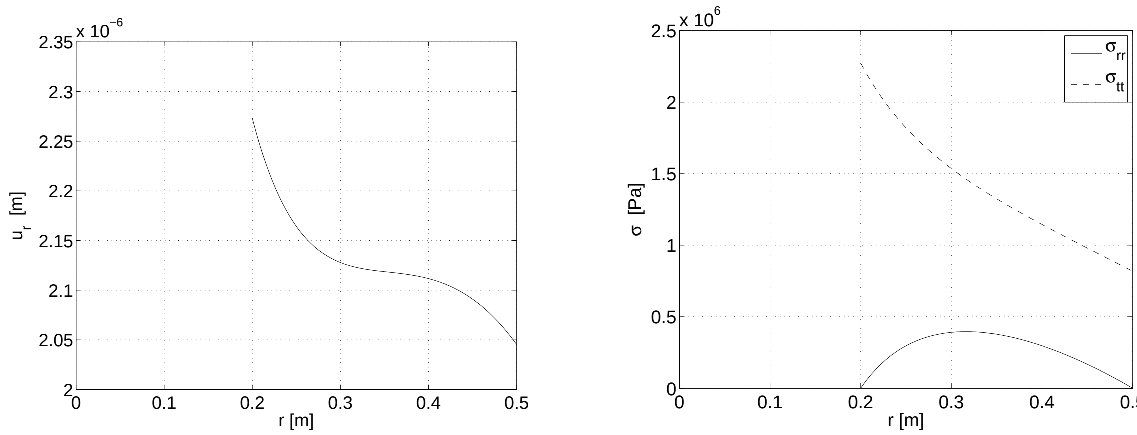
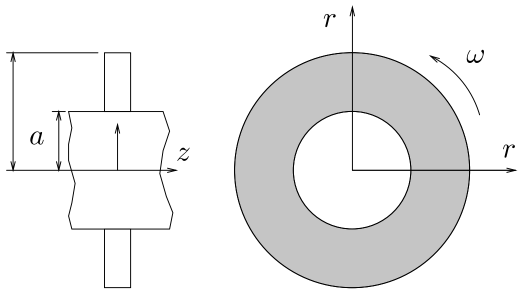


Fig. 9.16 : *Displacement and stresses in a rotating disc with a central hole*

Disc fixed on rigid axis

When the disc is fixed on an axis and the axis is assumed to be rigid, the displacement of the inner edge is suppressed. The radial stress at the outer edge is obviously zero.



$u_r(r = a) = 0$
$\sigma_{rr}(r = b) = 0$
c_1, c_2 : App. E

Fig. 9.17 : *Disc fixed on rigid axis*

For a plane stress state and with parameter values listed below the stresses and tha radial displacement is calculated and plotted.

$$\Big| \ \omega = 6 \text{ c/s} \ \Big| \ a = 0.2 \text{ m} \ \Big| \ b = 0.5 \text{ m} \ \Big| \ t = 0.05 \text{ m} \ \Big| \ \rho = 7500 \text{ kg/m}^3 \ \Big| \ E = 200 \text{ GPa} \ \Big| \ \nu = 0.3 \ \Big|$$

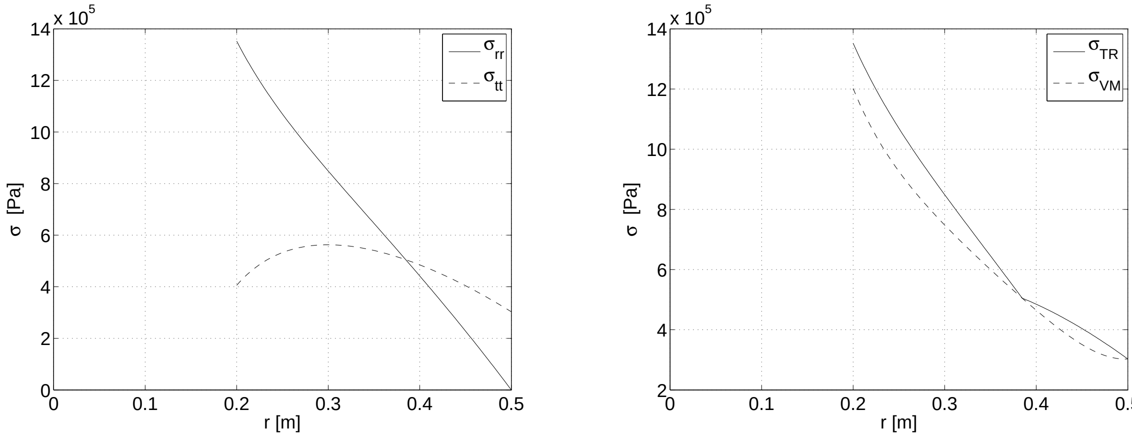


Fig. 9.18 : *Stresses in a rotating disc, fixed on an axis*

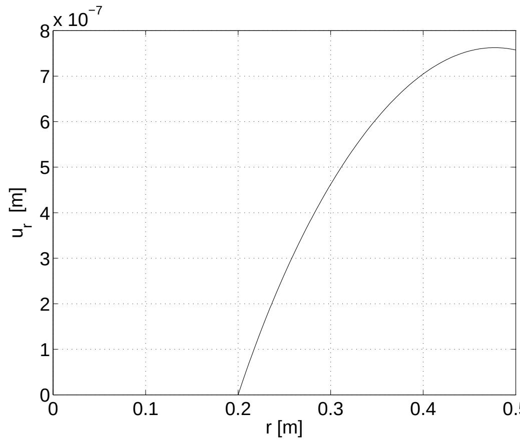


Fig. 9.19 : *Displacement in a rotating disc*

9.2.6 Rotating disc with variable thickness

For a rotating disc with variable thickness $t(r)$ the equation of motion in radial direction can be derived. For a disc with inner and outer radius a and b, respectively, and a thickness distribution $t(r) = \frac{t_a}{2}\frac{a}{r}$, a general solution for the stresses can be derived. The integration constants can be determined from the boundary conditions, e.g. $\sigma_{rr}(r=a) = \sigma_{rr}(r=b) = 0$.

equilibrium $$\frac{\partial(t(r)r\sigma_{rr})}{\partial r} - t(r)\sigma_{tt} = -\rho\omega^2 t(r) r^2 \quad \text{with} \quad t(r) = \frac{t_a}{2}\frac{a}{r}$$

general solution stresses

$$\sigma_{rr} = \frac{2c_1}{at_a}r^{d_1} + \frac{2c_2}{at_a}r^{d_2} - \frac{3+\nu}{5+\nu}\rho\omega^2 r^2 \quad ; \quad \sigma_{tt} = \frac{2c_1}{at_a}d_1 r^{d_1} + \frac{2c_2}{at_a}d_2 r^{d_2} - \frac{1+3\nu}{5+\nu}\rho\omega^2 r^2$$

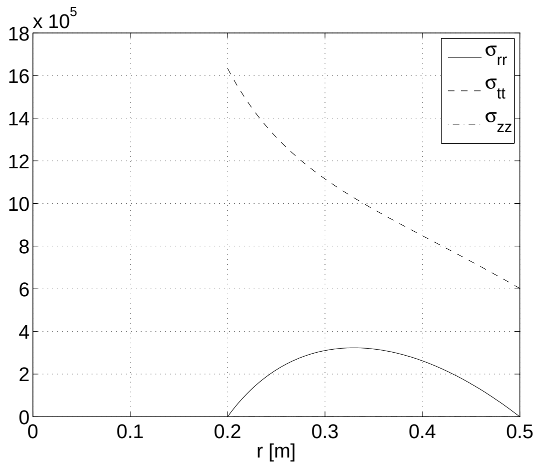
with $$d_1 = -\tfrac{1}{2} + \sqrt{\tfrac{5}{4}+\nu} \quad ; \quad d_2 = -\tfrac{1}{2} - \sqrt{\tfrac{5}{4}+\nu}$$

boundary conditions $\sigma_{rr}(r=a) = \sigma_{rr}(r=b) = 0 \quad \rightarrow$

$$\frac{2c_1}{at_a} = \frac{3+\nu}{5+\nu}\rho\omega^2 a^{-d_1}\left[a^2 - a^{d_2}\left(\frac{b^2 - a^{-d_1}b^{d_1}a^2}{b^{d_2} - a^{d_2}a^{-d_1}b^{d_1}}\right)\right]$$
$$\frac{2c_2}{at_a} = \frac{3+\nu}{5+\nu}\rho\omega^2\left(\frac{b^2 - a^{-d_1}b^{d_1}a^2}{b^{d_2} - a^{d_2}a^{-d_1}b^{d_1}}\right)$$

A disc with a central hole and a variable thickness rotates with an angular velocity of 6 cycles per second. The stresses are plotted as a function of the radius.

isotropic	plane stress	$\omega = 6$ c/s			
$a = 0.2$ m	$b = 0.5$ m	$t_a = 0.05$ m	$\rho = 7500$ kg/m^3	$E = 200$ GPa	$\nu = 0.3$



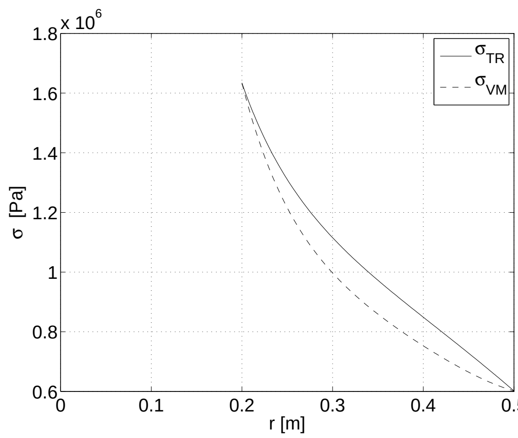


Fig. 9.20 : *Stresses in a rotating disc with variable thickness*

9.2.7 Thermal load

For a disc loaded with a distributed temperature $\Delta T(r)$ the external load $f(r)$ is due to thermal expansion. The coefficient of thermal expansion is α and the general material parameters for planar deformation are A_p and Q_p.

$$f(r) = \frac{\Theta_{p1}}{A_p}\,\alpha(\Delta T)_{,r}$$

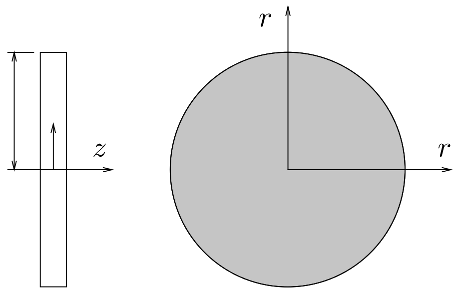
The part $\bar{u}_r$ has to be determined for a specific radial temperature loading. It is assumed here that the temperature is a third order function of the radius r.

$$\Delta T(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 \quad \rightarrow \quad f(r) = \frac{\Theta_{p1}}{A_p}\,\alpha\left(a_1 + 2a_2 r + 3a_3 r^2\right) \quad \rightarrow$$

$$\bar{u}_r(r) = \frac{\Theta_{p1}}{A_p}\,\alpha\left(\tfrac{1}{3}a_1 r^2 + \tfrac{1}{4}a_2 r^3 + \tfrac{1}{5}a_3 r^4\right)$$

Solid disc, free outer edge

As is always the case for a solid disc, the constant c_2 has to be zero to prevent the displacement to become infinitely large for $r = 0$. The constant c_1 must be calculated from the other boundary condition. It can be found in appendix E.



$u_r(r = 0) \neq \infty$
$\sigma_{rr}(r = b) = 0$
c_1, c_2 : App. E

Fig. 9.21 : *Solid disc with a radial temperature gradient*

An isotropic solid disc is subjected to the radial temperature profile, shown in the figure below. The temperature gradient is zero at the center and at the outer edge. For a plane stress state and with parameter values listed below the stresses are calculated and plotted as a function of the radius.

| $\underset{\sim}{a}^T = [100\ 20\ 0\ 0]$ | $a = 0$ m | $b = 0.5$ m | $E = 200$ GPa | $\nu = 0.3$ | $\alpha = 10^{-6}$ $1/^o$C |

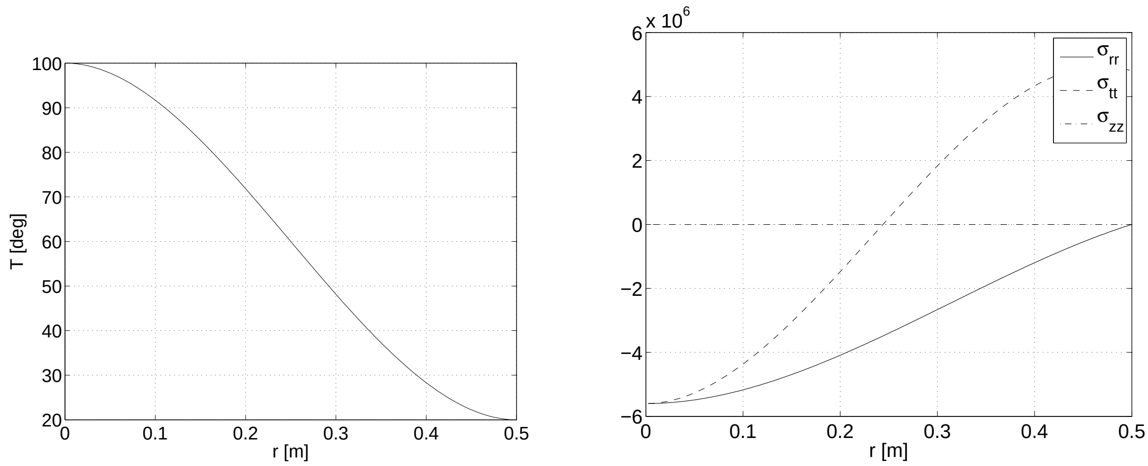


Fig. 9.22 : *Radial temperature profile and stresses in a solid disc in plane stress*

When the temperature field is uniform, $\Delta T(r) = a_0$, we have $\bar{u}_r = 0$, and for the solid disc $c_2 = 0$ to assure $u_r(r = 0) \neq \infty$. From the general solution for the stresses – see page a30 – it follows that both the radial and the tangential stresses are uniform. Because the outer edge is stress-free, they have to be zero. The integration constant c_1 can be determined to be $c_1 = \alpha a_0$, which gives for the radial displacement :

$$u_r = \alpha a_0 r$$

When the outer edge is clamped, the condition $u_r(r = b) = 0$ leads to $c_1 = 0$, so the radial displacement is uniformly zero. Radial and tangential stresses are uniform and equal :

$$\sigma_{rr} = \sigma_{tt} = -\alpha(A_\sigma + Q_\sigma)a_0$$

Different results for plane stress and plane strain emerge after substitution of the appropriate values for A_σ and Q_σ, see section 5.4.1 and appendix A. For plane stress, the thickness strain can be calculated (see section 5.5.1) :

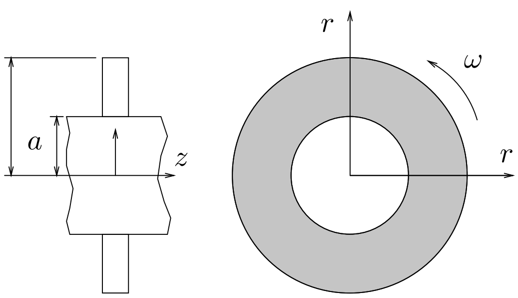
$$\varepsilon_{zz} = r\sigma_{11} + s\varepsilon_{22} + \alpha\Delta T$$

For plane strain, the axial stress can be calculated :

$$\sigma_{zz} = -\frac{r}{c}\,\sigma_{rr} - \frac{s}{c}\,\sigma_{tt} - \frac{\alpha}{c}\,\Delta T$$

Disc on a rigid axis

When the disc is mounted on a rigid axis, the radial displacement at the inner radius of the central hole is zero. The radial stress at the outer edge is again zero.



$u_r(r = a) = 0$
$\sigma_{rr}(r = b) = 0$
c_1, c_2 : App. E

Fig. 9.23 : *Disc on rigid axis subjected to a radial temperature gradient*

isotropic	plane stress	$\underset{\sim}{a}^T = [100\ 20\ 0\ 0]$		
$a = 0.2$ m	$b = 0.5$ m	$E = 200$ GPa	$\nu = 0.3$	$\alpha = 10^{-6}\ 1/^o\mathrm{C}$

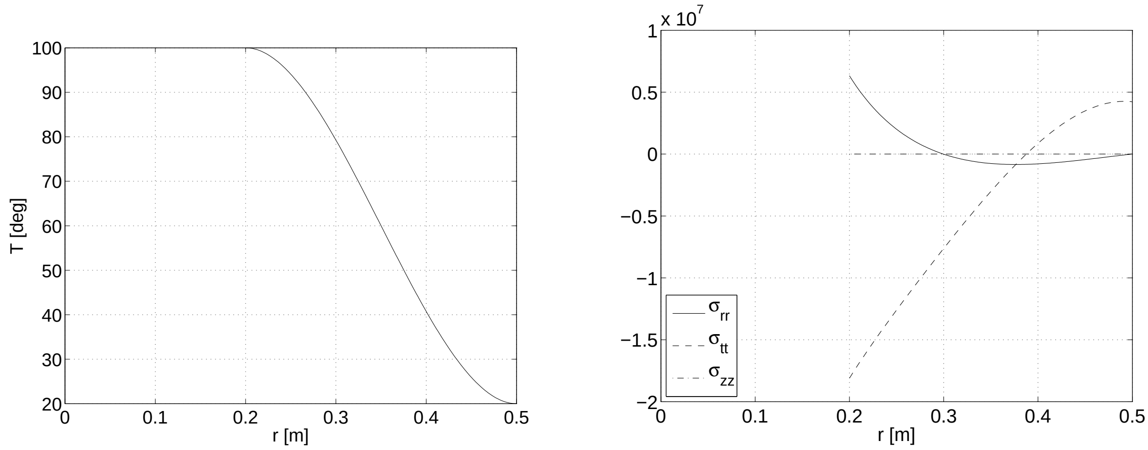


Fig. 9.24 : *Radial temperature profile and stresses in a disc which is fixed on a rigid axis*

9.2.8 Large thin plate with central hole

A large rectangular plate is loaded with a uniform stress $\sigma_{xx} = \sigma$. In the center of the plate is a hole with radius a, much smaller than the dimensions of the plate.

The stresses around the hole can be determined, using an Airy stress function approach. The relevant stresses are expressed as components in a cylindrical coordinate system, with coordinates r, measured from the center of the hole, and θ in the circumferential direction.

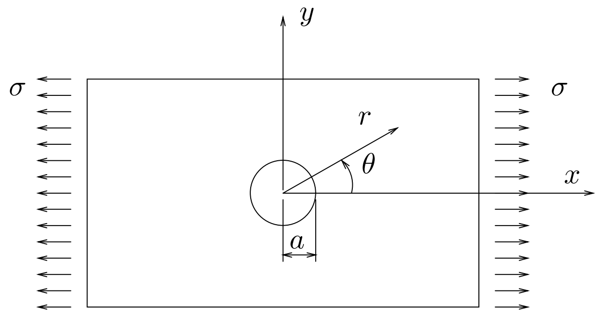


Fig. 9.25 : *Large thin plate with a central hole*

$$
\begin{aligned}
\sigma_{rr} &= \frac{\sigma}{2}\left[\left(1-\frac{a^2}{r^2}\right)+\left(1+3\,\frac{a^4}{r^4}-4\,\frac{a^2}{r^2}\right)\cos(2\theta)\right] \\
\sigma_{tt} &= \frac{\sigma}{2}\left[\left(1+\frac{a^2}{r^2}\right)-\left(1+3\,\frac{a^4}{r^4}\right)\cos(2\theta)\right] \\
\sigma_{rt} &= -\,\frac{\sigma}{2}\left[1-3\,\frac{a^4}{r^4}+2\,\frac{a^2}{r^2}\right]\sin(2\theta)
\end{aligned}
$$

At the inner edge of the hole, the tangential stress reaches a maximum value of 3σ for $\theta = 90^o$. For $\theta = 0^o$ a compressive tangential stress occurs. The stress concentration factor K_t is independent of material parameters and the hole diameter.

$$
\sigma_{tt}(r=a,\theta=\tfrac{\pi}{2}) = 3\sigma \quad ; \quad \sigma_{tt}(r=a,\theta=0) = -\sigma
$$

stress concentration factor $\quad K_t = \dfrac{\sigma_{max}}{\sigma} = 3$

At a large distance from the hole, so for $r \gg a$, the stress components are a function of the angle θ only.

$$
\begin{aligned}
\sigma_{rr} &= \frac{\sigma}{2}\,[1+\cos(2\theta)] = \sigma\,\cos^2(\theta) \\
\sigma_{tt} &= \frac{\sigma}{2}\,[1-\cos(2\theta)] = \sigma\left[1-\cos^2(\theta)\right] = \sigma\sin^2(\theta) \\
\sigma_{rt} &= -\,\frac{\sigma}{2}\,\sin(2\theta) = -\sigma\,\sin(\theta)\cos(\theta)
\end{aligned}
$$

For parameters values listed below stress components are calculated and plotted for $\theta = 0$ and for $\theta = \frac{\pi}{2}$ as a function of the radial distance r.

$a = 0.05$ m	$\sigma = 1000$ Pa

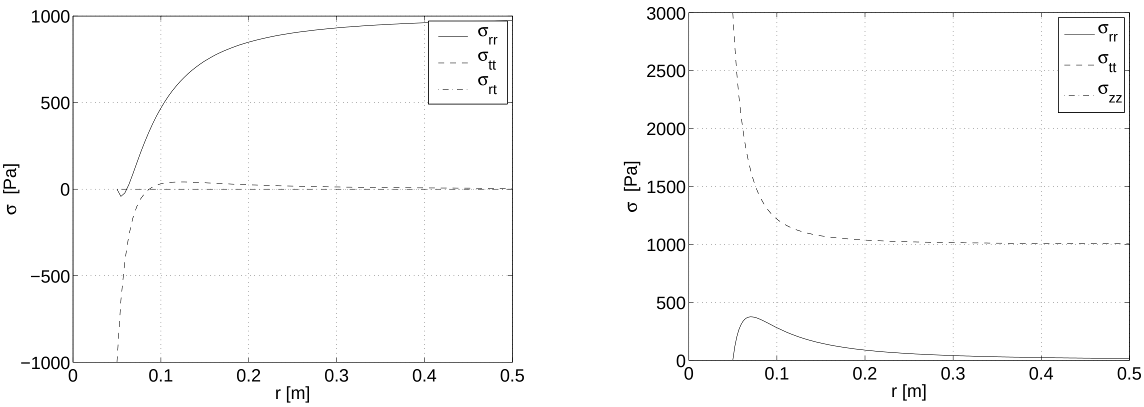


Fig. 9.26 : *Stresses in plate for* $\theta = 0$ *and* $\theta = \frac{\pi}{2}$

Chapter 10

Numerical solutions

In the following sections we present some problems and their numerical solutions. These solutions are determined with the MSC.Marc/Mentat FE-package. The numerical solutions can be compared with the analytical solutions, described in the previous section.

10.1 MSC.Marc/Mentat

The MSC.Mentat program is used to model the structure, which is subsequently analyzed by the FE-program MSC.Marc. Modeling the geometry – shape and dimensions – is the first step in this procedure. Dimensional units have to be chosen and consistently used in the entire analysis. In this stage it is already needed to decide whether the model is three-dimensional, planar or axi-symmetric. The finite element mesh is generated according to procedures which are described in the tutorial. In the examples discussed in this chapter, only linear elements are used, i.e. elements where the displacement is interpolated bi-linearly between the nodal point displacements. Quadratic elements will lead to more accurate results in most cases.

After defining the geometry, the material properties can be specified. Only linear elastic material behavior is considered, both isotropic and orthotropic. Boundary conditions can be : prescribed displacements, edge loads, gravitational loads and centrifugal loading due to rotation. Thermal loading is not shown here, but can be applied straightforwardly.

When the model is complete, it can be analyzed and the results can be observed and plotted. Contour bands of variables can be superposed on the geometry and variables can be plotted. In the next sections these plots will be presented.

10.2 Cartesian, planar

The most simple analysis, which can be done is the calculation of stress and strain in a tensile test and a shear test. Boundary conditions must be prescribed to ensure homogeneous deformation. As expected, the exact solution is reproduced with only one element.

10.2.1 Tensile test

A uni-axial tensile test, resulting in homogeneous deformation, can be modeled and analyzed with only one linear element, resulting in the exact solution. In the next example a square plate (length a, thickness h) is modeled with four equally sized elements, because the central node on the left edge must be fixed to prevent rigid body movement. The right-hand edge is loaded with an edge load p. The deformation is shown in the figure below with a magnification of 500.

When the left edge is clamped, the deformation and stress state is no longer homogeneous. An analytical solution does not exist for this case. An approximate solution can be determined rather easily. To model the inhomogeneous deformation, we need more elements, especially in the neighborhood of the clamped edge. Using more elements improves the accuracy of the result. Equal accuracy can be realized with fewer, but higher-order (quadratic) elements, as such elements interpolate the displacement field better than a linear element.

When subsequent analyses are done with decreasing element sizes, we will notice that at some point, further mesh refinement will not change the solution any more. This *convergence upon mesh refinement* is essential for good finite element modeling and analysis. When it does not occur, the results are always dependent on the element mesh and such a *mesh dependency* is not allowed. It may be found to occur when singularities are involved or when the (non-linear) material shows softening, i.e. decrease of stress at increasing strain.

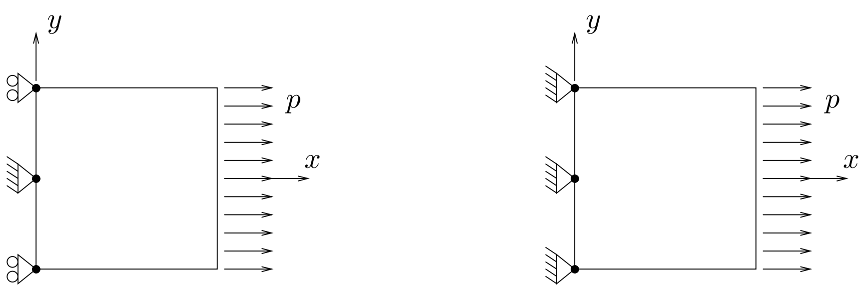


Fig. 10.1 : *Tensile test with different boundary conditions*

$p = 100$ MPa	$a = 0.5$ m	$h = 0.05$ m	$E = 200$ GPa	$\nu = 0.25$

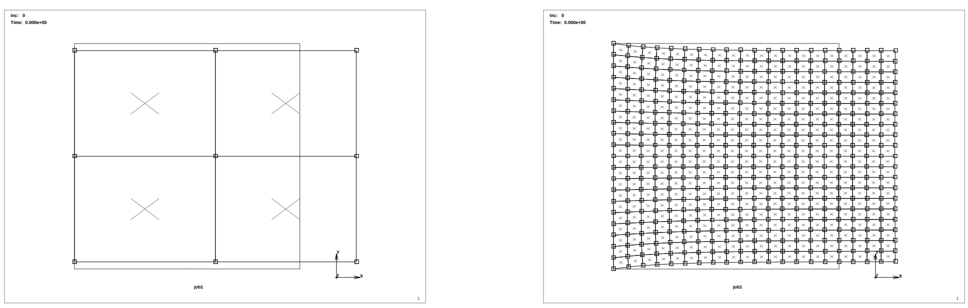


Fig. 10.2 : *Undeformed and deformed element mesh at 500 $\times$ magnification*

For the homogeneous case the displacement is

$$u_x(x = a) = 0.25 \times 10^{-3} \text{ m} \quad ; \quad u_y(y = a/2) = -0.3125 \times 10^{-4}$$

which, of course, is also the exact solution.

10.2.2 Shear test

The true shear test is another example of a homogeneous deformation. It is done here with a 20×20 mesh of square linear elements, but could have been done with only one element, leading to the same exact result. To prevent rigid body translation, the left-bottom node is fixed, i.e. its displacement is prevented. To prevent rigid body rotation, the nodes at the bottom are only allowed to move horizontally. Edges are loaded with a shear load p, leading to the deformation, which is shown in the figure, again with a magnification of 500.

Instead of this homogeneous shear test, often a so-called *simple shear test* is done experimentally. In that case the shear load p is only applied at the upper edge. The left- and right-edge is stress-free. Moreover, the displacement in y-direction of the upper edge is prevented, as is the case for the bottom-edge, which is clamped. The result is shown in the right-hand figure below with a magnification of 250. It is immediately clear that the deformation is no longer homogeneous.

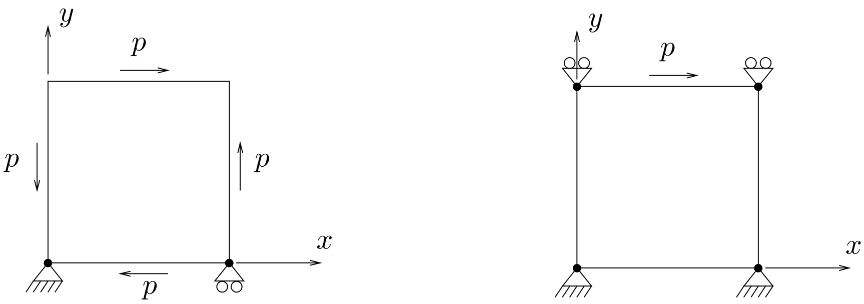


Fig. 10.3 : *Shear test with different boundary conditions*

isotropic	plane stress	$p = 100$ MPa	
$a = 0.5$ m	$h = 0.05$ m	$E = 200$ GPa	$\nu = 0.25$

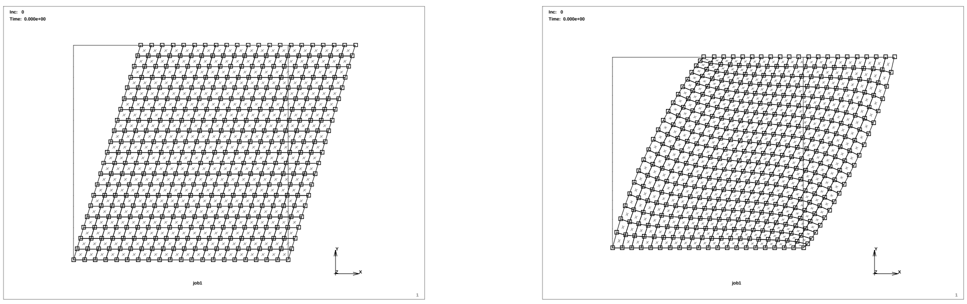


Fig. 10.4 : *Undeformed and deformed element mesh at 500 and 250 $\times$ magnification*

10.2.3 Orthotropic plate

The uni-axial tensile test and the real shear test are now carried out on a plate with orthotropic material behavior. For the tensile test, the center node on the left-edge is fixed, while the other nodes on this edge are restricted to move in y-direction. For the shear test, the center node is fixed and again the upper-right and lower-left corner nodes are restricted to move along the diagonal. The material coordinate system is oriented at an angle α w.r.t. the global x-axis. Material parameters are listed in the table. Because a plane stress state is assumed, the surface of the plate decreases and also the thickness will change.

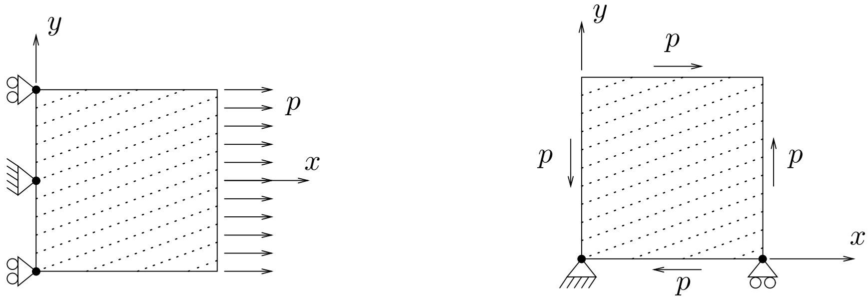


Fig. 10.5 : *Tensile test and shear test for orthotropic plate*

$p = 100$ MPa	$a = 0.5$ m	$h = 0.05$ m	$\alpha = 20^o$		
$E_{11} = 200$ GPa	$E_{22} = 50$ GPa	$E_{33} = 50$ GPa	$\nu_{12} = 0.4$	$\nu_{23} = 0.25$	$\nu_{31} = 0.25$
$G_{12} = 100$ GPa	$G_{23} = 20$ GPa	$G_{31} = 20$ GPa			

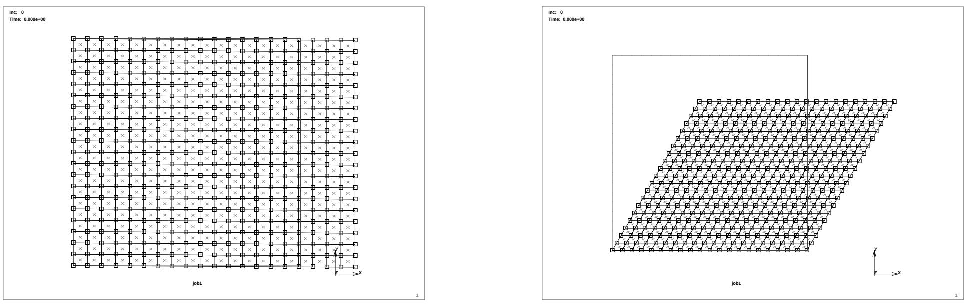


Fig. 10.6 : *Undeformed and deformed element mesh at 500 $\times$ magnification*

10.3 Axi-symmetric, $u_t = 0$

A tensile test on a cylindrical bar can be analyzed analytically when the material is isotropic. For orthotropic material, with principal material directions in radial, axial and tangential direction, the problem is analyzed numerically. Although the loading is uni-axially and uniform over the cross-section, the strain and stress distribition is not homogeneous.

The cylindrical tensile bar of length 0.5 m is modelled with axi-symmetric elements. The radius of the bar is 0.0892 m. The material coordinate system is $\{1, 2, 3\} = \{r, z, t\}$. Material parameters are listed in the table. The bar is loaded with an axial edge load p. The axial displacement is then about 0.001 m. The figure shows the stresses as a function of the radius.

$E_{11} = 200$ GPa	$E_{22} = 50$ GPa	$E_{33} = 50$ GPa	$\nu_{12} = 0.4$	$\nu_{23} = 0.25$	$\nu_{31} = 0.25$
$G_{12} = 100$ GPa	$G_{23} = 20$ GPa	$G_{31} = 20$ GPa	$p = 100$ MPa		

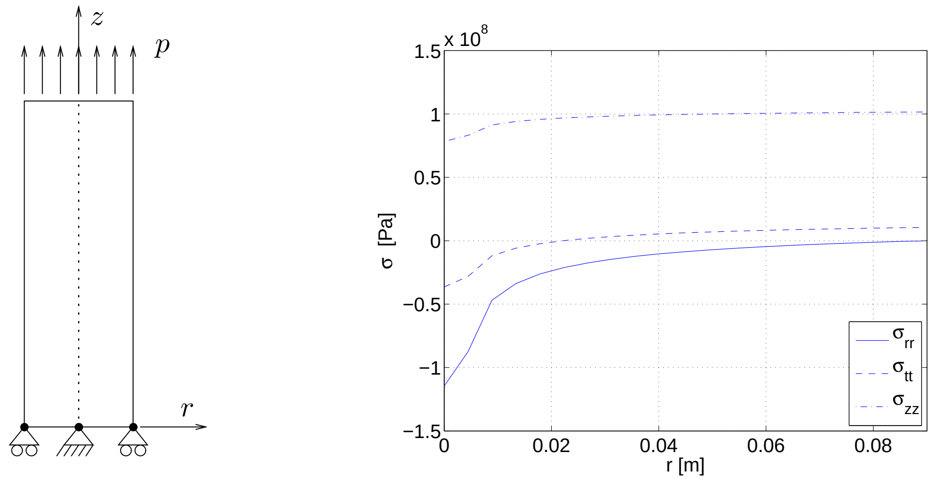


Fig. 10.7 : *Stresses in an orthotropic tensile bar*

10.4 Axi-symmetric, planar, $u_t = 0$

Axi-symmetric problems can be analyzed with planar elements – plane stress or plane strain – but also with axi-symmetric elements. In the latter case, the model is made in the zr-plane for $r > 0$.

10.4.1 Prescribed edge displacement

The model is made in the zr-plane with axi-symmetric elements. A plane stress state is modeled by choosing the proper boundary conditions.

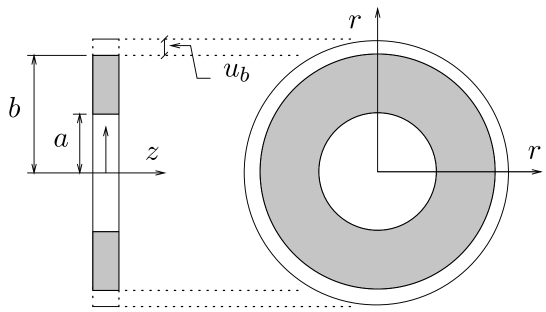


Fig. 10.8 : *Edge displacement of circular disc*

$u_b = 0.01$ m	$a = 0.25$ m	$b = 0.5$ m	$h = 0.05$ m	$E = 250$ GPa	$\nu = 0.33$

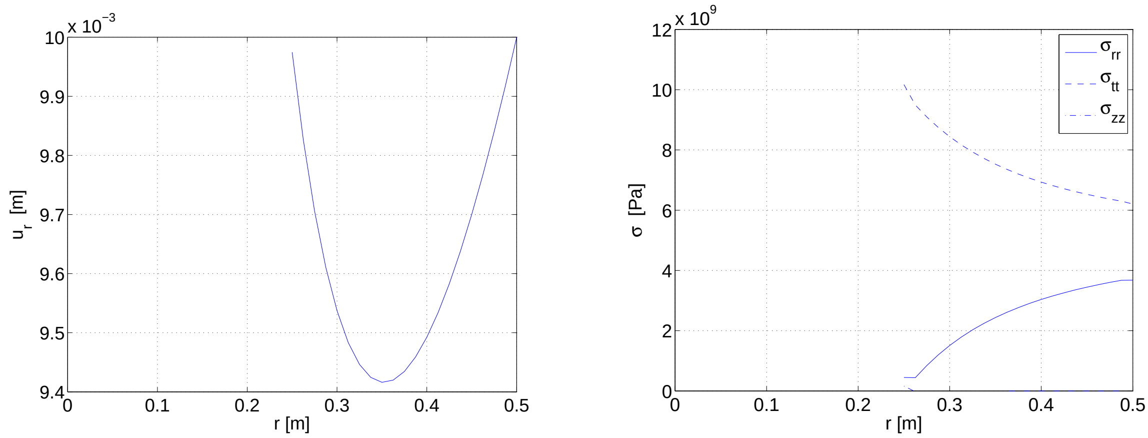


Fig. 10.9 : *Displacement and stresses for plane stress ($\sigma_{zz} = 0$)*

10.4.2 Edge load

A thick-walled cylinder is loaded with an internal pressure p_i. When the cylinder is open and its elongation unconfined, each cross-section over the axis is in a state of plane stress: $\sigma_{zz} = 0$. The plane strain situation, where the length of the cylinder is kept constant, is easily analyzed by choosing plane strain elements.

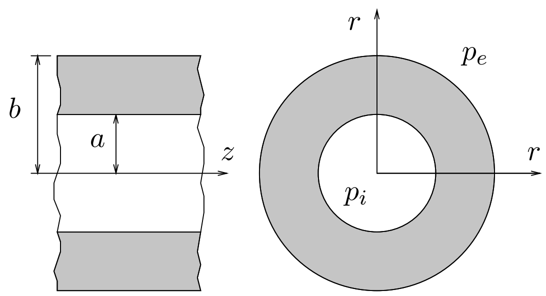


Fig. 10.10 : *Cross-section of a thick-walled circular cylinder*

The cylinder is open (plane stress) and made of isotropic material. Dimensions and material properties are listed in the table. The plots show the stresses as a function of the radius. There values coincide with the analytical solution, except near the edges. The reason is that stresses (and strains) are calculated in the *integration points*, which are located inside the element, and edge values are extrapolated. When more elements are used, the deviation will decrease.

$p_i = 100$ MPa	$a = 0.25$ m	$b = 0.5$ m	$h = 0.5$ m	$E = 250$ GPa	$\nu = 0.33$

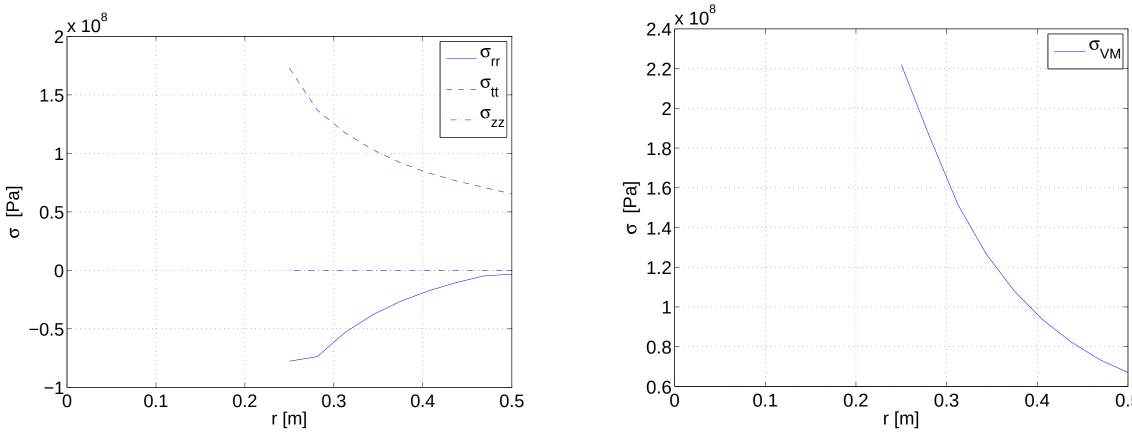


Fig. 10.11 : *Stresses in a thick-walled pressurized cylinder for plane stress ($\sigma_{zz} = 0$)*

For plane strain ($\varepsilon_{zz} = 0$) the length of the cylinder is kept constant. This will obviously lead to an axial stress σ_{zz}.

10.4.3 Centrifugal load

A centrifugal load is modeled to analyze rotating discs. The analysis can be done with an axi-symmetric model or with a plane stress model. Except for the location near the edges, the numerical solution coincides with the analytical solution, presented in chapter 9.

10.4.4 Large thin plate with a central hole

A rectangular plate with central hole is loaded uni-axially. The analytical solution for the stress distribution is known and given in chapter 9. To get a numerical solution, the plate is modeled with linear elements. Geometric and material data are listed in the table. The figures show the stresses as a function of the radial coordinate for the 0^o- and 90^o-direction.

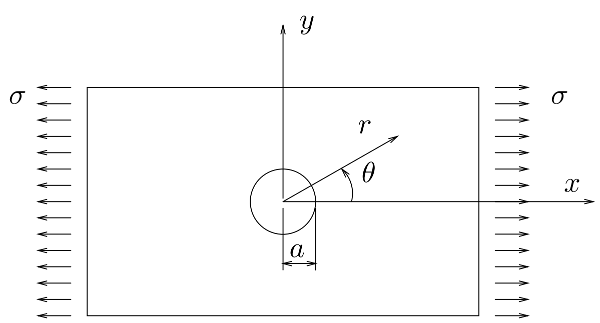


Fig. 10.12 : *Large thin plate with a central hole*

$a = 0.05$ m	$\sigma = 1$ kPa	$E = 250$ GPa	$\nu = 0.3$

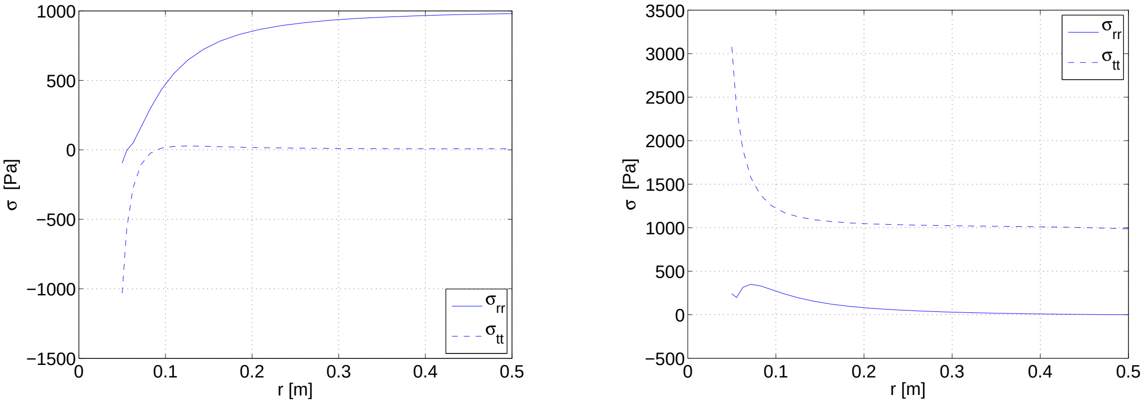


Fig. 10.13 : *Stresses in plate for* $\theta = 0$ *and* $\theta = \frac{\pi}{2}$

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APPENDICES

Appendix A

Stiffness and compliance matrices

In this appendix, the stiffness and compliance matrices for orthotropic, transversal isotropic and isotropic material are given.

A.1 Orthotropic

For an orthotropic material 9 material parameters are needed to characterize its mechanical behavior. Their names and formal definitions are :

$$
\begin{array}{lll}
\text{Young's moduli} & : & E_i = \dfrac{\partial \sigma_{ii}}{\partial \varepsilon_{ii}} \\
\text{Poisson's ratios} & : & \nu_{ij} = -\dfrac{\partial \varepsilon_{jj}}{\partial \varepsilon_{ii}} \\
\text{shear moduli} & : & G_{ij} = \dfrac{\partial \sigma_{ij}}{\partial \gamma_{ij}}
\end{array}
$$

The introduction of these parameters is easily accomplished in the compliance matrix $\underline{\underline{S}}$. The stiffness matrix $\underline{\underline{C}}$ can then be derived by inversion of $\underline{\underline{S}}$.

Due to the symmetry of the compliance matrix $\underline{\underline{S}}$, the material parameters must obey the three Maxwell relations.

$$
\underline{\underline{S}} = \begin{bmatrix}
E_1^{-1} & -\nu_{21}E_2^{-1} & -\nu_{31}E_3^{-1} & 0 & 0 & 0 \\
-\nu_{12}E_1^{-1} & E_2^{-1} & -\nu_{32}E_3^{-1} & 0 & 0 & 0 \\
-\nu_{13}E_1^{-1} & -\nu_{23}E_2^{-1} & E_3^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{12}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{23}^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{31}^{-1}
\end{bmatrix}
$$

$$
\text{with} \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \quad ; \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3} \quad ; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}
$$

$$\underline{\underline{C}} = \frac{1}{\Delta}\begin{bmatrix} \frac{1-\nu_{32}\nu_{23}}{E_2E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2E_3} & \frac{\nu_{21}\nu_{32}+\nu_{31}}{E_2E_3} & 0 & 0 & 0 \\ \frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1E_3} & \frac{\nu_{12}\nu_{31}+\nu_{32}}{E_1E_3} & 0 & 0 & 0 \\ \frac{\nu_{12}\nu_{23}+\nu_{13}}{E_1E_2} & \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1E_2} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta G_{31} \end{bmatrix}$$

$$\text{with} \qquad \Delta = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

A.1.1 Voigt notation

In composite mechanics the so-called Voigt notation is often used, where stress and strain components are simply numbered 1 to 6. Corresponding components of the compliance (and stiffness) matrix are numbered accordingly. However, there is more to it than that. The sequence of the shear components is changed. We will not use this changed sequence in the following.

stresses and strains

$$\underset{\sim}{\sigma}^T = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{31}] = [\sigma_1\ \sigma_2\ \sigma_3\ \sigma_6\ \sigma_4\ \sigma_5]$$
$$\underset{\sim}{\varepsilon}^T = [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \gamma_{12}\ \gamma_{23}\ \gamma_{31}] = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \varepsilon_6\ \varepsilon_4\ \varepsilon_5]$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

material parameters

$$\begin{array}{lll} S_{11} = \frac{1}{E_1} & S_{22} = \frac{1}{E_2} & S_{33} = \frac{1}{E_3} \\ S_{12} = -\frac{\nu_{21}}{E_2} & S_{13} = -\frac{\nu_{31}}{E_3} & S_{23} = -\frac{\nu_{32}}{E_3} \\ S_{44} = \frac{1}{G_{23}} & S_{55} = \frac{1}{G_{31}} & S_{66} = \frac{1}{G_{12}} \end{array}$$

A.1.2 Plane strain

For some geometries and loading conditions the strain in one direction is zero. Such deformation is referred to as plane strain. Here we take $\varepsilon_{33} = \gamma_{13} = \gamma_{23} = 0$. The stress σ_{33} is not zero but can be eliminated from the stress-strain relation and expressed in σ_{11} and σ_{22}.

For plane strain the stiffness matrix can be extracted directly from the three-dimensional stiffness matrix. The inverse of this 3x3 matrix is the plane strain compliance matrix.

$$\underline{\underline{C}} = \frac{1}{\Delta}\left[\begin{array}{ccc} \frac{1-\nu_{32}\nu_{23}}{E_2E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2E_3} & 0 \\ \frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1E_3} & 0 \\ 0 & 0 & \Delta G_{12} \end{array}\right]$$

with $\quad \Delta = \dfrac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$

$$\underline{\underline{S}} = \left[\begin{array}{ccc} \frac{1-\nu_{31}\nu_{13}}{E_1} & -\frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1} & \frac{1-\nu_{32}\nu_{23}}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{array}\right]$$

$$\sigma_{33} = \frac{1}{\Delta}\left\{\frac{\nu_{12}\nu_{32}+\nu_{13}}{E_1E_2}\,\varepsilon_{11} + \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1E_2}\,\varepsilon_{22}\right\} = \nu_{13}\,\frac{E_3}{E_1}\,\sigma_{11} + \nu_{23}\,\frac{E_3}{E_2}\,\sigma_{22}$$

A.1.3 Plane stress

When deformation in one direction is not restricted, the stress in that direction will be zero. This is called a plane stress situation. Here we assume $\sigma_{33} = \sigma_{13} = \sigma_{31} = 0$. The strain ε_{33} is not zero but can be eliminated from the stress-strain relation and expressed in the in-plane strains.

Such a plane stress state is often found in the deformation of thin plates, which are loaded in their plane.

For plane stress the compliance matrix can be extracted directly from the three-dimensional compliance matrix. The inverse of this 3x3 matrix is the plane strain stiffness matrix.

$$\underline{\underline{S}} = \left[\begin{array}{ccc} E_1^{-1} & -\nu_{21}E_2^{-1} & 0 \\ -\nu_{12}E_1^{-1} & E_2^{-1} & 0 \\ 0 & 0 & G_{12}^{-1} \end{array}\right]$$

$$\underline{\underline{C}} = \frac{1}{1-\nu_{21}\nu_{12}}\left[\begin{array}{ccc} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & (1-\nu_{21}\nu_{12})G_{12} \end{array}\right]$$

$$\varepsilon_{33} = -\nu_{13}E_1^{-1}\sigma_{11} - \nu_{23}E_2^{-1}\sigma_{22} = -\frac{1}{1-\nu_{12}\nu_{21}}\left\{(\nu_{12}\nu_{23}+\nu_{13})\varepsilon_{11} + (\nu_{21}\nu_{13}+\nu_{23})\varepsilon_{22}\right\}$$

A.2 Transversal isotropic

Considering an transversally isotropic material with the 12-plane isotropic, the Young's modulus E_p and the Poisson's ratio ν_p in this plane can be measured. The associated shear modulus is related by $G_p = \dfrac{E_p}{2(1+\nu_p)}$. In the perpendicular direction we have the Young's modulus E_3, the shear moduli $G_{3p} = G_{p3}$ and two Poisson ratio's, which are related by symmetry : $\nu_{p3}E_3 = \nu_{3p}E_p$.

$$\underline{\underline{S}} = \begin{bmatrix} E_p^{-1} & -\nu_p E_p^{-1} & -\nu_{3p} E_3^{-1} & 0 & 0 & 0 \\ -\nu_p E_p^{-1} & E_p^{-1} & -\nu_{3p} E_3^{-1} & 0 & 0 & 0 \\ -\nu_{p3} E_p^{-1} & -\nu_{p3} E_p^{-1} & E_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_p^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{p3}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{3p}^{-1} \end{bmatrix}$$

with $\qquad \dfrac{\nu_{p3}}{E_p} = \dfrac{\nu_{3p}}{E_3}$

$$\underline{\underline{C}} = \frac{1}{\Delta} \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_{3p}\nu_{p3}+\nu_p}{E_pE_3} & \frac{\nu_p\nu_{3p}+\nu_{3p}}{E_pE_3} & 0 & 0 & 0 \\ \frac{\nu_{p3}\nu_{3p}+\nu_p}{E_pE_3} & \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_p\nu_{3p}+\nu_{3p}}{E_pE_3} & 0 & 0 & 0 \\ \frac{\nu_p\nu_{p3}+\nu_{p3}}{E_pE_p} & \frac{\nu_p\nu_{p3}+\nu_{p3}}{E_pE_p} & \frac{1-\nu_p\nu_p}{E_pE_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta G_p & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta G_{p3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta G_{3p} \end{bmatrix}$$

with $\qquad \Delta = \dfrac{1 - \nu_p\nu_p - \nu_{p3}\nu_{3p} - \nu_{3p}\nu_{p3} - \nu_p\nu_{p3}\nu_{3p} - \nu_p\nu_{3p}\nu_{p3}}{E_pE_pE_3}$

A.2.1 Plane strain

For the plane strain case with $\varepsilon_{33} = \gamma_{13} = \gamma_{23} = 0$. The stress σ_{33} is not zero but can be eliminated from the stress-strain relation and expressed in σ_{11} and σ_{22}. The plane strain stiffness matrix can be extracted directly from the three-dimensional stiffness matrix. The inverse of this 3x3 matrix is the plane strain compliance matrix.

$$\underline{\underline{C}} = \frac{1}{\Delta} \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_{3p}\nu_{p3}+\nu_p}{E_pE_3} & 0 \\ \frac{\nu_{p3}\nu_{3p}+\nu_p}{E_pE_3} & \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & 0 \\ 0 & 0 & \Delta G_p \end{bmatrix}$$

with $\qquad \Delta = \dfrac{1 - \nu_p\nu_p - \nu_{p3}\nu_{3p} - \nu_{3p}\nu_{p3} - \nu_p\nu_{p3}\nu_{3p} - \nu_p\nu_{3p}\nu_{p3}}{E_pE_pE_3}$

$$\underline{\underline{S}} = \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_p} & -\frac{\nu_{3p}\nu_{p3}+\nu_p}{E_p} & 0 \\ -\frac{\nu_{p3}\nu_{3p}+\nu_p}{E_p} & \frac{1-\nu_{3p}\nu_{p3}}{E_p} & 0 \\ 0 & 0 & \frac{1}{G_p} \end{bmatrix}$$

$$\sigma_{33} = \frac{1}{\Delta}\frac{\nu_{3p}(\nu_p+1)}{E_p^2}\left(\varepsilon_{11}+\varepsilon_{22}\right)$$

A.2.2 Plane stress

For the plane stress state with $\sigma_{33} = \sigma_{13} = \sigma_{31} = 0$, the strain ε_{33} is not zero but can be eliminated from the stress-strain relation and expressed in the in-plane strains. For plane stress the compliance matrix can be extracted directly from the three-dimensional compliance matrix. The inverse of this 3x3 matrix is the plane strain stiffness matrix.

$$\underline{\underline{S}} = \begin{bmatrix} E_p^{-1} & -\nu_p E_p^{-1} & 0 \\ -\nu_p E_p^{-1} & E_p^{-1} & 0 \\ 0 & 0 & G_p^{-1} \end{bmatrix}$$

$$\underline{\underline{C}} = \frac{1}{1-\nu_p\nu_p} \begin{bmatrix} E_p & \nu_p E_p & 0 \\ \nu_p E_p & E_p & 0 \\ 0 & 0 & (1-\nu_p\nu_p)G_p \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu_{p3}}{E_p}(\sigma_{11} + \sigma_{22})$$

A.3 Isotropic

The linear elastic material behavior can be described with the material stiffness matrix $\underline{\underline{C}}$ or the material compliance matrix $\underline{\underline{S}}$. These matrices can be written in terms of the engineering elasticity parameters E and ν.

$$\underline{\underline{C}} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\underline{\underline{S}} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

A.3.1 Plane strain

For some geometries and loading conditions the strain in z-direction is zero : $\varepsilon_{zz} = 0$. Such deformation is referred to as plane strain. With $\gamma_{xz} = \gamma_{yz} = 0$ we have for the stresses $\sigma_{xz} = \sigma_{yz} = 0$. The stress σ_{zz} is not zero but can be eliminated from the stress-strain relation and expressed in σ_{xx} and σ_{yy}.

From the stress-strain relation for plane strain it is immediately clear that problems will occur for $\nu = 0.5$, which is the value for incompressible material behavior.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

$$\sigma_{zz} = \alpha\nu(\varepsilon_{xx} + \varepsilon_{yy}) = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\text{with} \quad : \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

A.3.2 Plane stress

When deformation in z-direction is not restricted, the stress σ_{zz} will be zero. This is called a plane stress situation. With additional $\sigma_{xz} = \sigma_{zx} = 0$, we have $\gamma_{xz} = \gamma_{zx} = 0$. The strain ε_{zz} is not zero but can be eliminated from the stress-strain relation and expressed in the in-plane strains.

Such a plane stress state is often found in the deformation of thin plates, which are loaded in their plane.

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

$$\varepsilon_{zz} = \frac{\Delta h}{h_0} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

A.3.3 Axi-symmetry

In each point of a cross section the displacement has two components : $\underset{\sim}{u}^T = [u_r u_z]$. The stress and strain components are :

$$\underset{\approx}{\sigma}^T = \begin{bmatrix} \sigma_{rr} & \sigma_{zz} & \sigma_{tt} & \sigma_{rz} \end{bmatrix} \quad ; \quad \underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{zz} & \varepsilon_{tt} & \gamma_{rz} \end{bmatrix}$$

The stress-strain relation according to Hooke's law can be derived from the general three-dimensional case.

With the well-known strain-displacement relations, the stress components can be related to the derivatives of the displacement components.

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{tt} \\ \sigma_{rz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{tt} \\ \gamma_{rz} \end{bmatrix}$$

a8

Appendix B

Matrix transformation

The rotation of an orthonormal vector base is described with a rotation matrix. The matrix components of a tensor w.r.t. the rotated base can be calculated from the components w.r.t. the initial base. This matrix transformation will be done for a general second-order tensor $\boldsymbol{A}$. It is repeated for the stress and strain tensors, which then results in the transformation of the material stiffness and compliance matrices.

B.1 Rotation of matrix with tensor components

A tensor $\boldsymbol{A}$ with matrix representation $\underline{A}$ w.r.t. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ has a matrix $\underline{A}^*$ w.r.t. basis $\{\vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3\}$. The matrix $\underline{A}^*$ can be calculated from $\underline{A}$ by multiplication with $\underline{Q}$, the rotation matrix w.r.t. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

$$
\begin{aligned}
\boldsymbol{A} &= \underset{\sim}{\vec{e}}^T \underline{A}\, \underset{\sim}{\vec{e}} = \underset{\sim}{\vec{\varepsilon}}^T\, \underline{A}^*\, \underset{\sim}{\vec{\varepsilon}} \quad \rightarrow \\
\underline{A}^* &= \underset{\sim}{\vec{\varepsilon}} \cdot \underset{\sim}{\vec{e}}^T \underline{A}\, \underset{\sim}{\vec{e}} \cdot \underset{\sim}{\vec{\varepsilon}}^T = \underline{Q}^T \underline{A}\, \underline{Q} \\
&= \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \\
&= \begin{bmatrix} A^*_{11} & A^*_{12} & A^*_{13} \\ A^*_{21} & A^*_{22} & A^*_{23} \\ A^*_{31} & A^*_{32} & A^*_{33} \end{bmatrix}
\end{aligned}
$$

B.2 Rotation of column with matrix components

The column $\underset{\approx}{A}$ with the 9 components of $\boldsymbol{A}$ can be transformed to $\underset{\approx}{A}^*$ by multiplication with the 9x9 transformation matrix $\underline{T}$ according to $\underset{\approx}{A}^* = \underline{T}\, \underset{\approx}{A}$.

The matrix $\underline{T}$ is not orthogonal, but its inverse can be calculated easily by reversing the rotation angles : $\underline{T}^{-1} = \underline{T}(-\alpha_1, -\alpha_2, -\alpha_3)$.

$$
\underset{\approx}{A}^T = \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{12} & A_{21} & A_{23} & A_{32} & A_{31} & A_{13} \end{bmatrix}
$$

$$\underline{T} = \underline{T}(\alpha_1, \alpha_2, \alpha_3) =$$

$$\begin{bmatrix}
Q_{11}^2 & Q_{21}^2 & Q_{31}^2 & Q_{21}Q_{11} & Q_{11}Q_{21} & Q_{31}Q_{21} & Q_{21}Q_{31} & Q_{11}Q_{31} & Q_{31}Q_{11} \\
Q_{12}^2 & Q_{22}^2 & Q_{32}^2 & Q_{22}Q_{12} & Q_{12}Q_{22} & Q_{32}Q_{22} & Q_{22}Q_{32} & Q_{12}Q_{32} & Q_{32}Q_{12} \\
Q_{13}^2 & Q_{23}^2 & Q_{33}^2 & Q_{23}Q_{13} & Q_{13}Q_{23} & Q_{33}Q_{23} & Q_{23}Q_{33} & Q_{13}Q_{33} & Q_{33}Q_{13} \\
Q_{12}Q_{11} & Q_{22}Q_{21} & Q_{32}Q_{31} & Q_{22}Q_{11} & Q_{12}Q_{21} & Q_{32}Q_{21} & Q_{22}Q_{31} & Q_{12}Q_{31} & Q_{32}Q_{11} \\
Q_{11}Q_{12} & Q_{21}Q_{22} & Q_{31}Q_{32} & Q_{21}Q_{12} & Q_{11}Q_{22} & Q_{31}Q_{22} & Q_{21}Q_{32} & Q_{11}Q_{32} & Q_{31}Q_{12} \\
Q_{13}Q_{12} & Q_{23}Q_{22} & Q_{33}Q_{32} & Q_{23}Q_{12} & Q_{13}Q_{22} & Q_{33}Q_{22} & Q_{23}Q_{32} & Q_{13}Q_{32} & Q_{33}Q_{12} \\
Q_{12}Q_{13} & Q_{22}Q_{23} & Q_{32}Q_{33} & Q_{22}Q_{13} & Q_{12}Q_{23} & Q_{32}Q_{23} & Q_{22}Q_{33} & Q_{12}Q_{33} & Q_{32}Q_{13} \\
Q_{11}Q_{13} & Q_{21}Q_{23} & Q_{31}Q_{33} & Q_{21}Q_{13} & Q_{11}Q_{23} & Q_{31}Q_{23} & Q_{21}Q_{33} & Q_{11}Q_{33} & Q_{31}Q_{13} \\
Q_{13}Q_{11} & Q_{23}Q_{21} & Q_{33}Q_{31} & Q_{23}Q_{11} & Q_{13}Q_{21} & Q_{33}Q_{21} & Q_{23}Q_{31} & Q_{13}Q_{31} & Q_{33}Q_{11}
\end{bmatrix}$$

When $\boldsymbol{A}$ is symmetric, the transformation matrix $\underline{T}$ is 6x6. Note that $\underline{T}$ is **not** the representation of a tensor.

$$\underset{\approx}{A}^T = \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{12} & A_{23} & A_{31} \end{bmatrix}$$

$$\underline{T} = \underline{T}(\alpha_1, \alpha_2, \alpha_3) =$$

$$\begin{bmatrix}
Q_{11}^2 & Q_{21}^2 & Q_{31}^2 & 2Q_{21}Q_{11} & 2Q_{31}Q_{21} & 2Q_{11}Q_{31} \\
Q_{12}^2 & Q_{22}^2 & Q_{32}^2 & 2Q_{22}Q_{12} & 2Q_{32}Q_{22} & 2Q_{12}Q_{32} \\
Q_{13}^2 & Q_{23}^2 & Q_{33}^2 & 2Q_{23}Q_{13} & 2Q_{33}Q_{23} & 2Q_{13}Q_{33} \\
Q_{12}Q_{11} & Q_{22}Q_{21} & Q_{32}Q_{31} & Q_{22}Q_{11} + Q_{12}Q_{21} & Q_{32}Q_{21} + Q_{22}Q_{31} & Q_{12}Q_{31} + Q_{32}Q_{11} \\
Q_{13}Q_{12} & Q_{23}Q_{22} & Q_{33}Q_{32} & Q_{23}Q_{12} + Q_{13}Q_{22} & Q_{33}Q_{22} + Q_{23}Q_{32} & Q_{13}Q_{32} + Q_{33}Q_{12} \\
Q_{11}Q_{13} & Q_{21}Q_{23} & Q_{31}Q_{33} & Q_{21}Q_{13} + Q_{11}Q_{23} & Q_{31}Q_{23} + Q_{21}Q_{33} & Q_{11}Q_{33} + Q_{31}Q_{13}
\end{bmatrix}$$

B.3 Transformation of material matrices

The material stiffness and compliance matrix change as a result of a rotation of the orthonormal vector base. This rotation is described by the rotation matrix $\underline{Q}$. The transformation of the stress and strain components is described by transformation matrices $\underline{T}_\sigma$ and $\underline{T}_\varepsilon$.

B.3.1 Rotation of stress and strain components

It is assumed that the components of a second order tensor w.r.t. an orthonormal vector basis are known and stored in a 3x3 matrix. A new orthonormal vector basis is the result of three subsequent rotations around three axes. The first rotation is over α_1 around the initial 1-axis. The second rotation is over α_2 around the new 2-axis. The final rotation is over α_3 around the resulting 3-axis.

Each of these rotations is described by a rotation tensor. After rotation the components of a tensor can be calculated using the resulting rotation matrix $\underline{Q}$. The column with components of the tensor can be transformed using the transformation matrix $\underline{T}$. For a symmetric tensor/matrix, this 6x6 transformation matrix is not symmetric and not orthogonal.

When in the column with strain components, the shear components γ_{ij} are used instead of the strain components ε_{ij}, the transformation matrix must be adapted.

For the stress column the transformation matrix $\underline{T}_\sigma = \underline{T}$ and for the strain column (with γ_{ij}) the transformation matrix is $\underline{T}_\varepsilon$. The inverse of the transformation matrix is easily calculated by using reversed rotation angles and sequence.

$$\underline{\sigma}^* = \underline{Q}^T \underline{\sigma}\, \underline{Q} \quad ; \quad \underline{\varepsilon}^* = \underline{Q}^T \underline{\varepsilon}\, \underline{Q} \qquad \rightarrow \qquad \underset{\sim}{\sigma}^* = \underline{T}_\sigma\, \underset{\sim}{\sigma} \quad ; \quad \underset{\sim}{\varepsilon}^* = \underline{T}_\varepsilon\, \underset{\sim}{\varepsilon}$$

$$\underline{T}_\sigma = \underline{T}_\sigma(\alpha_1, \alpha_2, \alpha_3) =$$

$$\begin{bmatrix}
Q_{11}^2 & Q_{21}^2 & Q_{31}^2 & 2Q_{21}Q_{11} & 2Q_{31}Q_{21} & 2Q_{11}Q_{31} \\
Q_{12}^2 & Q_{22}^2 & Q_{32}^2 & 2Q_{22}Q_{12} & 2Q_{32}Q_{22} & 2Q_{12}Q_{32} \\
Q_{13}^2 & Q_{23}^2 & Q_{33}^2 & 2Q_{23}Q_{13} & 2Q_{33}Q_{23} & 2Q_{13}Q_{33} \\
Q_{12}Q_{11} & Q_{22}Q_{21} & Q_{32}Q_{31} & Q_{22}Q_{11} + Q_{12}Q_{21} & Q_{32}Q_{21} + Q_{22}Q_{31} & Q_{12}Q_{31} + Q_{32}Q_{11} \\
Q_{13}Q_{12} & Q_{23}Q_{22} & Q_{33}Q_{32} & Q_{23}Q_{12} + Q_{13}Q_{22} & Q_{33}Q_{22} + Q_{23}Q_{32} & Q_{13}Q_{32} + Q_{33}Q_{12} \\
Q_{11}Q_{13} & Q_{21}Q_{23} & Q_{31}Q_{33} & Q_{21}Q_{13} + Q_{11}Q_{23} & Q_{31}Q_{23} + Q_{21}Q_{33} & Q_{11}Q_{33} + Q_{31}Q_{13}
\end{bmatrix}$$

$$\underline{T}_\varepsilon = \underline{T}_\varepsilon(\alpha_1, \alpha_2, \alpha_3) =$$

$$\begin{bmatrix}
Q_{11}^2 & Q_{21}^2 & Q_{31}^2 & Q_{21}Q_{11} & Q_{31}Q_{21} & Q_{11}Q_{31} \\
Q_{12}^2 & Q_{22}^2 & Q_{32}^2 & Q_{22}Q_{12} & Q_{32}Q_{22} & Q_{12}Q_{32} \\
Q_{13}^2 & Q_{23}^2 & Q_{33}^2 & Q_{23}Q_{13} & Q_{33}Q_{23} & Q_{13}Q_{33} \\
2Q_{12}Q_{11} & 2Q_{22}Q_{21} & 2Q_{32}Q_{31} & Q_{22}Q_{11} + Q_{12}Q_{21} & Q_{32}Q_{21} + Q_{22}Q_{31} & Q_{12}Q_{31} + Q_{32}Q_{11} \\
2Q_{13}Q_{12} & 2Q_{23}Q_{22} & 2Q_{33}Q_{32} & Q_{23}Q_{12} + Q_{13}Q_{22} & Q_{33}Q_{22} + Q_{23}Q_{32} & Q_{13}Q_{32} + Q_{33}Q_{12} \\
2Q_{11}Q_{13} & 2Q_{21}Q_{23} & 2Q_{31}Q_{33} & Q_{21}Q_{13} + Q_{11}Q_{23} & Q_{31}Q_{23} + Q_{21}Q_{33} & Q_{11}Q_{33} + Q_{31}Q_{13}
\end{bmatrix}$$

Inverse transformation

$$\underline{T}_\sigma^{-1} = \underline{T}_\sigma(-\alpha_1, -\alpha_2, -\alpha_3) \qquad ; \qquad \underline{T}_\varepsilon^{-1} = \underline{T}_\varepsilon(-\alpha_1, -\alpha_2, -\alpha_3)$$

B.3.2 Rotation of stiffness and compliance matrices

Using the transformation matrices, the rotated stiffness and compliance matrices can be calculated.

$$\begin{aligned}
\underset{\sim}{\sigma} = \underline{\underline{C}}\, \underset{\sim}{\varepsilon} \quad &\rightarrow \quad \underline{T}_\sigma^{-1} \underset{\sim}{\sigma}^* = \underline{\underline{C}}\, \underline{T}_\varepsilon^{-1}\, \underset{\sim}{\varepsilon}^* \qquad &\rightarrow \qquad \underset{\sim}{\sigma}^* = \underline{T}_\sigma\, \underline{\underline{C}}\, \underline{T}_\varepsilon^{-1}\, \underset{\sim}{\varepsilon}^* = \underline{\underline{C}}^*\, \underset{\sim}{\varepsilon}^* \\
\underset{\sim}{\varepsilon} = \underline{\underline{S}}\, \underset{\sim}{\sigma} \quad &\rightarrow \quad \underline{T}_\varepsilon^{-1} \underset{\sim}{\varepsilon}^* = \underline{\underline{S}}\, \underline{T}_\sigma^{-1}\, \underset{\sim}{\sigma}^* \qquad &\rightarrow \qquad \underset{\sim}{\varepsilon}^* = \underline{T}_\varepsilon\, \underline{\underline{S}}\, \underline{T}_\sigma^{-1}\, \underset{\sim}{\sigma}^* = \underline{\underline{S}}^*\, \underset{\sim}{\sigma}^*
\end{aligned}$$

B.3.3 Rotation about one axis

For the plane strain/stress situation we can easily derive the compliance and stiffness matrix w.r.t. a coordinate system $\{1^*, 2^*, 3\}$ which is rotated anticlockwise (right handed) over an angle α about the 3-axis. The transformation matrix $\underline{T}$ relates strain/stress components : $\underset{\sim}{\sigma}^* = \underline{T}_\sigma\, \underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}^* = \underline{T}_\varepsilon\, \underset{\sim}{\varepsilon}$. Its components are expressed in the cosine and sine of the angle α, ($s = \sin(\alpha)$ and $c = \cos(\alpha)$).

$$\underset{\approx}{\sigma}^T = \left[\begin{array}{ccc} \sigma_{11} & \sigma_{22} & \sigma_{12} \end{array} \right] \quad ; \quad \underset{\approx}{\sigma}^* = \underline{T}_\sigma \, \underset{\approx}{\sigma}$$

$$\underset{\approx}{\varepsilon}^T = \left[\begin{array}{ccc} \varepsilon_{11} & \varepsilon_{22} & \gamma_{12} \end{array} \right] \quad ; \quad \underset{\approx}{\varepsilon}^* = \underline{T}_\varepsilon \, \underset{\approx}{\varepsilon}$$

$$\underline{T}_\sigma = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \qquad \underline{T}_\varepsilon = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix}$$

$$\underline{T}_\sigma^{-1} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \qquad \underline{T}_\varepsilon^{-1} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{bmatrix}$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}} \, \underset{\approx}{\varepsilon} \quad \rightarrow \quad \underline{T}_\sigma^{-1} \underset{\approx}{\sigma}^* = \underline{\underline{C}} \, \underline{T}_\varepsilon^{-1} \, \underset{\approx}{\varepsilon}^* \quad \rightarrow \quad \underset{\approx}{\sigma}^* = \underline{T}_\sigma \, \underline{\underline{C}} \, \underline{T}_\varepsilon^{-1} \, \underset{\approx}{\varepsilon}^* = \underline{\underline{C}}^* \, \underset{\approx}{\varepsilon}^*$$

$$\underset{\approx}{\varepsilon} = \underline{\underline{S}} \, \underset{\approx}{\sigma} \quad \rightarrow \quad \underline{T}_\varepsilon^{-1} \underset{\approx}{\varepsilon}^* = \underline{\underline{S}} \, \underline{T}_\sigma^{-1} \, \underset{\approx}{\sigma}^* \quad \rightarrow \quad \underset{\approx}{\varepsilon}^* = \underline{T}_\varepsilon \, \underline{\underline{S}} \, \underline{T}_\sigma^{-1} \, \underset{\approx}{\sigma}^* = \underline{\underline{S}}^* \, \underset{\approx}{\sigma}^*$$

Rotation of stiffness matrix

The rotation of the planar stiffness matrix can be elaborated. The new components are functions of the original components and of cosine (c) and sine (s) of the rotation angle α. Remember that $\sigma_{12} = C_{44} \, \gamma_{12}$.

$$\begin{aligned}
\underline{\underline{C}}^* &= \underline{T}_\sigma \, \underline{\underline{C}} \, \underline{T}_\varepsilon^{-1} \\
&= \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{bmatrix} \\
&= \begin{bmatrix}
c^4 C_{11} + 2c^2 s^2 C_{12} + s^4 C_{22} + 4c^2 s^2 C_{44} & c^2 s^2 C_{11} + (c^4 + s^4) C_{12} + c^2 s^2 C_{22} - 4c^2 s^2 C_{44} & -c^3 s C_{11} + (c^3 s - cs^3) C_{12} + cs^3 C_{22} + 2cs(c^2 - s^2) C_{44} \\
c^2 s^2 C_{11} + (c^4 + s^4) C_{12} + c^2 s^2 C_{22} - 4c^2 s^2 C_{44} & s^4 C_{11} + 2c^2 s^2 C_{12} + c^4 C_{22} + 4c^2 s^2 C_{44} & -cs^3 C_{11} + (cs^3 - c^3 s) C_{12} + c^3 s C_{22} - 2cs(c^2 - s^2) C_{44} \\
-c^3 s C_{11} + (c^3 s - cs^3) C_{12} + cs^3 C_{22} + 2cs(c^2 - s^2) C_{44} & -cs^3 C_{11} + (cs^3 - c^3 s) C_{12} + c^3 s C_{22} - 2cs(c^2 - s^2) C_{44} & c^2 s^2 C_{11} - 2c^2 s^2 C_{12} + c^2 s^2 C_{22} + (c^2 - s^2)^2 C_{44}
\end{bmatrix} \\
&= \begin{bmatrix}
c^4 C_{11} + s^4 C_{22} + 2c^2 s^2 (C_{12} + 2C_{44}) & c^2 s^2 (C_{11} + C_{22} - 4C_{44}) + (c^4 + s^4) C_{12} & c^3 s (C_{12} - C_{11} + 2C_{44}) + cs^3 (C_{22} - C_{12} - 2C_{44}) \\
c^2 s^2 (C_{11} + C_{22} - 4C_{44}) + (c^4 + s^4) C_{12} & s^4 C_{11} + c^4 C_{22} + 2c^2 s^2 (C_{12} + 2C_{44}) & cs^3 (C_{12} - C_{11} + 2C_{44}) + c^3 s (C_{22} - C_{12} - 2C_{44}) \\
c^3 s (C_{12} - C_{11} + 2C_{44}) + cs^3 (C_{22} - C_{12} - 2C_{44}) & cs^3 (C_{12} - C_{11} + 2C_{44}) + c^3 s (C_{22} - C_{12} - 2C_{44}) & c^2 s^2 (C_{11} - 2C_{12} + C_{22}) + (c^2 - s^2)^2 C_{44}
\end{bmatrix}
\end{aligned}$$

Rotation of compliance matrix

The rotation of the planar compliance matrix can be elaborated. The new components are functions of the original components and of cosine (c) and sine (s) of the rotation angle α. Remember that $\gamma_{12} = S_{44}\,\sigma_{12}$.

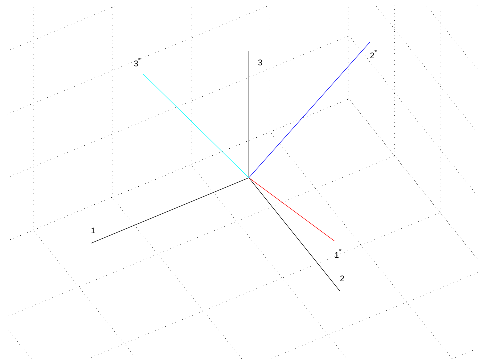
$$
\begin{aligned}
\underline{\underline{S}}^* &= \underline{T}_\varepsilon\, \underline{\underline{S}}\, \underline{T}_\sigma^{-1} \\
&= \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2-s^2 \end{bmatrix}
\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{44} \end{bmatrix}
\begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2-s^2 \end{bmatrix} \\
&= \begin{bmatrix}
c^4 S_{11} + 2c^2s^2 S_{12} + s^4 S_{22} + c^2 s^2 S_{44} \\
\quad c^2s^2 S_{11} + (c^4+s^4) S_{12} + c^2s^2 S_{22} - c^2s^2 S_{44} \\
\qquad -2c^3 s S_{11} + 2(c^3 s - cs^3) S_{12} + 2cs^3 S_{22} + cs(c^2-s^2) S_{44} \\
c^2s^2 S_{11} + (c^4+s^4) S_{12} + c^2s^2 S_{22} - c^2s^2 S_{44} \\
\quad s^4 S_{11} + 2c^2s^2 S_{12} + c^4 S_{22} + c^2s^2 S_{44} \\
\qquad -2cs^3 S_{11} + 2(cs^3 - c^3 s) S_{12} + 2c^3 s S_{22} - cs(c^2-s^2) S_{44} \\
-2c^3 s S_{11} + 2(c^3 s - cs^3) S_{12} + 2cs^3 S_{22} + cs(c^2-s^2) S_{44} \\
\quad -2cs^3 S_{11} + 2(cs^3 - c^3 s) S_{12} + 2c^3 s S_{22} - cs(c^2-s^2) S_{44} \\
\qquad 4c^2s^2 S_{11} - 8c^2s^2 S_{12} + 4c^2s^2 S_{22} + (c^2-s^2)^2 S_{44}
\end{bmatrix} \\
&= \begin{bmatrix}
c^4 S_{11} + s^4 S_{22} + c^2s^2(2S_{12} + S_{44}) \\
\quad c^2s^2(S_{11} + S_{22} - S_{44}) + (c^4+s^4) S_{12} \\
\qquad c^3 s(2S_{12} - 2S_{11} + S_{44}) + cs^3(2S_{22} - 2S_{12} - S_{44}) \\
c^2s^2(S_{11} + S_{22} - S_{44}) + (c^4+s^4) S_{12} \\
\quad s^4 S_{11} + c^4 S_{22} + c^2s^2(2S_{12} + S_{44}) \\
\qquad cs^3(2S_{12} - 2S_{11} + S_{44}) + c^3 s(2S_{22} - 2S_{12} - S_{44}) \\
c^3 s(2S_{12} - 2S_{11} + S_{44}) + cs^3(2S_{22} - 2S_{12} - S_{44}) \\
\quad cs^3(2S_{12} - 2S_{11} + S_{44}) + c^3 s(2S_{22} - 2S_{12} - S_{44}) \\
\qquad 4c^2s^2(S_{11} - 2S_{12} + S_{22}) + (c^2-s^2)^2 S_{44}
\end{bmatrix}
\end{aligned}
$$

B.3.4 Example

This matrix transformation can easily be done with a Matlab programs. The procedure is illustrated with an example for the stiffness matrix of a polyethylene crystal, which can be found in literature. In the chain direction – the 3-direction – the stiffness is very high because deformations involve primarily bending and stretching of covalent bonds. Perpendicular to the chain direction the stiffness is much lower because deformation is resisted by weak Van der Waals forces. It should be noted that experimental values are rather lower and depend strongly on temperature.

First the compliance matrix is calculated by inversion. Then both matrices are transformed to a rotated coordinate system with rotation angles $\{20^o 30^o 90^o\}$.

a14



Rotation of material coordinate system

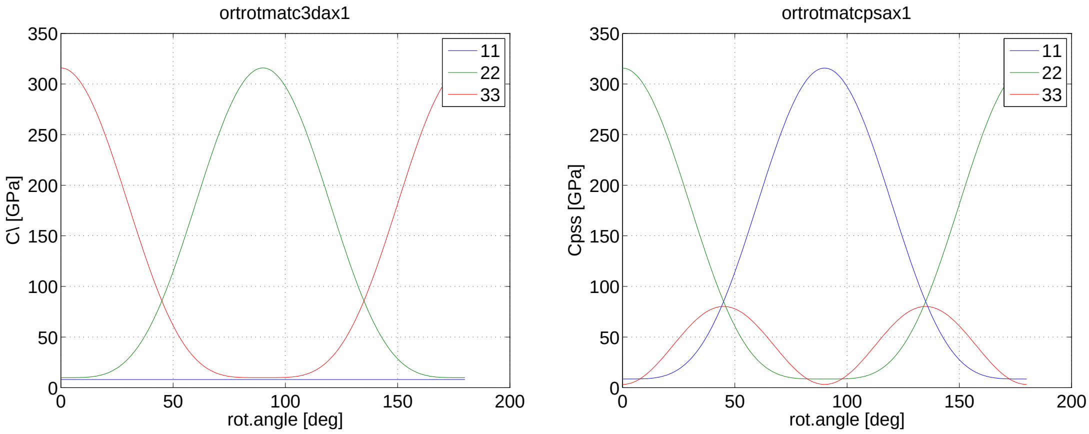
$$\underline{\underline{C}} = \begin{bmatrix} 7.99 & 3.28 & 1.13 & 0 & 0 & 0 \\ 3.28 & 9.92 & 2.14 & 0 & 0 & 0 \\ 1.13 & 2.14 & 315.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.62 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.19 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.62 \end{bmatrix} \text{ [GPa]}$$

$$\underline{\underline{S}} = \begin{bmatrix} 14.5 & -4.78 & -0.019 & 0 & 0 & 0 \\ -4.78 & 11.7 & -0.062 & 0 & 0 & 0 \\ -0.019 & -0.062 & 0.317 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 31.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 61.7 \end{bmatrix} \times 10^{-2} \text{ [GPa}^{-1}\text{]}$$

The resulting matrices are :

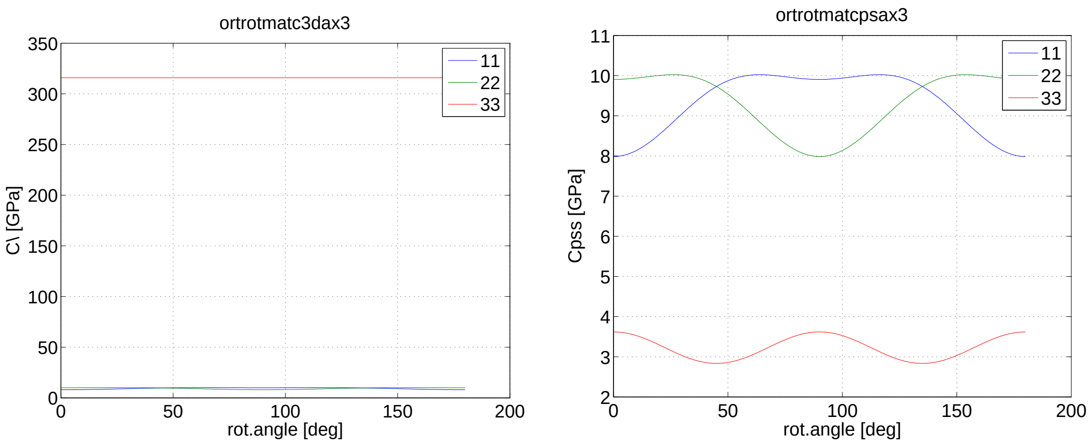
```
Ct =    14.6930    6.5535   29.6372   -4.0679    9.6848  -11.1766
        20.5236   12.2265   47.4614  -11.5658   15.5734  -27.1345
        56.0859   20.6217  139.1272  -29.3662   48.9284  -85.3252
        10.6101    5.4199   32.7978   -5.2719   13.9517  -21.2424
        30.7978    8.1694   78.4859  -18.8612   29.9577  -49.1406
        18.3772    6.5841   59.6109  -14.1472   20.8241  -34.9673

St =     0.0336   -0.0292    0.0562    0.0478    0.0717    0.1313
        -0.0134    0.1558   -0.0673   -0.1654    0.0701    0.0163
         0.0316   -0.0389    0.0390    0.1372   -0.1919   -0.0939
        -0.0800    0.1056   -0.0468    0.1060    0.2255   -0.0106
         0.0780   -0.0447   -0.1084   -0.1747    0.2436    0.0635
        -0.1385   -0.0180    0.1198   -0.0404   -0.0884    0.0632
```

Now we take the initial stiffness matrix $\underline{\underline{C}}$ again and rotate it about the material 1-axis. The left figure shows rotated components as a function of the rotation angle. When the 1-axis is assumed to be the axis perpendicular to the plane of plane stress state, the plane stress stiffness matrix $\underline{\underline{C}}_{\sigma}$ can be calculated and also rotated about the 1-axis. The right figure shows rotated components as a function of the rotation angle.



Three-dimensional and plane stress stiffness components

The same is done for totation about the material 3-axis, which is then also taken as the axis perpendicular to the panar plane.



Three-dimensional and plane stress stiffness components

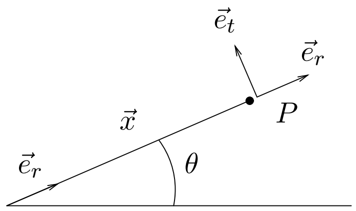
a16

Appendix C

Centrifugal load

When a point mass point rotates about the z-axis with radial velocity $\omega = \dot{\theta}$, the velocity and acceleration can be calculated.

$$
\begin{aligned}
&\vec{x} = r\vec{e}_r(\theta) + z\vec{e}_z \qquad\qquad \text{with} \qquad \dot{z} = 0 \\
&\dot{\vec{x}} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\,\frac{d\vec{e}_r}{d\theta}\,\omega = \dot{r}\vec{e}_r + r\omega\vec{e}_t \\
&\ddot{\vec{x}} = \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + \dot{r}\omega\vec{e}_t + r\dot{\omega}\vec{e}_t + r\omega\dot{\vec{e}}_t = \left(\ddot{r} - r\omega^2\right)\vec{e}_r + \left(2\dot{r}\omega + r\dot{\omega}\right)\vec{e}_t \\
&\qquad \text{constant } r \text{ and } \omega \qquad \rightarrow \qquad \ddot{\vec{x}} = -r\omega^2\vec{e}_r = \ddot{u}_r\vec{e}_r
\end{aligned}
$$



Rotation of a mass point

a18

Appendix D

Radial temperature field

An axi-symmetric material body may be subjected to a radial temperature field, resulting in non-homogeneous strains and stresses. The considered temperature is a cubic function of the radial coordinate. Its constants are a related to the temperatures and temperature gradients at the inner and the outer edges of the body.

Cubic temperature function

The radial temperature field is a third-order function of the radius.

$$\begin{aligned} T(r) &= a_0 + a_1 r + a_2 r^2 + a_3 r^3 \\ \frac{dT}{dr} &= a_1 + 2a_2 r + 3a_3 r^2 \end{aligned}$$

Boundary values

The coefficients in the temperature function are expressed in the boundary values of temperature T and its derivative $\frac{dT}{dr} = T_{,r}$. The boundaries are the inner and outer edge of the disc, with radius r_1 and r_2, respectively.

$$\begin{aligned} T(r = r_1) &= f_1 = a_0 + a_1 r_1 + a_2 r_1^2 + a_3 r_1^3 \\ T(r = r_2) &= f_2 = a_0 + a_1 r_2 + a_2 r_2^2 + a_3 r_2^3 \\ T_{,r}(r = r_1) &= f_3 = a_1 + 2a_2 r_1 + 3a_3 r_1^2 \\ T_{,r}(r = r_2) &= f_4 = a_1 + 2a_2 r_2 + 3a_3 r_2^2 \end{aligned}$$

in matrix notation

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_1^2 & r_1^3 \\ 1 & r_2 & r_2^2 & r_2^3 \\ 0 & 1 & 2r_1 & 3r_1^2 \\ 0 & 1 & 2r_2 & 3r_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{f} = \underline{M}\, \underset{\sim}{a}$$

a20

Coefficients

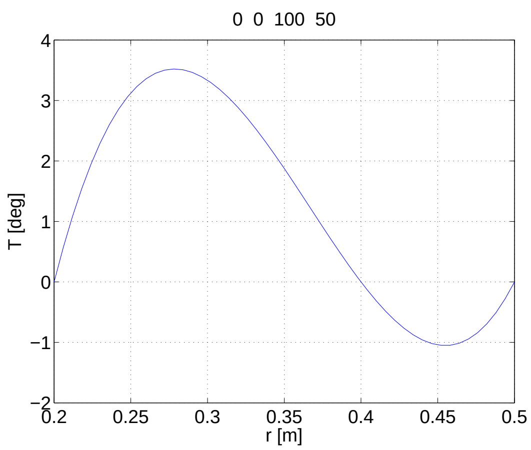
The coefficients can be expressed in the boundary values of temperature and temperature derivative. This can be done numerically by inversion of $\underset{\sim}{f} = \underline{M}\,\underset{\sim}{a}$. Here, also the analytical expressions are presented.

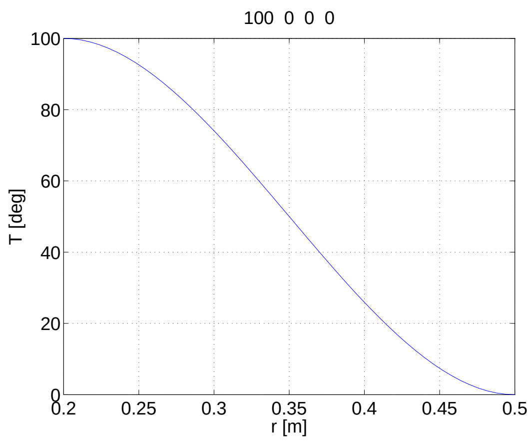
$$
\begin{aligned}
a_3 &= \frac{F_4 X_{12} - F_3 X_{22}}{X_{13} X_{22} - X_{23} X_{12}} \\
a_2 &= -\frac{F_3}{X_{12}} - \frac{X_{13}}{X_{12}}\, a_3 \\
a_1 &= \frac{f_2 - f_1}{r_2 - r_1} - (r_2 + r_1) a_2 - \frac{r_2^3 - r_1^3}{r_2 - r_1}\, a_3 \\
a_0 &= f_1 - r_1 a_1 - r_1^2 a_2 - r_1^3 a_3
\end{aligned}
$$

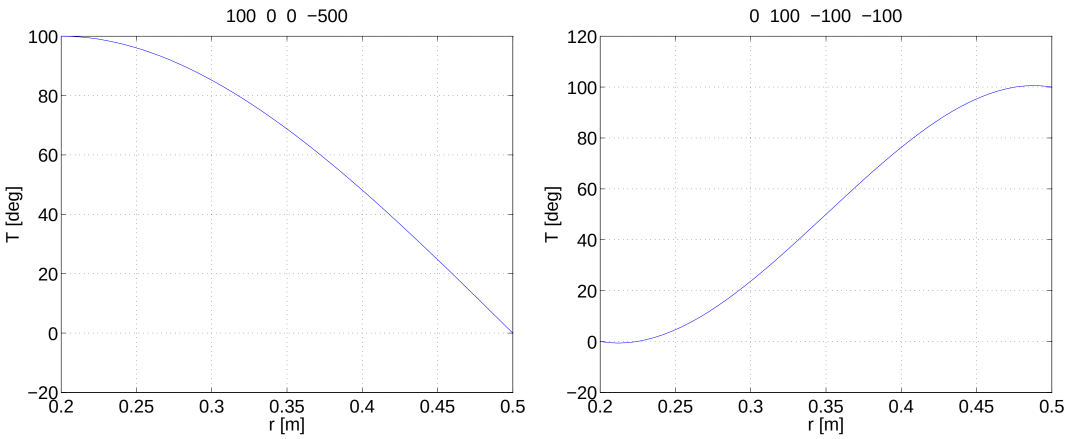
$$
\begin{aligned}
F_3 &= f_3(r_2 - r_1) - (f_2 - f_1) \\
F_4 &= f_4(r_2 - r_1) - (f_2 - f_1) \\
X_{12} &= r_2^2 - r_1^2 - 2r_1(r_2 - r_1) \\
X_{13} &= r_2^3 - r_1^3 - 3r_1^2(r_2 - r_1) \\
X_{22} &= r_2^2 - r_1^2 - 2r_2(r_2 - r_1) \\
X_{23} &= r_2^3 - r_1^3 - 3r_2^2(r_2 - r_1)
\end{aligned}
$$

Temperature fields

As an example, some temperature fields are plotted. The radius ranges between 0.2 and 0.5 m. The title of the plots gives the values of $T(r_1)$, $T(r_2)$, $T_{,r}(r1)$ and $T_{,r}(r2)$, respectively.







Radial temparature fields

a22

Appendix E

Examples

In this appendix a number of examples is shown. The general relations for the analytical solutions can be found in chapter 9 and are specified here. The integration constants are calculated for the specific boundary conditions and loading, and are given here.

E.1 Governing equations and general solution

$$\boxed{u_{r,rr} + \frac{1}{r}\, u_{r,r} - \zeta^2 \, \frac{1}{r^2}\, u_r = f(r)}$$

with $\quad \zeta = \sqrt{\dfrac{B_p}{A_p}}$

and $\quad f(r) = \dfrac{\rho}{A_p}\,(\ddot{u}_r - q_r) + \dfrac{A_p + Q_p}{A_p}\,\alpha(\Delta T)_{,r} + \dfrac{A_p - B_p}{A_p}\dfrac{1}{r}\,\alpha\Delta T$

orthotropic material :

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} + \bar{u}_r$

$\varepsilon_{rr} = c_1\zeta r^{\zeta-1} - c_2\zeta r^{-\zeta-1} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1} + \dfrac{\bar{u}_r}{r}$

$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} + A_p\bar{u}_{r,r} + Q_p\dfrac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$

$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} + Q_p\bar{u}_{r,r} + B_p\dfrac{\bar{u}_r}{r} - (B_p + Q_p)\alpha\Delta T$

isotropic material :

general solution $\quad u_r = c_1 r + \dfrac{c_2}{r} + \bar{u}_r$

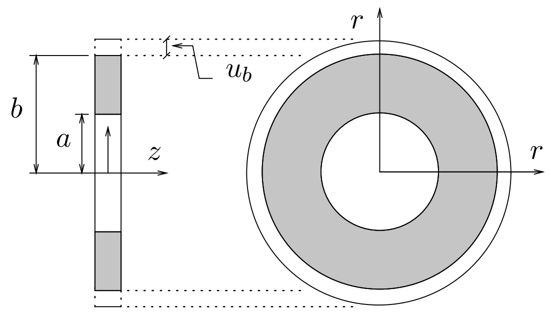
$\varepsilon_{rr} = c_1 - c_2 r^{-2} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2} + \dfrac{\bar{u}_r}{r}$

$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\dfrac{c_2}{r^2} + A_p\bar{u}_{r,r} + Q_p\,\dfrac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$

$\sigma_{tt} = (Q_p + A_p)c_1 - (Q_p - A_p)\dfrac{c_2}{r^2} + Q_p\bar{u}_{r,r} + A_p\,\dfrac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$

For plane strain and plane stress the material parameters can be found in appendix A.

E.2 Disc, edge displacement



$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$u_r(r = b) = u_b$$
$$\sigma_{rr}(r = a) = 0;$$

Edge displacement of circular disc

orthotropic material :

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta}$

$$\varepsilon_{rr} = c_1 \zeta r^{\zeta-1} - c_2 \zeta r^{-\zeta-1} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1}$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1}$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1}$$

$$c_1 = \frac{(A_p\zeta - Q_p)b^{\zeta}\, u_b}{(A_p\zeta + Q_p)a^{2\zeta} + (A_p\zeta - Q_p)b^{2\zeta}} \quad ; \quad c_2 = \frac{(A_p\zeta + Q_p)b^{\zeta}a^{2\zeta}\, u_b}{(A_p\zeta + Q_p)a^{2\zeta} + (A_p\zeta - Q_p)b^{2\zeta}}$$

isotropic material :

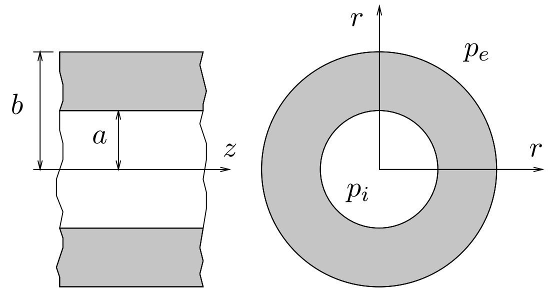
general solution $\quad u_r = c_1 r + \dfrac{c_2}{r}$

$$\varepsilon_{rr} = c_1 - c_2 r^{-2} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2}$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2}$$
$$\sigma_{tt} = (Q_p + A_p)c_1 - (Q_p - A_p)\frac{c_2}{r^2}$$

$$c_1 = \frac{(A_p - Q_p)b}{(A_p + Q_p)a^2 + (A_p - Q_p)b^2}\, u_b \quad ; \quad c_2 = \frac{(A_p + Q_p)ba^2}{(A_p + Q_p)a^2 + (A_p - Q_p)b^2}\, u_b$$

E.3 Disc/cylinder, edge load



$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$
$\sigma_{rr}(r = a) = -p_i$
$\sigma_{rr}(r = b) = -p_e$

Cross-section of a thick-walled circular cylinder

orthotropic material :

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta}$

$$\varepsilon_{rr} = c_1 \zeta r^{\zeta-1} - c_2 \zeta r^{-\zeta-1} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1}$$

$$\sigma_{rr} = (A_p \zeta + Q_p) c_1 r^{\zeta-1} - (A_p \zeta - Q_p) c_2 r^{-\zeta-1}$$
$$\sigma_{tt} = (Q_p \zeta + B_p) c_1 r^{\zeta-1} - (Q_p \zeta - B_p) c_2 r^{-\zeta-1}$$

$$c_1 = \frac{1}{A_p \zeta + Q_p} \frac{a^{\zeta+1} p_i - b^{\zeta+1} p_e}{b^{2\zeta} - a^{2\zeta}} \quad ; \quad c_2 = \frac{1}{A_p \zeta - Q_p} \frac{a^{\zeta+1} b^{2\zeta} p_i - b^{\zeta+1} a^{2\zeta} p_e}{b^{2\zeta} - a^{2\zeta}}$$

isotropic material :

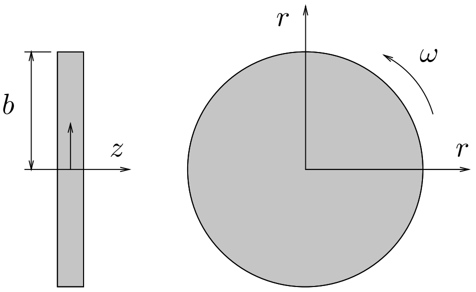
general solution $\quad u_r = c_1 r + \dfrac{c_2}{r}$

$$\varepsilon_{rr} = c_1 - c_2 r^{-2} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2}$$

$$\sigma_{rr} = (A_p + Q_p) c_1 - (A_p - Q_p) \frac{c_2}{r^2}$$
$$\sigma_{tt} = (Q_p + A_p) c_1 - (Q_p - A_p) \frac{c_2}{r^2}$$

$$c_1 = \frac{1}{A_p + Q_p} \frac{1}{b^2 - a^2} (p_i a^2 - p_e b^2) \quad ; \quad c_2 = \frac{1}{A_p - Q_p} \frac{a^2 b^2}{b^2 - a^2} (p_i - p_e)$$

E.4 Rotating solid disc



$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$
$$u_r(r=0) \neq \infty$$
$$\sigma_{rr}(r=b) = 0$$

A rotating solid disc

orthotropic material :

$$u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} - \frac{1}{A_p}\beta r^3 \quad \text{with} \quad \beta = \frac{1}{9-\zeta}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} - \frac{3A_p + Q_p}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} - \frac{3Q_p + B_p}{A_p}\,\beta r^2$$

$$c_2 = 0 \qquad ; \qquad c_1 = \frac{3A_p + Q_p}{A_p(A_p\zeta + Q_p)}\,\beta b^{-\zeta+3}$$

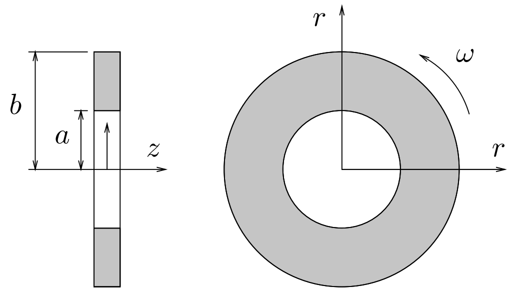
isotropic material :

$$u_r = c_1 r + \frac{c_2}{r} - \frac{1}{8}\,\frac{\rho}{A_p}\,\omega^2 r^3 = c_1 r + \frac{c_2}{r} - \frac{1}{A_p}\,\beta r^3 \quad \text{with} \quad \beta = \frac{1}{8}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} - \frac{(3A_p + Q_p)}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (A_p + Q_p)c_1 + (A_p - Q_p)\frac{c_2}{r^2} - \frac{(A_p + 3Q_p)}{A_p}\,\beta r^2$$

$$c_2 = 0 \qquad ; \qquad c_1 = \frac{(3A_p + Q_p)}{A_p(A_p + Q_p)}\,\beta b^2$$

E.5 Rotating disc with central hole



$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$
$$\sigma_{rr}(r=a) = 0$$
$$\sigma_{rr}(r=b) = 0$$

A rotating disc with central hole

orthotropic material :

$$u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} - \frac{1}{A_p}\beta r^3 \quad \text{with} \quad \beta = \frac{1}{9-\zeta}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} - \frac{3A_p + Q_p}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} - \frac{3Q_p + B_p}{A_p}\,\beta r^2$$

$$c_1 = \frac{3A_p + Q_p}{A_p(A_p\zeta + Q_p)}\left(\frac{b^{\zeta+3} - a^{\zeta+3}}{b^{2\zeta} - a^{2\zeta}}\right)\beta$$
$$c_2 = \frac{3A_p + Q_p}{A_p(A_p\zeta - Q_p)}\left(\frac{a^{2\zeta-2}b^{\zeta+1} - a^{\zeta+1}b^{2\zeta-2}}{b^{2\zeta} - a^{2\zeta}}\right)(a^2b^2)\beta$$

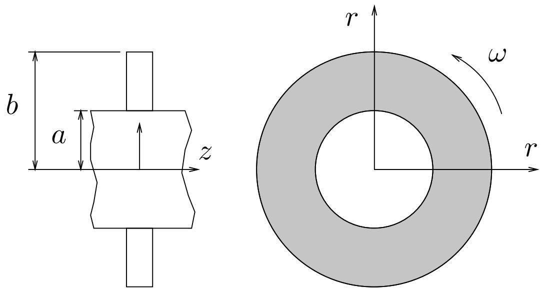
isotropic material :

$$u_r = c_1 r + \frac{c_2}{r} - \frac{1}{8}\,\frac{\rho}{A_p}\,\omega^2 r^3 = c_1 r + \frac{c_2}{r} - \frac{1}{A_p}\,\beta r^3 \quad \text{with} \quad \beta = \frac{1}{8}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} - \frac{(3A_p + Q_p)}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (A_p + Q_p)c_1 + (A_p - Q_p)\frac{c_2}{r^2} - \frac{(A_p + 3Q_p)}{A_p}\,\beta r^2$$

$$c_1 = \frac{(3A_p + Q_p)}{A_p(A_p + Q_p)}\,(a^2 + b^2)\beta \quad ; \quad c_2 = \frac{(3A_p + Q_p)}{A_p(A_p - Q_p)}\,(a^2b^2)\beta$$

E.6 Rotating disc fixed on rigid axis



$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$
$$u_r(r=a) = 0$$
$$\sigma_{rr}(r=b) = 0$$

Disc fixed on rigid axis

orthotropic material :

$$u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} - \frac{1}{A_p}\beta r^3 \quad \text{with} \quad \beta = \frac{1}{9-\zeta}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} - \frac{3A_p + Q_p}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} - \frac{3Q_p + B_p}{A_p}\,\beta r^2$$

$$c_1 = \frac{\beta}{(A_p\zeta + Q_p)b^{\zeta+1}a^{-\zeta+1} + (A_p\zeta - Q_p)b^{-\zeta+1}a^{\zeta+1}} \left\{\frac{3A_p + Q_p}{A_p}\,b^4 a^{-\zeta+1} + \frac{A_p\zeta - Q_p}{A_p}\,b^{-\zeta+1}a^4\right\}$$
$$c_2 = \frac{\beta}{(A_p\zeta + Q_p)b^{\zeta+1}a^{-\zeta+1} + (A_p\zeta - Q_p)b^{-\zeta+1}a^{\zeta+1}} \left\{\frac{A_p\zeta + Q_p}{A_p}\,b^{\zeta+1}a^4 - \frac{3A_p + Q_p}{A_p}\,b^4 a^{\zeta+1}\right\}$$

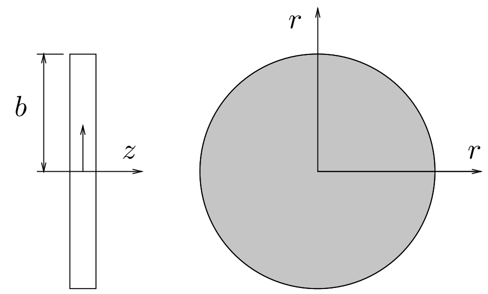
isotropic material :

$$u_r = c_1 r + \frac{c_2}{r} - \frac{1}{8}\,\frac{\rho}{A_p}\,\omega^2 r^3 = c_1 r + \frac{c_2}{r} - \frac{1}{A_p}\,\beta r^3 \quad \text{with} \quad \beta = \frac{1}{8}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} - \frac{(3A_p + Q_p)}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (A_p + Q_p)c_1 + (A_p - Q_p)\frac{c_2}{r^2} - \frac{(A_p + 3Q_p)}{A_p}\,\beta r^2$$

$$c_1 = \frac{\beta}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2}\left\{\frac{3A_p + Q_p}{A_p}\,b^4 + \frac{A_p - Q_p}{A_p}\,a^4\right\}$$
$$c_2 = \frac{\beta}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2}\left\{\frac{A_p + Q_p}{A_p}\,a^4 b^2 - \frac{3A_p + Q_p}{A_p}\,a^2 b^4\right\}$$

E.7 Thermal load



$$f(r) = \frac{A_p + Q_p}{A_p}\,\alpha(\Delta T)_{,r}$$
$$u_r(r=0) \neq \infty$$
$$\sigma_{rr}(r=b) = 0$$

Solid disc with a radial thermal load

orthotropic material :

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} + \bar{u}_r$

$$\varepsilon_{rr} = c_1\zeta r^{\zeta-1} - c_2\zeta r^{-\zeta-1} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1} + \frac{\bar{u}_r}{r}$$
$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} + A_p\bar{u}_{r,r} + Q_p\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} + Q_p\bar{u}_{r,r} + B_p\frac{\bar{u}_r}{r} - (B_p + Q_p)\alpha\Delta T$$

isotropic material :

general solution $\quad u_r = c_1 r + \frac{c_2}{r} + \bar{u}_r$

$$\varepsilon_{rr} = c_1 - c_2 r^{-2} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2} + \frac{\bar{u}_r}{r}$$
$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} + A_p\bar{u}_{r,r} + Q_p\,\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$$
$$\sigma_{tt} = (Q_p + A_p)c_1 - (Q_p - A_p)\frac{c_2}{r^2} + Q_p\bar{u}_{r,r} + A_p\,\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$$