COMPUTATIONAL MATERIAL MODELS

Piet Schreurs

Department of Mechanical Engineering
Eindhoven University of Technology

http://www.mate.tue.nl/~piet

2013/2014

June 19, 2014

INDEX

back to index

- Homogeneous truss
- Finite element method
- Nonlinear deformation
- Weighted residual formulation
- Finite element method

- One-dimensional material behavior
- Elastic
- Elastoplastic
- Linear viscoelastic
- Creep
- Viscoplastic
- Nonlinear viscoelastic

- Vectors and tensors
- Kinematics
- Small (linear) deformation
- Stress
- Balance laws
- Constitutive equations
- Linear elastic material
- Material symmetry
- Linear elastic isotropic material : engineering parameters
- Linear elastic isotropic material : tensorial form
- Thermo-elasticity
- Planar deformation
- Elastic limit

- Governing equations
- Solution strategies
- Weighted residual formulation

- Finite element method
- Numerical solutions

- Three-dimensional material models

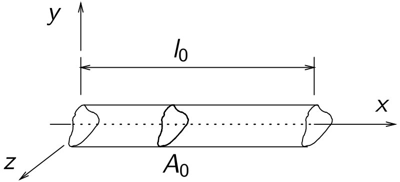
- Elastic
- Elastoplastic
- Linear viscoelastic
- Viscoplastic
- Nonlinear viscoelastic

- APPENDICES
- Tutorial ; tr2d.m
- Utilities m2cc.m and m2mm.m
- Stiffness and compliance matrices
- WR for axi-symmetric deformation
- FEM for axi-symmerric deformation
- FEM for planar deformation

HOMOGENEOUS TRUSS

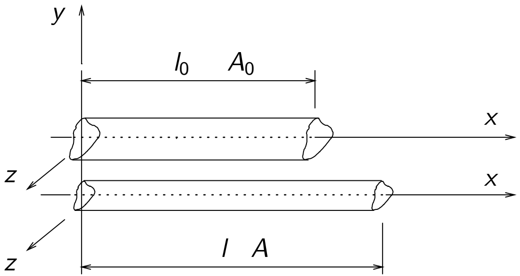
back to index

Homogeneous truss



length		l_0
truss cylindrical	$\longrightarrow$	
cross-sectional area uniform		A_0

Elongation and contraction



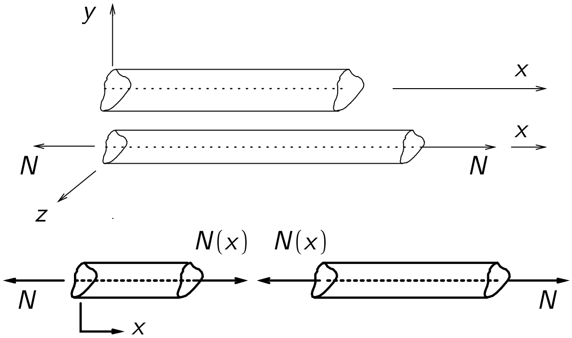
elongation factor $\qquad \lambda = \dfrac{l}{l_0} = \dfrac{l_0 + \Delta l}{l_0} = 1 + \dfrac{\Delta l}{l_0}$

contraction $\qquad \mu = \sqrt{\dfrac{A}{A_0}}$

volume change $\qquad J = \dfrac{lA}{l_0 A_0} = \lambda\mu^2$

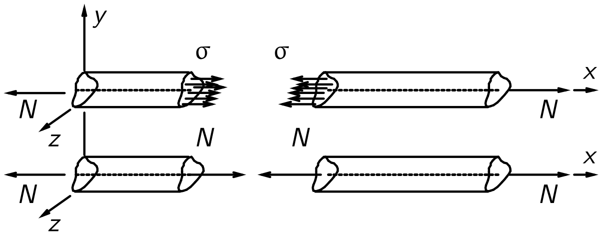
exampl. circular cross section $\quad \rightarrow \quad \mu = \dfrac{d}{d_0} = \sqrt{\dfrac{A}{A_0}}$

Stress



axial tensile force (external) N
cross-sectional force (internal) $N(x) = N$

Axial stress



axial stress $\sigma = \sigma(y,z)$

cross-sectional force $N(x) = N = \int_A \sigma(y,z)\, dA$

stress uniform in cross-section $N = \int_A \sigma\, dA = \sigma A$

true stress $\sigma = \dfrac{N}{A}$

engineering stress $\sigma_n = \dfrac{N}{A_0}$

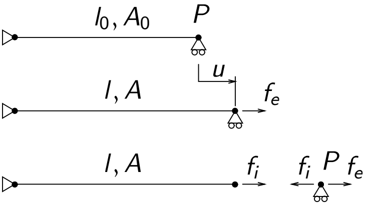
relation $\sigma = \dfrac{N}{A} = \dfrac{A_0}{A}\,\dfrac{N}{A_0} = \dfrac{1}{\mu^2}\,\sigma_n$

Linear elastic behavior

axial stress ~ strain	$\sigma = E\varepsilon$
contraction strain	$\varepsilon = \lambda - 1 \quad \rightarrow$
	$\varepsilon_d = \mu - 1 = -\nu\varepsilon = -\nu(\lambda - 1)$
volume change	$J = (\varepsilon + 1)(-\nu\varepsilon + 1)^2 \approx \varepsilon(1 - 2\nu) + 1$

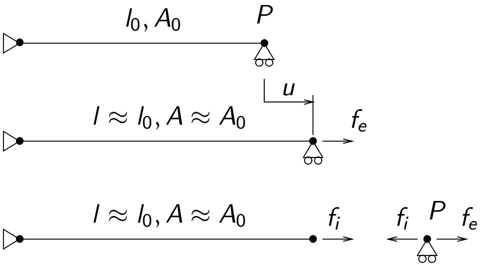
material	E [GPa]	ν [-]	material	E [GPa]	ν [-]
Aluminum	69 - 79	0.31 - 0.34	Copper	105 - 150	0.33 - 0.35
Cast iron	105 - 150	0.21 - 0.30	Steel	200	0.33
Stainless steel	190 - 200	0.28	Lead	14	0.43
Magnesium	41 - 45	0.29 - 0.35	Nickel	180 - 215	0.31
Titanium	80 - 130	0.31 - 0.34	Tungsten	400	0.27
Diamond	820 - 1050	-	Graphite	240 - 390	-
Glass	70 - 80	0.24	Epoxy	3.5 - 17	0.34
Nylon	1.4 - 2.8	0.32 - 0.40	Rubber	0.01 - 0.1	0.5

Equilibrium



external force	f_e
internal force	$f_i = f_i(u)$
equilibrium of point P	$f_i(u) = f_e$

Linear deformation



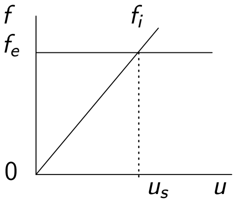
external force	f_e
internal force	$f_i = \sigma_n A_0$
equilibrium of point P	$f_i(u) = f_e$

$f_i(u)$ linear — direct solution possible

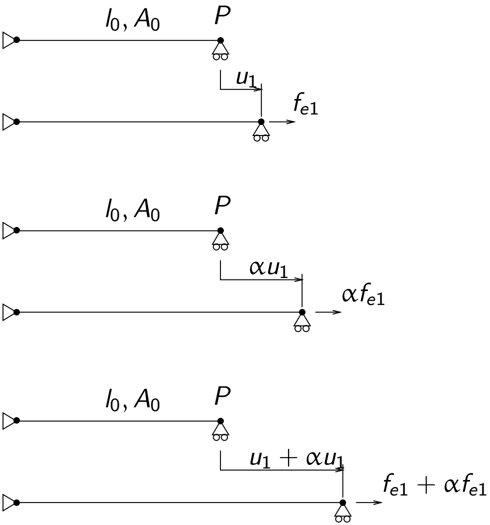
Direct solution

$$f_i = \sigma_n A_0 = E\,\varepsilon A_0 = \frac{EA_0}{l_0}\,u = Ku$$

$$f_i = f_e \quad \rightarrow \quad Ku = f_e \quad \rightarrow \quad u = u_s = \frac{f_e}{K} = \frac{l_0}{EA_0}\,f_e$$



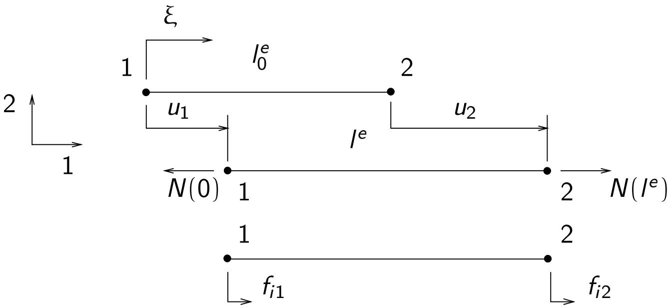
Proportionality and superposition



FINITE ELEMENT METHOD

back to index

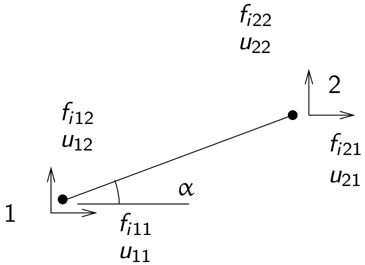
Truss element



$$\underset{\sim}{u}^e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u(0) \\ u(l^e) \end{bmatrix}$$

$$\underset{\sim}{f}_i^e = \begin{bmatrix} f_{i1} \\ f_{i2} \end{bmatrix} = \begin{bmatrix} -N(0) \\ N(l^e) \end{bmatrix} = \begin{bmatrix} -k(u_2 - u_1) \\ k(u_2 - u_1) \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

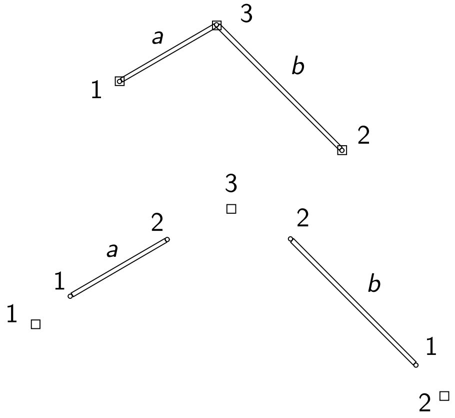
Two-dimensional element



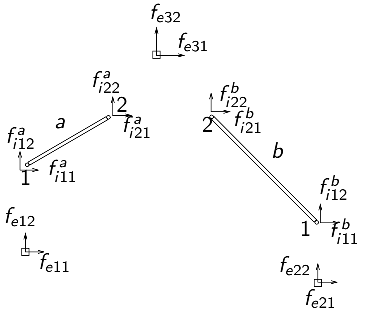
$$\begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \end{bmatrix} = \begin{bmatrix} cf_{i1}^L \\ sf_{i1}^L \\ cf_{i2}^L \\ sf_{i2}^L \end{bmatrix} = k(u_2^L - u_1^L) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix} = k(u_{21}c + u_{22}s - u_{11}c - u_{12}s) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}$$

$$= k \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} = \underline{K}^e \, \underset{\sim}{u}^e \qquad \begin{cases} c = \cos(\alpha) \\ s = \sin(\alpha) \end{cases}$$

Assembling



Assembling : internal forces



$$\begin{bmatrix} f_{e11} \\ f_{e12} \\ f_{e21} \\ f_{e22} \\ f_{e31} \\ f_{e32} \end{bmatrix} = \begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \\ f_{i31} \\ f_{i32} \end{bmatrix} = \begin{bmatrix} f^a_{i11} \\ f^a_{i12} \\ f^b_{i11} \\ f^b_{i12} \\ f^a_{i21} + f^b_{i21} \\ f^a_{i22} + f^b_{i22} \end{bmatrix} = \begin{bmatrix} f^a_{i11} \\ f^a_{i12} \\ 0 \\ 0 \\ f^a_{i21} \\ f^b_{i22} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f^b_{i11} \\ f^b_{i12} \\ f^b_{i21} \\ f^b_{i22} \end{bmatrix}$$

Assembling : nodal displacements

$$
\begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \\ f_{i31} \\ f_{i32} \end{bmatrix} =
\begin{bmatrix}
k^a c^2 & k^a cs & 0 & 0 & -k^a c^2 & -k^a cs \\
k^a cs & k^a s^2 & 0 & 0 & -k^a cs & -k^a s^2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-k^a c^2 & -k^a cs & 0 & 0 & k^a c^2 & k^a cs \\
-k^a cs & -k^a s^2 & 0 & 0 & k^a cs & k^a s^2
\end{bmatrix}
\begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{bmatrix} +
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k^b c^2 & k^b cs & -k^b c^2 & -k^b cs \\
0 & 0 & k^b cs & k^b s^2 & -k^b cs & -k^b s^2 \\
0 & 0 & -k^b c^2 & -k^b cs & k^b c^2 & k^b cs \\
0 & 0 & -k^b cs & -k^b s^2 & k^b cs & k^b s^2
\end{bmatrix}
\begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{bmatrix} =
$$

$$
\begin{bmatrix}
k^a c^2 & k^a cs & 0 & 0 & -k^a c^2 & -k^a cs \\
k^a cs & k^a s^2 & 0 & 0 & -k^a cs & -k^a s^2 \\
0 & 0 & k^b c^2 & k^b cs & -k^b c^2 & -k^b cs \\
0 & 0 & k^b cs & k^b s^2 & -k^b cs & -k^b s^2 \\
-k^a c^2 & -k^a cs & -k^b c^2 & -k^b cs & k^a c^2 + k^b c^2 & k^a cs + k^b cs \\
-k^a cs & -k^a s^2 & -k^b cs & -k^b s^2 & k^a cs + k^b cs & k^a s^2 + k^b s^2
\end{bmatrix}
\begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{bmatrix}
$$

Assembled system equations $\quad \utilde{f}_i = \underline{K}\,\utilde{u}$

Boundary conditions

equilibrium	$\underset{\sim}{f}_i = \underset{\sim}{f}_e \quad \rightarrow \quad \underline{K}\,\underset{\sim}{u} = \underset{\sim}{f}_e = \underset{\sim}{f}$
rigid translation	$\underset{\sim}{u} = \underset{\sim}{a}$
no forces needed	$\underline{K}\;\underset{\sim}{a} = \underset{\sim}{0}$ with $\underset{\sim}{a} \neq \underset{\sim}{0} \quad \rightarrow \quad \underline{K}$ singular

Prescribed nodal displacements and forces

reorganizing $\qquad \underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u}_u \\ \underset{\sim}{u}_p \end{bmatrix} \qquad ; \qquad \underset{\sim}{f} = \begin{bmatrix} \underset{\sim}{f}_u \\ \underset{\sim}{f}_p \end{bmatrix}$

Partitioning for boundary conditions

reorganizing $\quad \underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u}_u \\ \underset{\sim}{u}_p \end{bmatrix} \quad ; \quad \underset{\sim}{f} = \begin{bmatrix} \underset{\sim}{f}_u \\ \underset{\sim}{f}_p \end{bmatrix}$

equilibrium $\quad \underline{K}\,\underset{\sim}{u} = \underset{\sim}{f}$

partitioning
$$\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{pu} & \underline{K}_{pp} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}_u \\ \underset{\sim}{u}_p \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_u \\ \underset{\sim}{f}_p \end{bmatrix} \quad \rightarrow$$

$$\left.\begin{array}{l} \underline{K}_{uu}\underset{\sim}{u}_u + \underline{K}_{up}\underset{\sim}{u}_p = \underset{\sim}{f}_u \\ \underline{K}_{pu}\underset{\sim}{u}_u + \underline{K}_{pp}\underset{\sim}{u}_p = \underset{\sim}{f}_p \end{array}\right\}$$

solving $\underset{\sim}{u}_u$ $\quad \underline{K}_{uu}\underset{\sim}{u}_u = \underset{\sim}{f}_u - \underline{K}_{up}\underset{\sim}{u}_p \quad \rightarrow \quad \underset{\sim}{u}_u = \underline{K}_{uu}^{-1}(\underset{\sim}{f}_u - \underline{K}_{up}\underset{\sim}{u}_p)$

calculating $\underset{\sim}{f}_p$ $\quad \underset{\sim}{f}_p = \underline{K}_{pu}\underset{\sim}{u}_u + \underline{K}_{pp}\underset{\sim}{u}_p$

Links

equilibrium $\quad \underline{K}\,\tilde{u} = \tilde{f}$

partitioning

$$\begin{bmatrix} \underline{K}_{ff} & \underline{K}_{fr} & \underline{K}_{fl} \\ \underline{K}_{rf} & \underline{K}_{rr} & \underline{K}_{rl} \\ \underline{K}_{lf} & \underline{K}_{lr} & \underline{K}_{ll} \end{bmatrix} \begin{bmatrix} \tilde{u}_f \\ \tilde{u}_r \\ \tilde{u}_l \end{bmatrix} = \begin{bmatrix} \tilde{f}_f \\ \tilde{f}_r + \bar{\tilde{f}}_r \\ \tilde{f}_l + \bar{\tilde{f}}_l \end{bmatrix}$$

$$\begin{aligned} \underline{K}_{ff}\,\tilde{u}_f + \underline{K}_{fr}\,\tilde{u}_r + \underline{K}_{fl}\,\tilde{u}_l &= \tilde{f}_f \\ \underline{K}_{rf}\,\tilde{u}_f + \underline{K}_{rr}\,\tilde{u}_r + \underline{K}_{rl}\,\tilde{u}_l &= \tilde{f}_r + \bar{\tilde{f}}_r \\ \underline{K}_{lf}\,\tilde{u}_f + \underline{K}_{lr}\,\tilde{u}_r + \underline{K}_{ll}\,\tilde{u}_l &= \tilde{f}_l + \bar{\tilde{f}}_l \end{aligned}$$

Link relations

$$\underset{\sim}{u}_l = \underline{L}_{lr} \underset{\sim}{u}_r$$

$$\underset{\sim}{\bar{f}}_l^T \delta \underset{\sim}{u}_l + \underset{\sim}{\bar{f}}_r^T \delta \underset{\sim}{u}_r = 0 \quad \forall \quad \{\delta \underset{\sim}{u}_l, \delta \underset{\sim}{u}_r\} \qquad \rightarrow$$

$$\underset{\sim}{\bar{f}}_l^T \underline{L}_{lr} + \underset{\sim}{\bar{f}}_r^T = \underset{\sim}{0}^T \quad \rightarrow \quad \underline{L}_{lr}^T \underset{\sim}{\bar{f}}_l + \underset{\sim}{\bar{f}}_r = \underset{\sim}{0} \quad \rightarrow \quad \underset{\sim}{\bar{f}}_r = -\underline{L}_{lr}^T \underset{\sim}{\bar{f}}_l = -\underline{L}_{rl} \underset{\sim}{\bar{f}}_l$$

Partitioning for links

substitution of link relations $\quad \longrightarrow$

$$\left.\begin{array}{l} \underline{K}_{ff}\,\underset{\sim}{u}_f + (\underline{K}_{fr} + \underline{K}_{fl}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_f \\ \underline{K}_{rf}\,\underset{\sim}{u}_f + (\underline{K}_{rr} + \underline{K}_{rl}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_r - \underline{L}_{rl}\bar{\underset{\sim}{f}}_l \\ \underline{K}_{lf}\,\underset{\sim}{u}_f + (\underline{K}_{lr} + \underline{K}_{ll}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_l + \bar{\underset{\sim}{f}}_l \\ \\ \text{elimination of } \bar{\underset{\sim}{f}}_l \end{array}\right\} \quad \longrightarrow$$

$$\left.\begin{array}{l} \underline{K}_{ff}\,\underset{\sim}{u}_f + (\underline{K}_{fr} + \underline{K}_{fl}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_f \\ (\underline{K}_{rf} + \underline{L}_{rl}\underline{K}_{lf})\underset{\sim}{u}_f + \\ \qquad (\underline{K}_{rr} + \underline{K}_{rl}\underline{L}_{lr} + \underline{L}_{rl}\underline{K}_{lr} + \underline{L}_{rl}\underline{K}_{ll}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_r + \underline{L}_{rl}\underset{\sim}{f}_l \end{array}\right\} \quad \longrightarrow$$

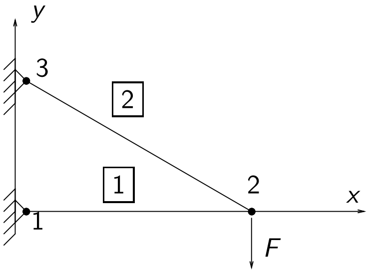
$$\begin{bmatrix} \underline{K}_{ff} & \underline{K}_{fr} + \underline{K}_{fl}\underline{L}_{rl} \\ \underline{K}_{rf} + \underline{L}_{rl}\underline{K}_{lf} & \underline{K}_{rr} + \underline{K}_{rl}\underline{L}_{lr} + \underline{L}_{rl}\underline{K}_{lr} + \underline{L}_{rl}\underline{K}_{ll}\underline{L}_{lr} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}_f \\ \underset{\sim}{u}_r \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_f \\ \underset{\sim}{f}_r + \underline{L}_{rl}\underset{\sim}{f}_l \end{bmatrix} \quad \longrightarrow$$

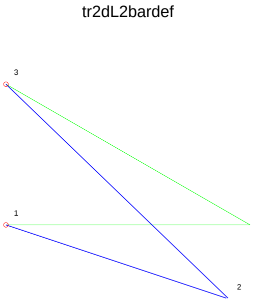
$$\underline{K}\,\underset{\sim}{u} = \underset{\sim}{f}$$

Program structure

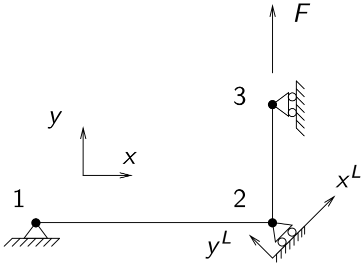
```
read input data from input file
calculate additional variables from input data
initialize values and arrays

for all elements
   calculate initial element stiffness matrix
   assemble global stiffness matrix
end element loop

determine external load from input

take tyings into account
take boundary conditions into account

calculate nodal displacements

for all elements
   calculate stresses from material behavior
   calculate element internal nodal forces
   assemble global internal load column
end element loop

store data for post-processing
```

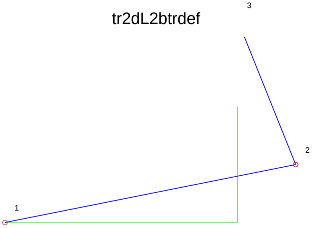
Simple two-dimensional truss structure



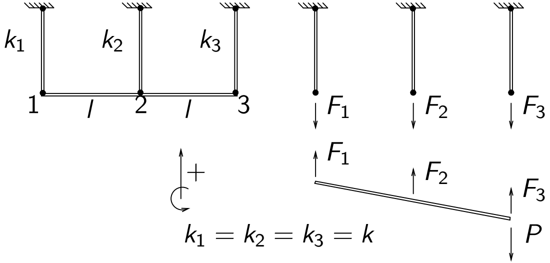


Transformation of nodal coordinate system





Tyings



truss stiffness

$$k_1 = k_2 = k_3 = k$$

equilibrium

$$F_1 + F_2 + F_3 - P = 0 \qquad ; \qquad -F_1 2l - F_2 l = 0$$

deformation

$$v_1 = -\frac{F_1}{k} \qquad ; \qquad v_2 = -\frac{F_2}{k} \qquad ; \qquad v_3 = -\frac{F_3}{k}$$

equilibrium equations in displacements

$$-kv_1 - kv_2 - kv_3 - P = 0 \qquad ; \qquad 2lkv_1 + lkv_2 = 0$$

Example : link relations

link relation

$$v_2 = \tfrac{1}{2}\,(v_1 + v_3) \qquad \rightarrow \qquad v_2 = \left[\begin{array}{cc} \tfrac{1}{2} & \tfrac{1}{2} \end{array}\right] \left[\begin{array}{c} v_1 \\ v_3 \end{array}\right]$$

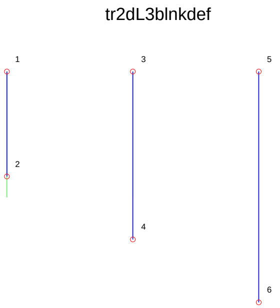
elimination of v_2 $\quad\rightarrow\quad$ equation for retained displacements

$$\left.\begin{array}{l} -\tfrac{3}{2}kv_1 - \tfrac{3}{2}kv_3 - P = 0 \\ \tfrac{5}{2}lkv_1 + \tfrac{1}{2}lkv_3 = 0 \qquad \rightarrow \qquad v_1 = -\tfrac{1}{5}v_3 \end{array}\right\} \quad \rightarrow$$

solving

$$\begin{array}{l} \tfrac{3}{10}kv_3 - \tfrac{3}{2}kv_3 - P = 0 \quad \rightarrow \quad -\tfrac{6}{5}kv_3 - P = 0 \quad \rightarrow \\ v_3 = -\tfrac{5}{6}\dfrac{P}{k} \qquad \rightarrow \qquad v_1 = \tfrac{1}{6}\dfrac{P}{k} \\ \text{link} \quad \rightarrow \quad v_2 = -\tfrac{1}{3}\dfrac{P}{k} \end{array}$$

FE solution

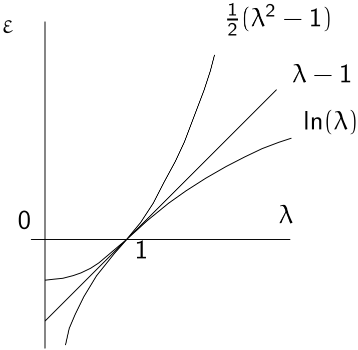


NONLINEAR DEFORMATION

back to index

Strains for large elongation

linear strain	$\varepsilon = \varepsilon_l = \lambda - 1$
logarithmic strain	$\varepsilon = \varepsilon_{ln} = \ln(\lambda)$
Green-Lagrange strain	$\varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$



Linear strain

linear strain $\quad \varepsilon = \varepsilon_l = \lambda - 1 = \dfrac{\Delta l}{l_0}$

contraction strain $\quad \varepsilon_d = \mu - 1 = -\nu \varepsilon_l = -\nu(\lambda - 1)$

change of cross-sectional area

$$\mu = \sqrt{\frac{A}{A_0}} = 1 - \nu(\lambda - 1) \quad \rightarrow \quad A = A_0\{1 - \nu(\lambda - 1)\}^2$$

restriction of elongation

$$1 - \nu(\lambda - 1) > 0 \quad \rightarrow \quad \lambda - 1 < \frac{1}{\nu} \quad \rightarrow \quad \lambda < \frac{1 + \nu}{\nu}$$

Logarithmic strain

logarithmic strain $\qquad \varepsilon = \varepsilon_{ln} = \ln(\lambda)$

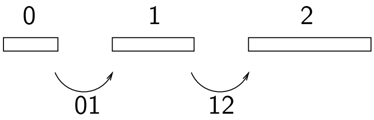
contraction strain $\qquad \varepsilon_d = \ln(\mu) = -\nu\varepsilon_{ln} = -\nu \ln\lambda$

change of cross-sectional area

$$\mu = \sqrt{\frac{A}{A_0}} = e^{-\nu\,\varepsilon_{ln}} = e^{-\nu\ln(\lambda)} = \left[e^{\ln(\lambda)}\right]^{-\nu} = \lambda^{-\nu} \qquad \rightarrow$$
$$A = A_0\lambda^{-2\nu}$$

NB $\quad : \quad \ln(x) = {}^{e}\log(x) = y \quad \rightarrow \quad x = e^{y}$

Advantage logarithmic strain



$l_0 \longrightarrow l_1$

$$\varepsilon_l(01) = \frac{l_1 - l_0}{l_0}$$
$$\varepsilon_{ln}(01) = \ln\left(\frac{l_1}{l_0}\right)$$

$l_1 \longrightarrow l_2$

$$\varepsilon_l(12) = \frac{l_2 - l_1}{l_1}$$
$$\varepsilon_{ln}(12) = \ln\left(\frac{l_2}{l_1}\right)$$

$l_0 \longrightarrow l_2$

$$\varepsilon_l(02) = \frac{l_2 - l_0}{l_0} \neq \varepsilon_l(01) + \varepsilon_l(12)$$
$$\varepsilon_{ln}(02) = \ln\left(\frac{l_2}{l_0}\right) = \ln\left(\frac{l_2}{l_1}\frac{l_1}{l_0}\right) = \varepsilon_{ln}(01) + \varepsilon_{ln}(12)$$

Green-Lagrange strain

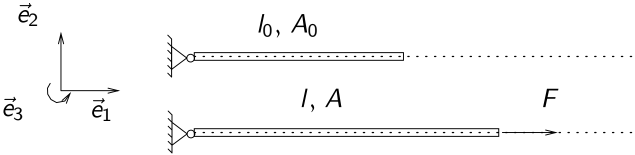
Green-Lagrange strain $\quad \varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$

contraction strain $\quad \varepsilon_d = \frac{1}{2}(\mu^2 - 1) = -\nu\varepsilon_{ln} = -\nu\frac{1}{2}(\lambda^2 - 1)$

change of cross-sectional area

$$1 - \nu(\lambda^2 - 1) > 0 \quad \rightarrow \quad \lambda < \sqrt{\frac{1+\nu}{\nu}}$$

Mechanical power for an axially loaded truss



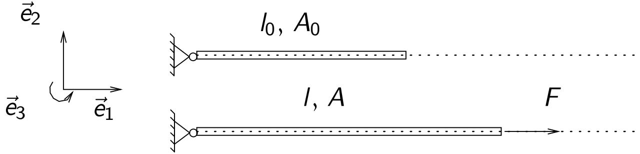
mechanical power $\qquad P = F\dot{l}$

$$\varepsilon_l = \lambda - 1 \qquad \rightarrow \qquad \dot{\varepsilon}_l = \dot{\lambda} = \frac{\dot{l}}{l_0}$$

$$\varepsilon_{ln} = \ln(\lambda) \qquad \rightarrow \qquad \dot{\varepsilon}_{ln} = \dot{\lambda}\lambda^{-1} = \frac{\dot{l}}{l}$$

$$\varepsilon_{gl} = \tfrac{1}{2}(\lambda^2 - 1) \qquad \rightarrow \qquad \dot{\varepsilon}_{gl} = \dot{\lambda}\lambda = \lambda\,\frac{\dot{l}}{l_0} = \lambda^2\,\frac{\dot{l}}{l}$$

Mechanical power for an axially loaded truss



$$
\begin{aligned}
P &= F\dot{\ell} = F\ell_0\dot{\varepsilon}_l = \frac{F}{A_0}A_0\ell_0\dot{\varepsilon}_l = \frac{F}{A_0}V_0\dot{\varepsilon}_l \\
P &= F\dot{\ell} = F\ell\dot{\varepsilon}_{ln} = \frac{F}{A}A\ell\dot{\varepsilon}_{ln} = \frac{F}{A}V\dot{\varepsilon}_{ln} \\
P &= F\dot{\ell} = F\ell_0\dot{\varepsilon}_l = \frac{F}{A}A\ell\frac{\ell_0}{\ell}\dot{\varepsilon}_l = \frac{F}{A}V\lambda^{-1}\dot{\varepsilon}_l \\
P &= F\dot{\ell} = F\ell\lambda^{-2}\dot{\varepsilon}_{gl} = \frac{F}{A}A\ell\lambda^{-2}\dot{\varepsilon}_{gl} = \frac{F}{A}V\lambda^{-2}\dot{\varepsilon}_{gl}
\end{aligned}
$$

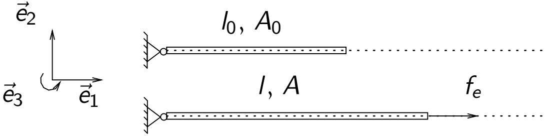
Mechanical power : stress ∼ strain

$$
\begin{array}{lclclcl}
P & = & & = & & = & V_0 \sigma_n \dot{\varepsilon}_l \\
P & = & V \sigma \dot{\varepsilon}_{ln} & = & V_0 (J\sigma) \dot{\varepsilon}_{ln} & = & V_0 \sigma_\kappa \dot{\varepsilon}_{ln} \\
P & = & V(\sigma\lambda^{-1}) \dot{\varepsilon}_l & = & V_0 (J\sigma\lambda^{-1}) \dot{\varepsilon}_l & = & V_0 \sigma_{p1} \dot{\varepsilon}_l \\
P & = & V(\sigma\lambda^{-2}) \dot{\varepsilon}_{gl} & = & V_0 (J\sigma\lambda^{-2}) \dot{\varepsilon}_{gl} & = & V_0 \sigma_{p2} \dot{\varepsilon}_{gl}
\end{array}
$$

specific mechanical power : $P = V_0 \dot{W}_0 = V \dot{W}$

$$
\begin{array}{lclclclcl}
\dot{W}_0 & = & \sigma_n \dot{\varepsilon}_l & = & \sigma_\kappa \dot{\varepsilon}_{ln} & = & \sigma_{p1} \dot{\varepsilon}_l & = & \sigma_{p2} \dot{\varepsilon}_{gl} \\
\dot{W} & = & & = & \sigma \dot{\varepsilon}_{ln} & = & \sigma\lambda^{-1} \dot{\varepsilon}_l & = & \sigma\lambda^{-2} \dot{\varepsilon}_{gl}
\end{array}
$$

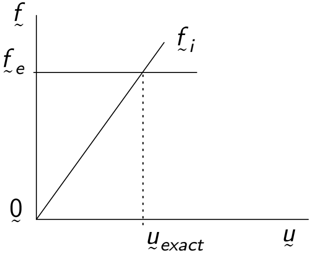
Equilibrium : linear



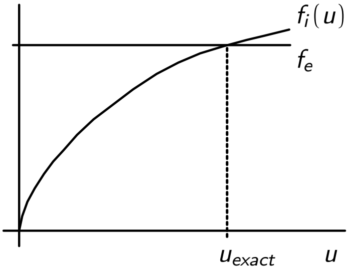
external force $\qquad f_e$

internal force $\qquad f_i = \sigma_n A_0 = E\,\varepsilon_l A_0 = EA_0\,\dfrac{u}{l_0}$

equilibrium of point P $\qquad f_i = \dfrac{EA_0}{l_0}\,u = f_e \quad \rightarrow \quad u_{exact} = \dfrac{l_0}{EA_0}\,f_e$



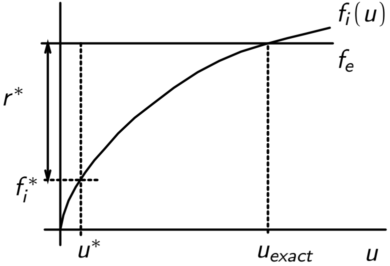
Equilibrium : nonlinear



external force	f_e
internal force	$f_i = \sigma A = f_i(u)$
equilibrium of point P	$f_i(u) = f_e$

$f_i(u)$ non-linear iterative solution process needed

Iterative solution procedure



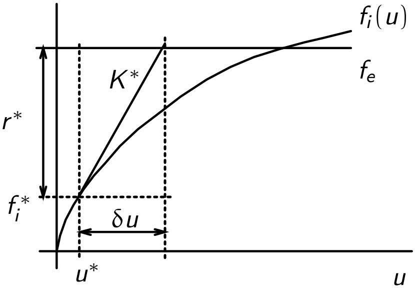
analytic solution	$f_i(u_{exact}) = f_e \quad \rightarrow \quad f_e - f_i(u_{exact}) = 0$
approximation u^*	$f_e - f_i(u^*) = r(u^*) \neq 0$
residual	$r^* = r(u^*)$

Newton-Raphson iteration procedure

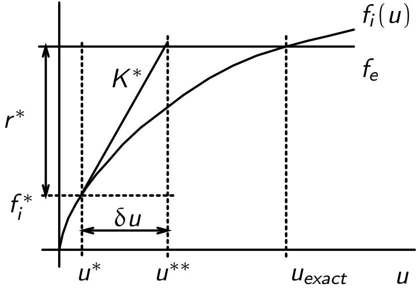
$$\left.\begin{aligned} f_i(u_{exact}) &= f_e \\ u_{exact} &= u^* + \delta u \end{aligned}\right\} \quad \rightarrow \quad f_i(u^* + \delta u) = f_e$$

$$f_i(u^*) + \left.\frac{df_i}{du}\right|_{u^*} \delta u = f_e \quad \rightarrow \quad f_i^* + K^* \delta u = f_e$$

$$K^* \, \delta u = f_e - f_i^* = r^* \quad \rightarrow \quad \delta u = \frac{1}{K^*} \, r^*$$

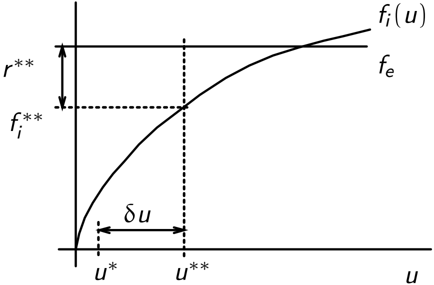


New approximate solution



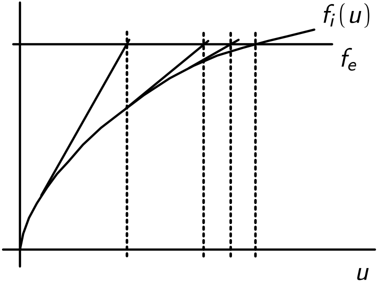
new approximation	$u^{**} = u^* + \delta u$
error	$u_{exact} - u^{**}$
error smaller	$\longrightarrow$ convergence

Convergence control



residual force	$\lvert r^{**}\rvert \leq c_r$	$\longrightarrow$	stop iteration
iterative displacement	$\lvert \delta u\rvert \leq c_u$	$\longrightarrow$	stop iteration

Convergence



Residual and tangential stiffness

internal nodal force $\qquad f_i^* = N(\lambda^*) = A^* \sigma^*$

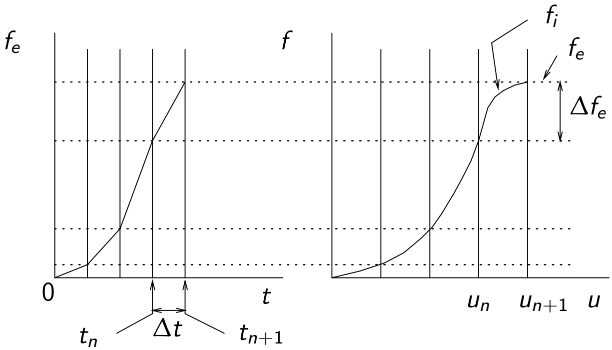
tangential stiffness $\qquad K^* = \left.\frac{\partial f_i}{\partial u}\right|_{u^*} = \left.\frac{\partial N(\lambda)}{\partial u}\right|_{u^*} = \left.\frac{dN}{d\lambda}\right|_{\lambda^*} \frac{d\lambda}{du}$

geometry $\qquad \lambda = 1 + \frac{\Delta l}{l_0} = 1 + \frac{1}{l_0}\, u \qquad \rightarrow \qquad \frac{d\lambda}{du} = \frac{1}{l_0}$

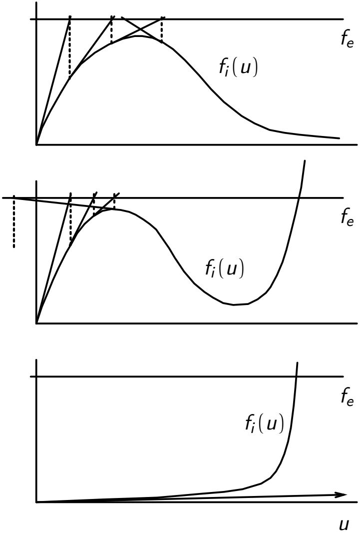
tangential stiffness $\qquad K^* = \left.\frac{dN}{d\lambda}\right|_{\lambda^*} \frac{\partial \lambda}{\partial u} = \left.\frac{dN}{d\lambda}\right|^* \frac{1}{l_0} = \frac{1}{l_0} \frac{d}{d\lambda}(\sigma A)\Big|^*$

$$K^* = \frac{1}{l_0}\left.\frac{d\sigma}{d\lambda}\right|^* A^* + \frac{1}{l_0}\,\sigma^* \left.\frac{dA}{d\lambda}\right|^*$$

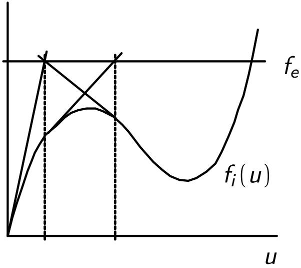
Incremental loading

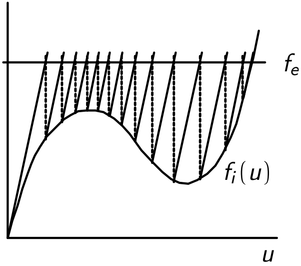


Non-converging solution process

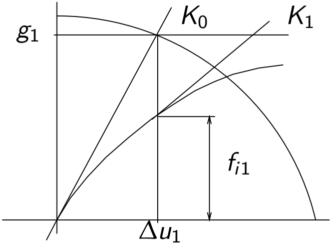


Modified Newton-Raphson procedure



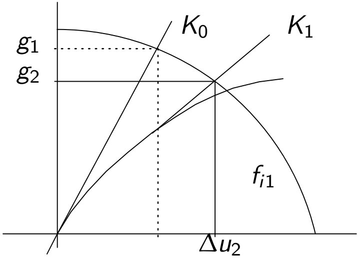


Path-following solution algorithm



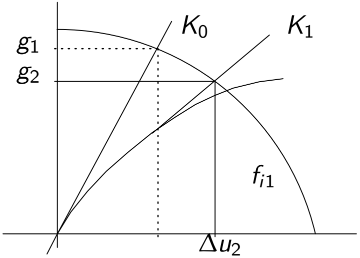
$$K_0 \delta u_1 = f_{e0} + \lambda_1 f_{ef} = g_1 \quad \rightarrow \quad \delta u_1 = K_0^{-1} g_1 \quad \rightarrow$$
$$\Delta u_1 = \delta u_1 \quad ; \quad u_1 = u_0 + \Delta u_1 \quad \rightarrow \quad f_{i1} \ , \ K_1$$

Path-following solution algorithm



$$K_1 \delta u_2 = g_2 - f_{i2} = f_{e0} + \lambda_2 f_{ef} - f_{i2} \quad \rightarrow$$
$$\delta u_2 = K_1^{-1} \left(f_{e0} + \lambda_2 f_{ef} - f_{i2} \right)$$

Path-following solution algorithm



$$\begin{bmatrix} \Delta u_2 & g_2 \end{bmatrix} \begin{bmatrix} \Delta u_2 \\ g_2 \end{bmatrix} = (\Delta u_2)^2 + (g_2)^2 = C^2 \quad \rightarrow$$

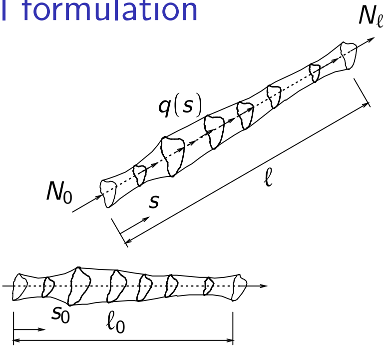
$$\left(\Delta u_1 + K_1^{-1} f_{e0} + K_1^{-1} f_{ef} \lambda_2 - K_1^{-1} f_{i2}\right)^2 + (f_{e0} + \lambda_2 f_{ef})^2 = C^2 \quad \rightarrow$$

$$\lambda_2 \quad \rightarrow \quad \delta u_2 \quad \rightarrow \quad \Delta u_2 \ , \ u_2 \quad \rightarrow \quad f_{i3} \ , \ K_2$$

WEIGHTED RESIDUAL FORMULATION

back to index

Weighted residual formulation



equilibrium $\quad \dfrac{d\vec{N}}{ds} + \vec{q}(s) = \vec{0} \quad \rightarrow \quad \dfrac{d(\sigma A \vec{n})}{ds} + \vec{q}(s) = \vec{0} \quad \forall\ s \in [0, \ell]$

approximation $\quad \dfrac{d(\sigma^* A^* \vec{\bar{n}})}{ds} + \vec{q}(s) = \vec{\Delta}(s) \neq \vec{0} \quad \forall\ s \in [0, \ell]$

weighted error $\quad \vec{\Delta}(s)$ is "smeared out" over$[0, \ell]$ $\quad \rightarrow \quad \displaystyle\int_{s=0}^{s=\ell} \vec{w}(s) \cdot \vec{\Delta}(s)\, ds$

Weighted residual formulation

$$\int_{s=0}^{s=\ell} \vec{w} \cdot \left\{ \frac{d(\sigma A \vec{n})}{ds} + \vec{q} \right\} ds = 0 \qquad \forall \quad \vec{w}(s)$$

partial integration of 1st term $\quad \longrightarrow \quad$ weak formulation

$$\int_{s=0}^{s=\ell} \frac{d\vec{w}}{ds} \cdot (\sigma A \vec{n})\, ds = \int_{s=0}^{s=\ell} \vec{w} \cdot \vec{q}\, ds + \left[\vec{w}(\ell) \cdot \vec{N}(\ell) - \vec{w}(0) \cdot \vec{N}(0) \right]$$

$$= f_e(\vec{w}) \quad \forall \quad \vec{w}(s)$$

State transformation

$$\int_{s=0}^{s=\ell} \frac{d\vec{w}}{ds} \cdot (\sigma A \vec{n})\, ds = f_e(\vec{w}) \quad \forall \quad \vec{w}(s)$$

$$\frac{d(\ )}{ds} = \frac{ds_0}{ds}\frac{d(\ )}{ds_0} = \frac{1}{\lambda}\frac{d(\ )}{ds_0} \qquad ; \qquad ds = \lambda ds_0$$

integral transformation

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot (\sigma A \vec{n})\, ds_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(s_0)$$

Iterative solution process

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot (\sigma^* + \delta\sigma)(A^* + \delta A)(\vec{n}^* + \delta\vec{n})\, ds_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(s_0)$$

Linearization

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot (\sigma^* + \delta\sigma)(A^* + \delta A)(\vec{n}^* + \delta\vec{n})\, ds_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(s_0)$$

$$\begin{aligned} &\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \delta\sigma A^* \vec{n}^*\, ds_0 + \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \delta\vec{n}\, ds_0 \\ &\qquad = f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \vec{n}^*\, ds_0 \qquad \forall \quad \vec{w}(s_0) \end{aligned}$$

Material model $\rightarrow$ iterative stress change

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot (\sigma^* + \delta\sigma)(A^* + \delta A)(\vec{n}^* + \delta\vec{n})\, ds_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(s_0)$$

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \delta\sigma A^* \vec{n}^*\, ds_0 + \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \delta\vec{n}\, ds_0$$
$$= f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \vec{n}^*\, ds_0 \qquad \forall \quad \vec{w}(s_0)$$

$$\sigma = \sigma(\lambda) \quad \rightarrow \quad \delta\sigma = \left.\frac{d\sigma}{d\lambda}\right|^* \delta\lambda = \left.\frac{d\sigma}{d\lambda}\right|^* \frac{d(\delta s)}{ds_0} = \left.\frac{d\sigma}{d\lambda}\right|^* \vec{n}^* \cdot \frac{d(\delta\vec{u})}{ds_0}$$

Rotation $\rightarrow$ iterative orientation change

$$\vec{n} = \frac{d\vec{x}}{ds} = \frac{ds_0}{ds}\frac{d\vec{x}}{ds_0} = \frac{1}{\lambda}\,\frac{d\vec{x}}{ds_0}$$

$$\begin{aligned}
\delta\vec{n} &= \left[-\frac{1}{\lambda^2}\,\frac{d\vec{x}}{ds_0}\right]^* \delta\lambda + \left[\frac{1}{\lambda}\right]^*\,\frac{d(\delta\vec{x})}{ds_0} = \left[-\frac{1}{\lambda}\,\vec{n}\right]^* \delta\lambda + \left[\frac{1}{\lambda}\right]^*\,\frac{d(\delta\vec{x})}{ds_0} \\
&= \left[-\frac{1}{\lambda}\,\vec{n}\vec{n}\right]^* \cdot \frac{d(\delta\vec{u})}{ds_0} + \left[\frac{1}{\lambda}\right]^*\,\frac{d(\delta\vec{u})}{ds_0} = \left[(\mathbf{I} - \vec{n}\vec{n})\,\frac{1}{\lambda}\right]^* \cdot \frac{d(\delta\vec{u})}{ds_0} \\
&= \left[\vec{m}\vec{m}\,\frac{1}{\lambda}\right]^* \cdot \frac{d(\delta\vec{u})}{ds_0}
\end{aligned}$$

Iterative weighted residual integral

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \left(\left. \frac{d\sigma}{d\lambda} \right|^* \vec{n}^* \cdot \frac{d(\delta\vec{u})}{ds_0} \right) A^* \vec{n}^* \, ds_0 +$$

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \left(\vec{m}^* \vec{m}^* \cdot \frac{1}{\lambda^*} \frac{d(\delta\vec{u})}{ds_0} \right) ds_0$$

$$= f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \vec{n}^* \, ds_0 \qquad \forall \quad \vec{w}(s_0)$$

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \vec{n}^* \left(\left. \frac{d\sigma}{d\lambda} \right|^* A^* \right) \vec{n}^* \cdot \frac{d(\delta\vec{u})}{ds_0} \, ds_0 +$$

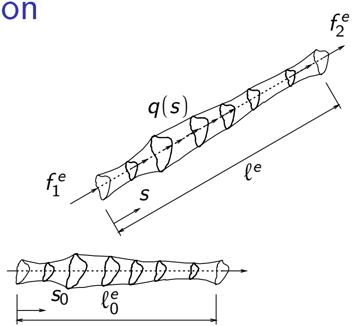
$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \vec{m}^* \left(\sigma^* A^* \frac{1}{\lambda^*} \right) \vec{m}^* \cdot \frac{d(\delta\vec{u})}{ds_0} \, ds_0$$

$$= f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \vec{n}^* \, ds_0 \qquad \forall \quad \vec{w}(s_0)$$

FINITE ELEMENT METHOD

back to index

Element equation



local coordinate : $\quad -1 \le \xi \le 1 \quad ; \quad ds_0 = \dfrac{l_0}{2}\, d\xi \quad ; \quad \dfrac{d(\;)}{ds_0} = \dfrac{2}{l_0}\,\dfrac{d(\;)}{d\xi}$

$$\int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \vec{n}^* \left(\left.\frac{d\sigma}{d\lambda}\right|^* A^* \, \frac{2}{l_0} \right) \vec{n}^* \cdot \frac{d(\delta\vec{u})}{d\xi}\, d\xi \, +$$

$$\int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \vec{m}^* \left(\sigma^* A^* \frac{1}{\lambda^*}\frac{2}{l_0} \right) \vec{m}^* \cdot \frac{d(\delta\vec{u})}{d\xi}\, d\xi = f_{e0}^e(\vec{w}) - \int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \sigma^* A^* \vec{n}^* \, d\xi$$

Components

$$\int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \underset{\sim}{n}^* \left(\left.\frac{d\sigma}{d\lambda}\right|^* A^* \, \frac{2}{l_0} \right) \underset{\sim}{n}^{*T} \, \frac{d(\delta \underset{\sim}{u})}{d\xi} \, d\xi +$$

$$\int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \underset{\sim}{m}^* \left(\sigma^* A^* \frac{1}{\lambda^*} \frac{2}{l_0} \right) \underset{\sim}{m}^{*T} \, \frac{d(\delta \underset{\sim}{u})}{d\xi} \, d\xi$$

$$= f_{e0}^{e}(\underset{\sim}{w}) - \int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \sigma^* A^* \underset{\sim}{n}^* \, d\xi$$

Interpolation

$$
\begin{aligned}
\delta \underset{\sim}{u}^T &= \left[\begin{array}{cc} \delta u_1 & \delta u_2 \end{array} \right] = \left[\begin{array}{cc} \delta u_{11}\psi^1 + \delta u_{21}\psi^2 & \delta u_{12}\psi^1 + \delta u_{22}\psi^2 \end{array} \right] \\
\underset{\sim}{w}^T &= \left[\begin{array}{cc} w_{11}\psi^1 + w_{21}\psi^2 & w_{12}\psi^1 + w_{22}\psi^2 \end{array} \right] \\
&\quad \text{with} \qquad \psi^1(\xi) = \tfrac{1}{2}(1-\xi) \quad ; \quad \psi^2(\xi) = \tfrac{1}{2}(1+\xi)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d(\delta \underset{\sim}{u})}{d\xi} &= \left[\begin{array}{c} \dfrac{d(\delta u_1)}{d\xi} \\ \dfrac{d(\delta u_2)}{d\xi} \end{array} \right] = \left[\begin{array}{cccc} \frac{d\psi^1}{d\xi} & 0 & \frac{d\psi^2}{d\xi} & 0 \\ 0 & \frac{d\psi^1}{d\xi} & 0 & \frac{d\psi^2}{d\xi} \end{array} \right] \left[\begin{array}{c} \delta u_{11} \\ \delta u_{12} \\ \delta u_{21} \\ \delta u_{22} \end{array} \right] \\
&= \frac{1}{2} \left[\begin{array}{cccc} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array} \right] \delta \underset{\sim}{u}^e
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\underset{\sim}{w}^T}{d\xi} &= \left[\begin{array}{cc} \dfrac{dw_1}{d\xi} & \dfrac{dw_2}{d\xi} \end{array} \right] = \left[\begin{array}{cccc} w_{11} & w_{12} & w_{21} & w_{22} \end{array} \right] \left[\begin{array}{cc} \frac{d\psi^1}{d\xi} & 0 \\ 0 & \frac{d\psi^1}{d\xi} \\ \frac{d\psi^2}{d\xi} & 0 \\ 0 & \frac{d\psi^2}{d\xi} \end{array} \right] \\
&= \underset{\sim}{w}^{eT} \frac{1}{2} \left[\begin{array}{cc} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{array} \right]
\end{aligned}
$$

Element equation

$$
\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix}^{*} \tfrac{1}{4} \left(\left. \frac{d\sigma}{d\lambda} \right|^{*} A^{*} \frac{2}{l_0} \right)
$$

$$
\begin{bmatrix} c & s \end{bmatrix}^{*} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} d\xi \, \delta \underset{\sim}{u}^{e} +
$$

$$
\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -s \\ c \end{bmatrix}^{*} \tfrac{1}{4} \left(\sigma^{*} A^{*} \frac{1}{\lambda^{*}} \frac{2}{l_0} \right)
$$

$$
\begin{bmatrix} -s & c \end{bmatrix}^{*} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} d\xi \, \delta \underset{\sim}{u}^{e}
$$

$$
= f_{e0}^{e}(\underset{\sim}{w}^{e}) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tfrac{1}{2} \begin{bmatrix} c \\ s \end{bmatrix}^{*} \left(\sigma^{*} A^{*} \right) \, d\xi
$$

Element equation

$$\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\frac{1}{2} \left. \frac{d\sigma}{d\lambda} \right|^{*} A^{*} \frac{1}{l_0} \right) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^{*} \begin{bmatrix} -c & -s & c & s \end{bmatrix}^{*} d\xi \, \delta \underset{\sim}{u}^{e} +$$

$$\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\frac{1}{2} \sigma^{*} A^{*} \frac{1}{\lambda^{*}} \frac{1}{l_0} \right) \begin{bmatrix} s \\ -c \\ -s \\ c \end{bmatrix}^{*} \begin{bmatrix} s & -c & -s & c \end{bmatrix}^{*} d\xi \, \delta \underset{\sim}{u}^{e}$$

$$= f_{e0}^{e}(\underset{\sim}{w}^{e}) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \frac{1}{2} \left(\sigma^{*} A^{*} \right) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^{*} d\xi$$

Element equation

$$
\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\frac{1}{2} \left. \frac{d\sigma}{d\lambda} \right|^{*} A^{*} \frac{1}{l_0} \right) \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}^{*} d\xi \, \delta \underset{\sim}{u}^{e} +
$$

$$
\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\frac{1}{2} \sigma^{*} A^{*} \frac{1}{\lambda^{*}} \frac{1}{l_0} \right) \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}^{*} d\xi \, \delta \underset{\sim}{u}^{e}
$$

$$
= f_{e0}^{e}(\underset{\sim}{w}^{e}) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \frac{1}{2} \, (\sigma^{*} A^{*}) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^{*} d\xi
$$

Element equation

$$
\underset{\sim}{w}^{eT} \left[\int_{\xi=-1}^{\xi=1} \left(\frac{1}{2} \, \frac{d\sigma}{d\lambda}\bigg|^{*} A^{*} \, \frac{1}{l_0} \right) \, d\xi \, \underline{M}_L^{*} \right] \delta \underset{\sim}{u}^{e} + \\
\underset{\sim}{w}^{eT} \left[\int_{\xi=-1}^{\xi=1} \left(\frac{1}{2} \sigma^{*} A^{*} \frac{1}{\lambda^{*}} \frac{1}{l_0} \right) \, d\xi \, \underline{M}_N^{*} \right] \delta \underset{\sim}{u}^{e} \\
= f_{e0}^{e} (\underset{\sim}{w}^{e}) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \frac{1}{2} \, (\sigma^{*} A^{*}) \, \underset{\sim}{V}^{*} \, d\xi
$$

$$
\underset{\sim}{w}^{eT} \underline{K}^{e^{*}} \delta \underset{\sim}{u}^{e} = \underset{\sim}{w}^{eT} \underset{\sim}{f}_{e0}^{e} - \underset{\sim}{w}^{eT} \underset{\sim}{f}_{i}^{e^{*}} = \underset{\sim}{w}^{eT} \underset{\sim}{r}^{e^{*}}
$$

Integration

tangential stiffness matrix

$$\underline{K}^{e^*} = \left(\frac{d\sigma}{d\lambda}\bigg|^* A^* \frac{1}{l_0} \right) \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}^* +$$

$$\left(\sigma^* A^* \frac{1}{l^*} \right) \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}^*$$

internal nodal forces

$$\underset{\sim}{f}_i^{e^*} = \sigma^* A^* \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^*$$

Assembling

element contribution $\quad \underset{\sim}{w}^{eT} \underline{K}^{e^*} \delta \underset{\sim}{u}^{e} = \underset{\sim}{w}^{eT} \underset{\sim}{f}^{e}_{e0} - \underset{\sim}{w}^{eT} \underset{\sim}{f}^{e^*}_{i} = \underset{\sim}{w}^{eT} \underset{\sim}{r}^{e^*}$

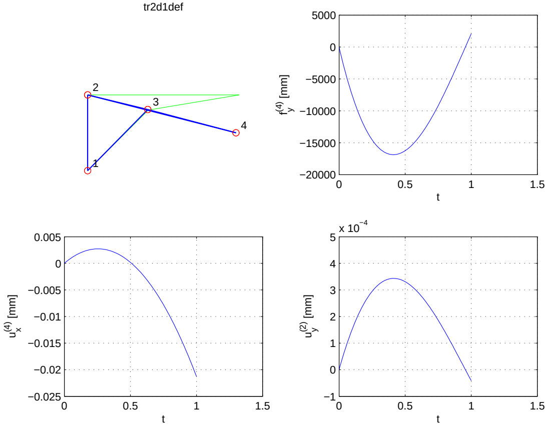
assembled equation $\quad \underset{\sim}{w}^{T} \underline{K}^{*} \delta \underset{\sim}{u} = \underset{\sim}{w}^{T} \underset{\sim}{f}_{e0} - \underset{\sim}{w}^{T} \underset{\sim}{f}^{*}_{i} = \underset{\sim}{w}^{T} \underset{\sim}{r}^{*} \qquad \forall\ \underset{\sim}{w}$

iterative equation system $\quad \underline{K}^{*} \delta \underset{\sim}{u} = \underset{\sim}{r}^{*}$

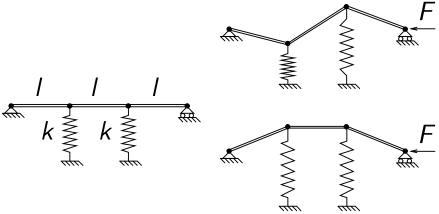
Program structure

```
read input data from input file
calculate additional variables from input data
initialize values and arrays

while load increments to be done

   for all elements
      calculate initial element stiffness matrix
      assemble global stiffness matrix
   end element loop

   determine external incremental load from input

   while non-converged iteration step

      take tyings into account
      take boundary conditions into account

      calculate iterative nodal displacements
      calculate total deformation

      for all elements
         calculate stresses from material behavior
         calculate material stiffness from material behavior
         calculate element internal nodal forces
         calculate element stiffness matrix
         assemble global stiffness matrix
         assemble global internal load column
      end element loop

      calculate residual load column
      calculate convergence norm

   end iteration step

   store data for post-processing
end load increment
```

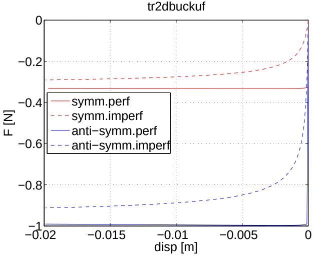
Large deformation of a truss structure

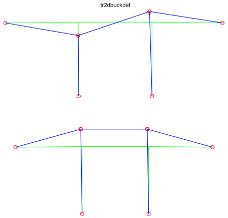


Buckling



symm : $F_c = \dfrac{kl}{3}$; anti-symm : $F_c = kl$



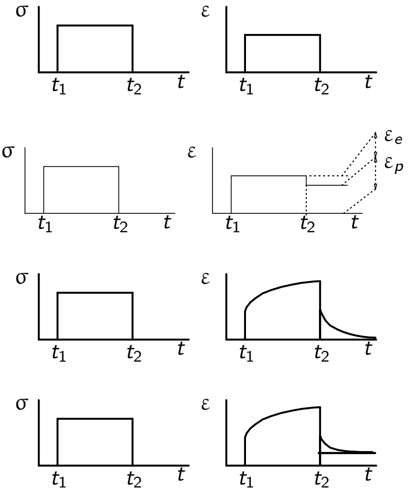


ONE-DIMENSIONAL MATERIAL BEHAVIOR

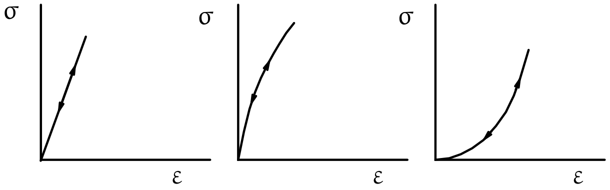
back to index

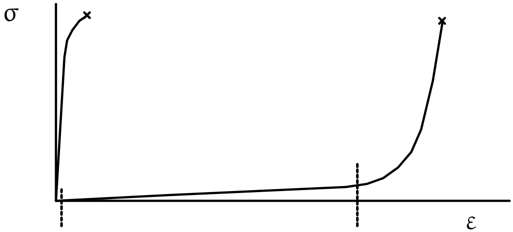
Time history plots

elastic, elastoplastic, viscoelastic, viscoplastic

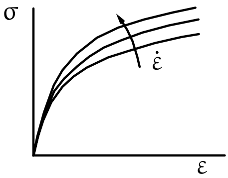


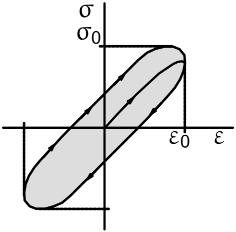
Tensile curve : elastic behavior





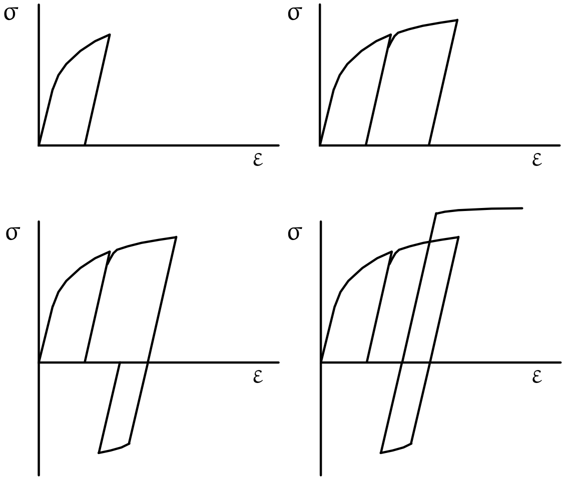
Tensile curve : viscoelastic behavior



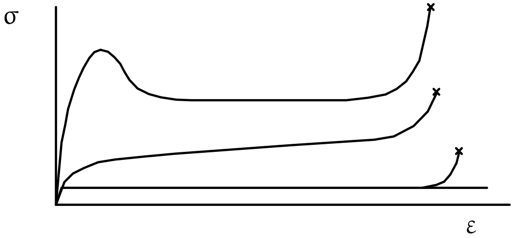


- energy dissipation $\longrightarrow$ heat

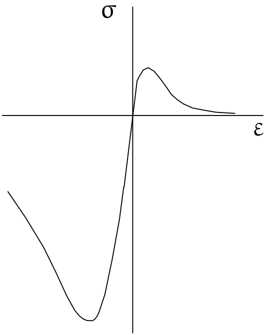
Tensile curve : elastoplastic behavior



Tensile curve : viscoplastic behavior



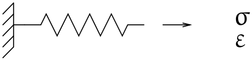
Tensile curve : damage



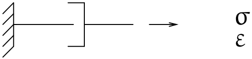
- necking / stable necking
- softening
- fracture
- ductile / brittle

Discrete material models

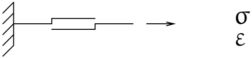
spring



dashpot



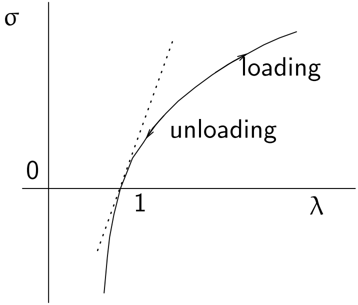
friction slider



ELASTIC

back to index

Elastic material behavior



- no permanent deformation after unloading
- no path- or time dependency
- no energy dissipation

Small strain elastic behavior

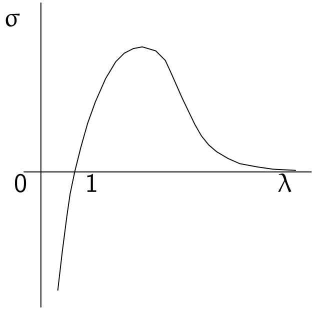
strain $$\varepsilon = \varepsilon_{gl} = \varepsilon_{ln} = \varepsilon_l = \lambda - 1$$

stress $$\sigma = \frac{F}{A} = \frac{F}{A_0} = \sigma_n$$

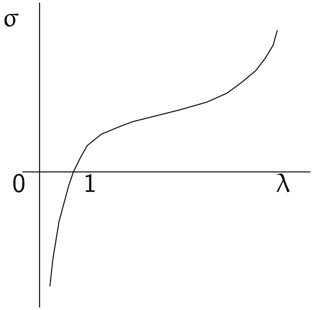
linear elastic behavior $$\sigma = E\varepsilon = E(\lambda - 1)$$

modulus $$E = \lim_{\lambda \longrightarrow 1} \frac{d\sigma}{d\lambda} = \lim_{\varepsilon \longrightarrow 0} \frac{d\sigma}{d\varepsilon}$$

Large strain elastic behavior



atomic bond



rubber

Elasticity models

constitutive equation $\sigma = \sigma(\lambda)$

stiffness $C_\lambda = \dfrac{d\sigma}{d\lambda} = \dfrac{d\sigma}{d\varepsilon}\dfrac{d\varepsilon}{d\lambda} = C_\varepsilon \dfrac{d\varepsilon}{d\lambda}$

elastic models (examples)

$$\begin{cases} \text{linear true-log.} & \sigma = C\ln(\lambda) = C\varepsilon_{ln} \\ \text{linear eng.-lin.} & \sigma_n = C(\lambda - 1) = C\varepsilon_l \end{cases}$$

Hyper-elastic models, incompressible

incompressible deformation

$$\frac{\Delta V}{V} = J = det(\mathbf{F}) = \lambda_1 \lambda_2 \lambda_3 = 1$$

specific energy

$$W = \sum_{i}^{n} \sum_{j}^{m} C_{ij} \left(I_1 - 3 \right)^i \left(I_2 - 3 \right)^j \qquad \text{with} \quad C_{00} = 0$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 = \frac{1}{\lambda_3^2} + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}$$

change of specific energy

$$dW = \sigma_1 d\varepsilon_{ln_1} + \sigma_2 d\varepsilon_{ln_2} + \sigma_3 d\varepsilon_{ln_3}$$

Mooney models

Neo-Hookean $W = C_{10}\,(I_1 - 3)$

Mooney-Rivlin $W = C_{10}\,(I_1 - 3) + C_{01}\,(I_2 - 3)$

Signiorini $W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2$

Yeoh $W = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$

Klosner-Segal $W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{03}(I_2 - 3)^3$

2-order invariant $W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{20}(I_1 - 3)^2$

Third-order model of James, Green and Simpson

$$
\begin{aligned}
W = {} & C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + \\
& C_{20}(I_1 - 3)^2 + C_{02}(I_2 - 3)^2 + C_{21}(I_1 - 3)^2(I_2 - 3) + \\
& C_{30}(I_1 - 3)^3 + C_{03}(I_2 - 3)^3 + C_{12}(I_1 - 3)(I_2 - 3)^2
\end{aligned}
$$

Ogden models

slightly compressible

$$W = \sum_{i=1}^{N} \frac{a_i}{b_i} \left[J^{-\frac{b_i}{3}} \left(\lambda_1^{b_i} + \lambda_2^{b_i} + \lambda_3^{b_i} \right) - 3 \right] + 4.5K \left(J^{\frac{1}{3}} - 1 \right)^2$$

K = bulk modulus
J = volume change factor = $\lambda_1 \lambda_2 \lambda_3$

highly compressible

$$W = \sum_{i=1}^{N} \frac{a_i}{b_i} \left(\lambda_1^{b_i} + \lambda_2^{b_i} + \lambda_3^{b_i} - 3 \right) + \sum_{i=1}^{N} \frac{a_i}{c_i} \left(1 - J^{c_i} \right)$$

One-dimensional models : Neo-Hookean

$$
\begin{aligned}
W &= C_{10}\left(\lambda^2 + \tfrac{2}{\lambda} - 3\right) \\
\sigma &= C_{10}\left(2\lambda - \frac{2}{\lambda^2}\right)\lambda = 2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) \\
C_\lambda &= \frac{\partial\sigma}{\partial\lambda} = 2C_{10}\left(2\lambda + \frac{1}{\lambda^2}\right) \\
E &= \lim_{\lambda\longrightarrow 1}\frac{\partial\sigma}{\partial\lambda} = 6C_{10} \\
F &= \sigma A = \sigma\frac{1}{\lambda}A_0 = 2C_{10}A_0\left(\lambda - \frac{1}{\lambda^2}\right)
\end{aligned}
$$

$$\sigma = \frac{\rho R T}{M}\left(\lambda^2 - \frac{1}{\lambda}\right)$$

with

ρ	:	density
R	:	gas constant $= 8.314\ \mathrm{JK^{-1}mol^{-1}}$
T	:	absolute temperature
M	:	average molecular weight

One-dimensional models : Mooney-Rivlin

$$
\begin{aligned}
W &= C_{10}\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + C_{01}\left(\frac{1}{\lambda^2} + 2\lambda - 3\right) \\
\sigma &= 2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) + 2C_{01}\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{1}{\lambda} \\
C_\lambda &= \frac{\partial\sigma}{\partial\lambda} = 2C_{10}\left(2\lambda + \frac{1}{\lambda^2}\right) + 2C_{01}\left(1 + \frac{2}{\lambda^3}\right) \\
E &= \lim_{\lambda\longrightarrow 1}\frac{\partial\sigma}{\partial\lambda} = 6(C_{10} + C_{01}) \\
F &= \sigma A = \sigma\frac{1}{\lambda}A_0 \\
&= A_0\frac{1}{\lambda}\left[2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) + 2C_{01}\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{1}{\lambda}\right]
\end{aligned}
$$

NUMERICAL IMPLEMENTATION

back to index

Stress update

stress update $\quad \sigma = \sigma(\lambda)$

Stiffness

stress update $\sigma = \sigma(t + \Delta t) = \sigma\left(\lambda(t + \Delta t)\right) = \sigma(\lambda)$

stiffness $C_\lambda = \dfrac{\partial \sigma}{\partial \lambda}$

Implementation

`tr2delas.m`

`tr2delam.m`

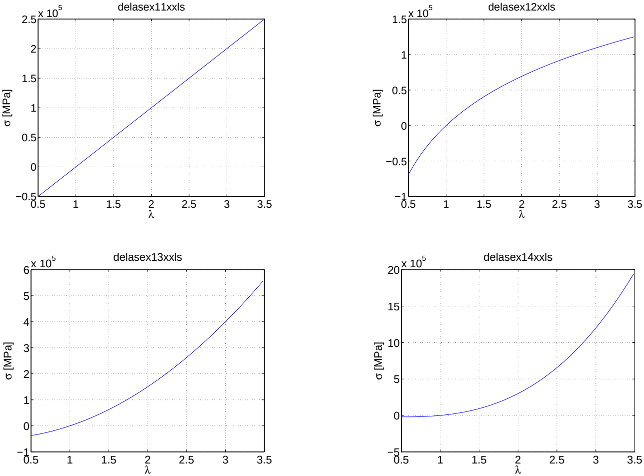
Strain excitation

initial length	l_0	100	mm
initial cross-sectional area	A_0	10	mm^2

Elastic models : stress

elastic constant	C	100000	MPa
Poisson's ratio	ν	0.3	-

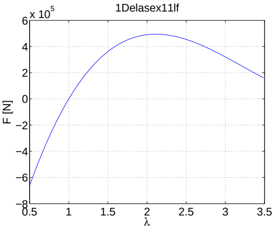
$\sigma \sim \varepsilon_l$ model	$\sigma \sim \varepsilon_{ln}$ model
$\sigma \sim \varepsilon_{gl}$ model	$P \sim \varepsilon_{Gl}$ model

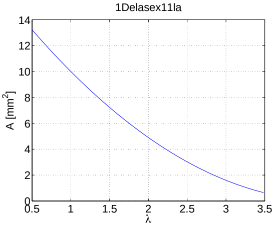


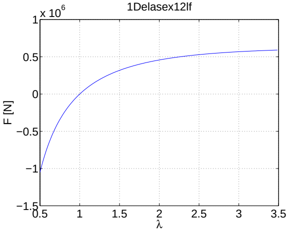
Elastic models : force and area

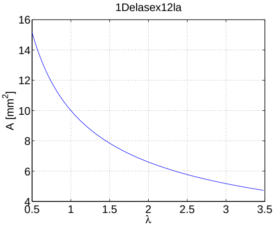
elastic constant	C	100000	MPa
Poisson's ratio	ν	0.3	-

$\sigma \sim \varepsilon_l$ model
$\sigma \sim \varepsilon_{ln}$ model



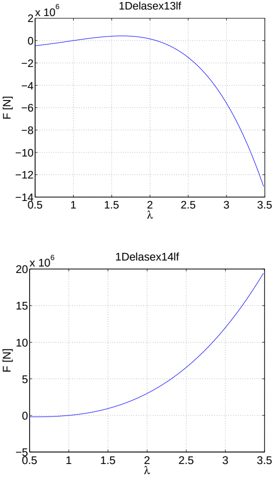






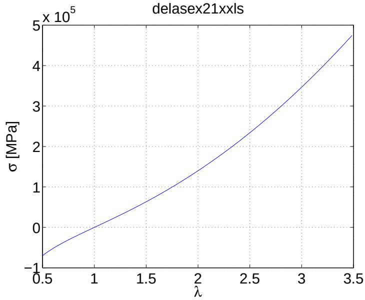
Elastic models : force and area

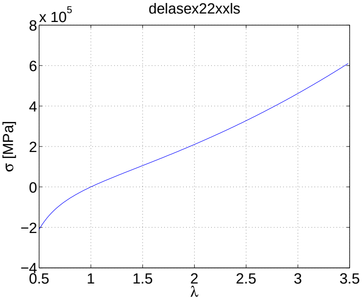
$\sigma \sim \varepsilon_{gl}$ model
$P \sim \varepsilon_{gl}$ model



Elastomeric models : stress

elastic constant	C_{01}	20000	MPa
elastic constant	C_{10}	20000	MPa



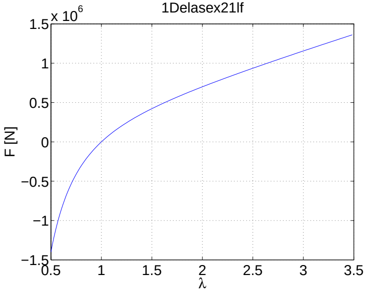


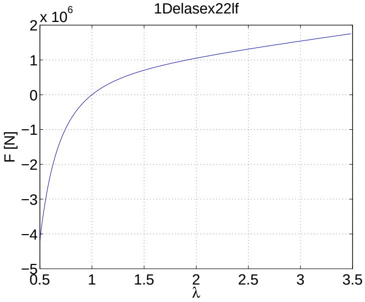
Neo-Hookean

Mooney-Rivlin

Elastomeric models : force and area

elastic constant	C_{01}	20000	MPa
elastic constant	C_{10}	20000	MPa





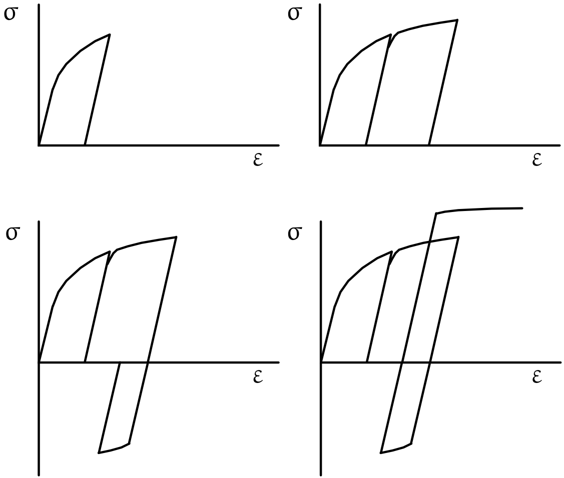
Neo-Hookean

Mooney-Rivlin

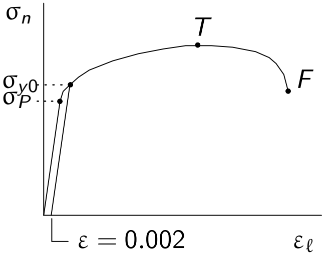
ELASTOPLASTIC

back to index

Elastoplastic material behavior



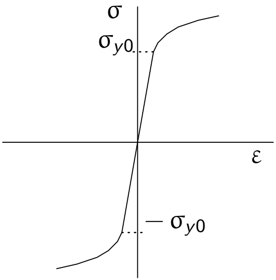
Tensile test



σ_P	proportional limit	
σ_{y0}	initial yield stress	
ε_{y0}	strain at σ_{y0}	: $\varepsilon_{y0} = \sigma_{y0}/E$
$\varepsilon_{0.2}$	0.2-strain	: $\varepsilon_p = 0.2\% = 0.002$
σ_T	tensile strength	
σ_F	fracture strength	
ε_F	fracture strain ($\approx 5\% = 0.05$ (metals))	

NB. : forming $\longrightarrow$ pressure $\longrightarrow$ larger strains

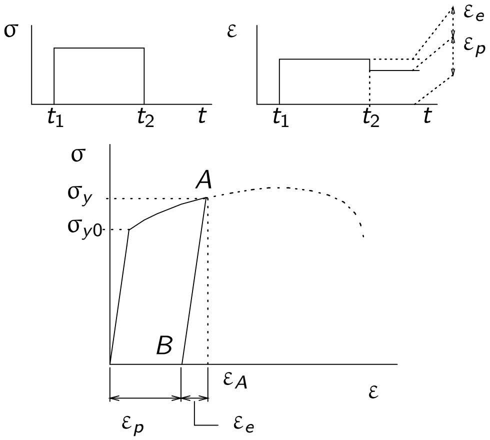
Compression test



yield in tensile test	$\sigma = \sigma_{y0}$
yield in compression test	$\sigma = -\sigma_{y0}$
yield general	$\sigma^2 = \sigma_{y0}^2$
elastic region	$-\sigma_{y0} < \sigma < \sigma_{y0}$

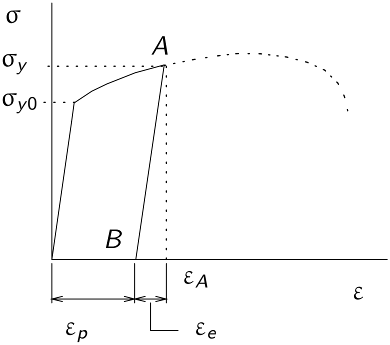
general yield criterion $\qquad f = \sigma^2 - \sigma_{y0}^2 = 0$

Interrupted tensile test



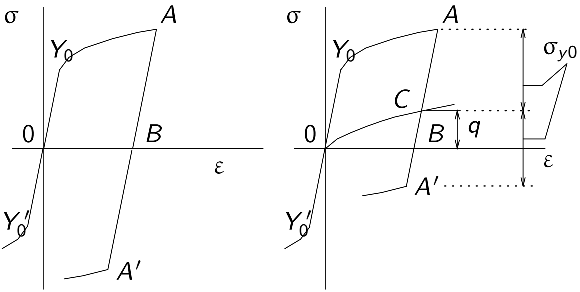
total strain	$\varepsilon = \varepsilon_A$	
plastic strain	ε_p	
elastic strain	ε_e	(springback)
assumptions	elastic parameters constant $\rightarrow \Delta\sigma = E\Delta\varepsilon = E\Delta\varepsilon_e$	

Resumed tensile test



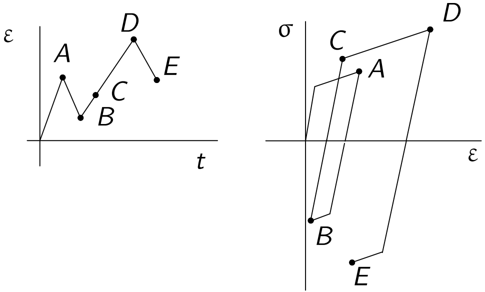
linear behavior $B \rightarrow A$	$\Delta\sigma = E\Delta\varepsilon = E\Delta\varepsilon_e$
current yield stress	$\sigma_y = \sigma_A$
hardening	σ_y increases $\quad\longrightarrow\quad \sigma_y > \sigma_{y0}$
hardening model	$\sigma_y \sim \varepsilon_p$
history parameter	ε_p

Hardening



isotropic hardening	:	elastic area larger & symmetric w.r.t. $\sigma = 0$		
tensile	:	$\sigma = \sigma_y$	$\rightarrow$	$f = \sigma^2 - \sigma_y^2 = 0$
compression	:	$\sigma = -\sigma_y$		
kinematic hardening	:	elastic area constant & symmetric w.r.t. $\sigma = q$		
tensile	:	$\sigma = q + \sigma_{y0}$	$\rightarrow$	$f = (\sigma - q)^2 - \sigma_{y0}^2 = 0$
compression	:	$\sigma = q - \sigma_{y0}$		
combined isotropic/kinematic hardening				
tensile	:	$\sigma = q + \sigma_y$	$\rightarrow$	$f = (\sigma - q)^2 - \sigma_y^2 = 0$
compression	:	$\sigma = q - \sigma_y$		

Effective plastic strain



$$\sigma_{yC} > \sigma_{yA} \qquad ; \qquad \varepsilon_{pC} < \varepsilon_{pA} \qquad \rightarrow$$

effective plastic strain (rate)

$$\bar{\varepsilon}_p = \sum_{\varepsilon} |\Delta\varepsilon_p| = \sum_{\tau=0}^{\tau=t} \frac{|\Delta\varepsilon_p|}{\Delta t}\,\Delta t = \int_{\tau=0}^{t} |\dot{\varepsilon}_p|\,d\tau = \int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p\,d\tau$$

Linear and power law hardening laws

linear hardening
$$\sigma_y = \sigma_{y0} + H\bar{\varepsilon}_p$$

Ludwik (1909)
$$\sigma_y = \sigma_{y0} + \sigma_{y0}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^n \quad (0 \leq n \leq 1) \quad \rightarrow$$
$$H = n\frac{\sigma_{y0}}{\varepsilon_{y0}}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^{n-1} = nE\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^{n-1}$$

mod. Ludwik
$$\sigma_y = \sigma_{y0}\left(1 + m\bar{\varepsilon}_p^n\right) \quad \rightarrow \quad H = \sigma_{y0} mn\bar{\varepsilon}_p^{n-1}$$

Swift (1952)
$$\sigma_y = C(m + \bar{\varepsilon}_p)^n \quad \text{with} \quad C = \frac{\sigma_{y0}}{m^n}$$
$$H = Cn\,(m + \bar{\varepsilon}_p)^{n-1}$$

Ramberg-Osgood (1943)
$$\bar{\varepsilon}_p = \frac{\sigma_y}{E}\left[1 + \alpha\left(\frac{\sigma_y}{\sigma_{y0}}\right)^{m-1}\right]$$
$$(m \geq 0; \alpha = \tfrac{3}{7})$$

Asymptotically perfect hardening laws

ideal plastic
$$\sigma_y = \sigma_{y0}$$

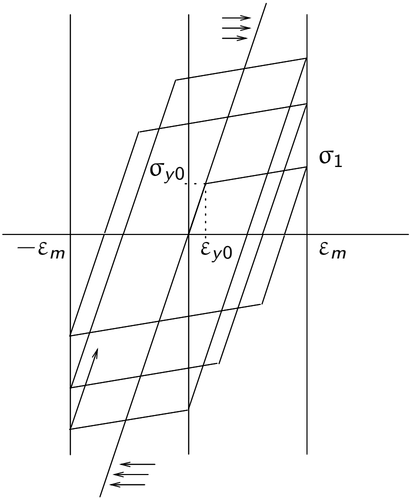
Prager (1938)
$$\sigma_y = \sigma_{y0} \tanh\left(\frac{E\bar{\varepsilon}_p}{\sigma_{y0}}\right)$$
$$H = \frac{\sigma_{y0}}{\varepsilon_{y0}}\left[\operatorname{sech}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)\right]^2 = E\left[\operatorname{sech}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)\right]^2$$

Betten I (1975)
$$\sigma_y = \sigma_{y0}\left[\tanh\left(\frac{E\bar{\varepsilon}_p}{\sigma_{y0}}\right)^m\right]^{1/m} \qquad (m > 1)$$
$$H = E\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^{m-1}\left[\tanh\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right]^{\frac{1}{m}-1}\left[\operatorname{sech}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right]^2$$

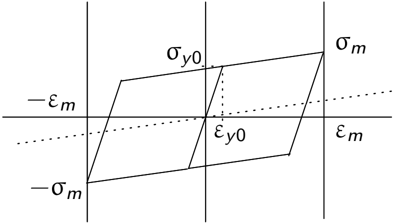
Voce (1949)
$$\sigma_y = C\left(1 - ne^{-m\bar{\varepsilon}_p}\right) \quad \text{with} \quad C = \frac{\sigma_{y0}}{1-n} \qquad (m > 1)$$
$$H = Cnme^{-m\bar{\varepsilon}_p}$$

Betten II (1975)
$$\sigma_y = \sigma_{y0} + (E\,\bar{\varepsilon}_p)\left[1 + \left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right]^{-1/m}$$
$$H = E\left[1 + \left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right]^{-\frac{1}{m}}\left[1 - \left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\left\{1 + \left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right\}^{-1}\right]$$

Cyclic load

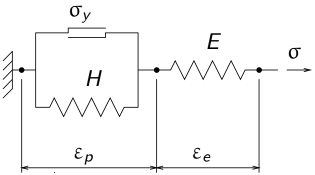


σ_{y0}
σ_1
$-\varepsilon_m$
ε_{y0}
ε_m



σ_{y0}
σ_m
$-\varepsilon_m$
ε_{y0}
ε_m
$-\sigma_m$

Elastoplastic model



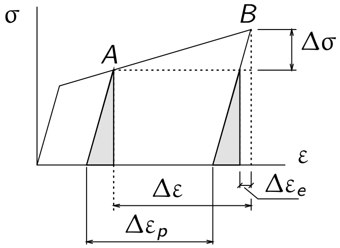
- $$f = (\sigma - q)^2 - \sigma_y^2 \quad \text{with} \quad \begin{array}{lll} f < 0 \quad | \quad f = 0 \wedge \dot{f} < 0 & \rightarrow & \text{el.} \\ f = 0 \wedge \dot{f} = 0 & \rightarrow & \text{elpl.} \end{array}$$
- $$\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p) \quad ; \quad q = q(\varepsilon_p)$$
- $$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p$$
- $$\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \frac{1}{E}\dot{\sigma}$$
- $$\dot{\varepsilon}_p = \dot{\lambda}\frac{\partial f}{\partial \sigma} = 2\dot{\lambda}(\sigma - q) \quad ; \quad \dot{\bar{\varepsilon}}_p = |\dot{\varepsilon}_p| = 2\dot{\lambda}\,|\sigma - q|$$
- $$\bar{\varepsilon}_p = \int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p \, d\tau = \sum_t |\Delta\varepsilon_p|$$

Constitutive equations

$$\left.\begin{array}{l}\dot{\sigma} = E\dot{\varepsilon}_e = E(\dot{\varepsilon} - \dot{\varepsilon}_p) = E\{\dot{\varepsilon} - 2\dot{\lambda}(\sigma - q)\} \\ f = 0\end{array}\right\} \quad \rightarrow$$

$$\left.\begin{array}{l}\dot{\sigma} + 2E(\sigma - q)\dot{\lambda} - E\dot{\varepsilon} = 0 \\ f = 0\end{array}\right\}$$

Isotropic hardening "monotonic" tensile test $\quad A \rightarrow B$



$$\Delta\sigma = E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p)$$

$$= E\left(\Delta\varepsilon - \frac{\Delta\sigma_y}{H}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{H}\right) \quad \rightarrow$$

$$\Delta\sigma = \frac{EH}{E+H}\,\Delta\varepsilon = S\Delta\varepsilon \qquad ; \qquad \Delta\varepsilon_p = \frac{\Delta\sigma}{H} = \frac{E}{E+H}\,\Delta\varepsilon$$

Kinematic hardening

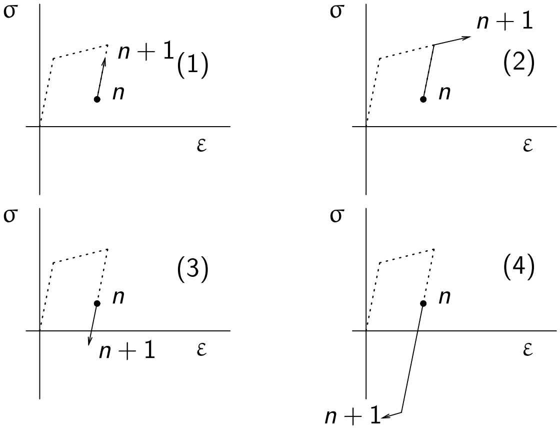
$$\Delta\sigma = \frac{EK}{E+K}\,\Delta\varepsilon \quad ; \quad \Delta\varepsilon_p = \frac{1}{K}\,\Delta\sigma = \frac{E}{E+K}\,\Delta\varepsilon$$

$$\begin{aligned}\Delta\sigma &= E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) \\ &= E\left(\Delta\varepsilon - \frac{\Delta q}{K}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{K}\right) \quad \rightarrow \\ \Delta\sigma &= \frac{EK}{E+K}\,\Delta\varepsilon = S\Delta\varepsilon \qquad ; \qquad \Delta\varepsilon_p = \frac{\Delta\sigma}{K} = \frac{E}{E+K}\,\Delta\varepsilon\end{aligned}$$

$$\lim_{H\longrightarrow\infty} \frac{EH}{E+H} = \lim_{H\longrightarrow\infty} \frac{E}{\frac{E}{H}+1} = E$$

Stress update

WHERE ARE WE ??



Elastic stress predictor

$$\sigma_e = \sigma_n + E(\varepsilon - \varepsilon_n)$$

- $f = (\sigma_e - q_n)^2 - \sigma_{y_n}^2 \leq 0 \quad \rightarrow \quad$ elastic increment
- $f = (\sigma_e - q_n)^2 - \sigma_{y_n}^2 > 0 \quad \rightarrow \quad$ elastoplastic increment

Elastic increment

- elastic solution is end-increment solution
- continue loading history

$$\sigma(t_{n+1}) = \sigma_e \quad ; \quad \bar{\varepsilon}_p(t_{n+1}) = \bar{\varepsilon}_p(t_n) = \bar{\varepsilon}_{p_n}$$

$$\sigma_y(t_{n+1}) = \sigma_y(t_n) = \sigma_{y_n} \quad ; \quad q(t_{n+1}) = q(t_n) = q_n$$

Implicit solution procedure

$$\left.\begin{aligned} &\sigma - \sigma_n + 2E(\sigma - q)(\lambda - \lambda_n) = E(\varepsilon - \varepsilon_n) \\ &f - f_n = f = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} &\sigma^* + \delta\sigma - \sigma_n + 2E(\sigma^* + \delta\sigma - q^* - \delta q)(\lambda^* + \delta\lambda - \lambda_n) = E(\varepsilon - \varepsilon_n) \\ &f^* + \delta f = 0 \quad \rightarrow \quad f^* + \frac{\partial f}{\partial \sigma}\delta\sigma + \frac{\partial f}{\partial \lambda}\delta\lambda = 0 \end{aligned}\right\}$$

$$\begin{aligned} \frac{\partial f}{\partial \sigma} &= 2(\sigma - q) \\ \frac{\partial f}{\partial \lambda} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial \varepsilon_p}\frac{\partial \varepsilon_p}{\partial \lambda} + \frac{\partial f}{\partial \sigma_y}\frac{\partial \sigma_y}{\partial \bar{\varepsilon}_p}\frac{\partial \bar{\varepsilon}_p}{\partial \lambda} \\ &= [-2(\sigma - q)][K][2(\sigma - q)] + [-2\sigma_y][H][2|\sigma - q|] \\ &= -4K(\sigma - q)^2 - 4H\sigma_y|\sigma - q| \end{aligned}$$

$$\left.\begin{aligned} &\sigma^* + \delta\sigma - \sigma_n + 2E(\sigma^* + \delta\sigma - q^* - \delta q)(\lambda^* + \delta\lambda - \lambda_n) = E(\varepsilon - \varepsilon_n) \\ &f^* + 2(\sigma^* - q^*)\delta\sigma - [4K^*(\sigma^* - q^*)^2 + 4H^*\sigma_y^*|\sigma^* - q^*|]\delta\lambda = 0 \end{aligned}\right\}$$

Implicit solution procedure

$$\left.\begin{array}{l} \sigma^* + \delta\sigma - \sigma_n + 2E(\sigma^* + \delta\sigma - q^* - \delta q)(\lambda^* + \delta\lambda - \lambda_n) = E(\varepsilon - \varepsilon_n) \\ f^* + 2(\sigma^* - q^*)\delta\sigma - [4K^*(\sigma^* - q^*)^2 + 4H^*\sigma_y^*|\sigma^* - q^*|]\delta\lambda = 0 \end{array}\right\}$$

$$\begin{aligned} &\sigma^* = \sigma^* + \delta\sigma \\ &\lambda^* = \lambda^* + \delta\lambda \\ &\Delta\varepsilon_p = 2(\lambda^* - \lambda_n)(\sigma^* - q_n) \quad \rightarrow \quad \varepsilon_p \quad \rightarrow \quad q^*, K^* \\ &\Delta\bar{\varepsilon}_p = |\Delta\varepsilon_p| \quad \rightarrow \quad \bar{\varepsilon}_p \quad \rightarrow \quad \sigma_y^*, H^* \end{aligned}$$

Stiffness : implicit

$$\begin{cases} \sigma - \sigma_n + 2E(\sigma - q)(\lambda - \lambda_n) - E(\varepsilon - \varepsilon_n) = 0 \\ f = 0 \end{cases}$$

$$\begin{cases} \delta\sigma + 2E\delta\sigma(\lambda - \lambda_n) + 2E(\sigma - q)\delta\lambda - E\delta\varepsilon = 0 \\ (\sigma - q)\delta\sigma - 2K(\sigma - q)^2\delta\lambda - 2H\sigma_y|\sigma - q|\delta\lambda = 0 \end{cases}$$

$$\left[1 + 2E(\lambda - \lambda_n) + \frac{2E(\sigma - q)^2}{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|}\right]\delta\sigma = E\delta\varepsilon$$

$$C_\varepsilon = \frac{E\{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|\}}{\{1 + 2E(\lambda - \lambda_n)\}\{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|\} + 2E(\sigma - q)^2}$$

yield at $\tau = t = t_{n+1} \quad \rightarrow \quad (\sigma - q)^2 = \sigma_y^2$ and $|\sigma - q| = \sigma_y \quad \rightarrow$

$$C_\varepsilon = \frac{E(K + H)}{E + K + H + 2E(K + H)(\lambda - \lambda_n)}$$

Explicit solution procedure

$$\left.\begin{array}{l} \Delta\sigma + 2E(\sigma_n - q_n)\Delta\lambda = E\Delta\varepsilon \\ \Delta f = 0 \quad \rightarrow \quad \left.\dfrac{\partial f}{\partial \sigma}\right|_n \Delta\sigma + \left.\dfrac{\partial f}{\partial \lambda}\right|_n \Delta\lambda = 0 \end{array}\right\}$$

$$\Delta\sigma + 2E(\sigma_n - q_n)\Delta\lambda = E\Delta\varepsilon$$

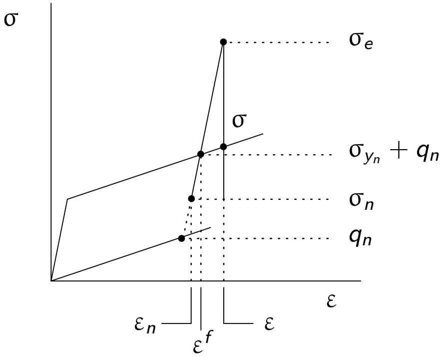
$$2(\sigma_n - q_n)\Delta\sigma - 4K_n(\sigma_n - q_n)^2\Delta\lambda - 4H_n\sigma_{yn}|\sigma_n - q_n|\Delta\lambda = 0 \quad \rightarrow$$

$$\Delta\lambda = \frac{(\sigma_n - q_n)}{2K_n(\sigma_n - q_n)^2 + 2H_n\sigma_{yn}|\sigma_n - q_n|}\Delta\sigma = \frac{1}{2K_n(\sigma_n - q_n) + 2H_n(\sigma_n - q_n)}\Delta\sigma$$

$$\Delta\sigma = \frac{E[K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n|]}{K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n| + E(\sigma_n - q_n)^2}\Delta\varepsilon$$

$$\Delta\varepsilon_p = 2(\sigma_n - q_n)\Delta\lambda = \frac{(\sigma_n - q_n)^2}{K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n|}$$

Increment splitting



$$\sigma_e = \sigma_n + E(\varepsilon - \varepsilon_n) \qquad \rightarrow \qquad \Delta\sigma_e = \sigma_e - \sigma_n = E(\varepsilon - \varepsilon_n)$$

$$\beta = \frac{|\ \mathrm{sign}(\varepsilon - \varepsilon_n)\sigma_{y_n} - (\sigma_n - q_n)|}{|\sigma_e - \sigma_n|}$$

$$\varepsilon^f = \varepsilon_n + \beta(\varepsilon - \varepsilon_n) \qquad \rightarrow \qquad \Delta\varepsilon^f = \varepsilon - \varepsilon^f = (1 - \beta)(\varepsilon - \varepsilon_n)$$

Explicit stress update

$$
\begin{cases}
\Delta\sigma^f + 2E(\sigma_n - q)\Delta\lambda = E\Delta\varepsilon^f \\
2(\sigma_n - q)\Delta\sigma^f - 4K_n(\sigma_n - q_n)^2\Delta\lambda - 4H_n\sigma_{yn}|\sigma_n - q_n|\Delta\lambda = 0
\end{cases}
$$

$$
\begin{aligned}
&\Delta\sigma = \beta\Delta\sigma_e + \Delta\sigma^f \quad \rightarrow \quad \sigma = \sigma_n + \Delta\sigma \\
&\lambda = \lambda_n + \Delta\lambda \\
&\Delta\varepsilon_p = 2(\lambda - \lambda_n)(\sigma - q_n) \quad \rightarrow \quad \varepsilon_p \quad \rightarrow \quad q, K \\
&\Delta\bar{\varepsilon}_p = |\Delta\varepsilon_p| \quad \rightarrow \quad \bar{\varepsilon}_p \quad \rightarrow \quad \sigma_y, H
\end{aligned}
$$

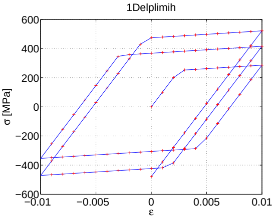
Implementation

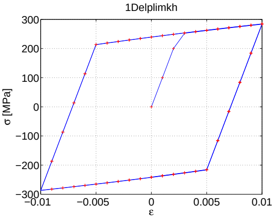
`tr2delpl.m`

Cyclic loading

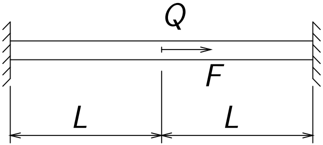
initial length	l_0	100	mm
initial cross-sectional area	A_0	10	mm^2

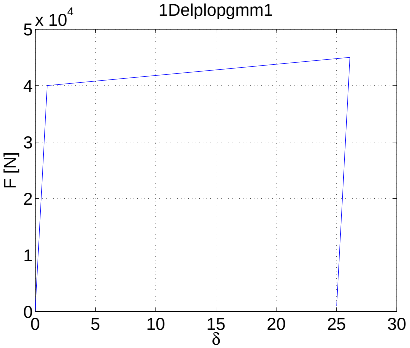
Young's modulus	E	100000	MPa
Poisson's ratio	ν	0.3	-
initial yield stress	σ_{y0}	250	MPa
isotropic hardening coefficient	H	5000	MPa
kinematic hardening coefficient	K	5000	MPa



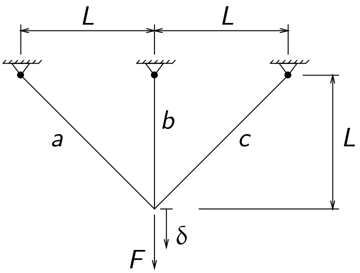


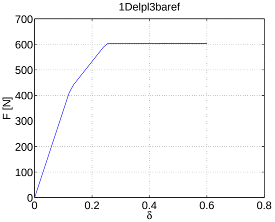
Clamped truss

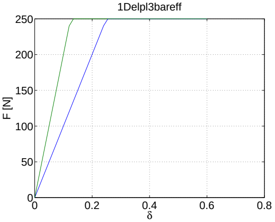




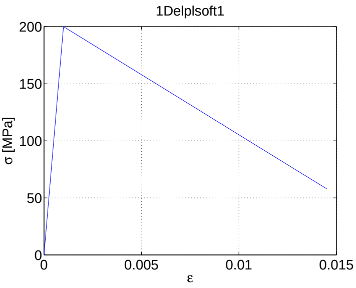
Truss structure

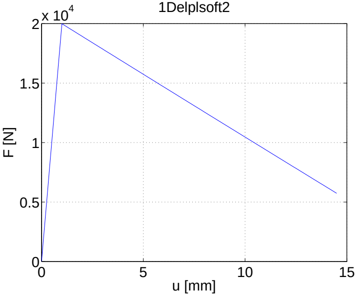






Softening material

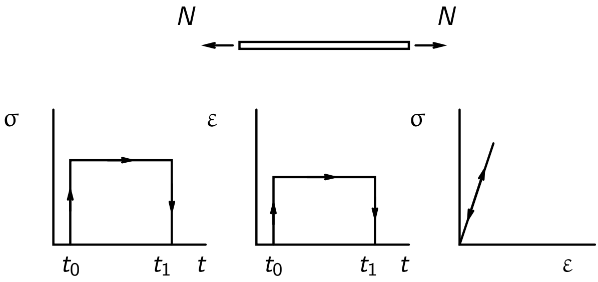




LINEAR VISCOELASTIC

back to index

Linear elastic material behavior



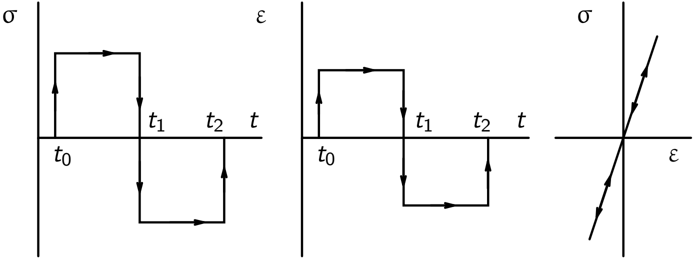
$$\varepsilon = \frac{1}{E}\,\sigma \qquad \rightarrow \qquad \sigma = E\,\varepsilon \qquad \rightarrow$$

$$N = \sigma A = EA\varepsilon = \frac{EA}{l}\,\Delta l = k\,\Delta l$$

constant Young's modulus $\quad E \quad : \quad$ Hooke's law

linear spring : spring stiffness $\quad k = \dfrac{EA}{l}$

Load cycle

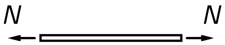


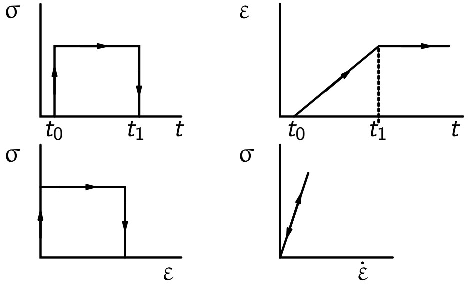
no dissipation : no area under (σ, ε)-curve

$$U_d = \int_{t_0}^{t_1} \sigma \, d\varepsilon + \int_{t_1}^{t_2} \sigma \, d\varepsilon = \int_{t_0}^{t_1} E\varepsilon \, d\varepsilon + \int_{t_1}^{t_2} E\varepsilon \, d\varepsilon$$
$$= \frac{1}{2} E[\varepsilon_1^2 - \varepsilon_0^2 + \varepsilon_2^2 - \varepsilon_1^2] = 0$$

NO ENERGY DISSIPATION

Linear viscous material behavior





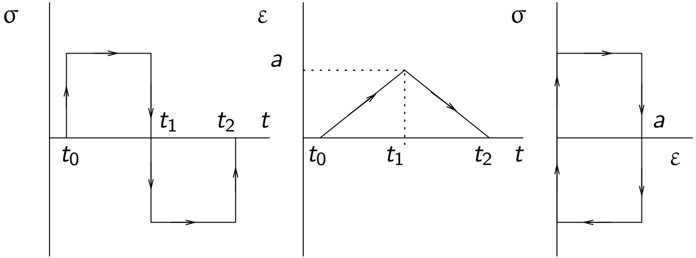
$$\dot{\varepsilon} = \frac{1}{\eta}\,\sigma \qquad \rightarrow \qquad \sigma = \eta\dot{\varepsilon} \qquad \rightarrow$$

$$N = \sigma A = \eta A\dot{\varepsilon} = \frac{\eta A}{l}\,\dot{\Delta l} = b\,\dot{\Delta l}$$

constant viscosity η : Newtonian fluid

linear dashpot : damping constant $b = \dfrac{\eta A}{l}$

Load cycle

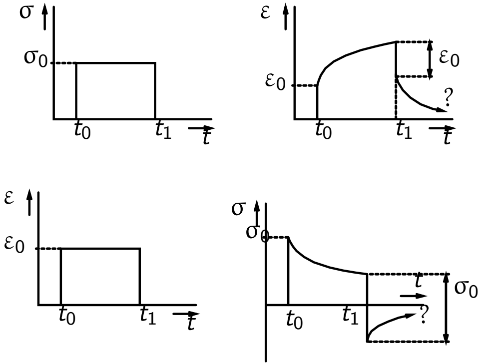


dissipated energy ∼ area

$$U_d = \int_{t_0}^{t_1} \sigma \, d\varepsilon + \int_{t_1}^{t_2} \sigma \, d\varepsilon = \int_{t_0}^{t_1} \eta \dot{\varepsilon} \, d\varepsilon + \int_{t_1}^{t_2} \eta \dot{\varepsilon} \, d\varepsilon = \int_{t_0}^{t_1} \eta c \, d\varepsilon - \int_{t_1}^{t_2} \eta c \, d\varepsilon$$
$$= \eta c [\varepsilon_1 - \varepsilon_0 - \varepsilon_2 + \varepsilon_1] = 2 \eta c a$$

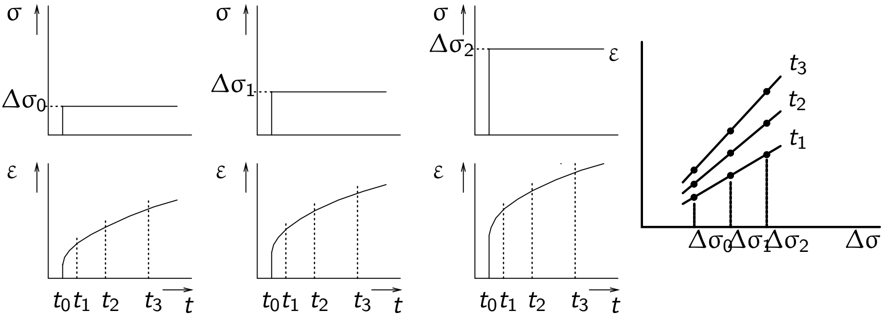
TOTAL ENERGY DISSIPATION

Viscoelastic material behavior



- small deformations !!
- description of experimental observations
- modeling the material behavior (a.o. with spring-dashpot models)

Proportionality

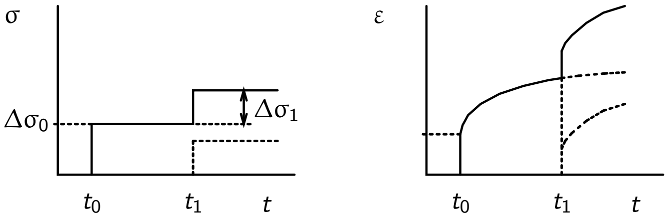


linear **isochrones** $\quad\rightarrow\quad$ proportionality

$$\varepsilon(t) = \Delta\sigma D(t - t_0) \qquad \text{for} \qquad \forall \quad t \geq t_0$$

$D(t - t_0)$ is no function of the stresses

Superposition



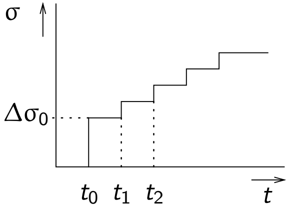
separate excitations

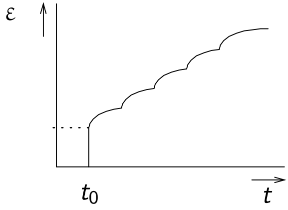
$$
\begin{array}{llll}
\Delta\sigma = \Delta\sigma_0 & \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) & \text{for} & t > t_0 \\
\Delta\sigma = \Delta\sigma_1 & \rightarrow \quad \varepsilon(t) = \Delta\sigma_1 D(t - t_1) & \text{for} & t > t_1
\end{array}
$$

subsequent excitations

$$
\begin{array}{llll}
\Delta\sigma = \Delta\sigma_0 & \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) & \text{for} & t_0 < t < t_1 \\
\Delta\sigma = \Delta\sigma_0 + \Delta\sigma_1 & \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1) & \text{for} & t > t_1
\end{array}
$$

Boltzmann integral : strain respons





$$
\begin{aligned}
\varepsilon(t) &= \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1) + \Delta\sigma_2 D(t - t_2) + .. \\
&= \sum_{i=1}^{n} \Delta\sigma_i D(t - t_i) \qquad \rightarrow \qquad \text{limit } n \rightarrow \infty \qquad (t \rightarrow \tau) \\
&= \int_{\tau=t_0^-}^{t} D(t-\tau)\, d\sigma(\tau) = \int_{\tau=t_0^-}^{t} D(t-\tau) \frac{d\sigma(\tau)}{d\tau}\, d\tau
\end{aligned}
$$

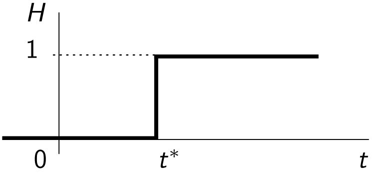
$$\varepsilon(t) = \int_{\tau=t_0^-}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau$$

Boltzmann integral : stress respons

$$\sigma(t) = \int_{\tau=t_0^-}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau$$

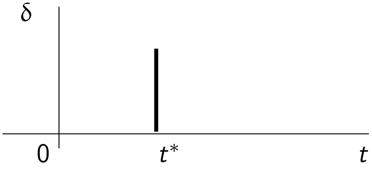
Step excitations

Heaviside function
$$H(t,t^*) \quad \left\{ \begin{array}{ccc} t < t^* & : & H(t,t^*) = 0 \\ t > t^* & : & H(t,t^*) = 1 \end{array} \right\}$$



Dirac function
$$\delta(t,t^*) = \frac{d}{dt}\{H(t,t^*)\}$$

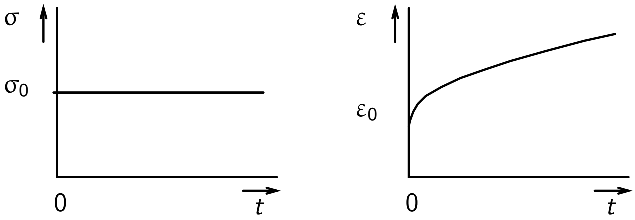
$$\int_{\tau=0}^{t>t^*} \delta(\tau,t^*)\, d\tau = 1 \quad ; \quad \int_{\tau=0}^{t>t^*} f(\tau)\delta(\tau,t^*)\, d\tau = f(t^*)$$



Creep (Retardation) $\longrightarrow$ creep function

$$\sigma(t) = \sigma_0 H(t,0) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0 \delta(t,0)$$

$$\varepsilon(t) = \int_{\tau=0^-}^{t} D(t-\tau)\dot{\sigma}(\tau)\,d\tau = \int_{\tau=0^-}^{t} D(t-\tau)\sigma_0\delta(\tau,0)\,d\tau = \sigma_0 D(t)$$

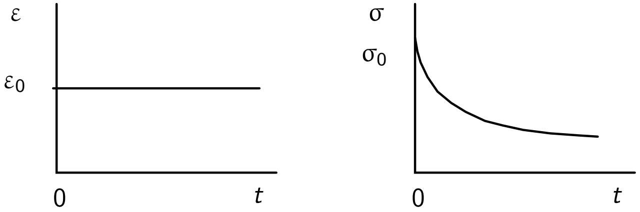


- $\dot{D}(t) \geq 0 \qquad \forall \quad t \geq 0$
- $\ddot{D}(t) < 0 \qquad \forall \quad t \geq 0$

Relaxation $\longrightarrow$ relaxation function

$$\varepsilon(t) = \varepsilon_0 H(t,0) \qquad \rightarrow \qquad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0)$$

$$\sigma(t) = \int_{\tau=0^-}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\,d\tau = \int_{\tau=0^-}^{t} E(t-\tau)\varepsilon_0\delta(\tau,0)\,d\tau = \varepsilon_0 E(t)$$

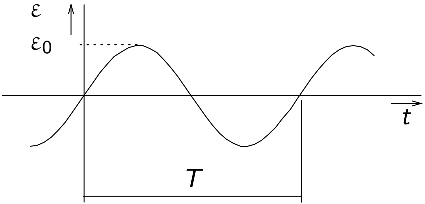


- $\dot{E}(t) \leq 0 \qquad \forall \quad t \geq 0$
- $\ddot{E}(t) > 0 \qquad \forall \quad t \geq 0$
- $\displaystyle\int_{t=0}^{\infty} \dot{E}(t)\,dt \geq 0 \quad \rightarrow \quad \lim_{t\rightarrow\infty} \dot{E}(t) = 0$

Harmonic strain excitation

harmonic strain (ω = angular frequency [rad s^{-1}])

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \quad \longrightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$



amplitude	ε_0
angular frequency	ω [rad s^{-1}]
period and frequency	$T = \dfrac{2\pi}{\omega}$ [s^{-1}] ; $f = \dfrac{1}{T}$

Stress response

$$
\begin{aligned}
\sigma(t) &= \int_{\tau=-\infty}^{t} E(t-\tau)\varepsilon_0\omega\cos(\omega\tau)\,d\tau \\
&= \varepsilon_0\omega\int_{\xi=-\infty}^{t} E(t-\tau)\cos(\omega\tau)\,d\tau \\
&\qquad t-\tau=s \quad\rightarrow\quad \tau=t-s \quad\rightarrow\quad d\tau=-ds \\
&= \varepsilon_0\omega\int_{s=0}^{\infty} E(s)\cos\{\omega(t-s)\}\,ds \\
&\qquad \cos(\omega t-\omega s)=\cos(\omega t)\cos(\omega s)+\sin(\omega t)\sin(\omega s) \\
&= \varepsilon_0\left[\omega\int_{s=0}^{\infty} E(s)\sin(\omega s)\,ds\right]\sin(\omega t)+\varepsilon_0\left[\omega\int_{s=0}^{\infty} E(s)\cos(\omega s)\,ds\right]\cos(\omega t) \\
&= \varepsilon_0 E'\sin(\omega t)+\varepsilon_0 E''\cos(\omega t)
\end{aligned}
$$

$$
\begin{aligned}
E'(\omega) &= \omega\int_{s=0}^{\infty} E(s)\sin(\omega s)\,ds &&: \text{ storage modulus} \\
E''(\omega) &= \omega\int_{s=0}^{\infty} E(s)\cos(\omega s)\,ds &&: \text{ loss modulus}
\end{aligned}
$$

Energy dissipation

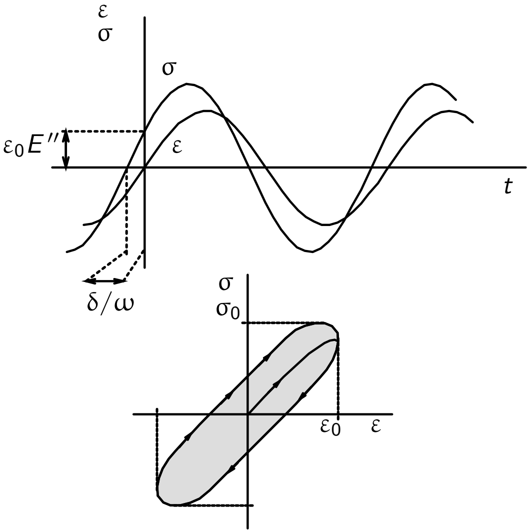
one period $\qquad 0 \le t \le \frac{2\pi}{\omega} = T = \frac{1}{f}$

dissipated energy per unit of volume

$$
\begin{aligned}
U_d &= \int_{\varepsilon(0)}^{\varepsilon(T)} \sigma \, d\varepsilon = \int_{t=0}^{T} \sigma\dot{\varepsilon} \, dt \\
&= \int_{t=0}^{T} \{\varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)\}\{\varepsilon_0 \omega \cos(\omega t)\} \, dt \\
&= \int_{t=0}^{T} \varepsilon_0^2 \omega \left\{ E' \sin(\omega t)\cos(\omega t) + E'' \cos^2(\omega t) \right\} \, dt \\
&= \int_{t=0}^{T} \varepsilon_0^2 \omega \left\{ \tfrac{1}{2} E' \sin(2\omega t) + \tfrac{1}{2} E'' + \tfrac{1}{2} E'' \cos(2\omega t) \right\} \, dt \\
&= \tfrac{1}{2}\varepsilon_0^2 \omega \left[-E' \frac{1}{2\omega} \cos(2\omega t) + E'' t + E'' \frac{1}{2\omega} \sin(2\omega t) \right]_0^{T=\frac{2\pi}{\omega}} \\
&= \tfrac{1}{2}\varepsilon_0^2 \omega \left[-E' \frac{1}{2\omega} + E' \frac{1}{2\omega} + E'' \frac{2\pi}{\omega} \right] = \pi \varepsilon_0^2 E'' \\
&> 0 \quad \Rightarrow \quad E'' > 0 \quad \rightarrow
\end{aligned}
$$

Phase difference

phase angle δ (phase difference $\dfrac{\delta}{\omega}$)



hysteresis

Relation between E', E'' and δ_σ

stress response $\qquad$ (δ = phase angle)

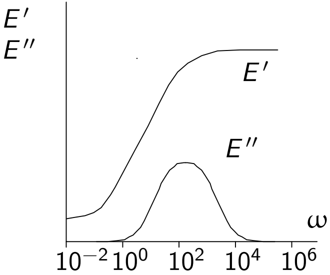
$$
\begin{aligned}
\sigma(t) &= \sigma_0 \sin(\omega t + \delta) \\
&= \sigma_0 \cos(\delta) \sin(\omega t) + \sigma_0 \sin(\delta) \cos(\omega t) \\
\sigma(t) &= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)
\end{aligned}
$$

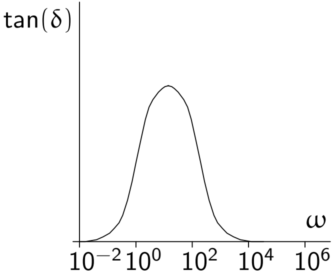
storage and loss modulus

$$
\left.
\begin{aligned}
E' &= \frac{\sigma_0}{\varepsilon_0} \cos(\delta) \\
E'' &= \frac{\sigma_0}{\varepsilon_0} \sin(\delta)
\end{aligned}
\right\}
\quad \rightarrow \quad
\left\{
\begin{aligned}
&\frac{E''}{E'} = \tan(\delta) \quad \rightarrow \\
&\qquad \delta = \arctan\left(\frac{E''}{E'}\right)
\end{aligned}
\right.
$$

amplitude $\qquad$ $\sigma_0 = \varepsilon_0 \sqrt{(E')^2 + (E'')^2}$

Measured E', E'' and $\tan(\delta)$





- measurement of E' and E'' can be done accurately
- $E'(\omega), E''(\omega) \quad \rightarrow \quad E(t)$ via fitting procedure
- range $\omega \quad \rightarrow \quad$ temperature $\quad \rightarrow \quad$ DMTA

- measurement of $E(t)$ in relaxation test is difficult
- fit is inaccurate

Harmonic stress excitation

$$\sigma(t) = \sigma_0 \sin(\omega t) \quad \longrightarrow \quad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

$$\begin{aligned}
\varepsilon(t) &= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau \\
&= \int_{\tau=-\infty}^{t} D(t-\tau)\sigma_0 \omega \cos(\omega \tau)\, d\tau \\
&= \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s) \sin(\omega s)\, ds\right] \sin(\omega t) + \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s) \cos(\omega s)\, ds\right] \cos(\omega t) \\
&= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}$$

$$\begin{aligned}
D'(\omega) &= \omega \int_{s=0}^{\infty} D(s) \sin(\omega s)\, ds && : \quad \text{storage compliance} \\
D''(\omega) &= -\omega \int_{s=0}^{\infty} D(s) \cos(\omega s)\, ds && : \quad \text{loss compliance}
\end{aligned}$$

Relation between D', D'' and δ_ε

strain response

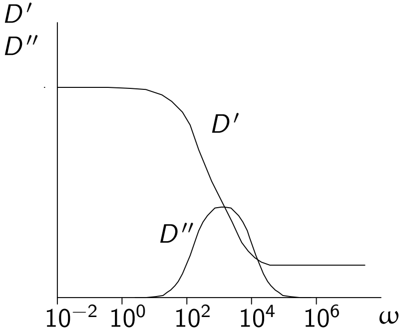
$$
\begin{aligned}
\varepsilon(t) &= \varepsilon_0 \sin(\omega t - \delta) \\
&= \varepsilon_0 \cos(\delta) \sin(\omega t) - \varepsilon_0 \sin(\delta) \cos(\omega t) \\
\varepsilon(t) &= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}
$$

storage and loss compliance

$$
\left.
\begin{aligned}
D' &= \frac{\varepsilon_0}{\sigma_0} \cos(\delta) \\
D'' &= \frac{\varepsilon_0}{\sigma_0} \sin(\delta)
\end{aligned}
\right\} \quad \rightarrow \quad
\left\{
\begin{aligned}
&\frac{D''}{D'} = \tan(\delta) \quad \rightarrow \\
&\qquad \delta = \arctan\left(\frac{D''}{D'}\right)
\end{aligned}
\right.
$$

amplitude $\qquad \varepsilon_0 = \sigma_0 \sqrt{(D')^2 + (D'')^2}$

Measured D' and D''



Relation between (D', D'') and (E', E'')

$$\left.\begin{array}{l}\sigma_0 = \varepsilon_0\sqrt{(E')^2 + (E'')^2} \\[2ex] \varepsilon_0 = \sigma_0\sqrt{(D')^2 + (D'')^2}\end{array}\right\} \quad \rightarrow$$

$$[(E')^2 + (E'')^2][(D')^2 + (D'')^2] = 1 \qquad (1)$$

$$\frac{D''}{D'} = \frac{E''}{E'} \quad \rightarrow \quad D'' = D'\,\frac{E''}{E'} \qquad (2)$$

(1) & (2) $\rightarrow \quad D' = \dfrac{E'}{(E')^2 + (E'')^2} \quad ; \quad D'' = \dfrac{E''}{(E')^2 + (E'')^2}$

idem $\quad E' = \dfrac{D'}{(D')^2 + (D'')^2} \quad ; \quad E'' = \dfrac{D''}{(D')^2 + (D'')^2}$

Complex variables

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) = \varepsilon_0 \cos(\omega t - \tfrac{\pi}{2}) = Re\left[\varepsilon_0 e^{-i\frac{\pi}{2}} e^{i\omega t}\right] = Re\left[\varepsilon^* e^{i\omega t}\right]$$

$$\sigma(t) = \sigma_0 \sin(\omega t + \delta) = \sigma_0 \cos(\omega t - \tfrac{\pi}{2} + \delta) = Re\left[\sigma_0 e^{i(\delta - \frac{\pi}{2})} e^{i\omega t}\right] = Re\left[\sigma^* e^{i\omega t}\right]$$

complex modulus and compliance

$$E^* = \frac{\sigma^*}{\varepsilon^*} = \frac{\sigma_0}{\varepsilon_0} e^{i\delta} = \frac{\sigma_0}{\varepsilon_0} \cos(\delta) + i \frac{\sigma_0}{\varepsilon_0} \sin(\delta) = E' + iE''$$

$$D^* = \frac{\varepsilon^*}{\sigma^*} = \frac{\varepsilon_0}{\sigma_0} e^{-i\delta} = \frac{\varepsilon_0}{\sigma_0} \cos(\delta) - i \frac{\varepsilon_0}{\sigma_0} \sin(\delta) = D' - iD''$$

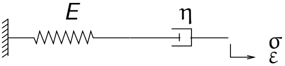
dynamic modulus en compliance

$$E_d = |E^*| = \sqrt{(E')^2 + (E'')^2} = \frac{\sigma_0}{\varepsilon_0}$$

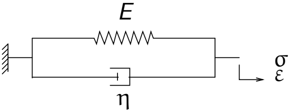
$$D_d = |D^*| = \sqrt{(D')^2 + (D'')^2} = \frac{\varepsilon_0}{\sigma_0}$$

Viscoelastic models

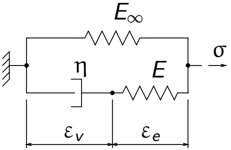
Maxwell



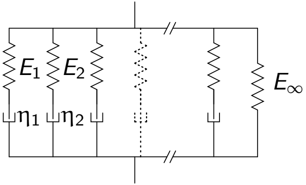
Kelvin-Voigt



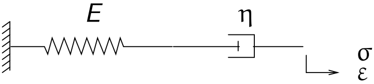
Standard Solid



Generalized Maxwell



Maxwell model



$$\varepsilon = \varepsilon_E + \varepsilon_\eta \quad \rightarrow$$

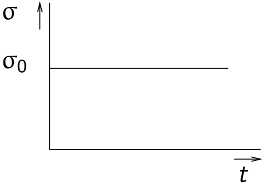
$$\dot{\varepsilon} = \dot{\varepsilon}_E + \dot{\varepsilon}_\eta = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

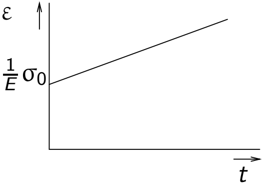
Maxwell : stress step excitation

$$\sigma(t) = \sigma_0 H(t,0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \delta(t,0)$$

$$\dot{\varepsilon}(t) = \frac{\sigma_0}{E}\,\delta(t,0) + \frac{\sigma_0}{\eta}$$

$$\varepsilon(t) = \frac{\sigma_0}{E} H(t,0) + \frac{\sigma_0}{\eta}\,t = \sigma_0 \left[\frac{1}{\eta}\left(t + \frac{\eta}{E}\right)\right] = \sigma_0 D(t)$$

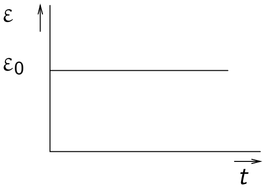


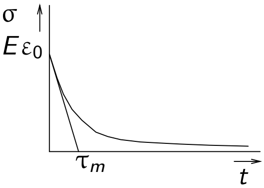


Maxwell : strain step excitation

$$\varepsilon(t) = \varepsilon_0 H(t,0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0)$$

$$\sigma(t) = \varepsilon_0 E\, e^{-\frac{E}{\eta}t} = \varepsilon_0 E\, e^{-\frac{t}{\tau_m}} = \varepsilon_0 E(t)$$





Maxwell : Boltzmann integrals

creep

$$\begin{aligned}\varepsilon(t) &= \frac{1}{\eta}\int_{\tau=-\infty}^{t} \left\{(t-\tau)+\frac{\eta}{E}\right\}\dot{\sigma}(\tau)\,d\tau \\ &= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\,d\tau\end{aligned}$$

relaxation

$$\begin{aligned}\sigma(t) &= \int_{\tau=-\infty}^{t} \left\{Ee^{-\frac{E}{\eta}(t-\tau)}\right\}\dot{\varepsilon}(\tau)\,d\tau \\ &= \int_{\tau=-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\,d\tau\end{aligned}$$

Maxwell : harmonic stress excitation

$$\sigma(t) = \sigma_0 \sin(\omega t) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

strain response

$$\begin{aligned}
\dot{\varepsilon}(t) &= = \frac{1}{E}\,\sigma_0\omega\cos(\omega t) + \frac{1}{\eta}\,\sigma_0\sin(\omega t) \\
\varepsilon(t) &= \sigma_0 \left[\frac{1}{E}\right] \sin(\omega t) - \sigma_0 \left[\frac{1}{\eta\omega}\right] \cos(\omega t) \\
&= \varepsilon_P(t) \qquad\qquad\qquad\qquad \varepsilon_H \text{ damps out} \\
&= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}$$

dynamic quantities

$$D' = \frac{1}{E} \qquad ; \qquad D'' = \frac{1}{\eta\omega}$$

$$\delta = \arctan\left(\frac{D''}{D'}\right) = \arctan\left(\frac{E}{\eta\omega}\right)$$

Maxwell : harmonic strain excitation

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \qquad \rightarrow \qquad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$

stress response

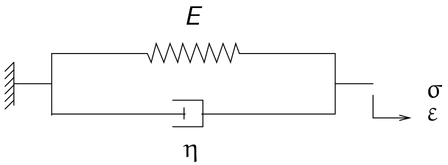
$$\begin{aligned}
\sigma(t) &= \int_{\tau=-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau = E\varepsilon_0\omega e^{-\frac{E}{\eta}t} \int_{\tau=0}^{t} e^{\frac{E}{\eta}\tau} \cos(\omega\tau)\, d\tau \\
&= \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2 + \omega^2} \, \frac{E}{\eta} \right] e^{-\frac{E}{\eta}t} + \\
&\qquad \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2 + \omega^2} \, \omega \right] \sin(\omega t) + \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2 + \omega^2} \, \frac{E}{\eta} \right] \cos(\omega t) \\
&= \varepsilon_0 \left[\frac{E\omega}{(\frac{E}{\eta})^2 + \omega^2} \, \omega \right] \sin(\omega t) + \varepsilon_0 \left[\frac{E\omega}{(\frac{E}{\eta})^2 + \omega^2} \, \frac{E}{\eta} \right] \cos(\omega t) \qquad \text{for } t \geq 0 \\
&= \varepsilon_0 \left[\frac{E\omega^2\tau_m^2}{1 + \omega^2\tau_m^2} \right] \sin(\omega t) + \varepsilon_0 \left[\frac{E\omega\tau_m}{1 + \omega^2\tau_m^2} \right] \cos(\omega t) \\
&= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)
\end{aligned}$$

Maxwell : harmonic strain excitation

dynamic quantities

$$E' = \frac{E\omega^2}{(\frac{E}{\eta})^2 + \omega^2} \quad ; \quad E'' = \frac{E\omega(\frac{E}{\eta})}{(\frac{E}{\eta})^2 + \omega^2} \quad ; \quad \tan(\delta) = \frac{E''}{E'} = \frac{1}{\omega\tau_m}$$

Kelvin-Voigt model



$$\sigma = \sigma_E + \sigma_\eta = E\varepsilon + \eta\dot{\varepsilon}$$

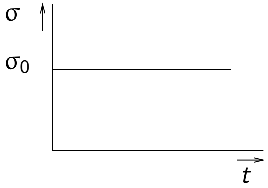
Kelvin-Voigt : stress step excitation

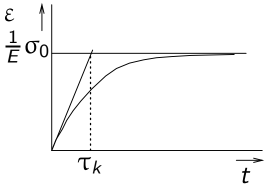
$$\sigma(t) = \sigma_0 H(t,0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \delta(t,0)$$

$$\eta \dot{\varepsilon}(t) + E\varepsilon(t) = \sigma(t) = \sigma_0 H(t,0)$$

$$\left.\begin{array}{l} \varepsilon(t) = \varepsilon_H(t) + \varepsilon_P = C\, e^{-\frac{E}{\eta}t} + \dfrac{\sigma_0}{E} \\ \varepsilon(t=0) = 0 \end{array}\right\} \quad \rightarrow \quad C = -\frac{\sigma_0}{E}$$

$$\varepsilon(t) = \frac{\sigma_0}{E}\left[1 - e^{-\frac{E}{\eta}t}\right] = \sigma_0 D(t)$$





Kelvin-Voigt : strain step excitation

$$\varepsilon(t) = \varepsilon_0 H(t,0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0)$$

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t)$$
$$\sigma(t) = E\varepsilon_0 + \eta\varepsilon_0\delta(t,0) = \varepsilon_0\left[E + \eta\delta(t,0)\right] = \infty$$

Kelvin-Voigt : Boltzmann integral

$$\begin{aligned}\varepsilon(t) &= \frac{1}{E}\int_{\tau=-\infty}^{t} \left\{1 - e^{-\frac{E}{\eta}(t-\tau)}\right\} \dot{\sigma}(\tau)\, d\tau \\ &= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau\end{aligned}$$

Kelvin-Voigt : harmonic stress excitation

$$\sigma(t) = \sigma_0 \sin(\omega t) \qquad \longrightarrow \qquad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

strain response

$$\begin{aligned}
\varepsilon(t) &= \int_{\tau=0}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau \\
&= \sigma_0 \left[\frac{1}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta^2} \right] \sin(\omega t) - \sigma_0 \left[\frac{\omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{1}{\eta} \right] \cos(\omega t) \\
&= \sigma_0 \left[\frac{1}{E(1+\omega^2\tau_k^2)} \right] \sin(\omega t) - \sigma_0 \left[\frac{\omega\tau_k}{E(1+\omega^2\tau_k^2)} \right] \cos(\omega t) \\
&= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}$$

Kelvin-Voigt : harmonic stress excitation

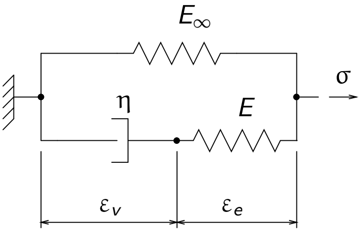
dynamic quantities

$$D'(\omega) = \frac{1}{\left(\frac{E}{\eta}\right)^2 + \omega^2}\,\frac{E}{\eta^2} = \frac{1}{E(1+\omega^2\tau_k^2)}$$

$$D''(\omega) = \frac{\omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2}\,\frac{1}{\eta} = \frac{\omega\tau}{E(1+\omega^2\tau_k^2)}$$

$$\tan(\delta) = \frac{D''}{D'} = \omega\tau_k \qquad \rightarrow \qquad \delta = arctan\left(\frac{\eta\omega}{E}\right)$$

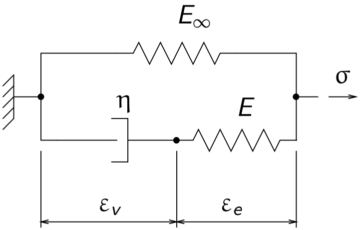
Standard Solid model



constitutive relations

- $\sigma = \sigma_\infty + \sigma_{ve}$
- $\dot{\varepsilon} = \dot{\varepsilon}_v + \dot{\varepsilon}_e$
- $\dot{\varepsilon}_v = \dfrac{1}{\eta}\,\sigma_{ve}$
- $\sigma_{ve} = E\,\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\,\dot{\sigma}_{ve}$
- $\varepsilon = \dfrac{1}{E_\infty}\,\sigma_\infty$

Standard Solid model

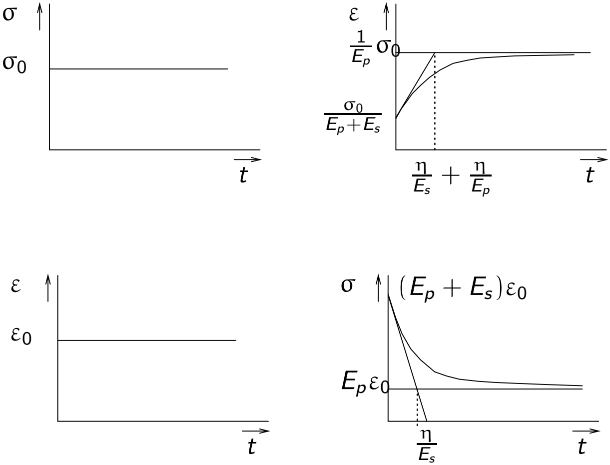


constitutive equation

$$
\begin{aligned}
\sigma &= \sigma_\infty + \sigma_{ve} = E_\infty\,\varepsilon + \eta\dot{\varepsilon}_v \\
&= E_\infty\,\varepsilon + \eta(\dot{\varepsilon} - \dot{\varepsilon}_e) = E_\infty\,\varepsilon + \eta\dot{\varepsilon} - \eta\,\frac{\dot{\sigma}_{ve}}{E} \\
&= E_\infty\,\varepsilon + \eta\dot{\varepsilon} - \frac{\eta}{E}\,(\dot{\sigma} - E_\infty\,\dot{\varepsilon}) \qquad \rightarrow
\end{aligned}
$$

$$
\sigma + \frac{\eta}{E}\,\dot{\sigma} = E_\infty\,\varepsilon + \frac{\eta(E + E_\infty)}{E}\,\dot{\varepsilon}
$$

Standard Solid : step excitations

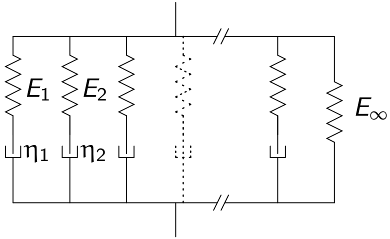


Standard Solid : Boltzmann integrals

$$
\begin{aligned}
\varepsilon(t) &= \int_{\tau=-\infty}^{t} \left\{ \frac{1}{E_\infty} - \frac{E}{E_\infty(E_\infty + E)} e^{-\frac{E_\infty E}{\eta(E_\infty + E)}(t-\tau)} \right\} \dot{\sigma}(\tau)\, d\tau \\
&= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
\sigma(t) &= \int_{\tau=-\infty}^{t} \left\{ E_\infty + E e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\varepsilon}(\tau)\, d\tau \\
&= \int_{\tau=-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau
\end{aligned}
$$

Generalized Maxwell model



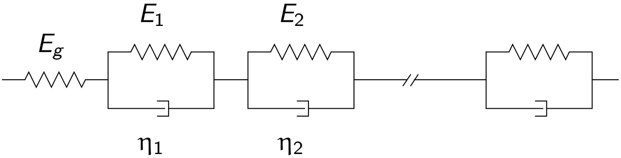
$$E(t) = E_\infty + \sum_i E_i e^{-\frac{t}{\tau_i}} \qquad ; \qquad \tau_i = \frac{\eta_i}{E_i}$$

equilibrium modulus $\qquad E_\infty = \lim_{t \longrightarrow \infty} E(t)$

glass modulus $\qquad E_g = \lim_{t \longrightarrow 0} E(t) = E_\infty + \sum_i E_i$

$i = 1 \qquad \rightarrow \qquad$ Standard Solid Model

Generalized Kelvin model



$$D(t) = \frac{1}{E_g} + \sum_i \frac{1}{E_i}(1 - e^{-\frac{t}{\tau_i}}) \qquad ; \qquad \tau_i = \frac{\eta_i}{E_i}$$

$$= D_g + \sum_i D_i(1 - e^{-\frac{t}{\tau_i}})$$

glass compliance $\qquad D_g = \frac{1}{E_g} = \lim_{t \longrightarrow 0} D(t)$

equilibrium compliance $\qquad D_\infty = \lim_{t \longrightarrow \infty} D(t) = D_g + \sum_i D_i$

viscoelastic liquid : extra serial dashpot with
end viscosity $\eta_v \quad \rightarrow \quad D(t) = .. + \frac{1}{\eta_v}\, t$

NUMERICAL IMPLEMENTATION

back to index

Stress update

stress must be updated $\longrightarrow$
Bolzmann integral must be calculated

we assume "Generalized Maxwell model"

Stress relaxation

$$\left.\begin{aligned} \sigma(t) &= \int\limits_{\tau=0}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\,d\tau \\ E(t) &= E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{t}{\tau_i}} \end{aligned}\right\} \quad \longrightarrow$$

$$\begin{aligned} \sigma(t) &= \int\limits_{\tau=0}^{t} \left[E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{t-\tau}{\tau_i}} \right] \dot{\varepsilon}(\tau)\,d\tau \\ &= E_\infty\,\varepsilon(t) + \sum_{i=1}^{N} \int\limits_{\tau=0}^{t} E_i e^{-\frac{t-\tau}{\tau_i}}\,\dot{\varepsilon}(\tau)\,d\tau \end{aligned}$$

To evaluate the integral, the time is discretized

Time discretization

$$
\begin{aligned}
\sigma(t) &= E_\infty \, \varepsilon(t) + \sum_{i=1}^{N} \int\limits_{\tau=0}^{t} E_i e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau) \, d\tau \\
&= E_\infty \, \varepsilon(t) + \\
&\qquad \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \int\limits_{\tau=0}^{t_n} E_i e^{-\frac{t_n-\tau}{\tau_i}} \dot{\varepsilon}(\tau) \, d\tau + E_i \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau) \, d\tau \right]
\end{aligned}
$$

Only the integral over the current increment has to be calculated
This can be done analytically after assumption for strain rate :

incremental strain rate is constant

Linear incremental strain

$$\int_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau)\, d\tau$$

$$= \frac{\Delta\varepsilon}{\Delta t} \int_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, d\tau$$

$$= \frac{\Delta\varepsilon}{\Delta t}\, \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)$$

Stress

$$
\begin{aligned}
\sigma(t) &= E_\infty\,\varepsilon(t) + \sum_{i=1}^{N} \sigma_i(t) \\
&= E_\infty\,\varepsilon(t) + \\
&\qquad \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \int\limits_{\tau=0}^{t_n} E_i e^{-\frac{t_n-\tau}{\tau_i}} \dot{\varepsilon}(\tau)\, d\tau + \right. \\
&\qquad\qquad \left. E_i \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta\varepsilon}{\Delta t} \right]
\end{aligned}
$$

$$
\sigma(t) = E_\infty\,\varepsilon(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \sigma_i(t_n) + E_i p_i \Delta\varepsilon \right]
$$

$$
\text{with} \qquad p_i = \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)
$$

Stiffness

$$\sigma(t) = E_\infty \, \varepsilon(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \sigma_i(t_n) + E_i p_i \Delta\varepsilon \right] \quad \rightarrow$$

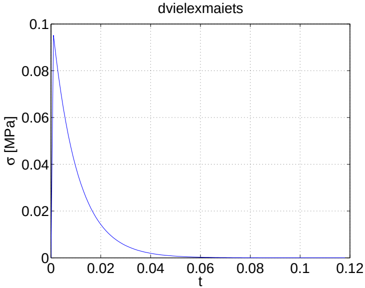
$$\frac{\partial \sigma}{\partial \lambda} = C_\lambda = C_\varepsilon = E_\infty + \sum_{i=1}^{N} E_i p_i$$

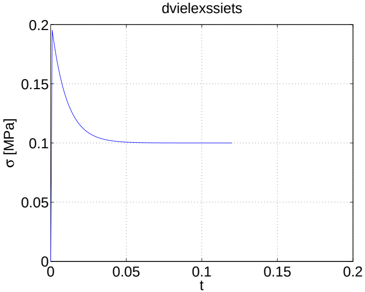
Implementation

`tr2dviel.m`

Strain step

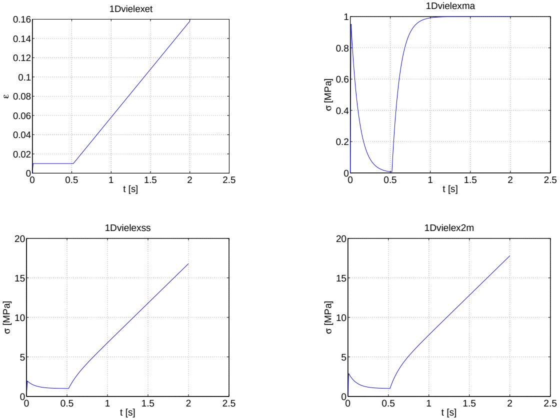
Maxwell	$E_\infty = 0$	$E_1 = 1$	$\tau_1 = 0.01$
Standard-Solid	$E_\infty = 1$	$E_1 = 1$	$\tau_1 = 0.01$





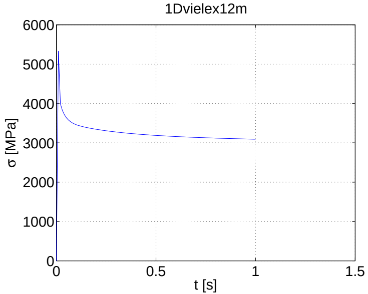
Linear viscoelastic models

	E_∞	E_1	τ_1	E_2	τ_2	ν
Maxwell	0	100	0.1	0	0	0
Kelvin-Voigt	100	10^{10}	10^{-12}	0	0	0
Standard-Solid	100	100	0.1	0	0	0
2-mode	100	100	0.1	100	0.1	0



Multi-mode model response

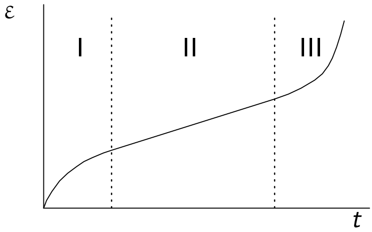
	E [MPa]	τ [s]		E [MPa]	τ [s]
1	3.0e6	3.1e-8	2	1.4e6	3.0e-7
3	3.9e6	3.0e-6	4	5.4e6	2.9e-5
5	1.3e6	2.8e-4	6	2.3e5	2.7e-3
7	7.6e4	2.6e-2	8	3.7e4	2.5e-1
9	3.3e4	2.5e+0	10	1.7e4	2.4e+1
11	8.0e3	2.3e+2	12	1.2e4	2.2e+3



CREEP

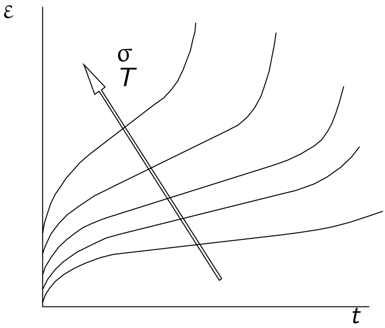
back to index

Creep behavior



- primary / stage I / transient creep (delayed elastic effect)
- secondary / steady-state / stage II creep (viscous flow)
- tertiary / stage III / accelerating creep

Creep strain rate



general model $\dot{\varepsilon}_c = A\, f_\sigma(\sigma) f_\varepsilon(\varepsilon_c) f_T(T) f_t(t)$

power law model $\dot{\varepsilon}_c = A\sigma^m \varepsilon_c^n T^p \left(qt^{q-1}\right)$

Primary creep

- $T < 0.4\,T_m$
- dislocation coalescence $\quad\longrightarrow\quad$ entanglement / pile-up $\quad\longrightarrow$
- hardening
- time-dependent plasticity

$$\dot{\varepsilon}_c = C(\sigma)\exp[-\alpha(\sigma)t]$$

Secondary creep

- $0.5\,T_m < T < 0.6\,T_m$
- vacancy movement (self diffusion)

$$\dot{\varepsilon}_c = A \exp\left(-\frac{Q_c}{RT}\right)\left(\frac{\sigma}{E}\right)^n \qquad (n \approx 5)$$

power-law-breakdown model

$$\dot{\varepsilon}_c = A \exp\left(-\frac{Q_c}{RT}\right)\left(\sinh\left(\alpha\,\frac{\sigma}{E}\right)\right)^5$$

Tertiary creep

- $0.6\,T_m < T < 0.8\,T_m$
- grain boundary sliding $\rightarrow$ void initiation/coalescence $\rightarrow$ inter granular cracks
- diffusional flow
- modelled with damage mechanics

tertiary creep damage model (Kachanov & Rabotnov)

$$\frac{\dot{\varepsilon}_c}{\dot{\varepsilon}_{c0}} = \frac{(\sigma/\sigma_0)^n}{(1-\omega)^m} \qquad ; \qquad \frac{\dot{\omega}}{\dot{\omega}_0} = \frac{(\sigma/\sigma_0)^\nu}{(1-\omega)^\mu} \qquad (n \geq \nu)$$

Stress functions

Norton; Bailey (1929)	$\dot{\varepsilon}_c = K\sigma^n$
Hooke-Norton	$\dot{\varepsilon}_c = \dfrac{\dot{\sigma}}{E} + K\sigma^n$
Johnson et.al. (1963)	$\dot{\varepsilon}_c = D_1\sigma^{n_1} + D_2\sigma^{n_2}$
Dorn (1955)	$\dot{\varepsilon}_c = B\exp(\beta\sigma)$
Soderberg (1936)	$\dot{\varepsilon}_c = B\left[\exp\left(\dfrac{\sigma}{\sigma_0}\right) - 1\right]$
Prandtl (1928)	$\dot{\varepsilon}_c = A\sinh\left(\dfrac{\sigma}{\sigma_0}\right)$
Garofalo (1965)	$\dot{\varepsilon}_c = A\left[\sinh\left(\dfrac{\sigma}{\sigma_0}\right)\right]^n$
Lemaitre, Chaboche (1985)	$\dot{\varepsilon}_c = \left(\dfrac{\sigma}{\lambda_0}\right)^{N_0}\exp\left(\alpha\sigma^{N_0+1}\right)$

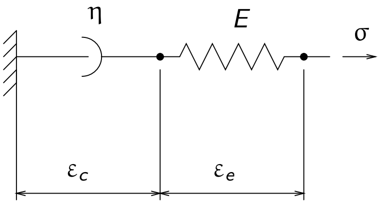
Temperature functions

Kauzmann (1941)	$\dot{\varepsilon}_c = A \exp\left(-\frac{\Delta H - \gamma\sigma}{RT}\right)$
Lifszic (1963)	$\dot{\varepsilon}_c = \frac{\sigma}{T} \exp\left(-\frac{\Delta H}{RT}\right)$
Dorn, Tietz (1949/55)	$\varepsilon_c = f\left(t \exp\left[-\frac{\Delta H}{RT}\right]\right) f_\sigma(\sigma)$
Penny, Marriott (1971)	$\varepsilon_c = \left(t \exp\left[-\frac{\Delta H}{RT}\right]\right)^n f_\sigma(\sigma)$
Boyle, Spence (1983)	$\varepsilon_c = C \exp\left(-\frac{\Delta H}{RT}\right) t^m \sigma^n$

Time functions

Andrade (1910)	$\varepsilon_c = \ln\left(1 + \beta t^{\frac{1}{3}}\right) + kt$
Andrade (small ε)	$\varepsilon_c = \beta t^{\frac{1}{3}} + kt \approx \beta t^{\frac{1}{3}}$
Bailey (1935)	$\varepsilon_c = Ft^n$
Graham, Walles (1955)	$\varepsilon_c = \sum_{j=1}^{M} a_j t^{m_j}$
McVetty (1934)	$\varepsilon_c = G\left(1 - \exp(-qt)\right) + Ht$
Findley et.al. (1944)	$\varepsilon_c = \varepsilon_1 + \varepsilon_2 t^n \qquad (n < 1)$
Pugh (1975)	$\varepsilon_c = \dfrac{a_1 t}{1 + b_1 t} + \dfrac{a_2 t}{1 + b_2 t} + \dot{\varepsilon}_m t$
Garofalo	$\varepsilon_c = \varepsilon_t(1 - e^{-rt}) + \dot{\varepsilon}_s t$

Creep model



constitutive relations

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_c$
- $\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\,\dot{\sigma}$
- $\dot{\varepsilon}_c = A\, f_\sigma(\sigma)\, f_{\varepsilon_c}(\varepsilon_c)\, f_T(T)\, f_t(t) = f(\sigma, \varepsilon_c, T, t)$

constitutive equation

$$\dot{\sigma} = E\dot{\varepsilon}_e = E\dot{\varepsilon} - E\dot{\varepsilon}_c = E\dot{\varepsilon} - Ef(\sigma, \varepsilon_c, T, t)$$

NUMERICAL IMPLEMENTATION

back to index

Stress update

$$
\begin{aligned}
\dot{\sigma} &= E\dot{\varepsilon} - Ef(\sigma, \varepsilon_c, T, t) \\
\Delta\sigma &= E\Delta\varepsilon - \Delta t E f(\sigma, \varepsilon_c, T, t) \\
\sigma - \sigma_n &= E(\varepsilon - \varepsilon_n) - \Delta t E f(\sigma, \varepsilon_c, T, t)
\end{aligned}
$$

- implicit procedure
- explicit procedure

Implicit stress update

$$
\begin{aligned}
\sigma - \sigma_n &= E(\varepsilon - \varepsilon_n) - \Delta t E f(\sigma, \varepsilon_c, T, t) \\
\sigma^* + \delta\sigma - \sigma_n &= E(\varepsilon - \varepsilon_n) - \Delta t E (f^* + \delta f) = E(\varepsilon - \varepsilon_n) - \Delta t E f^* - \Delta t E \delta f \\
&= E(\varepsilon - \varepsilon_n) - \Delta t E f^* - \Delta t E \frac{\partial f}{\partial \sigma}\, \delta\sigma \quad \rightarrow
\end{aligned}
$$

$$
\left[1 + \Delta t E\, \frac{\partial f}{\partial \sigma}\right] \delta\sigma = -\sigma^* + \sigma_n + E(\varepsilon - \varepsilon_n) - \Delta t E f^*
$$

Explicit stress update

$$\sigma = \sigma_n + E(\varepsilon - \varepsilon_n) - \Delta t\, Ef(\sigma_n, \varepsilon_{c_n}, T_n, t_n)$$

Stiffness

implicit

$$\sigma - \sigma_n - E\varepsilon + E\varepsilon_n + \Delta t\, Ef(\sigma, \varepsilon_c, T, t) = 0$$

$$\delta\sigma + \Delta t\, E\, \left.\frac{\partial f}{\partial \sigma}\right|^* \delta\sigma - E\delta\varepsilon = 0$$

$$C_\varepsilon = \left(1 + \Delta t\, E\, \left.\frac{\partial f}{\partial \sigma}\right|^*\right)^{-1} E$$

explicit

$$\sigma - \sigma_n - E\varepsilon + E\varepsilon_n + \Delta t\, Ef(\sigma_n, \varepsilon_{c_n}, T_n, t_n) = 0$$

$$\delta\sigma = E\delta\varepsilon \quad \rightarrow \quad C_\varepsilon = E$$

Implementation

`tr2delvi.m`

Creep versus viscoelasticity

Maxwell model (E, η)

$$\varepsilon = \varepsilon_e + \varepsilon_c \quad ; \quad E(t) = Ee^{t/\tau} \quad ; \quad \tau = \frac{\eta}{E}$$

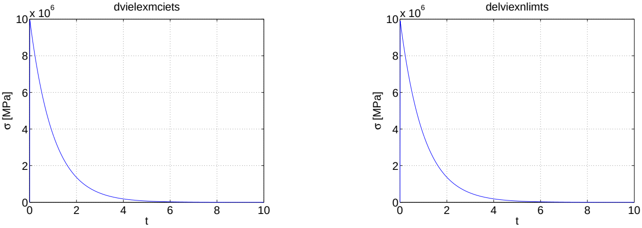
$$\dot{\varepsilon}_c = \frac{\sigma}{\eta} \quad ; \quad \varepsilon_e = \frac{\sigma}{E}$$

Norton model (A, m)

$$\varepsilon = \varepsilon_e + \varepsilon_c \quad ; \quad \dot{\varepsilon}_c = f(\sigma, \varepsilon_c, T, t)$$

$$\dot{\varepsilon}_c = A\sigma^m \quad ; \quad \varepsilon_e = \frac{\sigma}{E}$$

Maxwell	$E = 10^9$	$\eta = 10^9$	$\tau = 1$
Norton	$E = 10^9$	$A = \frac{1}{\eta} = 10^{-9}$	$m = 1$



General creep model for SnAg-solder

$$\varepsilon_c(t) = \varepsilon_0 + A(\sigma)\left[1 - e^{-\alpha(\sigma,T)t}\right] + B(\sigma,T)\left[e^{\alpha(\sigma,T)t} - 1\right]$$

$$\alpha(\sigma,T) = c_1\left[\sinh(\beta\sigma)\right]^{n_1} e^{-\frac{Q_1}{RT}}$$

$$A(\sigma) = c_2\sigma^{n_2}$$

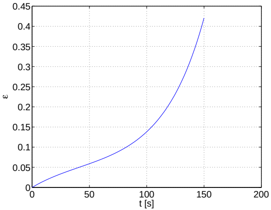
$$B(\sigma,T) = c_3\sigma^{n_3} e^{-\frac{Q_2}{RT}}$$

General creep model for SnAg-solder

$$\dot{\varepsilon}_c = A\alpha e^{-\alpha t} + B\alpha e^{\alpha t} \; ; \; \dot{\varepsilon}_{c,i} = \dot{\varepsilon}_c(t=0) = \alpha(A+B) \; ; \; t_m = \frac{1}{2\alpha} \ln\left(\frac{A}{B}\right)$$

$$\dot{\varepsilon}_{c,m} = \dot{\varepsilon}_c(t=t_m) = 2\alpha\sqrt{AB} \quad ; \quad \varepsilon_{c,m} = \varepsilon_c(t=t_m) = \varepsilon_0 + A - B$$

ε_0	0		
c_1 β	1.73×10^5 0.095	n_1 Q_1	4.66 70
c_2	2.06×10^{-3}	n_2	1.1
c_3 Q_2	9.65×10^{-4} 17.8	n_3	2.38

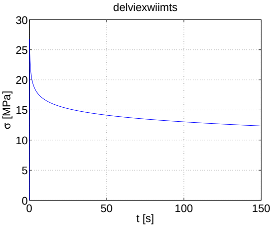


Special creep model for SnAg-solder

Wiese (2005) : 2-term model Sn4Ag0.5Cu

$$\dot{\varepsilon}_c = A_1 \sigma^{m_1} e^{e_1/T} + A_2 \sigma^{m_2} e^{e_2/T}$$

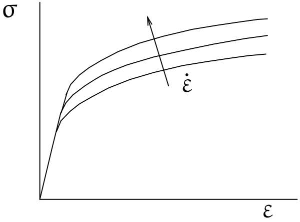
$E = 59.533 - 66.667\ T$		
$A_1 = 4.10^{-7}$	$m_1 = 3$	$e_1 = -3223$
$A_1 = 1.10^{-12}$	$m_1 = 12$	$e_1 = -7348$



VISCOPLASTIC

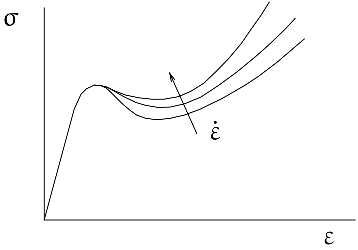
back to index

Viscoplastic material behavior



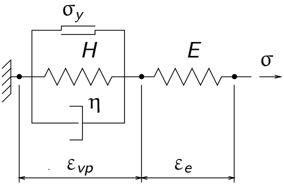
$$\left.\begin{array}{l}\text{viscoelastic (rate effects)}\\ \text{elastoplastic (permanent deformation)}\end{array}\right\} \rightarrow \text{viscoplastic}$$

Softening



- polymers

Viscoplastic (Perzyna) model



constitutive relations

- $f = \bar{\sigma} - \sigma_y$ with $f < 0 \;\rightarrow\;$ elastic
  $f \geq 0 \;\rightarrow\;$ viscoplastic
- $\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_{vp})$
- $\bar{\sigma} = |\sigma|$
- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_{vp}$
- $\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\,\dot{\sigma}$
- $\dot{\varepsilon}_{vp} = \dot{\lambda}\,\dfrac{\partial f}{\partial \sigma} = \dot{\lambda}\,\dfrac{\sigma}{\bar{\sigma}} \quad ; \quad \dot{\bar{\varepsilon}}_{vp} = |\dot{\varepsilon}_{vp}|$
- $\bar{\varepsilon}_{vp} = \displaystyle\int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_{vp}\, d\tau$
- $\dot{\lambda} = \gamma\phi(f) = \gamma\,(f/\sigma_{y0})^N$

Hardening laws

$$\sigma_y = \sigma_{y0} + H\bar{\varepsilon}_{vp} + a\bar{\varepsilon}_{vp}^2 + b\bar{\varepsilon}_{vp}^3 + c\bar{\varepsilon}_{vp}^4 + d\bar{\varepsilon}_{vp}^7$$

- parameters fitted with compression tests to prevent instability
- 7th-order polynomial is used for polycarbonate

Constitutive equations

$$\begin{cases} \dot{\sigma} = E\dot{\varepsilon}_e = E(\dot{\varepsilon} - \dot{\varepsilon}_{vp}) = E\{\dot{\varepsilon} - \dot{\lambda}\left(\dfrac{\sigma}{\bar{\sigma}}\right)\} \\ \dot{\lambda} = \gamma\phi \end{cases}$$

$$\begin{cases} \Delta\sigma = E\Delta\varepsilon - E\Delta\lambda\left(\dfrac{\sigma}{\bar{\sigma}}\right) \\ \Delta\lambda = \gamma\phi\Delta t \end{cases}$$

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\dfrac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \gamma\phi\Delta t \end{cases}$$

- $f > 0$ is allowed $\rightarrow$ "overstress model"
  (no consistency equation)
- viscoplastic multiplier λ cannot be eliminated

NUMERICAL IMPLEMENTATION

back to index

Elastic stress predictor

$$\sigma_e = \sigma_n + E(\varepsilon - \varepsilon_n)$$

- $f = \bar{\sigma}_e - \sigma_{y_n} \leq 0 \quad \rightarrow$ elastic increment
- $f = \bar{\sigma}_e - \sigma_{y_n} > 0 \quad \rightarrow$ elastoviscoplastic increment

Elastic increment

$$\sigma(t_{n+1}) = \sigma_e$$
$$\bar{\varepsilon}_{vp}(t_{n+1}) = \bar{\varepsilon}_{vp}(t_n) = \bar{\varepsilon}_{vp_n}$$
$$\sigma_y(t_{n+1}) = \sigma_y(t_n) = \sigma_{y_n}$$

Elastoviscoplastic increment

$$\left\{\begin{array}{l} \Delta\sigma = E\Delta\varepsilon - E\Delta\lambda\left(\frac{\sigma}{\bar{\sigma}}\right)\} \\ \Delta\lambda = \gamma\phi\Delta t \end{array}\right.$$

$$\left\{\begin{array}{l} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\frac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \Delta t\gamma\phi \end{array}\right.$$

- implicit
- explicit

Implicit stress update

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\frac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \Delta t\,\gamma\phi \end{cases}$$

$$\begin{cases} \sigma^* + \delta\sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda^* + \delta\lambda - \lambda_n)\left\{\left(\frac{\sigma}{\bar{\sigma}}\right)^* + \delta\left(\frac{\sigma}{\bar{\sigma}}\right)\right\} \\ \lambda^* + \delta\lambda - \lambda_n = \Delta t\,\gamma(\phi^* + \delta\phi) \end{cases}$$

linearization and reorganization

$$\begin{cases} \delta\sigma + \left[E\left(\frac{\sigma}{\bar{\sigma}}\right)^*\right]\delta\lambda \\ \qquad = -\sigma^* + \sigma_n + E\varepsilon - E\varepsilon_n - E(\lambda^* - \lambda_n)\left(\frac{\sigma}{\bar{\sigma}}\right)^* \\ \left[-\Delta t\,\gamma\frac{\partial\phi}{\partial\sigma}\right]\delta\sigma + \left[1 - \Delta t\,\gamma\frac{\partial\phi}{\partial\lambda}\right]\delta\lambda \\ \qquad = -\lambda^* + \lambda_n + \Delta t\gamma\phi^* \end{cases}$$

Implicit stress update

$$\frac{\partial\phi}{\partial\lambda} = \frac{d\phi}{df}\frac{df}{d\sigma_y}\frac{d\sigma_y}{d\bar{\varepsilon}_{vp}}\frac{d\bar{\varepsilon}_{vp}}{d\lambda} = \frac{d\phi}{df}(-1)\,H\left(\frac{\sigma}{\bar{\sigma}}\right)^* = -\frac{d\phi}{df}\,H\left(\frac{\sigma}{\bar{\sigma}}\right)^*$$

$$\frac{\partial\phi}{\partial\sigma} = \frac{d\phi}{df}\frac{df}{d\sigma} = \frac{d\phi}{df}\left(\frac{\sigma}{\bar{\sigma}}\right)^*$$

$$\frac{d\phi}{df} = N\left(\frac{f}{\sigma_{y0}}\right)^{N-1}\frac{1}{\sigma_{y0}}$$

Explicit stress update

$$
\begin{cases}
\sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\dfrac{\sigma}{\bar{\sigma}}\right) \\
\lambda - \lambda_n = \Delta t\,\gamma\phi
\end{cases}
$$

$$
\begin{cases}
\sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \\
\lambda - \lambda_n = \Delta t\,\gamma\phi_n
\end{cases}
$$

$$
\begin{cases}
\sigma + E\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right)\lambda = \sigma_n + E\varepsilon - E\varepsilon_n + E\lambda_n\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \\
\lambda = \lambda_n + \Delta t\,\gamma\phi_n\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right)
\end{cases}
$$

Stiffness : implicit

$$\left.\begin{aligned}
&\sigma - \sigma_n = E(\varepsilon - \varepsilon_n) - E(\lambda - \lambda_n)\left(\frac{\sigma}{\bar{\sigma}}\right) \\
&\lambda - \lambda_n = \Delta t\,\gamma\,\phi
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\delta\sigma = E\,\delta\varepsilon - E\,\delta\lambda\left(\frac{\sigma}{\bar{\sigma}}\right) - E(\lambda - \lambda_n)\left(\frac{1}{\bar{\sigma}}\right)\delta\sigma \\
&\delta\lambda = \Delta t\gamma\delta\phi = \Delta t\gamma\frac{\partial\phi}{\partial\sigma}\delta\sigma + \Delta t\gamma\frac{\partial\phi}{\partial\lambda}\delta\lambda
\end{aligned}\right\}$$

$$\delta\sigma = E\delta\varepsilon - E\left(\frac{\sigma}{\bar{\sigma}}\right)\frac{\gamma\Delta t\frac{\partial\phi}{\partial\sigma}}{1-\gamma\Delta t\frac{\partial\phi}{\partial\lambda}}\delta\sigma - E(\lambda - \lambda_n)\left(\frac{1}{\bar{\sigma}}\right)\delta\sigma$$

$$C_\varepsilon = \frac{E\left\{1-\gamma\Delta t\frac{\partial\phi}{\partial\lambda}\right\}}{\left\{1-\gamma\Delta t\frac{\partial\phi}{\partial\lambda}\right\} + E\left(\frac{\sigma}{\bar{\sigma}}\right)\gamma\Delta t\frac{\partial\phi}{\partial\sigma} + E(\lambda-\lambda_n)\frac{1}{\bar{\sigma}}\left\{1-\gamma\Delta t\frac{\partial\phi}{\partial\lambda}\right\}}$$

Stiffness : explicit

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \\ \lambda - \lambda_n = \Delta t\, \gamma \phi_n \end{cases}$$

$$\begin{cases} \delta\sigma = E\delta\varepsilon - E\delta\lambda\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \\ \delta\lambda = 0 \end{cases}$$

$$C_\varepsilon = E$$

Implementation

`tr2dperz.m`

Prescribed constant strain rates

$$\Delta l(t) = u(t) = u_0 f(t) \quad \rightarrow \quad \lambda(t) = 1 + \frac{u_0}{l_0} f(t) \quad \rightarrow \quad f(t) = \frac{l_0}{u_0}\left(\lambda(t) - 1\right)$$

linear strain

$$\dot{\varepsilon}_l = \dot{\lambda} = c \quad \rightarrow \quad \lambda(t) = ct + 1 \quad \rightarrow \quad f(t) = \frac{l_0}{u_0}\, c\, t$$

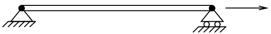
$$\lambda_e = \lambda(t_e) = ct_e + 1 \quad \rightarrow \quad t_e = \frac{1}{c}\left(\lambda_e - 1\right)$$

logarithmic strain

$$\dot{\varepsilon}_{ln} = \frac{\dot{\lambda}}{\lambda} = c \quad \rightarrow \quad \ln(\lambda) = ct \quad \rightarrow \quad \lambda(t) = e^{ct} \quad \rightarrow \quad f(t) = \frac{l_0}{u_0}\left(e^{ct} - 1\right)$$

$$\lambda_e = \lambda(t_e) = e^{ct_e} \quad \rightarrow \quad t_e = \frac{1}{c}\ln(\lambda_e)$$

Tensile test at various strain rates



initial length	l_0	100	mm
initial cross-sectional area	A_0	10	mm^2

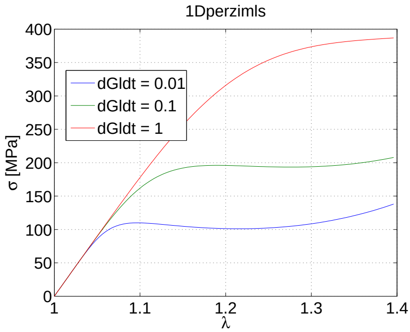
$$\sigma_y = \sigma_{v0} + H\bar{\varepsilon}_{vp} + a\bar{\varepsilon}_{vp}^2 + b\bar{\varepsilon}_{vp}^3 + c\bar{\varepsilon}_{vp}^4 + d\bar{\varepsilon}_{vp}^7$$

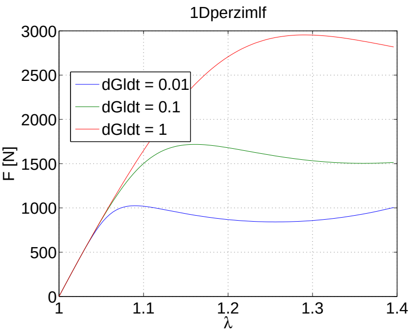
E	1800	MPa	ν	0.37	-
σ_{y0}	37	MPa	H	-200	MPa
γ	0.001	1/s	N	3	-
a	500	MPa	b	700	MPa
c	800	MPa	d	30000	MPa

$$\dot{\varepsilon}_l = \{\, 0.01,\ 0.1,\ 1 \,\}$$

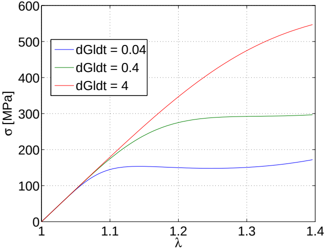
Tensile test at various strain rates

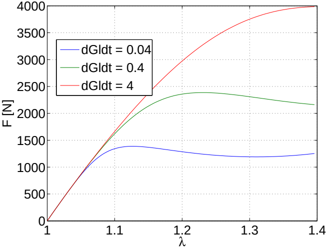
NB: linear strain is used : $\dot{\varepsilon} = \dot{\lambda} = \text{dGldt}$



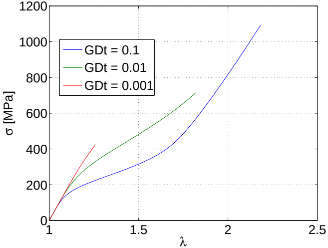


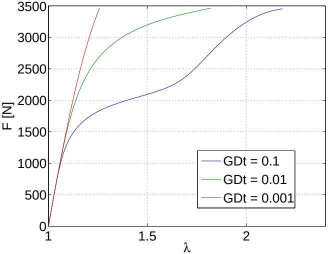
Tensile test at various time steps





Precribed elongation; nic $= 100 \;\; \rightarrow t_e = \{10, 1, 0.1\}$;



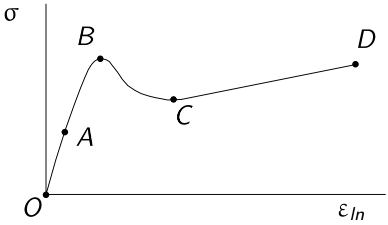


Precribed force; $F_{max} = 3500$; nic $= 100 \;\; \rightarrow t_e = \{10, 1, 0.1\}$;

NONLINEAR VISCOELASTIC

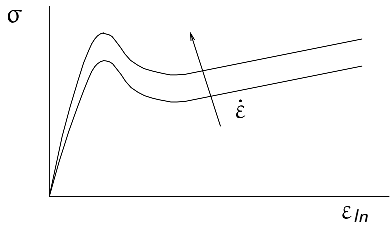
back to index

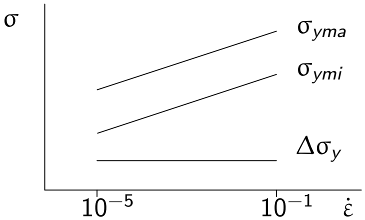
Nonlinear viscoelastic material behavior



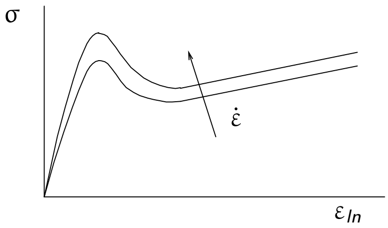
- compression test of polymers $\longrightarrow$
- OA : linear viscoelastic
- AB : nonlinear viscoelastic
- B : "(maximum) yield stress" (σ_{yma}) follwed by flow (creep)
- BC : softening
- C : "minimum yield stress" (σ_{ymi})
- CD : hardening
- different strain rate dependency

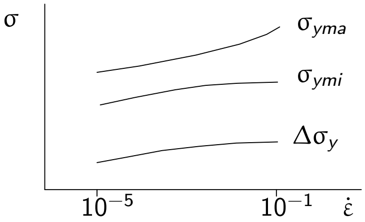
Strain rate dependency : PC



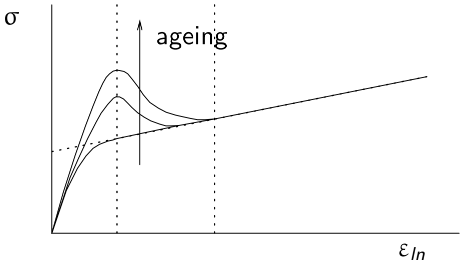


Strain rate dependency : PMMA





Aging

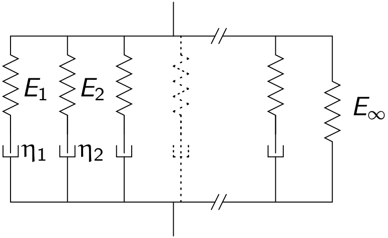


- rejuvenation $\longrightarrow$ no softening
- aging $\longrightarrow$ softening

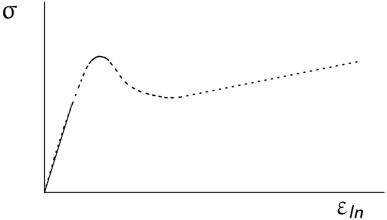
Nonlinear viscoelastic model

- Linear viscoelastic behavior
- Non-linear viscoelastic behavior
- Softening
- Hardening

Linear viscoelastic behavior



$$\sigma(t) = \int\limits_{\xi=-\infty}^{t} E(t-\xi)\dot{\varepsilon}(\xi)\,d\xi \quad ; \quad E(x) = E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{x}{\tau_i}} \quad ; \quad \tau_i = \frac{\eta_i}{E_i}$$

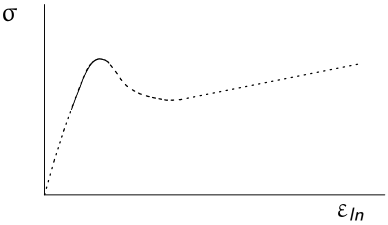


Nonlinear viscoelastic behavior

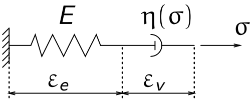
$$\sigma(t) = \int\limits_{\xi=-\infty}^{t} E(\psi - \psi')\dot{\varepsilon}(\xi)\, d\xi$$

$$\psi = \int\limits_{\zeta=-\infty}^{t} \frac{d\zeta}{a_\sigma\{\sigma(\zeta)\}} \qquad ; \qquad \psi' = \int\limits_{\zeta=-\infty}^{\xi} \frac{d\zeta}{a_\sigma\{\sigma(\zeta)\}}$$

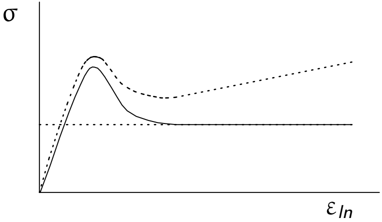
$$E(x) = E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{x}{\tau_i(\sigma)}}$$



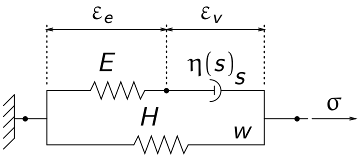
Softening $=$ tertiary creep



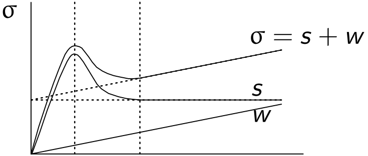
- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\,\dot{\sigma}$
- $\dot{\varepsilon}_v = \frac{1}{\eta(\bar{s},T,D)}\,\sigma \qquad ; \qquad \bar{s} = |s|$
- $\dot{D} = \left(1 - \frac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v \qquad ; \qquad \dot{\bar{\varepsilon}}_v = |\dot{\varepsilon}_v|$



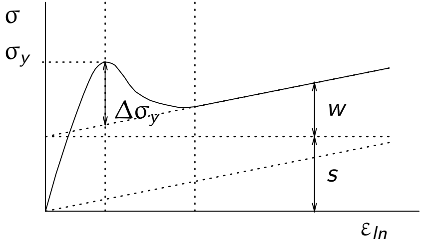
Hardening



- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = s + w = E\varepsilon_e + H\varepsilon$
- $\dot{\varepsilon}_v = \frac{1}{\eta(\bar{s},T,D)}\, s \qquad ; \qquad \bar{s} = |s|$
- $\dot{D} = \left(1 - \frac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v \quad ; \quad \dot{\bar{\varepsilon}}_v = |\dot{\varepsilon}_v|$



Aging and hardening (newest model)



- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = s + \Delta\sigma_y + w = E\varepsilon_e + \Delta\sigma_y + H\varepsilon$
- $\dot{\varepsilon}_v = \dfrac{1}{\eta(\bar{s}, T, S)}\, s \qquad ; \qquad \bar{s} = |s|$

Aging and hardening (newest model)

- $S(t, \bar{\varepsilon}_v) = S_a(t) R_\gamma(\bar{\varepsilon}_v)$
- $R_\gamma(\bar{\varepsilon}_v) = \left[\left\{1 + \left(r_0 e^{\bar{\varepsilon}_v}\right)^{r_1}\right\} / \left\{1 + r_0^{r_1}\right\}\right]^{\frac{r_2-1}{r_1}} \quad ; \quad 0 < R < 1$
- $S_\alpha(t) = S_a(t_{eff}) = c_0 + c_1 \ln\left[\frac{t_{eff} + t_a}{t_0}\right]$
- $t_{eff}(T, \bar{s}) = \int_{\xi=0}^{t} \frac{d\xi}{\alpha_T(T(\xi))\alpha_\sigma(\bar{s}(\xi))}$
- $t_a = exp\left(\frac{S_\alpha(0) - c_0}{c_1}\right)$
- $\Delta\sigma_y = \sigma_y(t) - \sigma_{y0} = \frac{c}{c_1}\{S_\alpha(t) - c_0\}$

Eyring viscosity

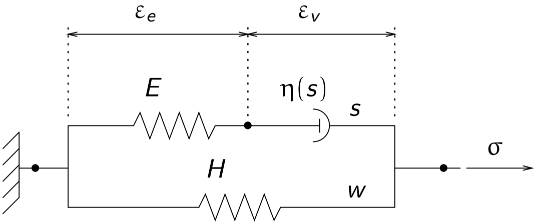
$$\eta = A_0 \, \frac{\bar{s}}{\sqrt{3}\,\sinh\left(\bar{s}/(\sqrt{3}\tau_0)\right)} \, \exp\left[\frac{\Delta H}{RT} + \frac{\mu p}{\tau_0} - D\right]$$

$$\bar{s} = |s| \quad ; \quad p = -\tfrac{1}{3}s \quad ; \quad \tau_0 = \frac{RT}{V}$$

Bodner-Partom viscosity

$$\eta = \frac{\bar{s}}{\sqrt{12\Gamma_0}} \exp\left[\tfrac{1}{2}\left(\frac{Z}{\bar{s}}\right)^{2n}\right]$$
$$Z = Z_1 + (Z_0 - Z_1)\exp\left[-m\bar{\varepsilon}_p\right]$$

Nonlinear viscoelastic model older model



- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = s + w = E\varepsilon_e + H\varepsilon$
- $\dot{\varepsilon}_v = \dfrac{1}{\eta(\bar{s}, T, D)}\, s \qquad ; \qquad \bar{s} = |s|$
- $\dot{D} = \left(1 - \dfrac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v \qquad ; \qquad \dot{\bar{\varepsilon}}_v = |\dot{\varepsilon}_v|$

Nonlinear viscoelastic model older model

constitutive equations

$$\left.\begin{aligned}
\dot{\varepsilon}_e &= \dot{\varepsilon} - \dot{\varepsilon}_v = \dot{\varepsilon} - \frac{1}{\eta(\bar{s}, T, D)}\, s = \dot{\varepsilon} - \frac{E}{\eta(\bar{s}, T, D)}\, \varepsilon_e \\
\sigma &= s + w = E\varepsilon_e + H\varepsilon \\
\dot{D} &= \left(1 - \frac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v
\end{aligned}\right\}$$

NUMERICAL IMPLEMENTATION

back to index

Stress update

$$\begin{cases} \dot{\varepsilon}_e = \dot{\varepsilon} - E\zeta(\bar{s}, T, D)\varepsilon_e \\ \dot{D} = \left(1 - \dfrac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v \end{cases}$$

$$\begin{cases} \Delta\varepsilon_e = \Delta\varepsilon - \Delta t E\zeta(\bar{s}, T, D)\varepsilon_e \\ \Delta D = \left(1 - \dfrac{D}{D_\infty}\right) h\Delta\bar{\varepsilon}_v \end{cases}$$

Implicit stress update

$$\left\{\begin{aligned} & \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}, T, D)\varepsilon_e \\ & D - D_n = \left(1 - \frac{D}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{aligned}\right.$$

$$\left\{\begin{aligned} & \varepsilon_e^* + \delta\varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}, T, D^* + \delta D)(\varepsilon_e^* + \delta\varepsilon_e) \\ & D^* + \delta D - D_n = \left(1 - \frac{D^* + \delta D}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{aligned}\right.$$

$$\left\{\begin{aligned} & \delta\varepsilon_e + \Delta t E \zeta(\bar{s}, T, D^*)\delta\varepsilon_e + \Delta t E \frac{\partial \zeta}{\partial D} \delta D \varepsilon_e^* \\ & \qquad = -\varepsilon_e^* + \varepsilon_{en} + \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}, T, D^*)\varepsilon_e^* \\ & \left[1 + \frac{h \Delta \bar{\varepsilon}_v}{D_\infty}\right] \delta D \\ & \qquad = -D^* + D_n + \left(1 - \frac{D^*}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{aligned}\right.$$

Explicit stress update

$$\begin{cases} \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}_n, T, D_n)\varepsilon_{en} \\ D - D_n = \left(1 - \dfrac{D_n}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{cases}$$

$$\begin{cases} \varepsilon_e = \varepsilon - \varepsilon_n + \{1 - \Delta t E \zeta(\bar{s}_n, T, D_n)\} \varepsilon_{en} \\ D = D_n + \left(1 - \dfrac{D_n}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{cases}$$

Stiffness : implicit

$$\left\{\begin{array}{l}\sigma = s + w = E\varepsilon_e + H\varepsilon \\ \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E\zeta(\bar{s}, T, D)\varepsilon_e \\ D - D_n = \left(1 - \dfrac{D}{D_\infty}\right) h\Delta\bar{\varepsilon}_v\end{array}\right.$$

$$\left\{\begin{array}{l}\delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \Delta t E\dfrac{\partial\zeta}{\partial D}\delta D\varepsilon_e - \Delta t E\zeta(\bar{s}, T, D)\delta\varepsilon_e \\ \delta D = -\dfrac{\delta D}{D_\infty} h\Delta\bar{\varepsilon}_v \qquad \rightarrow \quad \delta D = 0\end{array}\right.$$

$$\left\{\begin{array}{l}\delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \Delta t\, E\zeta(\bar{s}, T, D)\delta\varepsilon_e \\ \delta D = 0\end{array}\right.$$

$$C_\varepsilon = \frac{E + H\{1 + \Delta t\, E\zeta(\bar{s}, T, D)\}}{1 + \Delta t\, E\zeta(\bar{s}, T, D)}$$

Stiffness : explicit

$$\left\{\begin{array}{l} \sigma = s + w = E\varepsilon_e + H\varepsilon \\ \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E\zeta(\bar{s}_n, T, D_n)\varepsilon_e \\ D = D_n + \left(1 - \dfrac{D_n}{D_\infty}\right) h\Delta\bar{\varepsilon}_v \end{array}\right.$$

$$\left\{\begin{array}{l} \delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \Delta t E\zeta(\bar{s}_n, T, D_n)\delta\varepsilon_e \\ \delta D = 0 \end{array}\right.$$

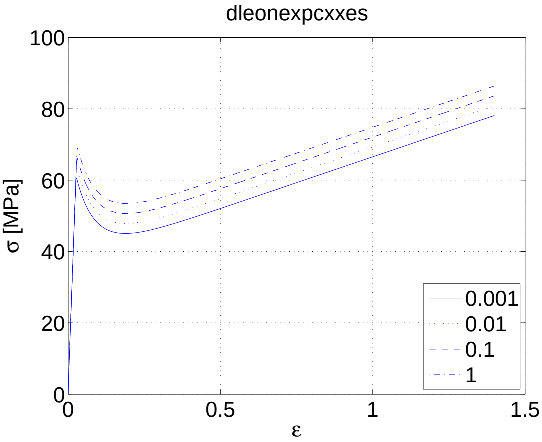
$$\delta\sigma = \frac{E}{1 + \Delta t E\zeta(\bar{s}_n, T, D_n)}\delta\varepsilon + H\delta\varepsilon$$

$$C_\varepsilon = \frac{E + H\{1 + \Delta t E\zeta(\bar{s}_n, T, D_n)\}}{1 + \Delta t E\zeta(\bar{s}_n, T, D_n)}$$

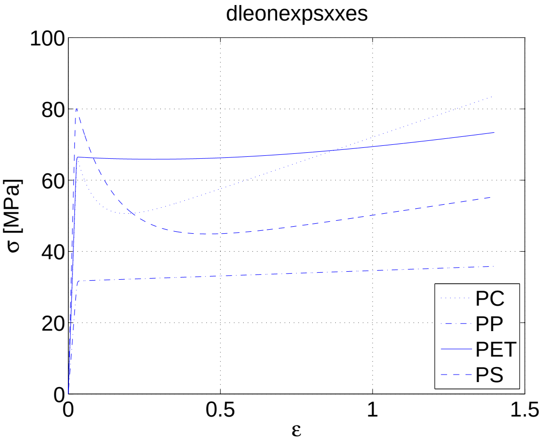
Examples

	PET	PC	PS	PP	
E	2400	2305	3300	1092	MPa
ν	0.35	0.37	0.37	0.4	-
H	15	29	13	3	MPa
h	13	270	100	0	-
D_∞	11	19	14	-	-
A_0	3.8568E-27	9.7573E-27	4.2619E-34	2.0319E-29	s
ΔH	2.617E+05	2.9E+05	2.6E+5	2.2E+5	J/mol
μ	0.0625	0.06984	0.294	0.23	-
τ_0	0.927	0.72	2.1	1.0	MPa

Tensile test at various strain rates



Tensile test for various polymers



VECTORS AND TENSORS

back to index

Vectors

$$\vec{a} = ||a||\vec{e} \quad ; \quad ||\vec{e}|| = 1$$

$$\alpha\vec{a} = \vec{b}$$

$$\vec{a} + \vec{b} = \vec{c}$$

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos(\phi)$$

$$\vec{c} = \vec{a} * \vec{b} = \left\{ ||\vec{a}|| ||\vec{b}|| \right\} \sin(\phi)\, \vec{n} \quad ; \quad ||\vec{n}|| = 1$$

$$\vec{a} * \vec{b} \cdot \vec{c} = \left\{ ||\vec{a}|| ||\vec{b}|| \sin(\phi) \right\} ||\vec{c}|| \cos(\psi)$$

$$\vec{a}\vec{b} = \text{dyad} \quad ; \quad \vec{q} = \vec{a}\vec{b} \cdot \vec{p} = \vec{p} \cdot (\vec{a}\vec{b})^c$$

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \quad ; \quad \vec{e}_i \cdot \vec{e}_{j \neq i} = 0 \quad ; \quad \vec{e}_i \cdot \vec{e}_i = 1$$

$$\vec{a} = \underset{\sim}{a}^T \underset{\sim}{\vec{e}} = \underset{\sim}{\vec{e}}^T \underset{\sim}{a}$$

Second-order tensors

$$\mathbf{A} = \sum_i \alpha_i \vec{a}_i \vec{b}_i \quad ; \quad \mathbf{A} \cdot \vec{p} = \vec{q} \quad ; \quad \mathbf{A} = \underset{\sim}{\vec{e}}^T \underline{A} \, \underset{\sim}{\vec{e}}$$

$$\mathbf{I} \cdot \vec{a} = \vec{a} \quad \forall \quad \vec{a} \qquad \rightarrow \qquad \mathbf{I} = \underset{\sim}{\vec{e}}^T \underline{I} \, \underset{\sim}{\vec{e}}$$

$$\mathbf{A}^c = \sum_i \alpha_i \vec{b}_i \vec{a}_i \quad ; \quad \mathbf{A} \cdot \vec{p} = \vec{p} \cdot \mathbf{A}^c$$

$$\alpha \mathbf{A} = \mathbf{B} \quad ; \quad \mathbf{A} + \mathbf{B} = \mathbf{C} \quad ; \quad \mathbf{B} \cdot \mathbf{A} = \mathbf{C}$$

$$\mathbf{A} : \mathbf{B} = \mathbf{A}^c : \mathbf{B}^c = \text{scalar}$$

$$J_1(\mathbf{A}) = \text{tr}(\mathbf{A})$$

$$J_2(\mathbf{A}) = \frac{1}{2} \left\{ \text{tr}^2(\mathbf{A}) - \text{tr}(\mathbf{A} \cdot \mathbf{A}) \right\}$$

$$J_3(\mathbf{A}) = \det(\mathbf{A}) \quad ; \quad \det(\mathbf{A}) = 0 \rightarrow \mathbf{A} = \text{singular}$$

$$\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I} \quad ; \quad \mathbf{A} = \text{regular}$$

$$\mathbf{A}^c = \mathbf{A} \quad ; \quad \mathbf{A}^c = -\mathbf{A}$$

$$\vec{a} \cdot \mathbf{A} \cdot \vec{a} > 0 \quad \forall \quad \vec{a} \neq \vec{0}$$

$$(\mathbf{A} \cdot \vec{a}) \cdot (\mathbf{A} \cdot \vec{b}) = \vec{a} \cdot \vec{b} \quad \forall \quad \vec{a}, \vec{b}$$

$$(\mathbf{A} \cdot \vec{a}) * (\mathbf{A} \cdot \vec{b}) = \mathbf{A}^a \cdot (\vec{a} * \vec{b}) \quad \forall \quad \vec{a}, \vec{b}$$

Fourth-order tensors

$$ {}^4\mathbf{A} = \sum_i \alpha_i \vec{a}_i \vec{b}_i \vec{c}_i \vec{d}_i \quad ; \quad {}^4\mathbf{A} : \mathbf{B} = \mathbf{C} $$

$$ {}^4\mathbf{I} : \mathbf{A} = \mathbf{A} \quad \forall \quad \mathbf{A} $$

$$ {}^4\mathbf{A} \cdot \mathbf{B} = {}^4\mathbf{C} $$

COLUMN AND MATRIX NOTATION

back to index

Matrix/column notation for second-order tensor

3×3 matrix of a second-order tensor

$$\mathbf{A} = \vec{e}_i A_{ij} \vec{e}_j \quad \rightarrow \quad \underline{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

column notation

$$\underset{\approx}{A}^T = \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{12} & A_{21} & A_{23} & A_{32} & A_{31} & A_{13} \end{bmatrix}$$
$$\underset{\approx}{A}_t^T = \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{21} & A_{12} & A_{32} & A_{23} & A_{13} & A_{31} \end{bmatrix}$$

conjugate tensor

$$\mathbf{A}^c \quad \rightarrow \quad A_{ji} \quad \rightarrow \quad \underline{A}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \quad \rightarrow \quad \underset{\approx}{A}_t$$

Column notation for **A** : **B**

$$
\begin{aligned}
C &= \mathbf{A} : \mathbf{B} \\
&= \vec{e}_i A_{ij} \vec{e}_j : \vec{e}_k B_{kl} \vec{e}_l = A_{ij} \delta_{jk} \delta_{il} B_{kl} = A_{ij} B_{ji} \\
&= A_{11} B_{11} + A_{12} B_{21} + A_{13} B_{31} + \\
&\qquad A_{21} B_{12} + A_{22} B_{22} + A_{23} B_{32} + \\
&\qquad A_{31} B_{13} + A_{32} B_{23} + A_{33} B_{33} \\
&= \left[\begin{array}{ccccccccc} A_{11} & A_{22} & A_{33} & A_{21} & A_{12} & A_{32} & A_{23} & A_{13} & A_{31} \end{array} \right] \\
&\qquad \left[\begin{array}{ccccccccc} B_{11} & B_{22} & B_{33} & B_{12} & B_{21} & B_{23} & B_{32} & B_{31} & B_{13} \end{array} \right]^T \\
&= \underset{\approx}{A}_t^T \underset{\approx}{B} = \underset{\approx}{A}^T \underset{\approx}{B}_t
\end{aligned}
$$

idem

$$
\begin{aligned}
C = \mathbf{A} : \mathbf{B}^c \qquad &\rightarrow \qquad C = \underset{\approx}{A}_t^T \underset{\approx}{B}_t = \underset{\approx}{A}^T \underset{\approx}{B} \\
C = \mathbf{A}^c : \mathbf{B} \qquad &\rightarrow \qquad C = \underset{\approx}{A}^T \underset{\approx}{B} = \underset{\approx}{A}_t^T \underset{\approx}{B}_t \\
C = \mathbf{A}^c : \mathbf{B}^c \qquad &\rightarrow \qquad C = \underset{\approx}{A}_t^T \underset{\approx}{B} = \underset{\approx}{A}^T \underset{\approx}{B}_t
\end{aligned}
$$

Matrix/column notation $\quad \mathbf{C} = \mathbf{A} \cdot \mathbf{B}$

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = \vec{e}_i A_{ik} \vec{e}_k \cdot \vec{e}_l B_{lj} \vec{e}_j = \vec{e}_i A_{ik} \delta_{kl} B_{lj} \vec{e}_j = \vec{e}_i A_{ik} B_{kj} \vec{e}_j \quad \rightarrow$$

$$\underline{C} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & & \\ & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} & \\ & & A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & & \\ & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} & \\ & & A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} \\ A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} & & \\ & A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} & \\ & & A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33} \end{bmatrix}$$

$$\underset{\approx}{C} = \begin{bmatrix} C_{11} \\ C_{22} \\ C_{33} \\ C_{12} \\ C_{21} \\ C_{23} \\ C_{32} \\ C_{31} \\ C_{13} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \\ A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33} \\ A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} \\ A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} \\ A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} \\ A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} \\ A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33} \end{bmatrix}$$

Matrix/column notation $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$

$$\underset{\approx}{C} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & A_{12} & 0 & 0 & A_{13} & 0 \\ 0 & A_{22} & 0 & A_{21} & 0 & 0 & A_{23} & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & A_{32} & 0 & 0 & A_{31} \\ 0 & A_{12} & 0 & A_{11} & 0 & 0 & A_{13} & 0 & 0 \\ A_{21} & 0 & 0 & 0 & A_{22} & 0 & 0 & A_{23} & 0 \\ 0 & 0 & A_{23} & 0 & 0 & A_{22} & 0 & 0 & A_{21} \\ 0 & A_{32} & 0 & A_{31} & 0 & 0 & A_{33} & 0 & 0 \\ A_{31} & 0 & 0 & 0 & A_{32} & 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{13} & 0 & 0 & A_{12} & 0 & 0 & A_{11} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{22} \\ B_{33} \\ B_{12} \\ B_{21} \\ B_{23} \\ B_{32} \\ B_{31} \\ B_{13} \end{bmatrix} = \underline{\underline{A}}\, \underset{\approx}{B}$$

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} \quad \rightarrow \quad \underset{\approx}{C} = \underline{\underline{A}}\, \underset{\approx}{B} = \underline{\underline{A}}_c\, \underset{\approx}{B}_t$$

$$\underset{\approx}{C}_t = \underline{\underline{A}}_r\, \underset{\approx}{B} = \underline{\underline{A}}_{rc}\, \underset{\approx}{B}_t$$

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B}^c \quad \rightarrow \quad \underset{\approx}{C} = \underline{\underline{A}}\, \underset{\approx}{B}_t = \underline{\underline{A}}_c\, \underset{\approx}{B}$$

$$\mathbf{C} = \mathbf{A}^c \cdot \mathbf{B} \quad \rightarrow \quad \underset{\approx}{C} = \underline{\underline{A}}_t\, \underset{\approx}{B} = \underline{\underline{A}}_{tc}\, \underset{\approx}{B}_t$$

$$\mathbf{C} = \mathbf{A}^c \cdot \mathbf{B}^c \quad \rightarrow \quad \underset{\approx}{C} = \underline{\underline{A}}_t\, \underset{\approx}{B}_t = \underline{\underline{A}}_{tc}\, \underset{\approx}{B}$$

Matrix notation of fourth-order tensor

$$ {}^4\mathbf{A} = \vec{e}_i \vec{e}_j A_{ijkl} \vec{e}_k \vec{e}_l \quad \rightarrow $$

$$ \underline{\underline{A}} = \begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} & A_{1123} & A_{1132} & A_{1131} & A_{1113} \\ A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} & A_{2223} & A_{2232} & A_{2231} & A_{2213} \\ A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} & A_{3323} & A_{3332} & A_{3331} & A_{3313} \\ A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} & A_{1223} & A_{1232} & A_{1231} & A_{1213} \\ A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121} & A_{2123} & A_{2132} & A_{2131} & A_{2113} \\ A_{2311} & A_{2322} & A_{2333} & A_{2312} & A_{2321} & A_{2323} & A_{2332} & A_{2331} & A_{2313} \\ A_{3211} & A_{3222} & A_{3233} & A_{3212} & A_{3221} & A_{3223} & A_{3232} & A_{3231} & A_{3213} \\ A_{3111} & A_{3122} & A_{3133} & A_{3112} & A_{3121} & A_{3123} & A_{3132} & A_{3131} & A_{3113} \\ A_{1311} & A_{1322} & A_{1333} & A_{1312} & A_{1321} & A_{1323} & A_{1332} & A_{1331} & A_{1313} \end{bmatrix} $$

$$ {}^4\mathbf{A}^c \rightarrow \underline{\underline{A}}^T \quad ; \quad {}^4\mathbf{A}^{rc} \rightarrow \underline{\underline{A}}_c \quad ; \quad {}^4\mathbf{A}^{lc} \rightarrow \underline{\underline{A}}_r $$

Matrix/column notation $\quad \mathbf{C} = {}^4\mathbf{A} : \mathbf{B}$

$$
\begin{aligned}
\mathbf{C} &= {}^4\mathbf{A} : \mathbf{B} \quad \rightarrow \\
\vec{e}_i C_{ij} \vec{e}_j &= \vec{e}_i \vec{e}_j A_{ijmn} \vec{e}_m \vec{e}_n : \vec{e}_p B_{pq} \vec{e}_q \\
&= \vec{e}_i \vec{e}_j A_{ijmn} \delta_{np} \delta_{mq} B_{pq} = \vec{e}_i \vec{e}_j A_{ijmn} B_{nm} \quad \rightarrow \\
\underset{\approx}{C} &= \underline{\underline{A}}_c \, \underset{\approx}{B} = \underline{\underline{A}} \, \underset{\approx}{B}_t
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{C} &= \mathbf{B} : {}^4\mathbf{A} \quad \rightarrow \\
\vec{e}_i C_{ij} \vec{e}_j &= \vec{e}_p B_{pq} \vec{e}_q : \vec{e}_m \vec{e}_n A_{mnij} \vec{e}_i \vec{e}_j \\
&= B_{pq} \delta_{qm} \delta_{pn} A_{mnij} \vec{e}_i \vec{e}_j = B_{nm} A_{mnij} \vec{e}_i \vec{e}_j \quad \rightarrow \\
\underset{\approx}{C}^T &= \underset{\approx}{B}^T \, \underline{\underline{A}}_r = \underset{\approx}{B}_t^T \, \underline{\underline{A}}
\end{aligned}
$$

Matrix notation $\quad {}^4\mathbf{C} = {}^4\mathbf{A} \cdot \mathbf{B}$

$$
\begin{aligned}
{}^4\mathbf{C} = {}^4\mathbf{A} \cdot \mathbf{B} &= \vec{e}_i \vec{e}_j A_{ijkl} \vec{e}_k \vec{e}_l \cdot \vec{e}_p B_{pq} \vec{e}_q \\
&= \vec{e}_i \vec{e}_j A_{ijkl} \vec{e}_k \delta_{lp} B_{pq} \vec{e}_q = \vec{e}_i \vec{e}_j A_{ijkl} B_{lq} \vec{e}_k \vec{e}_q \\
&= \vec{e}_i \vec{e}_j A_{ijkp} B_{pl} \vec{e}_k \vec{e}_l \quad \rightarrow
\end{aligned}
$$

$$
\underline{\underline{C}} = \begin{bmatrix}
A_{111p}B_{p1} & A_{112p}B_{p2} & A_{113p}B_{p3} & A_{111p}B_{p2} & A_{112p}B_{p1} \\
A_{221p}B_{p1} & A_{222p}B_{p2} & A_{223p}B_{p3} & A_{221p}B_{p2} & A_{222p}B_{p1} \\
A_{331p}B_{p1} & A_{332p}B_{p2} & A_{333p}B_{p3} & A_{331p}B_{p2} & A_{332p}B_{p1} \\
A_{121p}B_{p1} & A_{122p}B_{p2} & A_{123p}B_{p3} & A_{121p}B_{p2} & A_{122p}B_{p1} \\
A_{211p}B_{p1} & A_{212p}B_{p2} & A_{213p}B_{p3} & A_{211p}B_{p2} & A_{212p}B_{p1}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} \\
A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} \\
A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} \\
A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} \\
A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121}
\end{bmatrix}
\begin{bmatrix}
B_{11} & 0 & 0 & B_{12} & 0 \\
0 & B_{22} & 0 & 0 & B_{21} \\
0 & 0 & B_{33} & 0 & 0 \\
B_{21} & 0 & 0 & B_{22} & 0 \\
0 & B_{12} & 0 & 0 & B_{11}
\end{bmatrix}
$$

$$
= \underline{\underline{A}}\, \underline{\underline{B}}_{cr} = \underline{\underline{A}}_c\, \underline{\underline{B}}_c \qquad \rightarrow \qquad \underline{\underline{C}}_r = \underline{\underline{A}}_r\, \underline{\underline{B}}_{cr} = \underline{\underline{A}}_{cr}\, \underline{\underline{B}}_c
$$

Matrix notation ${}^4\mathbf{C} = \mathbf{B} \cdot {}^4\mathbf{A}$

$$\begin{aligned}
{}^4\mathbf{C} = \mathbf{B} \cdot {}^4\mathbf{A} &= \vec{e}_i B_{ij} \vec{e}_j \cdot \vec{e}_p \vec{e}_q A_{pqrs} \vec{e}_r \vec{e}_s \\
&= \vec{e}_i B_{ij} \delta_{jp} \vec{e}_q A_{pqrs} \vec{e}_r \vec{e}_s = \vec{e}_i \vec{e}_q B_{ij} A_{jqrs} \vec{e}_r \vec{e}_s \\
&= \vec{e}_i \vec{e}_j B_{ip} A_{pjkl} \vec{e}_k \vec{e}_l \quad \longrightarrow
\end{aligned}$$

$$\underline{\underline{C}} = \begin{bmatrix}
B_{1p}A_{p111} & B_{1p}A_{p122} & B_{1p}A_{p133} & B_{1p}A_{p112} & B_{1p}A_{p121} \\
B_{2p}A_{p211} & B_{2p}A_{p222} & B_{2p}A_{p233} & B_{2p}A_{p212} & B_{2p}A_{p221} \\
B_{3p}A_{p311} & B_{3p}A_{p322} & B_{3p}A_{p333} & B_{3p}A_{p312} & B_{3p}A_{p321} \\
B_{1p}A_{p211} & B_{1p}A_{p222} & B_{1p}A_{p233} & B_{1p}A_{p212} & B_{1p}A_{p221} \\
B_{2p}A_{p111} & B_{2p}A_{p122} & B_{2p}A_{p133} & B_{2p}A_{p112} & B_{2p}A_{p121}
\end{bmatrix}$$

$$= \begin{bmatrix}
B_{11} & 0 & 0 & 0 & B_{12} \\
0 & B_{22} & 0 & B_{21} & 0 \\
0 & 0 & B_{33} & 0 & 0 \\
0 & B_{12} & 0 & B_{11} & 0 \\
B_{21} & 0 & 0 & 0 & B_{22}
\end{bmatrix}
\begin{bmatrix}
A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} \\
A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} \\
A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} \\
A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} \\
A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121}
\end{bmatrix}$$

$$= \underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{B}}_c\,\underline{\underline{A}}_r \qquad \longrightarrow \qquad \underline{\underline{C}}_r = \underline{\underline{B}}_r\,\underline{\underline{A}}_c = \underline{\underline{B}}_{cr}\,\underline{\underline{A}}_{cr}$$

Matrix notation $\quad {}^4\mathbf{C} = {}^4\mathbf{A} : {}^4\mathbf{B}$

$$
\begin{aligned}
{}^4\mathbf{C} = {}^4\mathbf{A} : {}^4\mathbf{B} &= \vec{e}_i\vec{e}_j A_{ijkl}\vec{e}_k\vec{e}_l : \vec{e}_p\vec{e}_q B_{pqrs}\vec{e}_r\vec{e}_s \\
&= \vec{e}_i\vec{e}_j A_{ijkl}\delta_{lp}\delta_{kq} B_{pqrs}\vec{e}_r\vec{e}_s = \vec{e}_i\vec{e}_j A_{ijqp} B_{pqrs}\vec{e}_r\vec{e}_s \\
&= \vec{e}_i\vec{e}_j A_{ijqp} B_{pqkl}\vec{e}_k\vec{e}_l
\end{aligned}
$$

$$
\underline{\underline{C}} = \begin{bmatrix}
A_{11qp}B_{pq11} & A_{11qp}B_{pq22} & A_{11qp}B_{pq33} & A_{11qp}B_{pq12} & A_{11qp}B_{pq21} \\
A_{22qp}B_{pq11} & A_{22qp}B_{pq22} & A_{22qp}B_{pq33} & A_{22qp}B_{pq12} & A_{22qp}B_{pq21} \\
A_{33qp}B_{pq11} & A_{33qp}B_{pq22} & A_{33qp}B_{pq33} & A_{33qp}B_{pq12} & A_{33qp}B_{pq21} \\
A_{12qp}B_{pq11} & A_{12qp}B_{pq22} & A_{12qp}B_{pq33} & A_{12qp}B_{pq12} & A_{12qp}B_{pq21} \\
A_{21qp}B_{pq11} & A_{21qp}B_{pq22} & A_{21qp}B_{pq33} & A_{21qp}B_{pq12} & A_{21qp}B_{pq21}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} \\
A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} \\
A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} \\
A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} \\
A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121}
\end{bmatrix}
\begin{bmatrix}
B_{1111} & B_{1122} & B_{1133} & B_{1112} & B_{1121} \\
B_{2211} & B_{2222} & B_{2233} & B_{2212} & B_{2221} \\
B_{3311} & B_{3322} & B_{3333} & B_{3312} & B_{3321} \\
B_{2111} & B_{2122} & B_{2133} & B_{2112} & B_{2121} \\
B_{1211} & B_{1222} & B_{1233} & B_{1212} & B_{1221}
\end{bmatrix}
$$

$$
= \underline{\underline{A}}\,\underline{\underline{B}}_r = \underline{\underline{A}}_c\,\underline{\underline{B}}
$$

Matrix notation fourth-order unit tensor

$$^4\mathbf{I} = \vec{e}_i\vec{e}_j\delta_{il}\delta_{jk}\vec{e}_k\vec{e}_l \quad \rightarrow$$

$\underline{\underline{I}} =$

$$\begin{bmatrix} \delta_{11}\delta_{11} & \delta_{12}\delta_{12} & \delta_{13}\delta_{13} & \delta_{12}\delta_{11} & \delta_{11}\delta_{12} & \cdot \\ \delta_{21}\delta_{21} & \delta_{22}\delta_{22} & \delta_{23}\delta_{23} & \delta_{22}\delta_{21} & \delta_{21}\delta_{22} & \cdot \\ \delta_{31}\delta_{31} & \delta_{32}\delta_{32} & \delta_{33}\delta_{33} & \delta_{32}\delta_{31} & \delta_{31}\delta_{32} & \cdot \\ \delta_{11}\delta_{21} & \delta_{12}\delta_{22} & \delta_{13}\delta_{23} & \delta_{12}\delta_{21} & \delta_{11}\delta_{22} & \cdot \\ \delta_{21}\delta_{11} & \delta_{22}\delta_{12} & \delta_{23}\delta_{13} & \delta_{22}\delta_{11} & \delta_{21}\delta_{12} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

symmetric fourth-order tensor

$$^4\mathbf{I}^s = \frac{1}{2}\left(^4\mathbf{I} + {}^4\mathbf{I}^{rc}\right) \quad \rightarrow \quad \underline{\underline{I}}^s = \frac{1}{2}\left(\underline{\underline{I}} + \underline{\underline{I}}_c\right) = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdot \\ 0 & 2 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 2 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 1 & 1 & \cdot \\ 0 & 0 & 0 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

Matrix notation $\mathbf{II}$

$$\mathbf{II} = \vec{e}_i \delta_{ij} \vec{e}_j \vec{e}_k \delta_{kl} \vec{e}_l = \vec{e}_i \vec{e}_j \delta_{ij} \delta_{kl} \vec{e}_k \vec{e}_l \quad \longrightarrow$$

$$\underline{\underline{II}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \underset{\approx}{I}\,\underset{\approx}{I}^T$$

Matrix notation $\quad {}^4\mathbf{B} = {}^4\mathbf{I} \cdot \mathbf{A}$

$$
\begin{aligned}
{}^4\mathbf{B} = {}^4\mathbf{I} \cdot \mathbf{A} &= \vec{e}_i \vec{e}_j \delta_{il} \delta_{jk} \vec{e}_k \vec{e}_l \cdot \vec{e}_p A_{pq} \vec{e}_q \\
&= \vec{e}_i \vec{e}_j \delta_{il} \delta_{jk} \vec{e}_k \delta_{lp} A_{pq} \vec{e}_q = \vec{e}_i \vec{e}_j \delta_{il} \delta_{jk} A_{lq} \vec{e}_k \vec{e}_q \\
&= \vec{e}_i \vec{e}_j A_{iq} \delta_{jk} \vec{e}_k \vec{e}_q = \vec{e}_i \vec{e}_j A_{il} \delta_{jk} \vec{e}_k \vec{e}_l \\
&= \mathbf{A} \cdot {}^4\mathbf{I} \quad \rightarrow
\end{aligned}
$$

$$
\begin{aligned}
\underline{\underline{B}} &= \begin{bmatrix}
A_{11}\delta_{11} & A_{12}\delta_{12} & A_{13}\delta_{13} & A_{12}\delta_{11} & A_{11}\delta_{12} & .. \\
A_{21}\delta_{21} & A_{22}\delta_{22} & A_{23}\delta_{23} & A_{22}\delta_{21} & A_{21}\delta_{22} & .. \\
A_{31}\delta_{31} & A_{32}\delta_{32} & A_{33}\delta_{33} & A_{32}\delta_{31} & A_{31}\delta_{32} & .. \\
A_{11}\delta_{21} & A_{12}\delta_{22} & A_{13}\delta_{23} & A_{12}\delta_{21} & A_{11}\delta_{22} & .. \\
A_{21}\delta_{11} & A_{22}\delta_{12} & A_{23}\delta_{13} & A_{22}\delta_{11} & A_{21}\delta_{12} & .. \\
.. & .. & .. & .. & .. & ..
\end{bmatrix} \\
&= \begin{bmatrix}
A_{11} & 0 & 0 & A_{12} & 0 & .. \\
0 & A_{22} & 0 & 0 & A_{21} & .. \\
0 & 0 & A_{33} & 0 & 0 & .. \\
0 & A_{12} & 0 & 0 & A_{11} & .. \\
A_{21} & 0 & 0 & A_{22} & 0 & .. \\
.. & .. & .. & .. & .. & ..
\end{bmatrix} \\
&= \underline{\underline{A}}_c
\end{aligned}
$$

Summary and examples

$$
\begin{array}{ccc}
\vec{x} & \longrightarrow & \underset{\sim}{x} \\
\mathbf{A} & \longrightarrow & \underline{A} \quad ; \quad \underset{\approx}{A} \quad ; \quad \underline{\underline{A}} \\
{}^4\mathbf{A} & \longrightarrow & \underline{\underline{A}} \\
{}^4\mathbf{I} & \longrightarrow & \underline{\underline{I}}
\end{array}
$$

$$
\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ; \quad \underline{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}
$$

$$
\underset{\approx}{A} = \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{12} \\ A_{21} \\ .. \end{bmatrix} \quad ; \quad \underline{\underline{A}} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & A_{12} & .. \\ 0 & A_{22} & 0 & A_{21} & 0 & .. \\ 0 & 0 & A_{33} & 0 & 0 & .. \\ 0 & A_{12} & 0 & A_{11} & 0 & .. \\ A_{21} & 0 & 0 & 0 & A_{22} & .. \\ .. & .. & .. & .. & .. & .. \end{bmatrix}
$$

$$
\underline{\underline{A}} = \begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} & .. \\ A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} & .. \\ A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} & .. \\ A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} & .. \\ A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121} & .. \end{bmatrix}
$$

Manipulations

$$\mathbf{A} \longrightarrow \underline{A} \quad : \longrightarrow \texttt{mA}$$

$$\mathbf{A} \longrightarrow \underset{\approx}{A} \quad : \longrightarrow \texttt{ccA = m2cc(mA,9)}$$

$$\mathbf{A} \longrightarrow \underline{\underline{A}} \quad : \longrightarrow \texttt{mmA = m2mm(mA,9)}$$

$$\mathbf{A}^c \longrightarrow \underline{A}^T : \longrightarrow \texttt{mAt = mA'}$$

$$\mathbf{A}^c \longrightarrow \underset{\approx}{A}_t \quad : \longrightarrow \texttt{ccAt = m2cc(mAt,9)}$$

$$\mathbf{A}^c \longrightarrow \underline{\underline{A}}_t : \longrightarrow \texttt{mmAt = m2mm(mA')}$$

$$^4\mathbf{A}^{lc} \longrightarrow \underline{\underline{A}}_r : \longrightarrow \texttt{mmAr = mmA([1 2 3 5 4 7 6 9 8],:)}$$

$$^4\mathbf{A}^{rc} \longrightarrow \underline{\underline{A}}_c : \longrightarrow \texttt{mmAc = mmA(:,[1 2 3 5 4 7 6 9 8])}$$

Gradients

Cartesian

$$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = \begin{bmatrix} \vec{e}_x & \vec{e}_t & \vec{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \underset{\sim}{\vec{e}}^T \underset{\sim}{\nabla}$$

cylindrical

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_t \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z} = \begin{bmatrix} \vec{e}_r & \vec{e}_t & \vec{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{bmatrix} = \underset{\sim}{\vec{e}}^T \underset{\sim}{\nabla}$$

Gradient of a vector in Cartesian coordinates

$$
\begin{aligned}
\vec{\nabla}\vec{a} = \mathbf{L}_a^c &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)\,(a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \\
&= \vec{e}_x a_{x,x}\vec{e}_x + \vec{e}_x a_{y,x}\vec{e}_y + \vec{e}_x a_{z,x}\vec{e}_z + \vec{e}_y a_{x,y}\vec{e}_x + \\
&\qquad \vec{e}_y a_{y,y}\vec{e}_y + \vec{e}_y a_{z,y}\vec{e}_z + \vec{e}_z a_{x,z}\vec{e}_x + \vec{e}_z a_{y,z}\vec{e}_y + \vec{e}_z a_{z,z}\vec{e}_z \\
\underline{L}_a &= \begin{bmatrix} a_{x,x} & a_{x,y} & a_{x,z} \\ a_{y,x} & a_{y,y} & a_{y,z} \\ a_{z,x} & a_{z,y} & a_{z,z} \end{bmatrix} \\
\underset{\approx a}{L}^T &= \begin{bmatrix} a_{x,x} & a_{y,y} & a_{z,z} & a_{x,y} & a_{y,x} & a_{y,z} & a_{z,y} & a_{z,x} & a_{x,z} \end{bmatrix}
\end{aligned}
$$

Gradient of a vector in cylindrical coordinates

$$
\begin{aligned}
\vec{\nabla}\vec{a} = \mathbf{L}_a^c &= \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_t \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z} \right) \, (a_r \vec{e}_r + a_t \vec{e}_t + a_z \vec{e}_z) \\
&= \vec{e}_r a_{r,r} \vec{e}_r + \vec{e}_r a_{t,r} \vec{e}_t + \vec{e}_r a_{z,r} \vec{e}_z + \\
&\quad \vec{e}_t \frac{1}{r} a_{r,t} \vec{e}_r + \vec{e}_t \frac{1}{r} a_{t,t} \vec{e}_t + \vec{e}_t \frac{1}{r} a_{z,t} \vec{e}_z + \vec{e}_t \frac{1}{r} a_r \vec{e}_t - \vec{e}_t \frac{1}{r} a_t \vec{e}_r \\
&\quad \vec{e}_z a_{r,z} \vec{e}_r + \vec{e}_z a_{t,z} \vec{e}_t + \vec{e}_z a_{z,z} \vec{e}_z
\end{aligned}
$$

$$
\underline{L}_a = \begin{bmatrix} a_{r,r} & \frac{1}{r} a_{r,t} - \frac{1}{r} a_t & a_{r,z} \\ a_{t,r} & \frac{1}{r} a_{t,t} + \frac{1}{r} a_r & a_{t,z} \\ a_{z,r} & \frac{1}{r} a_{z,t} & a_{z,z} \end{bmatrix}
$$

$$
\underset{\approx}{L}{}_a^T = \begin{bmatrix} a_{r,r} & \frac{1}{r} a_{t,t} + \frac{1}{r} a_r & a_{z,z} & \frac{1}{r} a_{r,t} - \frac{1}{r} a_t & a_{t,r} & a_{t,z} & \frac{1}{r} a_{z,t} & a_{z,r} & a_{r,z} \end{bmatrix}
$$

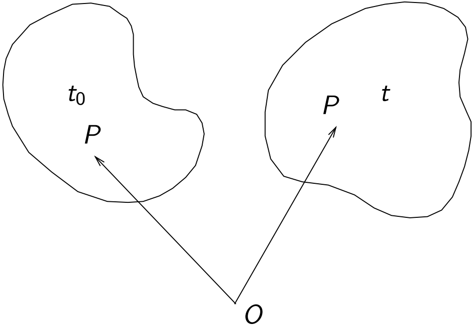
Divergence of tensor in cylindrical coordinates

$$
\begin{aligned}
\vec{\nabla} \cdot \mathbf{A} &= \vec{e}_i \cdot \nabla_i (\vec{e}_j A_{jk} \vec{e}_k) \\
&= \vec{e}_i \cdot (\nabla_i \vec{e}_j) A_{jk} \vec{e}_k + \vec{e}_i \cdot \vec{e}_j (\nabla_i A_{jk}) \vec{e}_k + \vec{e}_i \cdot \vec{e}_j A_{jk} (\nabla_i \vec{e}_k) \\
&= \vec{e}_i \cdot (\nabla_i \vec{e}_j) A_{jk} \vec{e}_k + \delta_{ij} (\nabla_i A_{jk}) \vec{e}_k + \delta_{ij} A_{jk} (\nabla_i \vec{e}_k) \\
&\qquad \nabla_i \vec{e}_j = \delta_{i2} \delta_{1j} \frac{1}{r} \vec{e}_t - \delta_{i2} \delta_{2j} \frac{1}{r} \vec{e}_r \\
&= \delta_{1j} \frac{1}{r} A_{jk} \vec{e}_k + (\nabla_j A_{jk}) \vec{e}_k + (\delta_{j2} \delta_{1k} \frac{1}{r} \vec{e}_t - \delta_{j2} \delta_{2k} \frac{1}{r} \vec{e}_r) A_{jk} \\
&= \frac{1}{r} A_{1k} \vec{e}_k + (\nabla_j A_{jk}) \vec{e}_k + \frac{1}{r} (A_{21} \vec{e}_t - A_{22} \vec{e}_r) \\
&= (\frac{1}{r} A_{11} - \frac{1}{r} A_{22}) \vec{e}_1 + (\frac{1}{r} A_{12} + \frac{1}{r} A_{21}) \vec{e}_2 + \frac{1}{r} A_{13} \vec{e}_3 + \\
&\qquad (\nabla_j A_{jk}) \vec{e}_k \\
&= g_k \vec{e}_k + \nabla_j A_{jk} \vec{e}_k \\
&= \underset{\sim}{g}^T \underset{\sim}{\vec{e}} + (\underset{\sim}{\nabla}^T \underline{A}) \underset{\sim}{\vec{e}} \\
&= (\underset{\sim}{\nabla}^T \underline{A}) \underset{\sim}{\vec{e}} + \underset{\sim}{g}^T \underset{\sim}{\vec{e}} \\
&\qquad \text{with} \qquad \underset{\sim}{g}^T = \frac{1}{r} \begin{bmatrix} (A_{11} - A_{22}) & (A_{12} + A_{21}) & A_{33} \end{bmatrix}
\end{aligned}
$$

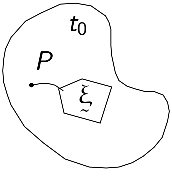
KINEMATICS

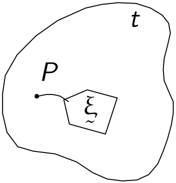
back to index

Kinematics



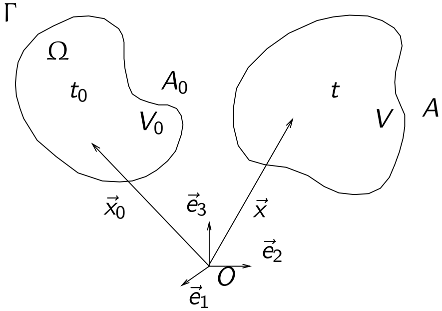
Material coordinates





$$\underset{\sim}{\xi}^T = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix}$$

Position vectors



undeformed configuration (t_0) $\quad \vec{x}_0 = \vec{\chi}(\underset{\sim}{\xi}, t_0) = x_{01}\vec{e}_1 + x_{02}\vec{e}_2 + x_{03}\vec{e}_3$

deformed configuration (t) $\quad \vec{x} = \vec{\chi}(\underset{\sim}{\xi}, t) = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$

Euler-Lagrange

Euler : "observer" is fixed in space

$$a = \mathcal{A}_E(\vec{x}, t)$$

$$da = a_Q - a_P = \mathcal{A}_E(\vec{x} + d\vec{x}, t) - \mathcal{A}_E(\vec{x}, t) = d\vec{x} \cdot \left. (\vec{\nabla} a) \right|_t$$

$$\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3}$$

Lagrange : "observer" follows the material

$$a = \mathcal{A}_L(\vec{x}_0, t)$$

$$da = a_Q - a_P = \mathcal{A}_L(\vec{x}_0 + d\vec{x}_0, t) - \mathcal{A}_L(\vec{x}_0, t) = d\vec{x}_0 \cdot \left. (\vec{\nabla}_0 a) \right|_t$$

$$\vec{\nabla}_0 = \vec{e}_1 \frac{\partial}{\partial x_{01}} + \vec{e}_2 \frac{\partial}{\partial x_{02}} + \vec{e}_3 \frac{\partial}{\partial x_{03}}$$

position vectors

$$\vec{\nabla}\vec{x} = \mathbf{I} \qquad ; \qquad \vec{\nabla}_0\vec{x}_0 = \mathbf{I}$$

Time derivatives

material time derivative

$$\frac{Da}{Dt} = \dot{a} = \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \left\{ A(\vec{x}_0, t + \Delta t) - A(\vec{x}_0, t) \right\}$$

velocity of a material point

$$\vec{v} = \vec{v}(\vec{x}_0) = \dot{\vec{x}}$$

spatial time derivative

$$\frac{\delta a}{\delta t} = \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \left\{ \mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t) \right\}$$

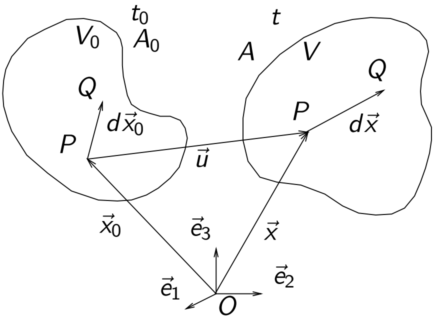
velocity *field*

$$\vec{v} = \vec{v}(\vec{x}, t)$$

Relation $\dot{\vec{a}}$ and $\frac{\delta \vec{a}}{\delta t}$

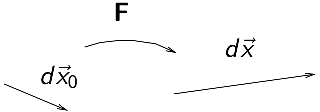
$$
\begin{aligned}
\frac{Da}{Dt} &= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{A(\vec{x}_0, t+\Delta t) - A(\vec{x}_0, t)\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{\mathcal{A}(\vec{x} + d\vec{x}, t+\Delta t) - \mathcal{A}(\vec{x}, t)\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{\mathcal{A}(\vec{x} + d\vec{x}, t+\Delta t) - \mathcal{A}(\vec{x}, t+\Delta t) + \\
&\qquad\qquad\qquad \mathcal{A}(\vec{x}, t+\Delta t) - \mathcal{A}(\vec{x}, t)\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{d\vec{x} \cdot \left. (\vec{\nabla} a)\right|_{t+\Delta t} + \mathcal{A}(\vec{x}, t+\Delta t) - \mathcal{A}(\vec{x}, t)\} \\
&= \lim_{\Delta t \longrightarrow 0} \{\frac{d\vec{x}}{\Delta t} \cdot \left. (\vec{\nabla} a)\right|_{t+\Delta t}\} + \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{\mathcal{A}(\vec{x}, t+\Delta t) - \mathcal{A}(\vec{x}, t)\} \\
&= \vec{v} \cdot (\vec{\nabla} a) + \frac{\delta a}{\delta t} \\
&= \text{(convective time derivative)} + \text{(spatial time derivative)} \\
&= \text{(material time derivative)}
\end{aligned}
$$

Deformation



displacement : $\vec{u} = \vec{x} - \vec{x}_0 = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$

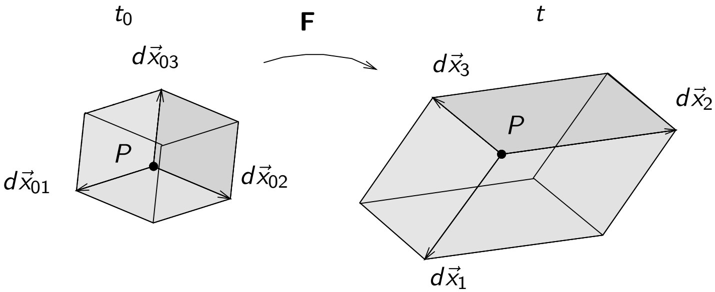
Deformation tensor



$$
\begin{aligned}
d\vec{x} &= \mathbf{F} \cdot d\vec{x}_0 \\
&= \vec{X}(\vec{x}_0 + d\vec{x}_0, \mathbf{t}) - \vec{X}(\vec{x}_0, \mathbf{t}) = d\vec{x}_0 \cdot \left(\vec{\nabla}_0 \vec{x}\right) \\
&= \left(\vec{\nabla}_0 \vec{x}\right)^c \cdot d\vec{x}_0 = \mathbf{F} \cdot d\vec{x}_0
\end{aligned}
$$

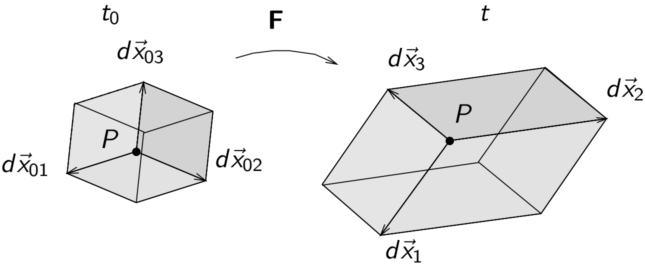
$$
\mathbf{F} = \left(\vec{\nabla}_0 \vec{x}\right)^c = \left[\left(\vec{\nabla}_0 \vec{x}_0\right)^c + \left(\vec{\nabla}_0 \vec{u}\right)^c\right] = \mathbf{I} + \left(\vec{\nabla}_0 \vec{u}\right)^c
$$

Deformation tensor



$$d\vec{x}_1 = \mathbf{F} \cdot d\vec{x}_{01} \qquad ; \qquad d\vec{x}_2 = \mathbf{F} \cdot d\vec{x}_{02} \qquad ; \qquad d\vec{x}_3 = \mathbf{F} \cdot d\vec{x}_{03}$$

Volume change



$$
\begin{aligned}
dV &= d\vec{x}_1 * d\vec{x}_2 \cdot d\vec{x}_3 \\
&= (\mathbf{F} \cdot d\vec{x}_{01}) * (\mathbf{F} \cdot d\vec{x}_{02}) \cdot (\mathbf{F} \cdot d\vec{x}_{03}) \\
&= \det(\mathbf{F})\,(d\vec{x}_{01} * d\vec{x}_{02} \cdot d\vec{x}_{03}) \\
&= \det(\mathbf{F})\, dV_0 \\
&= J\, dV_0
\end{aligned}
$$

Area change

$$
\begin{aligned}
dA\,\vec{n} &= d\vec{x}_1 * d\vec{x}_2 = (\mathbf{F}\cdot d\vec{x}_{01}) * (\mathbf{F}\cdot d\vec{x}_{02}) \\
dA\,\vec{n}\cdot(\mathbf{F}\cdot d\vec{x}_{03}) &= (\mathbf{F}\cdot d\vec{x}_{01}) * (\mathbf{F}\cdot d\vec{x}_{02})\cdot(\mathbf{F}\cdot d\vec{x}_{03}) \\
&= \det(\mathbf{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot d\vec{x}_{03} \qquad \forall \quad d\vec{x}_{03} \quad \longrightarrow \\
dA\,\vec{n}\cdot\mathbf{F} &= \det(\mathbf{F})(d\vec{x}_{01} * d\vec{x}_{02}) \\
dA\,\vec{n} &= \det(\mathbf{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot\mathbf{F}^{-1} \\
&= \det(\mathbf{F})dA_0\,\vec{n}_0\cdot\mathbf{F}^{-1} \\
&= dA_0\,\vec{n}_0\cdot\left(\mathbf{F}^{-1}\det(\mathbf{F})\right)
\end{aligned}
$$

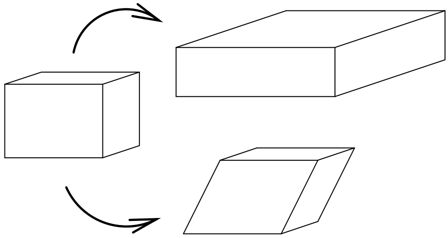
Inverse deformation

$$J = \frac{dV}{dV_0} = \det(\mathbf{F}) > 0 \quad \rightarrow \quad \mathbf{F} \text{ regular} \quad \rightarrow \quad d\vec{x}_0 = \mathbf{F}^{-1} \cdot d\vec{x}$$

relation between gradient operators

$$\mathbf{I} = \mathbf{F}^{-T} \cdot \mathbf{F}^{T} \rightarrow \left(\vec{\nabla}\vec{x}\right) = \mathbf{F}^{-T} \cdot \left(\vec{\nabla}_0\vec{x}\right) \quad \rightarrow \quad \vec{\nabla} = \mathbf{F}^{-T} \cdot \vec{\nabla}_0$$

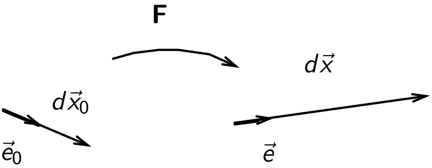
Homogeneous deformation



$$\vec{\nabla}_0 \vec{x} = \mathbf{F}^c = \text{uniform tensor} \quad \longrightarrow$$

$$\vec{x} = (\vec{x}_0 \cdot \mathbf{F}^c) + \vec{t} = \mathbf{F} \cdot \vec{x}_0 + \vec{t}$$

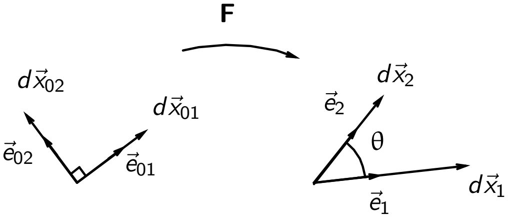
Elongation and shear



elongation factor in initial $\vec{e}_0$-direction

$$
\begin{aligned}
\lambda^2(\vec{e}_{01}) &= \frac{d\vec{x}_1 \cdot d\vec{x}_1}{d\vec{x}_{01} \cdot d\vec{x}_{01}} = \frac{d\vec{x}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\vec{x}_{01}}{d\vec{x}_{01} \cdot d\vec{x}_{01}} = \frac{||d\vec{x}_{01}||^2}{||d\vec{x}_{01}||^2} \left(\vec{e}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \vec{e}_{01} \right) \\
&= \vec{e}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \vec{e}_{01} = \vec{e}_{01} \cdot \mathbf{C} \cdot \vec{e}_{01}
\end{aligned}
$$

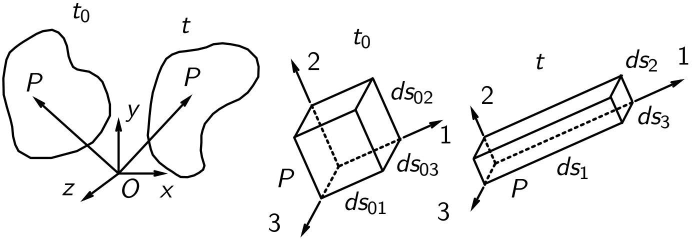
Elongation and shear



shear of initial $(\vec{e}_{01},\vec{e}_{02})$-directions

$$
\begin{aligned}
\gamma(\vec{e}_{01},\vec{e}_{02}) &= \sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta) = \frac{d\vec{x}_1 \cdot d\vec{x}_2}{\|d\vec{x}_1\|\|d\vec{x}_2\|} = \frac{d\vec{x}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\vec{x}_{02}}{\|d\vec{x}_1\|\|d\vec{x}_2\|} \\
&= \frac{\|d\vec{x}_{01}\|\|d\vec{x}_{02}\|(\vec{e}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \vec{e}_{02})}{\lambda(\vec{e}_{01})\|d\vec{x}_{01}\|\lambda(\vec{e}_{02})\|d\vec{x}_{02}\|} = \frac{\vec{e}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})} \\
&= \frac{\vec{e}_{01} \cdot \mathbf{C} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})}
\end{aligned}
$$

Principal directions of deformation



$$\lambda_1 = \frac{ds_1}{ds_{01}} \quad ; \quad \lambda_2 = \frac{ds_2}{ds_{02}} \quad ; \quad \lambda_3 = \frac{ds_3}{ds_{03}} \quad ; \quad \gamma_{12} = \gamma_{23} = \gamma_{31} = 0$$

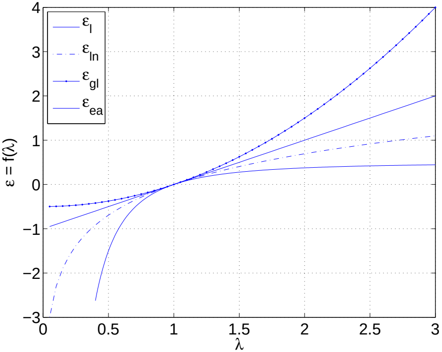
$$J = \frac{dV}{dV_0} = \frac{ds_1\,ds_2\,ds_3}{ds_{01}\,ds_{02}\,ds_{03}} = \lambda_1\lambda_2\lambda_3$$

Strains

$$\varepsilon = f(\lambda)$$

- $f(\lambda = 1) = 0$
- $\lim_{\lambda \longrightarrow 1} f(\lambda) = \lambda - 1$
- $f(\lambda)$ monotonic increasing
- $f(\lambda)$ differentiable

linear	$\varepsilon_l = \lambda - 1$
logarithmic	$\varepsilon_{ln} = \ln(\lambda)$
Green-Lagrange	$\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$
Euler-Almansi	$\varepsilon_{ea} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$



Strain tensor

$$\frac{1}{2}\left\{\lambda^2(\vec{e}_{01})-1\right\} = \vec{e}_{01}\cdot\left\{\frac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{I}\right)\right\}\cdot\vec{e}_{01} = \vec{e}_{01}\cdot\mathbf{E}\cdot\vec{e}_{01}$$

$$\gamma(\vec{e}_{01},\vec{e}_{02}) = \frac{\vec{e}_{01}\cdot(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{I})\cdot\vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})} = \left[\frac{2}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})}\right]\vec{e}_{01}\cdot\mathbf{E}\cdot\vec{e}_{02}$$

$$\left.\begin{aligned}\mathbf{E} &= \frac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{I}\right)\\ \mathbf{F} &= \left(\vec{\nabla}_0\vec{x}\right)^T = \mathbf{I}+\left(\vec{\nabla}_0\vec{u}\right)^T\end{aligned}\right\}\rightarrow\quad \begin{aligned}\mathbf{E} &= \frac{1}{2}\left[\left\{\mathbf{I}+\left(\vec{\nabla}_0\vec{u}\right)\right\}\cdot\left\{\mathbf{I}+\left(\vec{\nabla}_0\vec{u}\right)^T\right\} \right.\\ &= \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T+\left(\vec{\nabla}_0\vec{u}\right)+\left(\vec{\nabla}_0\vec{u}\right)\cdot\left(\right.\right.\end{aligned}$$

Right Cauchy-Green deformation tensor

$$\mathbf{C} = \mathbf{F}^c \cdot \mathbf{F}$$

1. symmetric $\quad \mathbf{C}^c = \mathbf{C}$
2. positive definite

$$\vec{a} \cdot \mathbf{C} \cdot \vec{a} = \vec{a} \cdot \mathbf{F}^c \cdot \mathbf{F} \cdot \vec{a} = (\mathbf{F} \cdot \vec{a}) \cdot (\mathbf{F} \cdot \vec{a})$$

$$\mathbf{F} \text{ is regular} \quad \rightarrow \quad \mathbf{F} \cdot \vec{a} \neq \vec{0} \quad \text{if} \quad \vec{a} \neq \vec{0} \quad \rightarrow$$

$$\vec{a} \cdot \mathbf{C} \cdot \vec{a} > 0 \quad \forall \quad \vec{a} \neq \vec{0}$$

3. $\left.\begin{array}{l} \text{eigenvalues and eigenvectors real} \\ \text{eigenvalues positive} \\ \text{eigenvectors} \perp \text{(choice)} \end{array}\right\} \rightarrow \quad$ spectral representation

$$\mathbf{C} = \mu_1 \vec{m}_1 \vec{m}_1 + \mu_2 \vec{m}_2 \vec{m}_2 + \mu_3 \vec{m}_3 \vec{m}_3$$

Eigenvectors and eigenvalues

$$\mathbf{C} = \mu_1 \vec{m}_1 \vec{m}_1 + \mu_2 \vec{m}_2 \vec{m}_2 + \mu_3 \vec{m}_3 \vec{m}_3$$

$$\lambda(\vec{e}_0) = \sqrt{\vec{e}_0 \cdot \mathbf{C} \cdot \vec{e}_0} \qquad ; \qquad \gamma(\vec{e}_{01}, \vec{e}_{02}) = \frac{\vec{e}_{01} \cdot \mathbf{C} \cdot \vec{e}_{02}}{\sqrt{\vec{e}_{01} \cdot \mathbf{C} \cdot \vec{e}_{01}} \sqrt{\vec{e}_{02} \cdot \mathbf{C} \cdot \vec{e}_{02}}} \quad \rightarrow$$

$$\mathbf{C} = \mu_1 \vec{n}_{01} \vec{n}_{01} + \mu_2 \vec{n}_{02} \vec{n}_{02} + \mu_3 \vec{n}_{03} \vec{n}_{03}$$

$$\lambda(\vec{n}_{01}) = \sqrt{\vec{n}_{01} \cdot \mathbf{C} \cdot \vec{n}_{01}} = \sqrt{\mu_1} \quad ; \quad \gamma(\vec{n}_{01}, \vec{n}_{02}) = \frac{\vec{n}_{01} \cdot \mathbf{C} \cdot \vec{n}_{02}}{\sqrt{\vec{n}_{01} \cdot \mathbf{C} \cdot \vec{n}_{01}} \sqrt{\vec{n}_{02} \cdot \mathbf{C} \cdot \vec{n}_{02}}} = 0$$

$$\mathbf{C} = \lambda_1^2 \, \vec{n}_{01} \vec{n}_{01} + \lambda_2^2 \, \vec{n}_{02} \vec{n}_{02} + \lambda_3^2 \, \vec{n}_{03} \vec{n}_{03}$$

Right stretch tensor

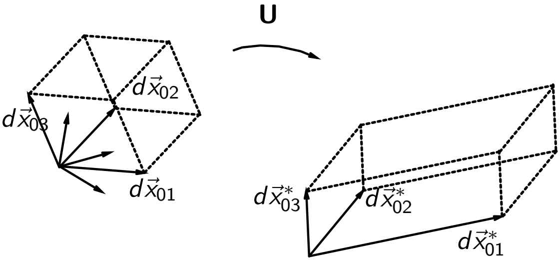
$$\mathbf{U} = \sqrt{\mathbf{C}} = \lambda_1 \vec{n}_{01}\vec{n}_{01} + \lambda_2 \vec{n}_{02}\vec{n}_{02} + \lambda_3 \vec{n}_{03}\vec{n}_{03}$$

properties

1. symmetric : $\mathbf{U}^c = \mathbf{U}$
2. pos. def. : $\vec{a} \cdot \mathbf{U} \cdot \vec{a} > 0 \qquad \forall \quad \vec{a}$
3. regular : $\mathbf{U}^{-1} = \dfrac{1}{\lambda_1}\vec{n}_{01}\vec{n}_{01} + \dfrac{1}{\lambda_2}\vec{n}_{02}\vec{n}_{02} + \dfrac{1}{\lambda_3}\vec{n}_{03}\vec{n}_{03}$
4. $\det(\mathbf{C}) = \det(\mathbf{U}\cdot\mathbf{U}) = \det(\mathbf{F}^c\cdot\mathbf{F}) = \det^2(\mathbf{F}) \quad \rightarrow$

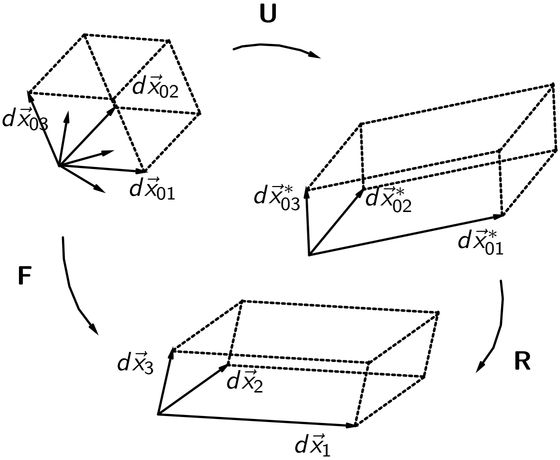
   $\det(\mathbf{U}) = \lambda_1\lambda_2\lambda_3 = \det(\mathbf{F}) = J$

Stretch tensor transformation



$$d\vec{x}^*_{01} = \mathbf{U} \cdot d\vec{x}_{01} \quad ; \quad d\vec{x}^*_{02} = \mathbf{U} \cdot d\vec{x}_{02} \quad ; \quad d\vec{x}^*_{03} = \mathbf{U} \cdot d\vec{x}_{03}$$

Total transformation



$$\left.\begin{array}{l} d\vec{x}^*_{01} = \mathbf{U} \cdot d\vec{x}_{01} \quad \rightarrow \quad d\vec{x}_{01} = \mathbf{U}^{-1} \cdot d\vec{x}^*_{01} \\ d\vec{x}_1 = \mathbf{F} \cdot d\vec{x}_{01} \end{array}\right\} \quad \rightarrow$$

$$d\vec{x}_1 = \mathbf{F} \cdot \mathbf{U}^{-1} \cdot d\vec{x}^*_{01} = \mathbf{R} \cdot d\vec{x}^*_{01} \quad \rightarrow \quad \mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$$

Rotation tensor

$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$$

1.

$$\begin{aligned} \mathbf{R}^c \cdot \mathbf{R} &= \mathbf{U}^{-c} \cdot \mathbf{F}^c \cdot \mathbf{F} \cdot \mathbf{U}^{-1} \\ &= \mathbf{U}^{-c} \cdot \mathbf{U} \cdot \mathbf{U} \cdot \mathbf{U}^{-1} \\ &= \mathbf{U}^{-c} \cdot \mathbf{U}^c \cdot \mathbf{U} \cdot \mathbf{U}^{-1} \\ &= \mathbf{I} \qquad \rightarrow \qquad \mathbf{R} \text{ is orthogonal} \end{aligned}$$

2.

$$\begin{aligned} \det(\mathbf{R}) &= \det(\mathbf{F} \cdot \mathbf{U}^{-1}) \\ &= \det(\mathbf{U}) \det(\mathbf{U}^{-1}) = \det(\mathbf{U} \cdot \mathbf{U}^{-1}) \\ &= \det(\mathbf{I}) = 1 \qquad \rightarrow \qquad \mathbf{R} \text{ is rotation tensor} \end{aligned}$$

Right polar decomposition

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$$

- $\mathbf{F}$ known
- calculate $\mathbf{C} = \mathbf{F}^c \cdot \mathbf{F}$
- calculate λ_i en $\vec{n}_{0i}$
- $\mathbf{U}$ known
- calculate $\mathbf{U}^{-1}$
- calculate $\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$

Strain tensors

stretch ratio $\lambda(\vec{e}_0) = \sqrt{\vec{e}_0 \cdot \mathbf{C} \cdot \vec{e}_0}$

strain tensor $\boldsymbol{\varepsilon}$

strain measure $\varepsilon(\vec{e}_0) = \vec{e}_0 \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_0 = f(\lambda(\vec{e}_0))$

shear measure $\gamma(\vec{e}_{01}, \vec{e}_{02}) = \vec{e}_{01} \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_{02}$

Linear strain tensor

$$\boldsymbol{\mathcal{E}} = \mathbf{U} - \mathbf{I}$$

$$\begin{aligned}
\vec{e}_0 \cdot \boldsymbol{\mathcal{E}} \cdot \vec{e}_0 &= \vec{e}_0 \cdot \mathbf{U} \cdot \vec{e}_0 - \vec{e}_0 \cdot \mathbf{I} \cdot \vec{e}_0 \\
&= \vec{e}_0 \cdot \mathbf{U} \cdot \vec{e}_0 - 1 \\
&\neq \lambda(\vec{e}_0) - 1
\end{aligned}$$

$$\begin{aligned}
\vec{n}_{0i} \cdot \boldsymbol{\mathcal{E}} \cdot \vec{n}_{0i} &= \vec{n}_{0i} \cdot \mathbf{U} \cdot \vec{n}_{0i} - 1 \\
&= \lambda(\vec{n}_{0i}) - 1 \\
&= \lambda_i - 1
\end{aligned}$$

Logarithmic strain tensor

$$\boldsymbol{\Lambda} = \ln(\mathbf{U})$$

$$\begin{aligned} \vec{e}_0 \cdot \boldsymbol{\Lambda} \cdot \vec{e}_0 &= \vec{e}_0 \cdot \ln(\mathbf{U}) \cdot \vec{e}_0 \\ &\neq \ln(\lambda(\vec{e}_0)) \end{aligned}$$

$$\begin{aligned} \vec{n}_{0i} \cdot \boldsymbol{\Lambda} \cdot \vec{n}_{0i} &= \vec{n}_{0i} \cdot \ln(\mathbf{U}) \cdot \vec{n}_{0i} \\ &= \ln(\lambda(\vec{n}_{0i})) \\ &= \ln(\lambda_i) \end{aligned}$$

Green-Lagrange strain tensor

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right)$$

$$\begin{aligned} \vec{e}_0 \cdot \mathbf{E} \cdot \vec{e}_0 &= \frac{1}{2}\left(\vec{e}_0 \cdot \mathbf{C} \cdot \vec{e}_0 - 1\right) \\ &= \frac{1}{2}\left(\lambda^2(\vec{e}_0) - 1\right) \end{aligned}$$

Infinitesimal linear strain tensor

$$\begin{aligned}\mathbf{E} &= \tfrac{1}{2}\left(\mathbf{F}^c \cdot \mathbf{F} - \mathbf{I}\right) \\ &= \tfrac{1}{2}\left\{(\vec{\nabla}_0\vec{u}) + (\vec{\nabla}_0\vec{u})^c + (\vec{\nabla}_0\vec{u})\cdot(\vec{\nabla}_0\vec{u})^c\right\}\end{aligned}$$

linearisation $\longrightarrow$ infinitesimal strain tensor

$$\begin{aligned}\boldsymbol{\varepsilon} &= \tfrac{1}{2}\left\{(\vec{\nabla}_0\vec{u}) + (\vec{\nabla}_0\vec{u})^c\right\} \\ &= \tfrac{1}{2}\left(\mathbf{F} + \mathbf{F}^c\right) - \mathbf{I} \\ &= \tfrac{1}{2}\left\{(\vec{\nabla}\vec{u}) + (\vec{\nabla}\vec{u})^c\right\}\end{aligned}$$

only correct for small strains AND small rotations

Deformation rate

$$d\dot{\vec{x}} = \dot{\mathbf{F}} \cdot d\vec{x}_0 = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \cdot d\vec{x} = \mathbf{L} \cdot d\vec{x} = (\vec{\nabla}\vec{v})^c \cdot d\vec{x}$$

$$= \tfrac{1}{2}\{\mathbf{L} + \mathbf{L}^c\} \cdot d\vec{x} + \tfrac{1}{2}\{\mathbf{L} - \mathbf{L}^c\} \cdot d\vec{x}$$

$$= \mathbf{D} \cdot d\vec{x} + \mathbf{\Omega} \cdot d\vec{x}$$

velocity gradient tensor	$\mathbf{L}$
deformation rate tensor	$\mathbf{D}$
rotation rate tensor or spin tensor	$\mathbf{\Omega}$

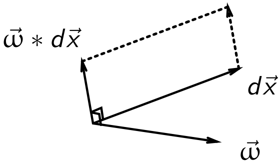
Spin tensor

$$\boldsymbol{\Omega} = \frac{1}{2}\left\{\dot{\mathbf{F}}\cdot\mathbf{F}^{-1} - (\dot{\mathbf{F}}\cdot\mathbf{F}^{-1})^c\right\} = \frac{1}{2}\left\{\left(\vec{\nabla}\vec{v}\right)^c - \left(\vec{\nabla}\vec{v}\right)\right\}$$

$\boldsymbol{\Omega} =$ skewsymmetric $\quad\rightarrow$

$\boldsymbol{\Omega}\cdot d\vec{x} = \vec{\omega} * d\vec{x} =$ velocity $\perp d\vec{x} =$ rotation rate

$\vec{\omega}\quad :\qquad$ axial vector



Axial vector

$$\vec{q} \cdot \mathbf{\Omega} \cdot \vec{q} = \vec{q} \cdot \mathbf{\Omega}^c \cdot \vec{q} = - \, \vec{q} \cdot \mathbf{\Omega} \cdot \vec{q} \quad \rightarrow$$
$$\vec{q} \cdot \mathbf{\Omega} \cdot \vec{q} = 0 \quad \rightarrow$$
$$\mathbf{\Omega} \cdot \vec{q} = \vec{p} \quad \rightarrow$$
$$\vec{q} \cdot \vec{p} = 0 \quad \rightarrow$$
$$\vec{q} \perp \vec{p} \quad \rightarrow$$
$$\exists \quad \vec{\omega} \quad \text{zdd} \quad \vec{p} = \vec{\omega} * \vec{q} \quad \rightarrow$$

$$\mathbf{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q}$$

Axial vector components

$$\mathbf{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q} \qquad\qquad \forall \quad \vec{q}$$

$$\mathbf{\Omega} \cdot \vec{q} = \tilde{\vec{e}}^T \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \tilde{\vec{e}}^T \begin{bmatrix} \Omega_{11} q_1 + \Omega_{12} q_2 + \Omega_{13} q_3 \\ \Omega_{21} q_1 + \Omega_{22} q_2 + \Omega_{23} q_3 \\ \Omega_{31} q_1 + \Omega_{32} q_2 + \Omega_{33} q_3 \end{bmatrix}$$

$$\begin{aligned} \vec{\omega} * \vec{q} &= (\omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3) * (q_1 \vec{e}_1 + q_2 \vec{e}_2 + q_3 \vec{e}_3) \\ &= \omega_1 q_2 (\vec{e}_3) + \omega_1 q_3 (-\vec{e}_2) + \omega_2 q_1 (-\vec{e}_3) + \omega_2 q_3 (\vec{e}_1) + \\ &\quad\ \omega_3 q_1 (\vec{e}_2) + \omega_3 q_2 (-\vec{e}_1) \\ &= [\vec{e}_1 \ \ \vec{e}_2 \ \ \vec{e}_3] \begin{bmatrix} \omega_2 q_3 - \omega_3 q_2 \\ \omega_3 q_1 - \omega_1 q_3 \\ \omega_1 q_2 - \omega_2 q_1 \end{bmatrix} \end{aligned}$$

$$\mathbf{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q} \qquad \forall \vec{q} \qquad \rightarrow \qquad \underline{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Deformation rate tensor

$$\mathbf{D} = \frac{1}{2}\left\{\dot{\mathbf{F}}\cdot\mathbf{F}^{-1} + (\dot{\mathbf{F}}\cdot\mathbf{F}^{-1})^c\right\} = \left\{\left(\vec{\nabla}\vec{v}\right)^c + \left(\vec{\nabla}\vec{v}\right)\right\}$$

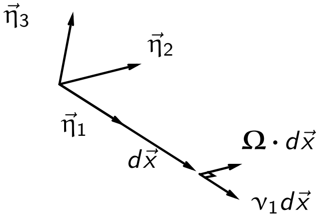
$$\mathbf{D} = \mathbf{D}^c \quad \rightarrow \quad \mathbf{D} = \nu_1\vec{\eta}_1\vec{\eta}_1 + \nu_2\vec{\eta}_2\vec{\eta}_2 + \nu_3\vec{\eta}_3\vec{\eta}_3$$

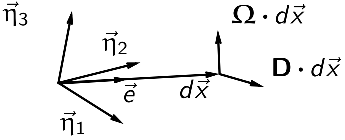
1. : vector $d\vec{x}$ along $\vec{\eta}_1$: $\quad d\vec{x} = dx_1\vec{\eta}_1$

$$\mathbf{D}\cdot d\vec{x} = dx_1\mathbf{D}\cdot\vec{\eta}_1 = dx_1\nu_1\vec{\eta}_1 = \nu_1 d\vec{x}$$

2. : random vector : $\quad d\vec{x} = dx_1\vec{\eta}_1 + dx_2\vec{\eta}_2 + dx_3\vec{\eta}_3$

$$\mathbf{D}\cdot d\vec{x} = dx_1\nu_1\vec{\eta}_1 + dx_2\nu_2\vec{\eta}_2 + dx_3\nu_3\vec{\eta}_3$$





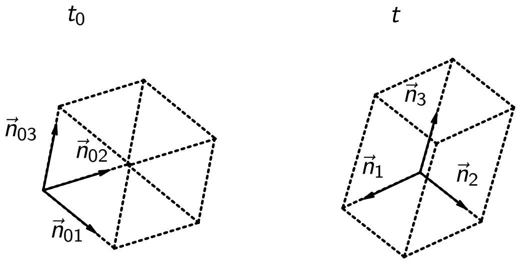
Elongation rate

$$\lambda^2 = \vec{e}_0 \cdot \mathbf{C} \cdot \vec{e}_0 \qquad \rightarrow \qquad \frac{D}{Dt}(\lambda^2) = \frac{D}{Dt}(\vec{e}_0 \cdot \mathbf{C} \cdot \vec{e}_0) \qquad \rightarrow$$

$$\begin{aligned}
2\lambda\dot{\lambda} &= \vec{e}_0 \cdot \frac{D}{Dt}(\mathbf{C}) \cdot \vec{e}_0 = \vec{e}_0 \cdot \frac{D}{Dt}(\mathbf{F}^c \cdot \mathbf{F}) \cdot \vec{e}_0 \\
&= \vec{e}_0 \cdot \{\dot{\mathbf{F}}^c \cdot \mathbf{F} + \mathbf{F}^c \cdot \dot{\mathbf{F}}\} \cdot \vec{e}_0 \\
&= \vec{e}_0 \cdot \mathbf{F}^c \cdot \{\mathbf{F}^{-c} \cdot \dot{\mathbf{F}}^c + \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}\} \cdot \mathbf{F} \cdot \vec{e}_0 \\
&= (\mathbf{F} \cdot \vec{e}_0) \cdot \{(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1})^c + \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}\} \cdot (\mathbf{F} \cdot \vec{e}_0) \\
&= (\lambda\vec{e}) \cdot (2\,\mathbf{D}) \cdot (\lambda\vec{e}) \qquad \rightarrow
\end{aligned}$$

$$\frac{\dot{\lambda}}{\lambda} = \vec{e} \cdot \mathbf{D} \cdot \vec{e}$$

Volume change rate



$$
\begin{aligned}
\mathrm{tr}(\mathbf{D}) &= \vec{n}_1 \cdot \mathbf{D} \cdot \vec{n}_1 + \vec{n}_2 \cdot \mathbf{D} \cdot \vec{n}_2 + \vec{n}_3 \cdot \mathbf{D} \cdot \vec{n}_3 = \frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{\lambda}_3}{\lambda_3} \\
&= \frac{D}{Dt}\left\{\ln(\lambda_1) + \ln(\lambda_2) + \ln(\lambda_3)\right\} = \frac{D}{Dt}\left\{\ln(\lambda_1\lambda_2\lambda_3)\right\} \\
&= \frac{D}{Dt}\left[\ln\{\det(\mathbf{U})\}\right] = \frac{D}{Dt}\left[\ln\{\det(\mathbf{F})\}\right] = \frac{D}{Dt}\{\ln(J)\} = \frac{\dot{J}}{J} \quad \rightarrow
\end{aligned}
$$

$$
\dot{J} = J\mathrm{tr}(\mathbf{D}) = J\left(\vec{\nabla} \cdot \vec{v}\right)
$$

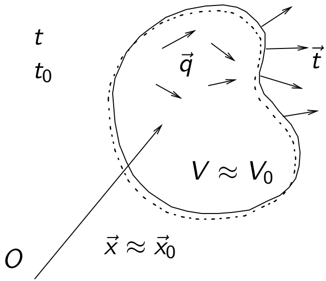
Area change rate

$$
\begin{aligned}
\frac{D}{Dt}\left(dA\,\vec{n}\right) &= \frac{D}{Dt}\left\{\det(\mathbf{F})\,dA_0\,\vec{n}_0\cdot\mathbf{F}^{-1}\right\}\\
&= \frac{D}{Dt}\left\{\det(\mathbf{F})\right\}dA_0\,\vec{n}_0\cdot\mathbf{F}^{-1} + \det(\mathbf{F})\,dA_0\,\vec{n}_0\cdot\dot{\mathbf{F}}^{-1}\\
&= \dot{J}\,dA_0\,\vec{n}_0\cdot\mathbf{F}^{-1} - J\,dA_0\,\vec{n}_0\cdot\mathbf{F}^{-1}\cdot\mathbf{L}\\
&= \mathrm{tr}(\mathbf{L})\,J dA_0\,\vec{n}_0\cdot\mathbf{F}^{-1} - J\,dA_0\,\vec{n}_0\cdot\mathbf{F}^{-1}\cdot\mathbf{L}\\
&= J\,\mathrm{tr}(\mathbf{L})\mathbf{F}^{-c}\cdot dA_0\,\vec{n}_0 - J\,\mathbf{L}^{c}\cdot\mathbf{F}^{-c}\cdot dA_0\,\vec{n}_0\\
&= J\,\left(\mathrm{tr}(\mathbf{L})\mathbf{I} - \mathbf{L}^{c}\right)\cdot\mathbf{F}^{-c}\cdot dA_0\,\vec{n}_0\\
&= \left(\mathrm{tr}(\mathbf{L})\mathbf{I} - \mathbf{L}^{c}\right)dA\,\vec{n}
\end{aligned}
$$

SMALL (LINEAR) DEFORMATION

back to index

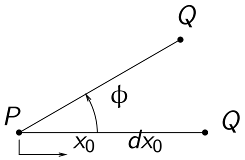
Linear deformation



$$\left.\begin{array}{l} \mathbf{E} = \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T + \left(\vec{\nabla}_0\vec{u}\right) + \left(\vec{\nabla}_0\vec{u}\right)\cdot\left(\vec{\nabla}_0\vec{u}\right)^T\right] \\ \text{small deformation} \quad \rightarrow \quad \left(\vec{\nabla}_0\vec{u}\right)^T = \mathbf{F} - \mathbf{I} \approx \mathbf{O} \end{array}\right\} \quad \rightarrow$$

$$\mathbf{E} \approx \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T + \left(\vec{\nabla}_0\vec{u}\right)\right] \approx \frac{1}{2}\left[\left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right)\right] = \boldsymbol{\varepsilon} \qquad \text{symm!}$$

Rigid rotation



$$\left.\begin{array}{l} u = u_Q = -[dx_0 - dx_0\cos(\phi)] = [\cos(\phi) - 1]dx_0 \\[2ex] v = v_Q = [\sin(\phi)]dx_0 \end{array}\right\} \quad \rightarrow$$

$$\frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \quad ; \quad \frac{\partial v}{\partial x_0} = \sin(\phi) \quad \rightarrow$$

$$\varepsilon_{gl} = \frac{\partial u}{\partial x_0} + \tfrac{1}{2}\left(\frac{\partial u}{\partial x_0}\right)^2 + \tfrac{1}{2}\left(\frac{\partial v}{\partial x_0}\right)^2 = 0$$

$$\varepsilon_l = \frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \neq 0 \quad !!$$

Elongational, shear and volume strain

elong. strain

$$\tfrac{1}{2}\left(\lambda^2(\vec{e}_{01})-1\right) = \vec{e}_{01}\cdot\mathbf{E}\cdot\vec{e}_{01}$$

$$\downarrow$$

$$\lambda(\vec{e}_{01})-1 = \vec{e}_{01}\cdot\varepsilon\cdot\vec{e}_{01}$$

shear strain

$$\gamma(\vec{e}_{01},\vec{e}_{02}) = \sin\left(\tfrac{\pi}{2}-\theta\right) = \left(\frac{2}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})}\right)\vec{e}_{01}\cdot\mathbf{E}\cdot\vec{e}_{02}$$

$$\downarrow$$

$$\tfrac{\pi}{2}-\theta = 2\,\vec{e}_{01}\cdot\varepsilon\cdot\vec{e}_{02}$$

volume change

$$J=\frac{dV}{dV_0} = \lambda_1\lambda_2\lambda_3 = (\varepsilon_1+1)(\varepsilon_2+1)(\varepsilon_2+1)$$

$$\downarrow$$

$$J = \varepsilon_1+\varepsilon_2+\varepsilon_3+1 = \mathrm{tr}(\varepsilon)+1$$

volume strain

$$J-1 = \mathrm{tr}(\varepsilon)$$

Linear strain matrix

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \qquad \text{with} \qquad \left\{ \begin{array}{l} \varepsilon_{21} = \varepsilon_{12} \\ \varepsilon_{32} = \varepsilon_{23} \\ \varepsilon_{31} = \varepsilon_{13} \end{array} \right.$$

principal strain matrix

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

spectral form

$$\boldsymbol{\varepsilon} = \varepsilon_1 \vec{n}_1 \vec{n}_1 + \varepsilon_2 \vec{n}_2 \vec{n}_2 + \varepsilon_3 \vec{n}_3 \vec{n}_3$$

Linear strain : Cartesian components

gradient operator $$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$

displacement vector $$\vec{u} = u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z$$

linear strain tensor $$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left\{(\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u})\right\} = \underset{\sim}{\vec{e}}^T \, \underline{\varepsilon} \, \underset{\sim}{\vec{e}}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2u_{x,x} & u_{x,y} + u_{y,x} & u_{x,z} + u_{z,x} \\ u_{y,x} + u_{x,y} & 2u_{y,y} & u_{y,z} + u_{z,y} \\ u_{z,x} + u_{x,z} & u_{z,y} + u_{y,z} & 2u_{z,z} \end{bmatrix}$$

Linear strain : cylindrical components

gradient operator $\vec{\nabla} = \vec{e}_r \dfrac{\partial}{\partial r} + \vec{e}_t \dfrac{1}{r} \dfrac{\partial}{\partial \theta} + \vec{e}_z \dfrac{\partial}{\partial z}$

displacement vector $\vec{u} = u_r \vec{e}_r(\theta) + u_t \vec{e}_t(\theta) + u_z \vec{e}_z$

linear strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left\{ (\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u}) \right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{rt} & \varepsilon_{rz} \\ \varepsilon_{tr} & \varepsilon_{tt} & \varepsilon_{tz} \\ \varepsilon_{zr} & \varepsilon_{zt} & \varepsilon_{zz} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ u_{z,r} + u_{r,z} & \frac{1}{r}u_{z,t} + u_{t,z} & 2u_{z,z} \end{bmatrix}$$

Compatibility relations

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\,\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$
$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\,\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}$$
$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\,\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}$$
$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial z}$$
$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial y^2} = \frac{\partial^2 \varepsilon_{yx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial x}$$
$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial z^2} = \frac{\partial^2 \varepsilon_{zy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial y}$$

$$\frac{1}{r^2}\,\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \varepsilon_{tt}}{\partial r^2} - \frac{2}{r}\,\frac{\partial^2 \varepsilon_{rt}}{\partial r \partial \theta} - \frac{1}{r}\,\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\,\frac{\partial \varepsilon_{tt}}{\partial r} - \frac{2}{r^2}\,\frac{\partial \varepsilon_{rt}}{\partial \theta} = 0$$

Planar deformation

$$u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = u_3(x_1, x_2, x_3)$$

Plane strain

planar deformation

$$u_1 = u_1(x_1, x_2) \ ; \ u_2 = u_2(x_1, x_2) \ ; \ u_3 = u_3(x_1, x_2, x_3)$$

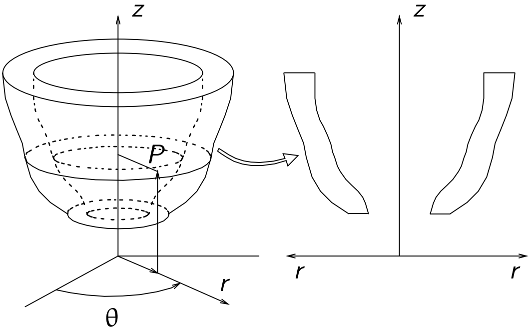
plane strain

$$u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = 0$$

$$\varepsilon_{33} = 0 \quad ; \quad \gamma_{13} = \gamma_{23} = 0$$

compatibility : $\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}$

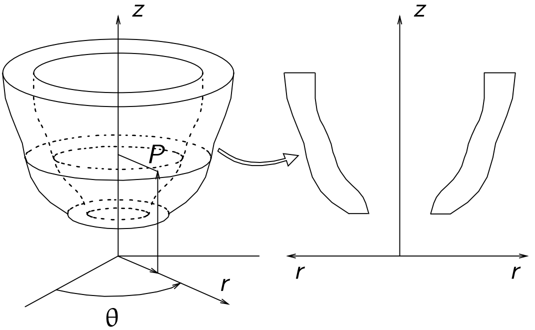
Axi-symmetric deformation



$$\frac{\partial}{\partial \theta}(\ ) = 0 \qquad \rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_t(r,z)\vec{e}_t(\theta) + u_z(r,z)\vec{e}_z$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & -\frac{1}{r}(u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ -\frac{1}{r}(u_t) + u_{t,r} & 2\frac{1}{r}(u_r) & u_{t,z} \\ u_{z,r} + u_{r,z} & u_{t,z} & 2u_{z,z} \end{bmatrix}$$

Axi-symmetric deformation with $u_t = 0$



$$\frac{\partial}{\partial \theta}(\ ) = 0 \text{ and } u_t = 0 \qquad \longrightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_z(r,z)\vec{e}_z$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & 0 & u_{r,z} + u_{z,r} \\ 0 & 2\frac{1}{r}(u_r) & 0 \\ u_{z,r} + u_{r,z} & 0 & 2u_{z,z} \end{bmatrix}$$

Axi-symmetric plane strain

plane strain deformation

$$\left.\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta) \\ u_z = 0 \end{array}\right\} \quad \rightarrow \quad \varepsilon_{zz} = \gamma_{rz} = \gamma_{tz} = 0$$

linear strain matrix

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & u_{t,r} - \frac{1}{r}(u_t) & 0 \\ u_{t,r} - \frac{1}{r}(u_t) & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

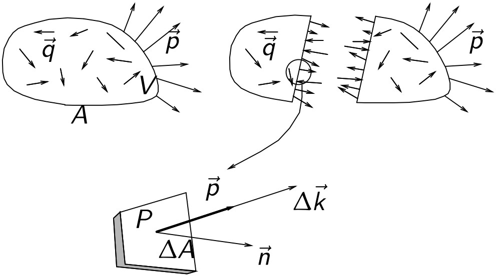
plane strain deformation with $u_t = 0$

$$\left.\begin{array}{l} u_r = u_r(r) \\ u_z = 0 \end{array}\right\} \quad \rightarrow \quad \underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & 0 & 0 \\ 0 & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

STRESS

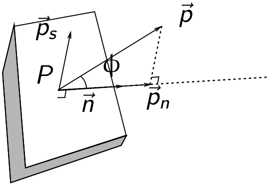
back to index

Stress vector



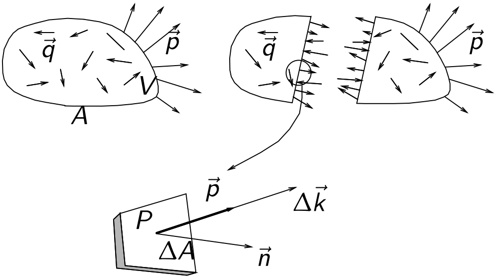
$$\vec{p} = \lim_{\Delta A \to 0} \frac{\Delta\vec{k}}{\Delta A}$$

Normal stress and shear stress



normal stress	:	$p_n = \vec{p} \cdot \vec{n}$
tensile stress	:	positive $(\phi < \frac{\pi}{2})$
compression stress	:	negative $(\phi > \frac{\pi}{2})$
normal stress vector	:	$\vec{p}_n = p_n \vec{n}$
shear stress vector	:	$\vec{p}_s = \vec{p} - \vec{p}_n$
shear stress	:	$p_s = \|\vec{p}_s\| = \sqrt{\|\vec{p}\|^2 - p_n^2}$

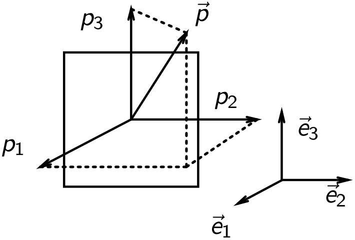
Cauchy stress tensor



Theorem of Cauchy :

$$\exists! \quad \text{tensor} \quad \boldsymbol{\sigma} \quad \text{such that} \quad : \quad \vec{p} = \boldsymbol{\sigma} \cdot \vec{n}$$

Cauchy stress matrix

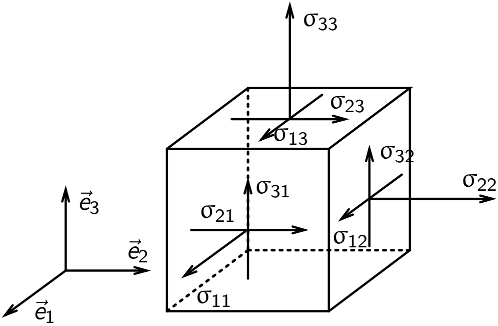


$$\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} \quad \longrightarrow \quad \tilde{\vec{e}}^T \tilde{p} = \tilde{\vec{e}}^T \underline{\sigma}\ \tilde{\vec{e}} \cdot \tilde{\vec{e}}^T \tilde{n} = \tilde{\vec{e}}^T \underline{\sigma}\ \tilde{n}$$

$$\vec{n} = \vec{e}_1 \quad \longrightarrow$$

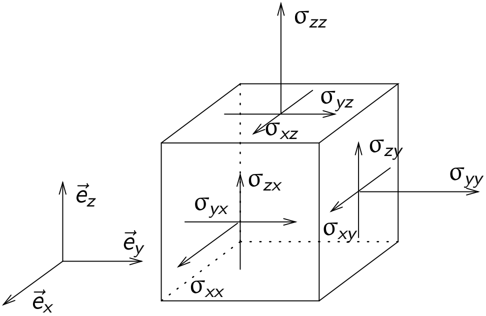
$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}$$

Stress cube



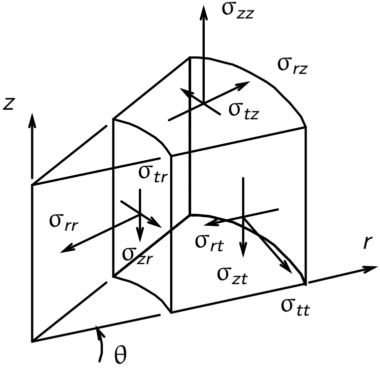
$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Cartesian components



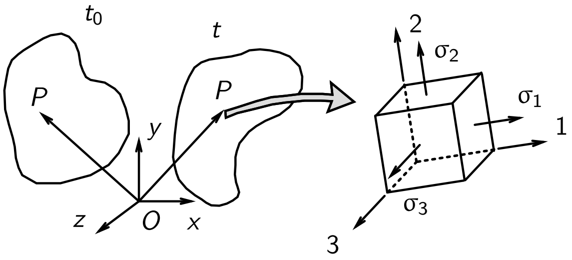
$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Cylindrical components



$$\underline{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{rt} & \sigma_{rz} \\ \sigma_{tr} & \sigma_{tt} & \sigma_{tz} \\ \sigma_{zr} & \sigma_{zt} & \sigma_{zz} \end{bmatrix}$$

Principal stresses and directions



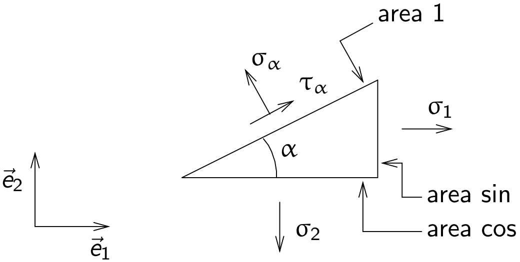
spectral form

$$\left.\begin{aligned} \boldsymbol{\sigma} \cdot \vec{n}_1 &= \sigma_1 \vec{n}_1 \\ \boldsymbol{\sigma} \cdot \vec{n}_2 &= \sigma_2 \vec{n}_2 \\ \boldsymbol{\sigma} \cdot \vec{n}_3 &= \sigma_3 \vec{n}_3 \end{aligned}\right\} \rightarrow \boldsymbol{\sigma} = \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3$$

principal stress matrix

$$\underline{\sigma}_P = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Stress transformation



$$\boldsymbol{\sigma} = \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2$$

$$\vec{n} = -\sin(\alpha)\vec{e}_1 + \cos(\alpha)\vec{e}_2$$

$$\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} = -\sigma_1 \sin(\alpha)\vec{e}_1 + \sigma_2 \cos(\alpha)\vec{e}_2$$

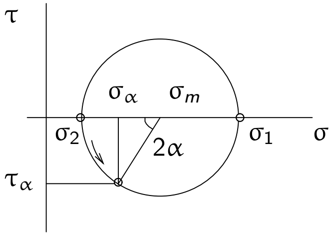
$$\sigma_\alpha = \sigma_1 \sin^2(\alpha) + \sigma_2 \cos^2(\alpha)$$

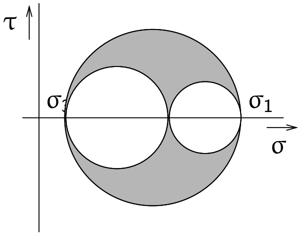
$$\tau_\alpha = (\sigma_2 - \sigma_1) \sin(\alpha) \cos(\alpha)$$

Mohr's circles of stress

$$\sigma_\alpha = \sigma_1 \sin^2(\alpha) + \sigma_2 \cos^2(\alpha) = \sigma_1(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\alpha)) + \sigma_2(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\alpha))$$
$$= \tfrac{1}{2}(\sigma_1 + \sigma_2) - \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos(2\alpha) \quad \rightarrow$$
$$(1) \qquad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2 \cos^2(2\alpha)$$
$$\tau_\alpha = -\cos(\alpha)\sin(\alpha)\sigma_1 + \cos(\alpha)\sin(\alpha)\sigma_2 = \tfrac{1}{2}(\sigma_2 - \sigma_1)\sin(2\alpha) \quad \rightarrow$$
$$(2) \qquad \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_2 - \sigma_1)\right\}^2 \sin^2(2\alpha)$$

$$(1) + (2) \quad \rightarrow \quad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 + \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2$$





Mohr's circles of stress

inside σ_1, σ_3-circle

$$\{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_3)\}^2 + \tau^2 = \sigma^2 + \tau^2 = \|\vec{p}\|^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n}$$
$$= n_1^2 \alpha^2 + n_2^2 \beta^2 + n_3^2 \alpha^2$$

with $\quad \beta^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right)^2 \leq \alpha^2 = \left(\sigma_1 - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \leq \alpha^2$

outside σ_2, σ_3-circle

$$\{\sigma - \tfrac{1}{2}(\sigma_3 + \sigma_2)\}^2 + \tau^2 = \sigma^2 + \tau^2 = \|\vec{p}\|^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n}$$
$$= n_1^2 \beta^2 + n_2^2 \alpha^2 + n_3^2 \alpha^2$$

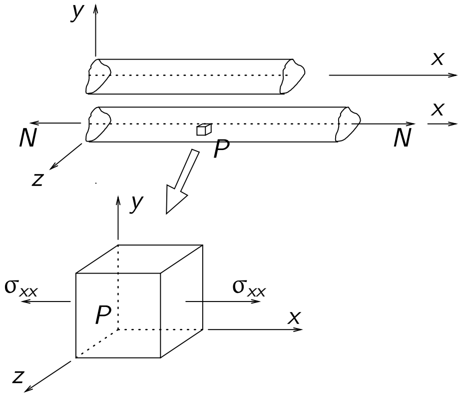
with $\quad \beta^2 = \left(\sigma_1 - \tfrac{1}{2}(\sigma_3 + \sigma_2)\right)^2 \geq \alpha^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_3 + \sigma_2)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \geq \alpha^2$

outside σ_1, σ_2-circle

$$\{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_2)\}^2 + \tau^2 = \sigma^2 + \tau^2 = \|\vec{p}\|^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n}$$
$$= n_1^2 \alpha^2 + n_2^2 \alpha^2 + n_3^2 \beta^2$$

with $\quad \beta^2 = \left(\sigma_3 - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right)^2 \geq \alpha^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \geq \alpha^2$

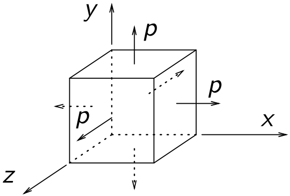
Uni-axial stress



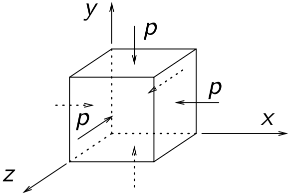
true or Cauchy stress $\sigma = \dfrac{N}{A} = \sigma_{xx} \quad \rightarrow \quad \boldsymbol{\sigma} = \sigma_{xx}\vec{e}_x\vec{e}_x$

engineering stress $\sigma_n = \dfrac{N}{A_0}$

Hydrostatic stress

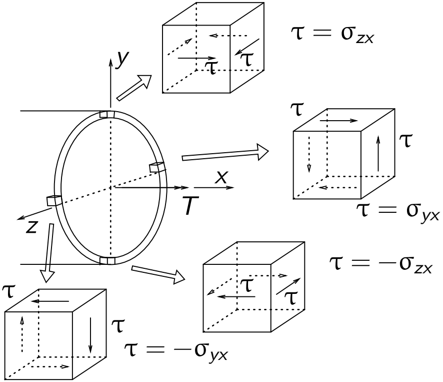


$$\sigma_{xx} = p$$
$$\sigma_{yy} = p$$
$$\sigma_{zz} = p$$



$$\sigma_{xx} = -p$$
$$\sigma_{yy} = -p$$
$$\sigma_{zz} = -p$$

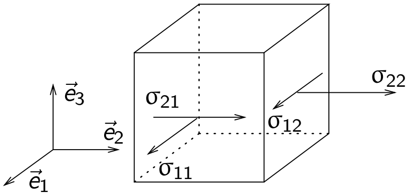
Shear stress



$$\tau = \frac{T}{2\pi R^2 t}$$

$$\boldsymbol{\sigma} = \tau(\vec{e}_i\vec{e}_j + \vec{e}_j\vec{e}_i) \text{ with } i \neq j$$

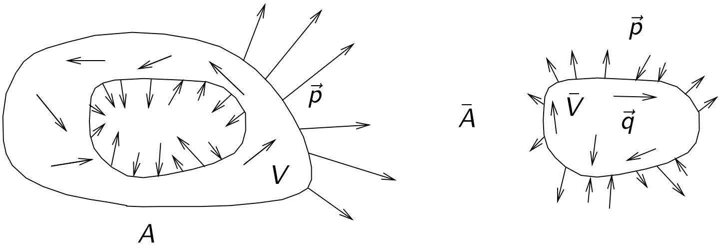
Plane stress



$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \quad \rightarrow \quad \boldsymbol{\sigma} \cdot \vec{e}_3 = \vec{0} \quad \rightarrow$$

relevant stresses : $\quad \sigma_{11}, \sigma_{22}, \sigma_{12}$

Resulting force on arbitrary material volume

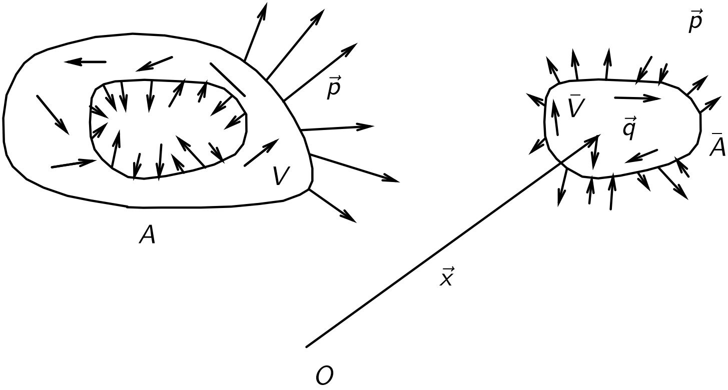


$$\vec{K} = \int_{\bar{V}} \rho\vec{q}\, dV + \int_{\bar{A}} \vec{p}\, dA = \int_{\bar{V}} \rho\vec{q}\, dV + \int_{\bar{A}} \vec{n}\cdot\boldsymbol{\sigma}^T\, dA$$

Gauss theorem : $\int_{\bar{A}} \vec{n}\cdot(\,)\, dA = \int_{\bar{V}} \vec{\nabla}\cdot(\,)\, dV \quad \rightarrow$

$$\vec{K} = \int_{\bar{V}} [\rho\vec{q} + \vec{\nabla}\cdot\boldsymbol{\sigma}^T]\, dV$$

Resulting moment on arbitrary material volume



$$\vec{M}_O = \int_{\bar{V}} \vec{x} * \rho \vec{q} \, dV + \int_{\bar{A}} \vec{x} * \vec{p} \, dA$$

Resulting moment on total body

$$
\begin{aligned}
\vec{M}_O &= \int\limits_V \vec{x} * \rho\vec{q}\, dV + \int\limits_A \vec{x} * \vec{p}\, dA \\
&= \int\limits_V (\vec{x}_R + \vec{r}) * \rho\vec{q}\, dV + \int\limits_A (\vec{x}_R + \vec{r}) * \vec{p}\, dA \\
&= \vec{x}_R * \int\limits_V \rho\vec{q}\, dV + \vec{x}_R * \int\limits_A \vec{p}\, dA + \int\limits_V \vec{r} * \rho\vec{q}\, dV + \int\limits_A \vec{r} * \vec{p}\, dA \\
&= \vec{x}_R * \vec{K} + \vec{M}_R \\
&= \vec{x}_M * \vec{K} + \vec{M}_M \quad \rightarrow
\end{aligned}
$$

$$
\vec{M}_R = (\vec{x}_M - \vec{x}_R) * \vec{K} + \vec{M}_M = \vec{r}_M * \vec{K} + \vec{M}_M
$$

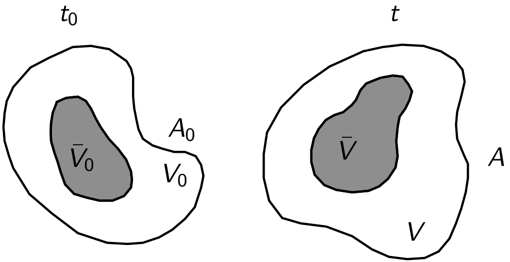
BALANCE LAWS

back to index

Balance or conservation laws

- mass
- momentum
- moment of momentum
- energy

Balance of mass

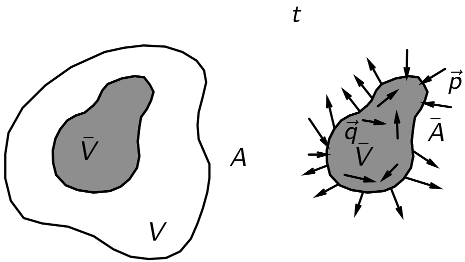


$$\int_{\bar{V}} \rho \, dV = \int_{\bar{V}_0} \rho_0 \, dV_0 \quad \forall \ \bar{V} \quad \rightarrow \quad \int_{\bar{V}_0} (\rho J - \rho_0) \, dV_0 = 0 \quad \forall \ \bar{V}_0 \quad \rightarrow$$

$$\rho J = \rho_0 \qquad \forall \quad \vec{x} \in V(t)$$

$$dM = dM_0 \quad \rightarrow \quad \rho dV = \rho_0 dV_0 \quad \rightarrow \quad \rho J = \rho_0 \quad \rightarrow \quad \dot{\rho} J + \rho \dot{J} = 0$$

Balance of momentum : global



$$\vec{K} = \frac{D\vec{i}}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \rho\vec{v}\,dV = \frac{D}{Dt}\int_{\bar{V}_0} \rho\vec{v}J\,dV_0 = \int_{\bar{V}_0} \frac{D}{Dt}\left(\rho\vec{v}J\right)\,dV_0 \qquad \forall \quad \bar{V}_0$$

$$= \int_{\bar{V}_0} \left(\dot{\rho}\vec{v}J + \rho\dot{\vec{v}}J + \rho\vec{v}\dot{J}\right)\,dV_0 \qquad \forall \quad \bar{V}_0$$

$$\text{mass balance} \quad : \quad \dot{\rho}J + \rho\dot{J} = 0 \quad \rightarrow$$

$$= \int_{\bar{V}_0} \rho\dot{\vec{v}}J\,dV_0 = \int_{\bar{V}} \rho\dot{\vec{v}}\,dV \qquad \forall \quad \bar{V}$$

Balance of momentum : local

$$\int_{\bar{V}} \left(\rho \vec{q} + \vec{\nabla} \cdot \boldsymbol{\sigma}^T \right) \, dV = \int_{\bar{V}} \rho \dot{\vec{v}} \, dV \qquad \forall \quad \bar{V} \quad \rightarrow$$

$$\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \rho \dot{\vec{v}} = \rho \frac{\delta \vec{v}}{\delta t} + \rho \vec{v} \cdot \left(\vec{\nabla} \vec{v} \right) \qquad \forall \quad \vec{x} \in V(t)$$

stationary $\left(\frac{\delta \vec{v}}{\delta t} = 0 \right)$ $\qquad$ $\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \rho \vec{v} \cdot \left(\vec{\nabla} \vec{v} \right)$

static : equilibrium equation $\qquad$ $\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \vec{0}$

Equilibrium equations : Cartesian components

$$\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \vec{0}$$
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^c$$

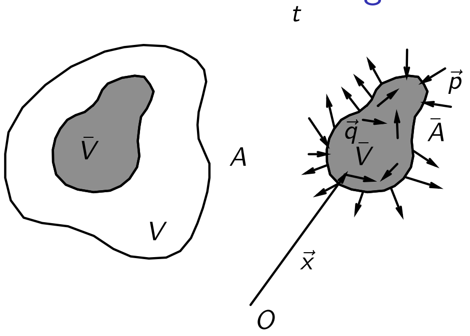
$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x = 0$$
$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y = 0$$
$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z = 0$$

Equilibrium equations : cylindrical components

$$\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \vec{0}$$
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^c$$

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = 0$$
$$\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t = 0$$
$$\sigma_{zr,r} + \frac{1}{r}\sigma_{zt,t} + \frac{1}{r}\sigma_{zr} + \sigma_{zz,z} + \rho q_z = 0$$

Balance of moment of momentum : global



$$\vec{M}_O = \frac{D\vec{L}_O}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \vec{x} * \rho\vec{v}\, dV = \frac{D}{Dt}\int_{\bar{V}_0} \vec{x} * \rho\vec{v}J\, dV_0 = \int_{\bar{V}_0} \frac{D}{Dt}\left(\vec{x} * \rho\vec{v}J\right)\, dV_0$$

$$= \int_{\bar{V}_0} \left(\dot{\vec{x}} * \rho\vec{v}J + \vec{x} * \dot{\rho}\vec{v}J + \vec{x} * \rho\dot{\vec{v}}J + \vec{x} * \rho\vec{v}\dot{J}\right)\, dV_0 \qquad \forall \quad \bar{V}_0$$

$$\text{mass balance} \quad : \quad \dot{\rho}J + \rho\dot{J} = 0$$

$$= \int_{\bar{V}_0} \vec{x} * \rho\dot{\vec{v}}J\, dV_0 = \int_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \qquad \forall \quad \bar{V}$$

Balance of moment of momentum : local

$$\int_{\bar{V}} \vec{x} * \rho \vec{q} \, dV + \int_{\bar{A}} \vec{x} * \vec{p} \, dA = \int_{\bar{V}} \vec{x} * \rho \dot{\vec{v}} \, dV \qquad \forall \quad \bar{V}$$

Transformation of surface integral with $\vec{x} * \vec{p} = {}^3\boldsymbol{\epsilon} : (\vec{x}\,\vec{p})$

$$\begin{aligned}
\int_{\bar{A}} \vec{x} * \vec{p} \, dA &= \int_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x}\,\vec{p}) \, dA = \int_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x}\,\boldsymbol{\sigma} \cdot \vec{n}) \, dA = \int_{\bar{A}} \vec{n} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x}\boldsymbol{\sigma})\}^c \, dA \\
&= \int_{\bar{V}} \vec{\nabla} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x}\boldsymbol{\sigma})\}^c \, dV \\
&= \int_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} \cdot (\vec{\nabla} \cdot \vec{x}) : {}^3\boldsymbol{\epsilon}^c \right] dV \\
&= \int_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} : {}^3\boldsymbol{\epsilon}^c \right] dV \\
&= \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c + \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) \, dV
\end{aligned}$$

Balance of moment of momentum : local

$$\int_{\bar{V}} \vec{x} * \rho \vec{q} \, dV + \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c \, dV + \int_{\bar{V}} \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) \, dV = \int_{\bar{V}} \vec{x} * \rho \dot{\vec{v}} \, dV \quad \forall \quad \bar{V} \quad \rightarrow$$

$$\int_{\bar{V}} \vec{x} * \left[\rho \vec{q} + (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) - \rho \dot{\vec{v}} \right] \, dV + \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c \, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow$$

$$\int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c \, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow \quad {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c = \vec{0} \quad \forall \quad \vec{x} \in \bar{V}$$

$$\epsilon_{ijk} = -1|0|1 \quad \rightarrow \quad \begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow$$

$$\boldsymbol{\sigma}^c = \boldsymbol{\sigma} \qquad\qquad \forall \quad \vec{x} \in V(t)$$

Cartesian and cylindrical components

$$\underline{\sigma} = \underline{\sigma}^T \qquad \rightarrow$$

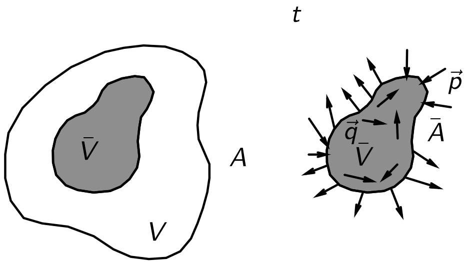
$$\begin{array}{llcccccc} \text{Cartesian} & : & \sigma_{xy} = \sigma_{yx} & ; & \sigma_{yz} = \sigma_{zy} & ; & \sigma_{zx} = \sigma_{xz} \\ \text{cylindrical} & : & \sigma_{rt} = \sigma_{tr} & ; & \sigma_{tz} = \sigma_{zt} & ; & \sigma_{zr} = \sigma_{rz} \end{array}$$

Balance of energy

$$\frac{D}{Dt}\left(U_e + U_t\right) = \frac{D}{Dt}\left(U_k + U_i\right)$$

U_e	:	mechanical energy
U_t	:	thermal energy
U_k	:	kinetic energy
U_i	:	internal energy

Mechanical energy



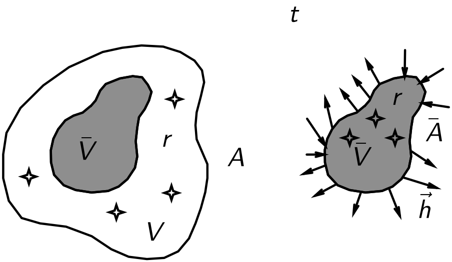
$$\dot{U}_e = \int_{\bar{V}} \rho\vec{q}\cdot\vec{v}\,dV + \int_{\bar{A}} \vec{p}\cdot\vec{v}\,dA = \int_{\bar{V}} \{\rho\vec{q}\cdot\vec{v} + \vec{\nabla}\cdot(\boldsymbol{\sigma}^c\cdot\vec{v})\}\,dV$$

$$\vec{\nabla}\cdot(\boldsymbol{\sigma}^c\cdot\vec{v}) = (\vec{\nabla}\cdot\boldsymbol{\sigma}^c)\cdot\vec{v} + \boldsymbol{\sigma} : (\vec{\nabla}\vec{v})$$

$$= \rho\dot{\vec{v}}\cdot\vec{v} - \rho\vec{q}\cdot\vec{v} + \boldsymbol{\sigma} : \mathbf{D} + \boldsymbol{\sigma} : \boldsymbol{\Omega}$$

$$= \int_{\bar{V}} (\rho\dot{\vec{v}}\cdot\vec{v} + \boldsymbol{\sigma} : \mathbf{D})\,dV$$

Thermal energy



heat flux density $$\vec{h} = \lim_{\Delta A \longrightarrow 0} \frac{\vec{H}}{\Delta A} \qquad [\mathrm{J\ m^{-2}}]$$

$$\dot{U}_t = \int_{\bar{V}} \rho r \, dV - \int_{\bar{A}} \vec{n} \cdot \vec{h} \, dA = \int_{\bar{V}} (\rho r - \vec{\nabla} \cdot \vec{h}) \, dV$$

Kinetic energy

$$U_k(t) = \int_{\bar{V}} \tfrac{1}{2}\,\rho\,\vec{v}\cdot\vec{v}\,dV$$

$$\dot{U}_k = \frac{D}{Dt}\int_{\bar{V}} \tfrac{1}{2}\,\rho\,\vec{v}\cdot\vec{v}\,dV = \frac{D}{Dt}\int_{\bar{V}_0} \tfrac{1}{2}\,\rho\,\vec{v}\cdot\vec{v}J\,dV_0$$

$$= \tfrac{1}{2}\int_{\bar{V}_0} \left\{\dot{\rho}\,\vec{v}\cdot\vec{v}J + 2\rho\,\dot{\vec{v}}\cdot\vec{v}J + \rho\,\vec{v}\cdot\vec{v}\dot{J}\right\}\,dV_0$$

$$= \int_{\bar{V}_0} \rho\,\dot{\vec{v}}\cdot\vec{v}J\,dV_0 = \int_{\bar{V}} \rho\,\dot{\vec{v}}\cdot\vec{v}\,dV$$

Internal energy

$$U_i(t) = \int\limits_{\bar{V}} \rho\phi \, dV$$

$$\dot{U}_i = \frac{D}{Dt} \int\limits_{\bar{V}} \rho\phi \, dV = \frac{D}{Dt} \int\limits_{\bar{V}_0} \rho\phi J \, dV_0$$

$$= \int\limits_{\bar{V}_0} \left\{ \dot{\rho}\phi J + \rho\dot{\phi} J + \rho\phi\dot{J} \right\} \, dV_0$$

$$= \int\limits_{\bar{V}} \rho\dot{\phi} \, dV$$

Energy balance

$$\dot{U}_e + \dot{U}_t = \dot{U}_k + \dot{U}_i$$

$$\int\limits_{\bar{V}} (\rho \dot{\vec{v}} \cdot \vec{v} + \boldsymbol{\sigma} : \mathbf{D} + \rho r - \vec{\nabla} \cdot \vec{h})\, dV = \int\limits_{\bar{V}} (\rho \dot{\vec{v}} \cdot \vec{v} + \rho \dot{\phi})\, dV \qquad \forall \quad \bar{V}$$

$$\int\limits_{\bar{V}} \rho \dot{\phi}\, dV = \int\limits_{\bar{V}} (\boldsymbol{\sigma} : \mathbf{D} + \rho r - \vec{\nabla} \cdot \vec{h})\, dV \qquad \forall \quad \bar{V}$$

Energy equation

$$\left.\begin{array}{ll}\rho\dot{\phi} = \boldsymbol{\sigma} : \mathbf{D} + \rho r - \vec{\nabla} \cdot \vec{h} & \forall \quad \vec{x} \in V(t) \\ \dot{\phi} = C_p \dot{T} \quad (C_p : \text{ specific heat}) & \end{array}\right\} \quad \rightarrow$$

$$\left.\begin{array}{ll}\rho C_p \dot{T} = \boldsymbol{\sigma} : \mathbf{D} + \rho r - \vec{\nabla} \cdot \vec{h} & \forall \quad \vec{x} \in V(t) \\ \vec{h} = -k \vec{\nabla} T \quad (k : \text{ thermal conductivity}) & \end{array}\right\} \quad \rightarrow$$

$$\rho C_p \dot{T} - k \nabla^2 T = \boldsymbol{\sigma} : \mathbf{D} + \rho r \qquad \forall \quad \vec{x} \in V(t)$$

Mechanical power for three-dimensional deformation

$$\dot{W} = \boldsymbol{\sigma} : \mathbf{D} \qquad \boldsymbol{\sigma} = \text{Cauchy stress tensor}$$

$$\begin{aligned} \dot{W}_0 &= [J\boldsymbol{\sigma}] : \mathbf{D} \\ &= \boldsymbol{\kappa} : \mathbf{D} \qquad \boldsymbol{\kappa} = \text{Kirchhoff stress tensor} \end{aligned}$$

$$\begin{aligned} \dot{W}_0 &= J\boldsymbol{\sigma} : \mathbf{D} = J\boldsymbol{\sigma} : \tfrac{1}{2}\left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1} + (\dot{\mathbf{F}} \cdot \mathbf{F}^{-1})^c\right) = \\ &= J\boldsymbol{\sigma} : \left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1}\right) = J\left(\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}\right) : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{F}} = \mathbf{S} : \dot{\mathbf{U}} \\ &= \mathbf{S} : \dot{\boldsymbol{\mathcal{E}}} \qquad \mathbf{S} = \text{1st-Piola-Kirchhoff stress tensor} \end{aligned}$$

$$\begin{aligned} \dot{W}_0 &= J\boldsymbol{\sigma} : \mathbf{D} = J\boldsymbol{\sigma} : \left(\mathbf{F}^{-c} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{-1}\right) = J\left(\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-c}\right) : \dot{\mathbf{E}} \\ &= \mathbf{P} : \dot{\mathbf{E}} \qquad \mathbf{P} = \text{2nd-Piola-Kirchhoff stress tensor} \end{aligned}$$

Planar deformation

Cartesian components

$$
\begin{aligned}
&\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = 0 \\
&\sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = 0 \\
&\sigma_{xy} = \sigma_{yx}
\end{aligned}
$$

cylindrical components

$$
\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \rho q_t = 0 \\
&\\
&\sigma_{rt} = \sigma_{tr}
\end{aligned}
$$

Axisymmetric deformation

$$
\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{2}{r}(\sigma_{tr}) + \sigma_{tz,z} + \rho q_t = 0 \qquad\qquad (\text{if } u_t \neq 0) \\
&\sigma_{zr,r} + \frac{1}{r}\sigma_{zr} + \sigma_{zz,z} + \rho q_z = 0 \\
&\sigma_{rt} = \sigma_{tr} \quad ; \quad \sigma_{tz} = \sigma_{zt} \qquad\qquad (\text{if } u_t \neq 0) \\
&\sigma_{zr} = \sigma_{rz}
\end{aligned}
$$

planar

$$
\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{2}{r}(\sigma_{tr}) + \rho q_t = 0 \qquad\qquad (\text{if } u_t \neq 0) \\
&\sigma_{rt} = \sigma_{tr} \qquad\qquad (\text{if } u_t \neq 0)
\end{aligned}
$$

THREE-DIMENSIONAL MATERIAL MODELS

back to index

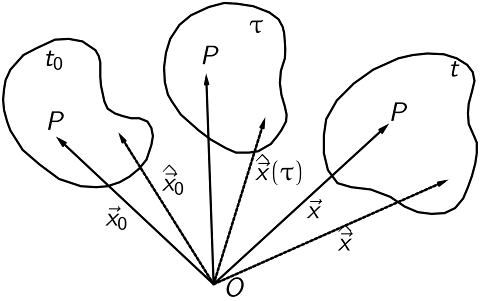
Equations and unknowns

mass	$\rho J = \rho_0$
momentum	$\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \rho \dot{\vec{v}}$
moment of momentum	$\boldsymbol{\sigma}^c = \boldsymbol{\sigma}$

density	ρ
position vector	$\vec{x}$
Cauchy stress tensor	$\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \mathbf{N}(\vec{x})$$

General constitutive equation

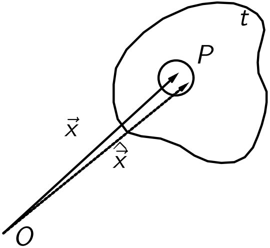


$$\boldsymbol{\sigma}(\vec{x}, t) = \mathbf{N}\{\hat{\vec{x}}, \tau \mid \forall\ \hat{\vec{x}} \in V;\ \forall\ \tau \leq t\}$$

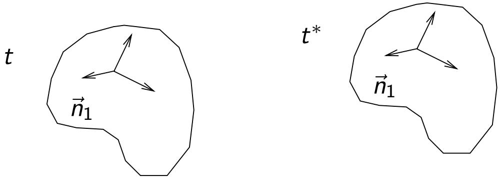
Locality

$$\left.\begin{array}{l} \boldsymbol{\sigma}(\vec{x},t) = \mathbf{N}\{\hat{\vec{x}},\tau \mid \forall\ \hat{\vec{x}} \in V\ ;\ \forall\ \tau \leq t\} \\ \hat{\vec{x}} = \vec{x} + d\vec{x} = \vec{x} + \mathbf{F}(\vec{x}) \cdot d\vec{x}_0 \end{array}\right\} \quad \rightarrow$$

$$\boldsymbol{\sigma}(\vec{x},t) = \mathbf{N}(\vec{x}, \mathbf{F}(\vec{x},\tau), \tau \mid \forall\ \tau \leq t)$$

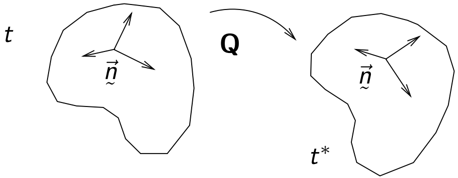


Rigid body translation



$$\boldsymbol{\sigma}(\vec{x},t) = \mathbf{N}(\mathbf{F}(\vec{x},\tau),\tau \mid \forall\ \tau \leq t)$$

Rigid body rotation



$$\vec{n}_1^* = \mathbf{Q} \cdot \vec{n}_1$$
$$\vec{n}_2^* = \mathbf{Q} \cdot \vec{n}_2$$
$$\vec{n}_3^* = \mathbf{Q} \cdot \vec{n}_3$$

$$\begin{aligned}
\boldsymbol{\sigma} &= \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3 \\
\boldsymbol{\sigma}^* &= \sigma_1 \vec{n}_1^* \vec{n}_1^* + \sigma_2 \vec{n}_2^* \vec{n}_2^* + \sigma_3 \vec{n}_3^* \vec{n}_3^* \\
&= \sigma_1 \mathbf{Q} \cdot \vec{n}_1 \vec{n}_1 \cdot \mathbf{Q}^c + \sigma_2 \mathbf{Q} \cdot \vec{n}_2 \vec{n}_2 \cdot \mathbf{Q}^c + \sigma_3 \mathbf{Q} \cdot \vec{n}_3 \vec{n}_3 \cdot \mathbf{Q}^c \\
&= \mathbf{Q} \cdot [\sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3] \cdot \mathbf{Q}^c = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^c
\end{aligned}$$

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} \quad \rightarrow \quad \mathbf{F}^* = \mathbf{R}^* \cdot \mathbf{U} = \mathbf{Q} \cdot \mathbf{R} \cdot \mathbf{U} \quad \rightarrow \quad \mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F}$$

objectivity requirement

$$\mathbf{Q}(t) \cdot \mathbf{N}\left(\mathbf{F}(\tau) \mid \forall\ \tau \leq t\right) \cdot \mathbf{Q}^c(t) = \mathbf{N}\left(\mathbf{Q} \cdot \mathbf{F}(\tau) \mid \forall\ \tau \leq t\right) \qquad \forall \quad \mathbf{Q}$$

Example

$$\sigma = C\mathbf{E} = C\frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) = C\frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right)$$

$$\boldsymbol{\sigma}^* = \mathbf{Q} \cdot \sigma \cdot \mathbf{Q}^T$$

$$\mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F}$$
$$\mathbf{E}^* = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}\right) = \mathbf{E}$$

$$\boldsymbol{\sigma}^* = C\mathbf{E}$$

NOT OBJECTIVE

Example

$$\sigma = C\mathbf{A} = C\frac{1}{2}\left(\mathbf{B} - \mathbf{I}\right) = C\frac{1}{2}\left(\mathbf{F}\cdot\mathbf{F}^T - \mathbf{I}\right)$$

$$\boldsymbol{\sigma}^* = \mathbf{Q}\cdot\sigma\cdot\mathbf{Q}^T$$

$$\mathbf{F}^* = \mathbf{Q}\cdot\mathbf{F}$$

$$\mathbf{A}^* = \frac{1}{2}\left(\mathbf{Q}\cdot\mathbf{F}\cdot\mathbf{F}^T\cdot\mathbf{Q}^T - \mathbf{I}\right) = \frac{1}{2}\mathbf{Q}\cdot\left(\mathbf{F}\cdot\mathbf{F}^T - \mathbf{I}\right)\cdot\mathbf{Q}^T = \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^c$$

$$\boldsymbol{\sigma}^* = C\mathbf{A}^*$$

OBJECTIVE

Example

$$\sigma = -p\mathbf{I} + 2\eta\mathbf{D}$$
$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^c) \qquad \text{with} \qquad \mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$$

$$\sigma^* = \mathbf{Q} \cdot \sigma \cdot \mathbf{Q}^T$$

$$\mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F} \quad ; \quad {\mathbf{F}^*}^{-1} = \mathbf{F}^{-1} \cdot \mathbf{Q}^c \quad ; \quad \dot{\mathbf{F}}^* = \dot{\mathbf{Q}} \cdot \mathbf{F} + \mathbf{Q} \cdot \dot{\mathbf{F}}$$
$$\mathbf{L}^* = (\dot{\mathbf{Q}} \cdot \mathbf{F} + \mathbf{Q} \cdot \dot{\mathbf{F}}) \cdot \mathbf{F}^{-1} \cdot \mathbf{Q}^c = \dot{\mathbf{Q}} \cdot \mathbf{Q}^c + \mathbf{Q} \cdot \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \cdot \mathbf{Q}^c$$
$$\mathbf{D}^* = \tfrac{1}{2}\left[\dot{\mathbf{Q}} \cdot \mathbf{Q}^c + \mathbf{Q} \cdot \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \cdot \mathbf{Q}^c + \mathbf{Q} \cdot \dot{\mathbf{Q}}^c + \mathbf{Q} \cdot (\dot{\mathbf{F}} \cdot \mathbf{F}^{-1})^c \cdot \mathbf{Q}^c\right]$$
$$\mathbf{Q} \cdot \mathbf{Q}^c = \mathbf{I} \quad \rightarrow \quad \dot{\mathbf{Q}} \cdot \mathbf{Q}^c + \mathbf{Q} \cdot \dot{\mathbf{Q}}^c$$
$$= \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^c$$

$$\sigma^* = -p\mathbf{I} + 2\eta\mathbf{D}^*$$

OBJECTIVE

Special stress tensors

- choose invariant stress tensor
- choose invariant rate of stress tensor

Invariant stress tensor

$$\mathbf{S} = \mathbf{A} \cdot \boldsymbol{\sigma} \cdot \mathbf{A}^c$$

$$\left.\begin{array}{ll} \mathbf{S}^* = \mathbf{A}^* \cdot \boldsymbol{\sigma}^* \cdot \mathbf{A}^{*c} = \mathbf{A}^* \cdot \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^c \cdot \mathbf{A}^{*c} & \\ \text{define} \qquad \mathbf{A}^* = \mathbf{A} \cdot \mathbf{Q}^c & \end{array}\right\} \quad \rightarrow$$

$$\mathbf{S}^* = \mathbf{A} \cdot \mathbf{Q}^c \cdot \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^c \cdot \mathbf{Q} \cdot \mathbf{A}^c = \mathbf{A} \cdot \boldsymbol{\sigma} \cdot \mathbf{A}^c = \mathbf{S}$$

$\mathbf{S}$ = invariant for rigid rotation

Invariant rate of stress tensor

$$\dot{\mathbf{S}} = \dot{\mathbf{A}} \cdot \sigma \cdot \mathbf{A}^c + \mathbf{A} \cdot \dot{\sigma} \cdot \mathbf{A}^c + \mathbf{A} \cdot \sigma \cdot \dot{\mathbf{A}}^c$$

$$\begin{aligned}
\dot{\mathbf{S}}^* &= \dot{\mathbf{A}}^* \cdot \sigma^* \cdot \mathbf{A}^{*c} + \mathbf{A}^* \cdot \dot{\sigma}^* \cdot \mathbf{A}^{*c} + \mathbf{A}^* \cdot \sigma^* \cdot \dot{\mathbf{A}}^{*c} \\
&= (\dot{\mathbf{A}} \cdot \mathbf{Q}^c + \mathbf{A} \cdot \dot{\mathbf{Q}}^c) \cdot \mathbf{Q} \cdot \sigma \cdot \mathbf{Q}^c \cdot \mathbf{Q} \cdot \mathbf{A}^c + \\
&\qquad \mathbf{A} \cdot \mathbf{Q}^c \cdot (\dot{\mathbf{Q}} \cdot \sigma \cdot \mathbf{Q}^c + \mathbf{Q} \cdot \dot{\sigma} \cdot \mathbf{Q}^c + \mathbf{Q} \cdot \sigma \cdot \dot{\mathbf{Q}}^c) \cdot \mathbf{Q} \cdot \mathbf{A}^c + \\
&\qquad \mathbf{A} \cdot \mathbf{Q}^c \cdot \mathbf{Q} \cdot \sigma \cdot \mathbf{Q}^c \cdot (\mathbf{Q} \cdot \dot{\mathbf{A}}^c + \dot{\mathbf{Q}} \cdot \mathbf{A}^c) \\
&= \dot{\mathbf{A}} \cdot \sigma \cdot \mathbf{A}^c + \mathbf{A} \cdot \dot{\mathbf{Q}}^c \cdot \mathbf{Q} \cdot \sigma \cdot \mathbf{A}^c + \mathbf{A} \cdot \mathbf{Q}^c \cdot \dot{\mathbf{Q}} \cdot \sigma \cdot \mathbf{A}^c + \\
&\qquad \mathbf{A} \cdot \dot{\sigma} \cdot \mathbf{A}^c + \mathbf{A} \cdot \sigma \cdot \dot{\mathbf{Q}}^c \cdot \mathbf{Q} \cdot \mathbf{A}^c + \mathbf{A} \cdot \sigma \cdot \dot{\mathbf{A}}^c + \\
&\qquad \mathbf{A} \cdot \sigma \cdot \mathbf{Q}^c \cdot \dot{\mathbf{Q}} \cdot \mathbf{A}^c \\
&= \dot{\mathbf{A}} \cdot \sigma \cdot \mathbf{A}^c + \mathbf{A} \cdot \dot{\sigma} \cdot \mathbf{A}^c + \mathbf{A} \cdot \sigma \cdot \mathbf{A}^c + \\
&\qquad \mathbf{A} \cdot \dot{\mathbf{Q}}^c \cdot \mathbf{Q} \cdot \sigma \cdot \mathbf{A}^c + \mathbf{A} \cdot \mathbf{Q}^c \cdot \dot{\mathbf{Q}} \cdot \sigma \cdot \mathbf{A}^c + \\
&\qquad \mathbf{A} \cdot \sigma \cdot \dot{\mathbf{Q}}^c \cdot \mathbf{Q} \cdot \mathbf{A}^c + \mathbf{A} \cdot \sigma \cdot \mathbf{Q}^c \cdot \dot{\mathbf{Q}} \cdot \mathbf{A}^c \\
&= \dot{\mathbf{S}} \qquad \rightarrow \qquad \dot{\mathbf{S}} = \text{invariant for rigid rotation}
\end{aligned}$$

Rate of Cauchy stress tensor

$$
\begin{aligned}
\mathbf{S} &= \mathbf{A} \cdot \sigma \cdot \mathbf{A}^c \\
\dot{\mathbf{S}} &= \dot{\mathbf{A}} \cdot \sigma \cdot \mathbf{A}^c + \mathbf{A} \cdot \dot{\sigma} \cdot \mathbf{A}^c + \mathbf{A} \cdot \sigma \cdot \dot{\mathbf{A}}^c \\
&= \mathbf{A} \cdot \left\{ (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}}) \cdot \sigma + \sigma \cdot (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}})^c + \dot{\sigma} \right\} \cdot \mathbf{A}^c = \mathbf{A} \cdot \overset{\odot}{\sigma} \cdot \mathbf{A}^c
\end{aligned}
$$

$$
\overset{\odot}{\sigma} = \dot{\sigma} + (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}}) \cdot \sigma + \sigma \cdot (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}})^c
$$

$$
\begin{aligned}
\overset{\odot}{\sigma}{}^* &= \dot{\sigma}^* + (\mathbf{A}^{-1^*} \cdot \dot{\mathbf{A}}^*) \cdot \sigma^* + \sigma^* \cdot (\mathbf{A}^{-1^*} \cdot \dot{\mathbf{A}}^*)^c \\
&\qquad \mathbf{A}^* = \mathbf{A} \cdot \mathbf{Q}^c \quad \rightarrow \quad \mathbf{A}^{*-1} = \mathbf{A}^{-1^*} = \mathbf{Q} \cdot \mathbf{A}^{-1} \\
&\qquad \dot{\mathbf{A}}^* = \dot{\mathbf{A}} \cdot \mathbf{Q}^c + \mathbf{A} \cdot \dot{\mathbf{Q}}^c \\
&\qquad \mathbf{A}^{-1^*} \cdot \dot{\mathbf{A}}^* = \mathbf{Q} \cdot \mathbf{A}^{-1} \cdot \dot{\mathbf{A}} \cdot \mathbf{Q}^c + \mathbf{Q} \cdot \dot{\mathbf{Q}}^c \\
&= \dot{\sigma}^* + \mathbf{Q} \cdot \mathbf{A}^{-1} \cdot \dot{\mathbf{A}} \cdot \mathbf{Q}^c \cdot \sigma^* + \mathbf{Q} \cdot \dot{\mathbf{Q}}^c \cdot \sigma^* + \\
&\qquad \sigma^* \cdot \mathbf{Q} \cdot (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}})^c \cdot \mathbf{Q}^c + \sigma^* \cdot \dot{\mathbf{Q}} \cdot \mathbf{Q}^c \\
&= \mathbf{Q} \cdot \overset{\odot}{\sigma} \cdot \mathbf{Q}^c \qquad \rightarrow \qquad \overset{\odot}{\sigma} = \text{objective}
\end{aligned}
$$

Objective rates and associated tensors

general tensor	$\mathbf{S} = \sigma_O = \mathbf{A} \cdot \sigma \cdot \mathbf{A}^c$
	$\dot{\mathbf{S}} = \dot{\sigma}_O = \mathbf{A} \cdot \overset{\odot}{\sigma}_O \cdot \mathbf{A}^c$
general rate	$\overset{\odot}{\sigma}_O = \dot{\sigma} + (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}}) \cdot \sigma + \sigma \cdot (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}})^c$
Truesdell tensor	$\sigma_T = \mathbf{F}^{-1} \cdot \sigma \cdot \mathbf{F}^{-c}$
	$\dot{\sigma}_T = \mathbf{F}^{-1} \cdot \overset{\odot}{\sigma}_T \cdot \mathbf{F}^{-c}$
Truesdell rate	$\overset{\odot}{\sigma}_T = \overset{\triangledown}{\sigma} = \dot{\sigma} - \mathbf{L} \cdot \sigma - \sigma \cdot \mathbf{L}^c$
Jaumann tensor	$\sigma_J = \mathbf{Q}^{-1} \cdot \sigma \cdot \mathbf{Q}^{-c} \quad \text{with} \quad \dot{\mathbf{Q}} = \mathbf{\Omega} \cdot \mathbf{Q}$
	$\dot{\sigma}_J = \mathbf{Q}^{-1} \cdot \overset{\odot}{\sigma}_J \cdot \mathbf{Q}^{-c}$
Jaumann rate	$\overset{\odot}{\sigma}_J = \overset{\circ}{\sigma} = \dot{\sigma} - \mathbf{\Omega} \cdot \sigma - \sigma \cdot \mathbf{\Omega}^c$
Cotter-Rivlin tensor	$\sigma_C = \mathbf{F}^c \cdot \sigma \cdot \mathbf{F}$
	$\dot{\sigma}_C = \mathbf{F}^c \cdot \overset{\odot}{\sigma}_C \cdot \mathbf{F}$
Cotter-Rivlin rate	$\overset{\odot}{\sigma}_C = \overset{\triangle}{\sigma} = \dot{\sigma} + \mathbf{L}^c \cdot \sigma + \sigma \cdot \mathbf{L}$
Dienes tensor	$\sigma_D = \mathbf{R}^c \cdot \sigma \cdot \mathbf{R} \quad \text{with} \quad \mathbf{F} = \mathbf{R} \cdot \mathbf{U}$
	$\dot{\sigma}_D = \mathbf{R}^c \cdot \overset{\odot}{\sigma}_D \cdot \mathbf{R}$
Dienes rate	$\overset{\odot}{\sigma}_D = \overset{\diamond}{\sigma} = \dot{\sigma} - (\dot{\mathbf{R}} \cdot \mathbf{R}^c) \cdot \sigma - \sigma \cdot (\dot{\mathbf{R}} \cdot \mathbf{R}^c)^c$

LINEAR ELASTIC MATERIAL

back to index

Linear elastic material

tensor notation	$\boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\varepsilon}$			
index notation	$\sigma_{ij} = C_{ijkl}\,\varepsilon_{lk}$	;	$i, j, k, l \in \{1, 2, 3\}$	
matrix notation	$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon}$			

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

Symmetry

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

specific energy $\quad W = \frac{1}{2} \underset{\approx}{\varepsilon}^T \underline{\underline{C}}\, \underset{\approx}{\varepsilon} \quad \rightarrow$

symmetry $\quad \boxed{\underline{\underline{C}} = \underline{\underline{C}}^T}$

Symmetric stresses

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

Symmetric strains

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}
$$

$$
\boxed{\varepsilon_{ij} = \varepsilon_{ji}}
$$

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & [C_{1121} + C_{1112}] & [C_{1132} + C_{1123}] & [C_{1113} + C_{1131}] \\ C_{2211} & C_{2222} & C_{2233} & [C_{2221} + C_{2212}] & [C_{2232} + C_{2223}] & [C_{2213} + C_{2231}] \\ C_{3311} & C_{3322} & C_{3333} & [C_{3321} + C_{3312}] & [C_{3332} + C_{3323}] & [C_{3313} + C_{3331}] \\ C_{1211} & C_{1222} & C_{1233} & [C_{1221} + C_{1212}] & [C_{1232} + C_{1223}] & [C_{1213} + C_{1231}] \\ C_{2311} & C_{2322} & C_{2333} & [C_{2321} + C_{2312}] & [C_{2332} + C_{2323}] & [C_{2313} + C_{2331}] \\ C_{3111} & C_{3122} & C_{3133} & [C_{3121} + C_{3112}] & [C_{3132} + C_{3123}] & [C_{3113} + C_{3131}] \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}
$$

Symmetric material parameters

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} =
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & [C_{1121}+C_{1112}] & [C_{1132}+C_{1123}] & [C_{1113}+C_{1131}] \\
C_{2211} & C_{2222} & C_{2233} & [C_{2221}+C_{2212}] & [C_{2232}+C_{2223}] & [C_{2213}+C_{2231}] \\
C_{3311} & C_{3322} & C_{3333} & [C_{3321}+C_{3312}] & [C_{3332}+C_{3323}] & [C_{3313}+C_{3331}] \\
C_{1211} & C_{1222} & C_{1233} & [C_{1221}+C_{1212}] & [C_{1232}+C_{1223}] & [C_{1213}+C_{1231}] \\
C_{2311} & C_{2322} & C_{2333} & [C_{2321}+C_{2312}] & [C_{2332}+C_{2323}] & [C_{2313}+C_{2331}] \\
C_{3111} & C_{3122} & C_{3133} & [C_{3121}+C_{3112}] & [C_{3132}+C_{3123}] & [C_{3113}+C_{3131}]
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}
$$

$$\boxed{C_{ijkl} = C_{ijlk}}$$

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} =
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\
C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\
C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\
C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\
C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\
C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}
$$

Shear strain

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\boxed{2\varepsilon_{ij} = \gamma_{ij}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

Material symmetry

monoclinic $\longrightarrow$ orthotropic $\longrightarrow$ quadratic $\longrightarrow$ transversal isotropic $\longrightarrow$ cubic $\longrightarrow$ isotropic

MATERIAL SYMMETRY

back to index

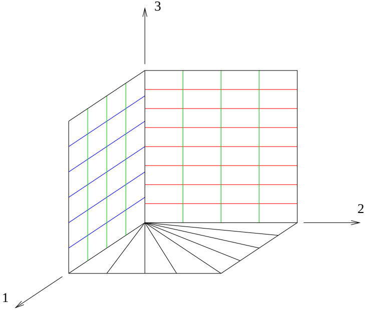
Triclinic : no symmetry

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

$\boxed{\text{21 \quad material parameters}}$

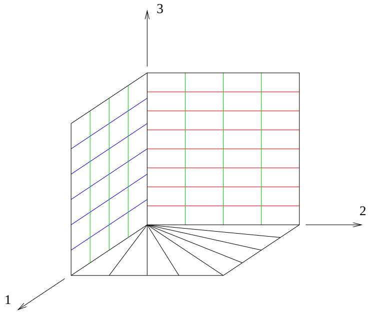
Monoclinic : 1 symmetry plane (here 12)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$



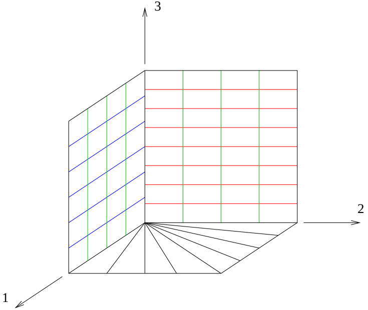
Monoclinic : tensile test

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

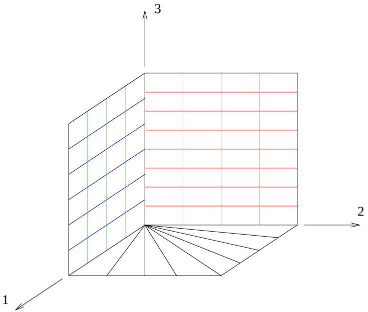


Monoclinic : tensile test

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\
C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\
C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\
C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\
0 & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\
0 & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$



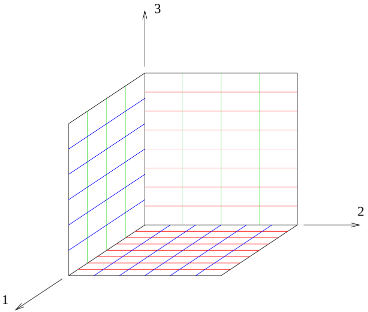
Monoclinic : 1 symmetry plane (here 12)



$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & 0 & 0 \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2332} & C_{2313} \\ 0 & 0 & 0 & 0 & C_{3132} & C_{3113} \end{bmatrix}$$

13	material parameters

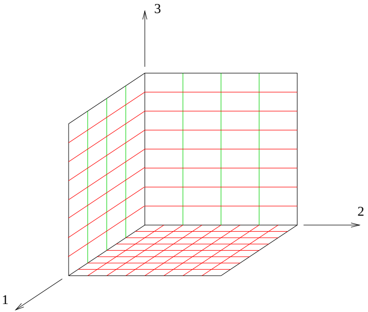
Orthotropic : 3 symmetry planes (12, 23, 31)



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix}$$

$\boxed{9 \quad \text{material parameters}}$

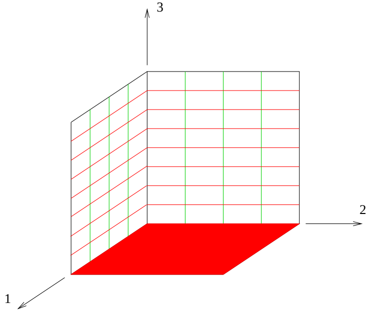
Quadratic : 2 isotropic directions (here 1 and 2)



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$\boxed{6 \quad \text{material parameters}}$

Transversal isotropic : 1 isotropic plane (here 12)



$$\begin{bmatrix} A & Q & R & & & \\ Q & A & R & & & \\ R & R & C & & & \\ & & & K & & \\ & & & & L & \\ & & & & & L \end{bmatrix}$$

Transversal isotropic : shear test in 12-plane

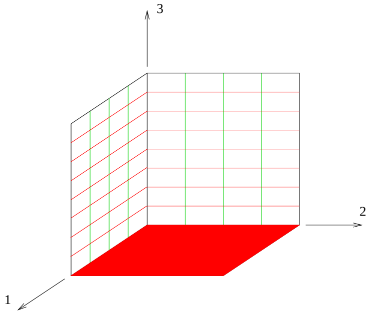
$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \quad \rightarrow \; \det(\underline{\sigma} - \sigma \underline{I}) = 0 \; \rightarrow \; \begin{cases} \sigma_1 = \tau \\ \sigma_2 = -\tau \end{cases}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} \rightarrow \; \det(\underline{\varepsilon} - \varepsilon \underline{I}) = 0 \; \rightarrow \; \begin{cases} \varepsilon_1 = \frac{1}{2}\gamma \\ \varepsilon_2 = -\frac{1}{2}\gamma \end{cases}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} A & Q \\ Q & A \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \rightarrow \quad \begin{array}{l} \sigma_1 = A\varepsilon_1 + Q\varepsilon_2 = \tau = K\gamma \\ \sigma_2 = Q\varepsilon_1 + A\varepsilon_2 = -\tau = -K\gamma \end{array} \quad \rightarrow$$

$$\left. \begin{array}{l} (A - Q)(\varepsilon_1 - \varepsilon_2) = 2K\gamma \\ \\ \varepsilon_1 = \frac{1}{2}\gamma \quad ; \quad \varepsilon_1 = -\frac{1}{2}\gamma \end{array} \right\} \quad \rightarrow \quad \boxed{K = \frac{1}{2}(A - Q)}$$

Transversal isotropic

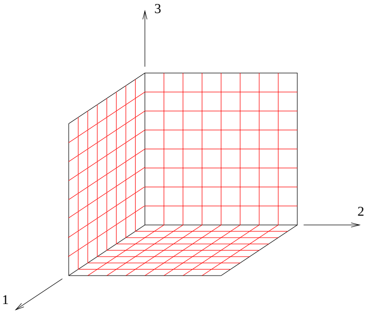


$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$K = \frac{1}{2}(A - Q)$$

$$\boxed{5 \quad \text{material parameters}}$$

Cubic : 3 isotropic directions (here 1, 2 and 3)

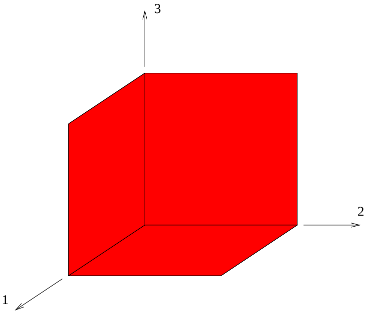


$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$L \neq \frac{1}{2}(A - Q)$$

$\boxed{3 \quad \text{material parameters}}$

Isotropic



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$L = \frac{1}{2}(A - Q)$$

$\boxed{2 \quad \text{material parameters}}$

LINEAR ELASTIC ISOTROPIC MATERIAL

ENGINEERING PARAMETERS

back to index

Tensile test

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon & \varepsilon_d & \varepsilon_d & 0 & 0 & 0 \end{bmatrix} \; ; \; \underset{\approx}{\sigma}^T = \begin{bmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\left.\begin{array}{l} \sigma = A\varepsilon + 2Q\varepsilon_d \\ 0 = Q\varepsilon + (A+Q)\varepsilon_d \rightarrow \varepsilon_d = -\dfrac{Q}{A+Q}\,\varepsilon \end{array}\right\} \rightarrow \sigma = \dfrac{A^2 + AQ - 2Q^2}{A+Q}\,\varepsilon \\ \varepsilon_d = -\nu\varepsilon \qquad ; \qquad \sigma = E\varepsilon \right\} \rightarrow$$

$$A = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \qquad Q = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad L = \frac{E}{2(1+\nu)}$$

Shear test

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \frac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = [\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \gamma\ ] \ ; \ \underset{\approx}{\sigma}^T = [\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \tau\ ]$$

$$\tau = L\gamma = \frac{E}{2(1+\nu)}\gamma = G\gamma$$

Volume change

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} J - 1 \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} &= \frac{1-2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &= \frac{3(1-2\nu)}{E}\,\tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{K}\,\tfrac{1}{3}\mathrm{tr}(\underline{\sigma}) \end{aligned}$$

Isotropic compliance and stiffness matrix

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

$$\text{with} \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}$$

LINEAR ELASTIC ISOTROPIC MATERIAL

TENSORIAL FORM

back to index

Column/matrix notation of Hooke's law

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\text{with} \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}$$

Isotropic stiffness matrix

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \frac{E}{(1+\nu)} \begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$= \frac{E}{(1+\nu)} \left[\begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} - 1 & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} - 1 & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$= \left[\frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \right.$$

$$\left. \frac{E}{(1+\nu)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Isotropic stiffness tensor

$$\boldsymbol{\sigma} = \left[\frac{E\nu}{(1+\nu)(1-2\nu)}\right] \mathbf{I}\,\text{tr}(\boldsymbol{\varepsilon}) + \left[\frac{E}{(1+\nu)}\right] \boldsymbol{\varepsilon}$$

$$= Q\,\text{tr}(\boldsymbol{\varepsilon})\,\mathbf{I} + 2L\boldsymbol{\varepsilon}$$

$$= c_0\,\text{tr}(\boldsymbol{\varepsilon})\,\mathbf{I} + c_1\,\boldsymbol{\varepsilon}$$

$$= \left[c_0\mathbf{I}\mathbf{I} + c_1\,{}^4\mathbf{I}^s\right] : \boldsymbol{\varepsilon} \qquad \text{with} \qquad {}^4\mathbf{I}^s = \tfrac{1}{2}\left({}^4\mathbf{I} + {}^4\mathbf{I}^{rc}\right)$$

$$= {}^4\mathbf{C} : \boldsymbol{\varepsilon}$$

Stiffness and compliance tensor

$$
\begin{aligned}
\boldsymbol{\sigma} &= {}^4\mathbf{C} : \boldsymbol{\varepsilon} \\
&= \left[c_0 \mathbf{I}\mathbf{I} + c_1\, {}^4\mathbf{I}^s \right] : \boldsymbol{\varepsilon} \\
& \quad \text{with} \quad {}^4\mathbf{I}^s = \tfrac{1}{2}\left({}^4\mathbf{I} + {}^4\mathbf{I}^{rc} \right) \\
&= c_0 \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + c_1 \boldsymbol{\varepsilon} \\
&= c_0 \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + c_1 \left\{ \boldsymbol{\varepsilon}^d + \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} \right\} \\
&= (c_0 + \tfrac{1}{3}c_1)\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + c_1 \boldsymbol{\varepsilon}^d \\
&= (3c_0 + c_1)\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + c_1 \boldsymbol{\varepsilon}^d \\
&= (3c_0 + c_1)\boldsymbol{\varepsilon}^h + c_1 \boldsymbol{\varepsilon}^d \\
&= \boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^d \\
&= \frac{1}{3c_0 + c_1}\, \boldsymbol{\sigma}^h + \frac{1}{c_1}\, \boldsymbol{\sigma}^d \\
&= \frac{1}{3c_0 + c_1}\, \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} + \frac{1}{c_1}\, \left\{ \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} \right\} \\
&= -\frac{c_0}{(3c_0 + c_1)c_1}\, \mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} + \frac{1}{c_1}\, \boldsymbol{\sigma} \\
&= \left[-\frac{c_0}{(3c_0 + c_1)c_1}\mathbf{I}\mathbf{I} + \frac{1}{c_1}\, {}^4\mathbf{I}^s \right] : \boldsymbol{\sigma} \\
&= \left[\gamma_0 \mathbf{I}\mathbf{I} + \gamma_1\, {}^4\mathbf{I}^s \right] : \boldsymbol{\sigma} \\
&= {}^4\mathbf{S} : \boldsymbol{\sigma}
\end{aligned}
$$

$$
c_0 = \frac{\nu E}{(1+\nu)(1-2\nu)} = Q \quad ; \quad c_1 = \frac{E}{1+\nu} = 2L
$$

$$
\gamma_0 = -\frac{c_0}{(3c_0 + c_1)c_1} = -\frac{\nu}{E} = q \quad ; \quad \gamma_1 = \frac{1}{c_1} = \frac{1+\nu}{E} = \tfrac{1}{2}l
$$

Stiffness and compliance components

$$\boldsymbol{\sigma} = \left[c_0 \mathbf{II} + c_1\, {}^4\mathbf{I}^s \right] : \boldsymbol{\varepsilon}$$

$$\begin{aligned}
\sigma_{ij} &= \left[c_0 \delta_{ij}\delta_{kl} + c_1 \tfrac{1}{2} \left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \right) \right] \varepsilon_{lk} \\
&= c_0 \delta_{ij} \varepsilon_{kk} + c_1 \varepsilon_{ij} \\
&= c_1 \left(\varepsilon_{ij} + \frac{c_0}{c_1} \delta_{ij} \varepsilon_{kk} \right) \\
&= \frac{E}{1+\nu} \left(\varepsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} \varepsilon_{kk} \right)
\end{aligned}$$

$$\boldsymbol{\varepsilon} = \left[- \frac{c_0}{(3c_0 + c_1)c_1} \mathbf{II} + \frac{1}{c_1}\, {}^4\mathbf{I}^s \right] : \boldsymbol{\sigma}$$

$$\begin{aligned}
\varepsilon_{ij} &= \left[- \frac{c_0}{(3c_0 + c_1)c_1} \delta_{ij}\delta_{kl} + \right. \\
&\qquad \left. \frac{1}{c_1} \tfrac{1}{2} \left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \right) \right] \sigma_{lk} \\
&= - \frac{c_0}{(3c_0 + c_1)c_1} \delta_{ij} \sigma_{kk} + \frac{1}{c_1} \sigma_{ij} \\
&= \frac{1}{c_1} \left(\sigma_{ij} - \frac{c_0}{3c_0 + c_1} \delta_{ij} \sigma_{kk} \right) \\
&= \frac{1+\nu}{E} \left(\sigma_{ij} - \frac{\nu}{1+\nu} \delta_{ij} \sigma_{kk} \right)
\end{aligned}$$

Specific elastic energy

$$W = \tfrac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\sigma} : {}^4\mathbf{S} : \boldsymbol{\sigma} = \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : {}^4\mathbf{S} : (\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d)$$

$$= \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : \left(\gamma_0 \mathbf{I}\mathbf{I} + \gamma_1 {}^4\mathbf{I}^s\right) : (\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d)$$

$$\gamma_0 \mathbf{I}\left[\mathbf{I} : \boldsymbol{\sigma}^h\right] = \gamma_0 \mathbf{I}\left[\mathbf{I} : \mathbf{I}\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\right] = \gamma_0 \mathbf{I}\left[\mathrm{tr}(\boldsymbol{\sigma})\right] = 3\gamma_0 \boldsymbol{\sigma}^h$$

$$\gamma_0 \mathbf{I}\left[\mathbf{I} : \boldsymbol{\sigma}^d\right] = \gamma_0 \mathbf{I}\left[\mathrm{tr}(\boldsymbol{\sigma}^d)\right] = \gamma_0 \mathbf{I}\left[0\right] = 0$$

$$= \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : (3\gamma_0 \boldsymbol{\sigma}^h + \gamma_1 \boldsymbol{\sigma}^h + \gamma_1 \boldsymbol{\sigma}^d)$$

$$\boldsymbol{\sigma}^h : \boldsymbol{\sigma}^h = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} : \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} = \tfrac{1}{9}\mathrm{tr}^2(\boldsymbol{\sigma})(3) = \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma})$$

$$\boldsymbol{\sigma}^h : \boldsymbol{\sigma}^d = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} : [\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}] = \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) - \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) = 0$$

$$= \left[\tfrac{1}{2}(\gamma_0 + \tfrac{1}{3}\gamma_1)\right]\mathrm{tr}^2(\boldsymbol{\sigma}) + \left[\tfrac{1}{2}\gamma_1\right]\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d$$

$$= \left[\tfrac{1}{2}\frac{1-2\nu}{3E}\right]\mathrm{tr}^2(\boldsymbol{\sigma}) + \left[\tfrac{1}{2}\frac{1+\nu}{E}\right]\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{18K}\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d$$

$$= W^h + W^d$$

THERMO-ELASTICITY

back to index

Thermoelasticity

Anisotropic

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_T = {}^4\mathbf{S} : \boldsymbol{\sigma} + \boldsymbol{A}\Delta T \quad \rightarrow$$

$$\underset{\approx}{\varepsilon} = \underset{\approx}{\varepsilon}_m + \underset{\approx}{\varepsilon}_T = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \underline{\underline{A}}\,\Delta T$$

$$\boldsymbol{\sigma} = {}^4\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{A}\Delta T) \quad \rightarrow$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\left(\underset{\approx}{\varepsilon} - \underline{\underline{A}}\,\Delta T\right)$$

Isotropic

$$\boldsymbol{\varepsilon} = {}^4\mathbf{S} : \boldsymbol{\sigma} + \alpha\Delta T\mathbf{I} \quad \rightarrow \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \alpha\,\Delta T\,\underset{\approx}{I}$$

$$\boldsymbol{\sigma} = {}^4\mathbf{C} : (\boldsymbol{\varepsilon} - \alpha\Delta T\mathbf{I}) \quad \rightarrow \quad \underset{\approx}{\sigma} = \underline{\underline{C}}(\underset{\approx}{\varepsilon} - \alpha\,\Delta T\,\underset{\approx}{I})$$

Orthotropic thermo-elasticity

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} A+Q+R \\ Q+B+S \\ R+S+C \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

PLANAR DEFORMATION

back to index

Plane strain

$$\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0 \quad \longrightarrow \quad \sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\varepsilon \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{AB - Q^2} \begin{bmatrix} B & -Q & 0 \\ -Q & A & 0 \\ 0 & 0 & \dfrac{AB - Q^2}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$
$$= \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\varepsilon \underset{\sim}{\sigma}$$

$$\sigma_{33} = \frac{1}{AB^2 - Q^2} \left[(BR - QS)\sigma_{11} + (AS - QR)\sigma_{22} \right]$$

Plane strain

$$\varepsilon_{33} = 0 = r\sigma_{11} + s\sigma_{22} + c\sigma_{33} \quad \rightarrow \quad \sigma_{33} = -\frac{r}{c}\sigma_{11} - \frac{s}{c}\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} - \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \begin{bmatrix} \frac{r}{c} & \frac{s}{c} & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} ac - r^2 & qc - rs & 0 \\ qc - sr & bc - s^2 & 0 \\ 0 & 0 & kc \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\varepsilon \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & 0 \\ -qc + rs & ac - r^2 & 0 \\ 0 & 0 & \frac{\Delta_s}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$$

$$= \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \underline{\underline{C}}_\varepsilon \underset{\sim}{\sigma}$$

$$\sigma_{33} = -\frac{1}{\Delta_s}\left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22}\right]$$

Plane stress

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \quad \longrightarrow \quad \varepsilon_{33} = r\sigma_{11} + s\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{ab - q^2} \begin{bmatrix} b & -q & 0 \\ -q & a & 0 \\ 0 & 0 & \dfrac{ab - q^2}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

$$\varepsilon_{33} = \frac{1}{ab - q^2} \left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

Plane stress

$$\sigma_{33} = 0 = R\varepsilon_{11} + S\varepsilon_{22} + C\varepsilon_{33} \quad \rightarrow \quad \varepsilon_{33} = -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \begin{bmatrix} R \\ S \\ 0 \end{bmatrix} \begin{bmatrix} \frac{R}{C} & \frac{S}{C} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \frac{1}{C} \begin{bmatrix} AC - R^2 & QC - RS & 0 \\ QC - SR & BC - S^2 & 0 \\ 0 & 0 & KC \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{\Delta_c} \begin{bmatrix} BC - S^2 & -QC + RS & 0 \\ -QC + RS & AC - R^2 & 0 \\ 0 & 0 & \frac{\Delta_c}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_c = ABC - AS^2 - BR^2 - CQ^2 + 2QRS$$

$$= \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

Plane strain thermo-elastic

$$\sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22} - \alpha(R+S+C)\,\Delta T \qquad \text{(from } \underline{\underline{C}}\text{)}$$
$$= -\frac{r}{c}\,\sigma_{11} - \frac{s}{c}\,\sigma_{22} - \frac{\alpha}{c}\,\Delta T \qquad \text{(from } \underline{\underline{S}}\text{)}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} A+Q+R \\ B+Q+S \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} A+Q+R \\ B+Q+S \\ 0 \end{bmatrix} \right\}$$
$$= \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 + q_\varepsilon S + a_\varepsilon R \\ 1 + q_\varepsilon R + b_\varepsilon S \\ 0 \end{bmatrix}$$

Plane stress thermo-elastic

$$\varepsilon_{33} = r\sigma_{11} + s\sigma_{22} + \alpha\Delta T \qquad \text{(from } \underline{\underline{S}}\text{)}$$

$$= -\frac{R}{C}\,\varepsilon_{11} - \frac{S}{C}\,\varepsilon_{22} + \frac{1}{C}\,(R + S + C)\,\alpha\,\Delta T \qquad \text{(from } \underline{\underline{C}}\text{)}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} A_\sigma + Q_\sigma \\ B_\sigma + Q_\sigma \\ 0 \end{bmatrix}$$

General planar material laws

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} - \alpha \Delta T \begin{bmatrix} \Theta_{p1} \\ \Theta_{p2} \\ 0 \end{bmatrix}$$

$$\underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix} + \alpha \Delta T \begin{bmatrix} \theta_{p1} \\ \theta_{p2} \\ 0 \end{bmatrix}$$

plane strain : $(\ )_p = (\ )_\varepsilon$
plane stress : $(\ )_p = (\ )_\sigma$

ELASTIC LIMIT

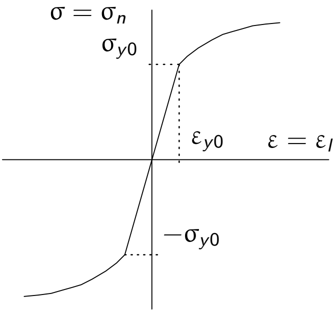
back to index

Elastic limit criteria

failure mode	**mechanism**
plastic yielding	crystallographic slip (metals)
brittle fracture	(sudden) breakage of bonds
progressive damage	micro-cracks $\rightarrow$ growth $\rightarrow$ coalescence
fatigue	damage/fracture under cyclic loading
dynamic failure	vibration $\rightarrow$ resonance
thermal failure	creep / melting
elastic instabilities	buckling $\rightarrow$ plastic deformation

Yield function : one-dimensional

$$f(\sigma) = \sigma^2 - \sigma_{y0}^2 = 0 \quad \rightarrow$$
$$g(\sigma) = \sigma^2 = \sigma_{y0}^2 = g_t$$
$$g_t = \text{limit in tensile test}$$



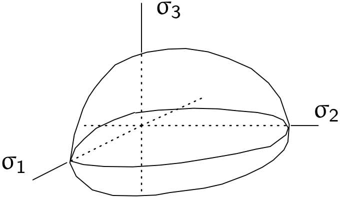
Yield function : three-dimensional

$$f(\boldsymbol{\sigma}) = 0 \qquad\qquad \rightarrow \quad g(\boldsymbol{\sigma}) = g_t$$

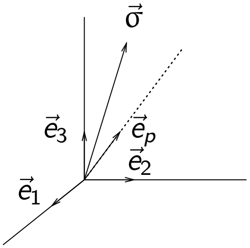
yield surface in 6D stress space

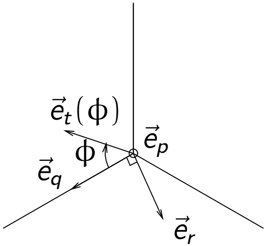
$$f(\sigma_1, \sigma_2, \sigma_3) = 0 \quad \rightarrow \quad g(\sigma_1, \sigma_2, \sigma_3) = g_t$$

yield surface in 3D principal stress space



Principal stress space





hydrostatic axis $\quad \vec{e}_p = \frac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \quad$ with $\quad \|\vec{e}_p\| = 1$

plane $\perp$ hydrostatic axis

$$\vec{e}_q^{\,*} = \vec{e}_1 - (\vec{e}_p \cdot \vec{e}_1)\vec{e}_p = \vec{e}_1 - \frac{1}{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) = \frac{1}{3}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3)$$

$$\vec{e}_q = \frac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3)$$

$$\vec{e}_r = \vec{e}_p * \vec{e}_q = \frac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) * \frac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) = \frac{1}{2}\sqrt{2}(\vec{e}_2 - \vec{e}_3)$$

vector in Π-plane $\quad \vec{e}_t(\phi) = \cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r$

Principal stress space

$$
\begin{aligned}
\vec{\sigma} &= \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3 = \vec{\sigma}^h + \vec{\sigma}^d \\
\vec{\sigma}^h &= (\vec{\sigma} \cdot \vec{e}_p)\vec{e}_p = \sigma^h \vec{e}_p = \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3)\vec{e}_p = \sqrt{3}\sigma_m \vec{e}_p \\
\sigma^h &= \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3) \\
\vec{\sigma}^d &= \vec{\sigma} - (\vec{\sigma} \cdot \vec{e}_p)\vec{e}_p \\
&= \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3 - \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3)\tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \\
&= \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3 - \\
&\qquad \tfrac{1}{3}(\sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_1 + \sigma_3 \vec{e}_1 + \sigma_1 \vec{e}_2 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_2 + \sigma_1 \vec{e}_3 + \sigma_2 \vec{e}_3 + \sigma_3 \vec{e}_3) \\
&= \tfrac{1}{3}\{(2\sigma_1 - \sigma_2 - \sigma_3)\vec{e}_1 + (-\sigma_1 + 2\sigma_2 - \sigma_3)\vec{e}_2 + (-\sigma_1 - \sigma_2 + 2\sigma_3)\vec{e}_3\} \\
\sigma^d &= \|\vec{\sigma}^d\| = \sqrt{\vec{\sigma}^d \cdot \vec{\sigma}^d} \\
&= \tfrac{1}{3}\sqrt{(2\sigma_1 - \sigma_2 - \sigma_3)^2 + (-\sigma_1 + 2\sigma_2 - \sigma_3)^2 + (-\sigma_1 - \sigma_2 + 2\sigma_3)^2} \\
&= \sqrt{\tfrac{2}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)} \\
&= \sqrt{\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d}
\end{aligned}
$$

Principal stress space

$$
\begin{aligned}
\vec{\sigma} &= \vec{\sigma}^h + \vec{\sigma}^d = \sigma^h \vec{e}_p + \sigma^d \vec{e}_t(\phi) \\
&= \sigma^h \vec{e}_p + \sigma^d \{\cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r\} \\
&= \sigma^h \tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) + \sigma^d \{\cos(\phi)\tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) - \sin(\phi)\tfrac{1}{2}\sqrt{2}(\vec{e}_2 - \vec{e}_3)\} \\
&= \{\tfrac{1}{3}\sqrt{3}\,\sigma^h + \tfrac{1}{3}\sqrt{6}\,\sigma^d \cos(\phi)\}\vec{e}_1 + \\
&\quad \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d \cos(\phi) - \tfrac{1}{2}\sqrt{2}\,\sigma^d \sin(\phi)\}\vec{e}_2 + \\
&\quad \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d \cos(\phi) + \tfrac{1}{2}\sqrt{2}\,\sigma^d \sin(\phi)\}\vec{e}_3 \\
&= \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3
\end{aligned}
$$

Maximum stress/strain

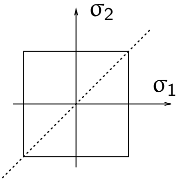
$$\sigma_{ij} = \sigma_{max} \quad | \quad \varepsilon_{ij} = \varepsilon_{max} \quad ; \quad \{i,j\} = \{1,2,3\}$$ (orthotropic materials)

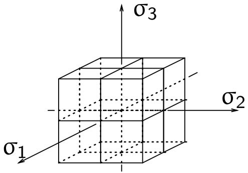
Rankine

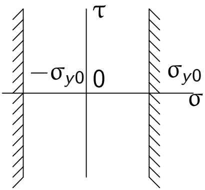
$$|\sigma_{max}| = max(|\sigma_i| \;;\; i = 1, 2, 3) = \sigma_{max,t} = \sigma_{y0} \qquad \text{(brittle materials; cast iron)}$$

Rankine

$|\sigma_{max}| = max(|\sigma_i| \; ; \; i = 1, 2, 3) = \sigma_{max,t} = \sigma_{y0}$ (brittle materials; cast iron)

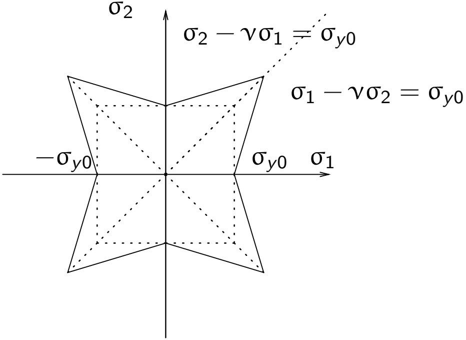






De Saint Venant

$$\varepsilon_{max} = max(|\varepsilon_i| \; ; \; i = 1,2,3) = \varepsilon_{max_t} = \varepsilon_{y0} = \frac{\sigma_{y0}}{E}$$ (brittle materials; cast iron)

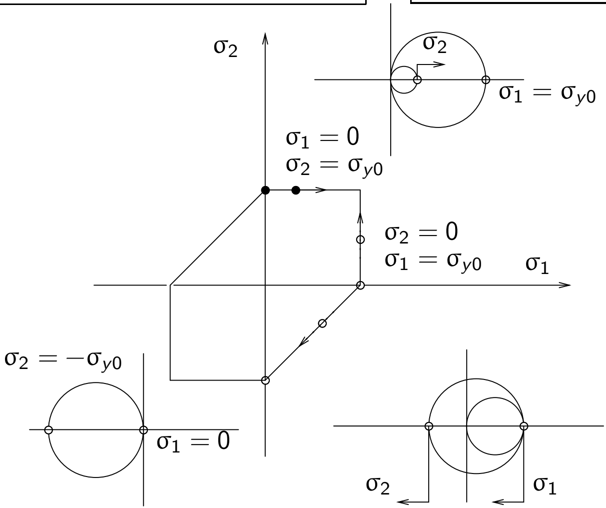


Tresca

$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \tau_{max,t} = \frac{1}{2}\sigma_{y0} \longrightarrow \bar{\sigma}_{TR} = \sigma_{max} - \sigma_{min} = \sigma_{y0}$$

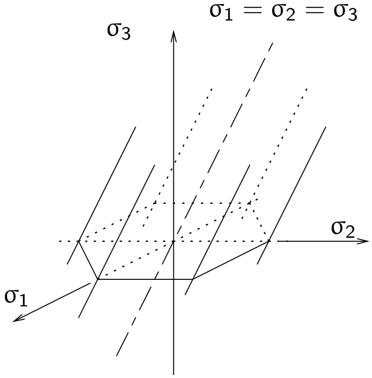
Tresca : 2D yield contour

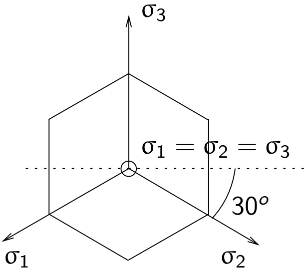
$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \tau_{max,t} = \frac{1}{2}\sigma_{y0} \quad \rightarrow \quad \bar{\sigma}_{TR} = \sigma_{max} - \sigma_{min} = \sigma_{y0}$$



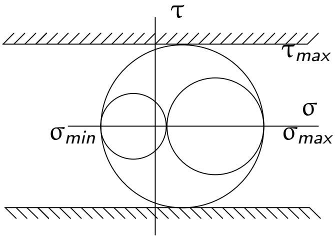
Tresca : 3D yield surface

Mohr $\quad\rightarrow\quad$ invariant for hydrostatic stress $\quad\rightarrow$
yield surface // hydrostatic axis
$\Pi -$ plane $\perp$ hydrostatic axis





Tresca : st-plane



Von Mises

$$W^d = W_t^d$$

$$\begin{aligned} W^d &= \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{4G}\left\{\boldsymbol{\sigma} : \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma})\right\} \quad \left(= -\frac{1}{2G}\,J_2(\boldsymbol{\sigma}^d)\right) \\ &= \frac{1}{4G}\,(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{12G}\,(\sigma_1 + \sigma_2 + \sigma_3)^2 \\ &= \frac{1}{4G}\,\tfrac{1}{3}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\} \end{aligned}$$

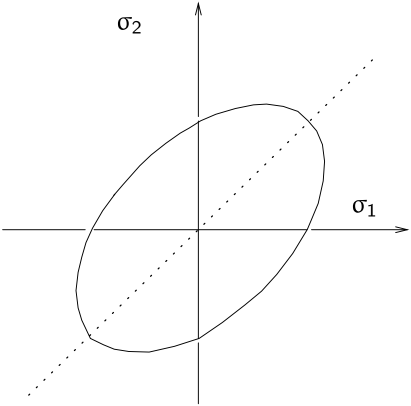
$$W_t^d = \frac{1}{4G}\,\tfrac{1}{3}\left\{(\sigma - 0)^2 + (0 - 0)^2 + (0 - \sigma)^2\right\} = \frac{1}{4G}\,\tfrac{1}{3}2\sigma^2 = \frac{1}{4G}\,\tfrac{1}{3}\,2\sigma_{y0}^2$$

$$\bar{\sigma}_{VM} = \sqrt{\tfrac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}} = \sigma_{y0}$$

Von Mises : Cartesian stress components

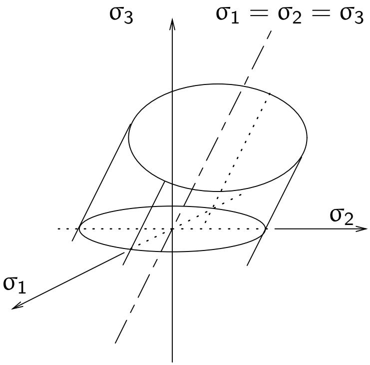
$$
\begin{aligned}
\bar{\sigma}_{VM}^2 &= \tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = 3J_2 \\
&= \tfrac{3}{2}\mathrm{tr}(\underline{\sigma}^d \underline{\sigma}^d) \quad \text{with } \underline{\sigma}^d = \underline{\sigma} - \tfrac{1}{3}\mathrm{tr}(\underline{\sigma})\underline{I} \\
&= \tfrac{3}{2}\Big\{ \left(\tfrac{2}{3}\sigma_{xx} - \tfrac{1}{3}\sigma_{yy} - \tfrac{1}{3}\sigma_{zz}\right)^2 + \sigma_{xy}^2 + \sigma_{xz}^2 + \\
&\qquad\quad \left(\tfrac{2}{3}\sigma_{yy} - \tfrac{1}{3}\sigma_{zz} - \tfrac{1}{3}\sigma_{xx}\right)^2 + \sigma_{yz}^2 + \sigma_{yx}^2 + \\
&\qquad\quad \left(\tfrac{2}{3}\sigma_{zz} - \tfrac{1}{3}\sigma_{xx} - \tfrac{1}{3}\sigma_{yy}\right)^2 + \sigma_{zx}^2 + \sigma_{zy}^2 \Big\} \\
&= \left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2\right) - \left(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}\right) + 2\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right) \\
&= \sigma_{y0}^2
\end{aligned}
$$

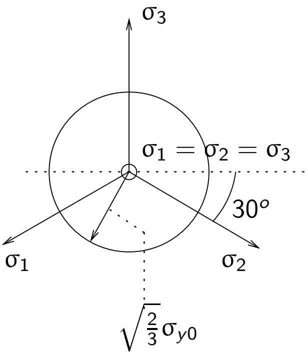
Von Mises : 2D yield surface



Von Mises : 3D yield surface

$$\boxed{\tfrac{1}{2}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}=\sigma_{y0}^2}$$





Beltrami-Haigh

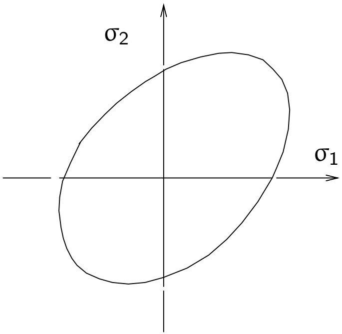
$$W = W_t$$

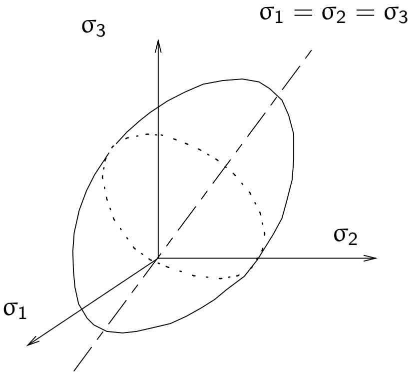
$$\begin{aligned} W &= W^h + W^d = \frac{1}{18K}\,\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d \\ &= \left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 + \frac{1}{4G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \end{aligned}$$

$$W_t = \left(\frac{1}{18K} - \frac{1}{12G}\right)\sigma^2 + \frac{1}{4G}\,\sigma^2 = \frac{1}{2E}\,\sigma^2 = \frac{1}{2E}\,{\sigma_{y0}}^2$$

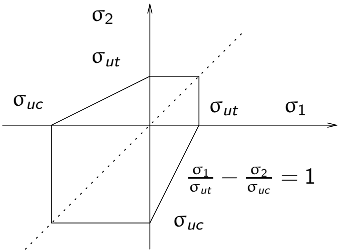
$$2E\left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 + \frac{2E}{4G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) = \sigma_{y0}^2$$

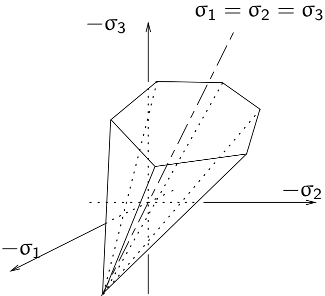
Beltrami-Haigh : 2D/3D yield surface

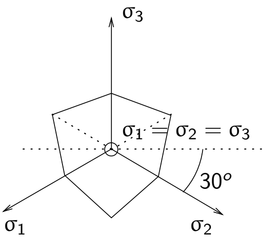




Mohr-Coulomb : 2D/3D yield surface

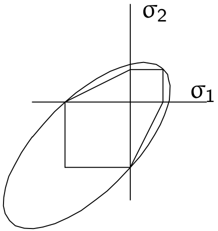


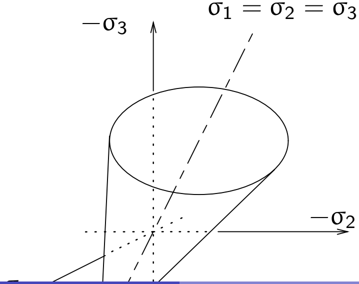


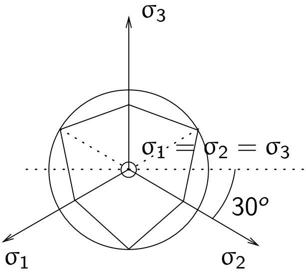


Drucker-Prager

$$\sqrt{\tfrac{2}{3}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} + \frac{6\sin(\phi)}{3-\sin(\phi)}\, p = \frac{6\cos(\phi)}{3-\sin(\phi)}\, C$$







Other yield criteria

parabolic Drucker-Prager $\left(3J_2 + \sqrt{3}\,\beta\,\sigma_{y0}J_1\right)^{\frac{1}{2}} = \sigma_{y0}$

Buyokozturk $\left(3J_2 + \sqrt{3}\,\beta\,\sigma_{y0}J_1 - 0.2J_1^2\right)^{\frac{1}{2}} = \sigma_{y0}$

Hill $\dfrac{\sigma_{11}^2}{X^2} - \dfrac{\sigma_{11}\sigma_{22}}{XY} + \dfrac{\sigma_{22}^2}{Y^2} + \dfrac{\sigma_{12}^2}{S^2}$

Hoffman

$$\left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_{11} + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_{22} + \left(\frac{1}{X_tX_c}\right)\sigma_{11}^2 + \left(\frac{1}{Y_tY_c}\right)\sigma_{22}^2 + \left(\frac{1}{S^2}\right)\sigma_{12}^2 - \left(\frac{1}{X_tX_c}\right)\sigma_{11}\sigma_{22} = 0$$

Tsai-Wu

$$\left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_{11} + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_{22} + \left(\frac{1}{X_tX_c}\right)\sigma_{11}^2 + \left(\frac{1}{Y_tY_c}\right)\sigma_{22}^2 + \left(\frac{1}{S^2}\right)\sigma_{12}^2 + 2F_{12}\;\sigma_{11}\sigma_{22} = 0$$

with $F_{12}^2 > \dfrac{1}{X_tX_c}\;\dfrac{1}{Y_tY_c}$

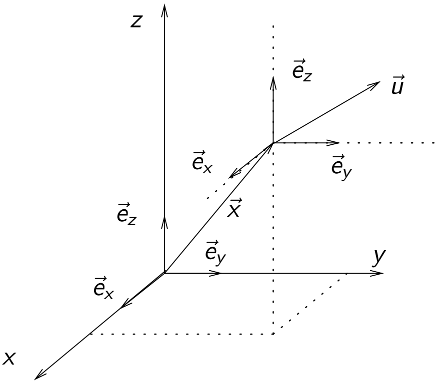
GOVERNING EQUATIONS

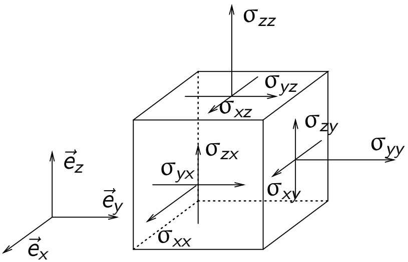
back to index

Vector/tensor equations

gradient operator	:	$\vec{\nabla} = \underset{\sim}{\nabla}^T \underset{\sim}{\vec{e}}$
position	:	$\vec{x} = \underset{\sim}{x}^T \underset{\sim}{\vec{e}}$
displacement	:	$\vec{u} = \underset{\sim}{u}^T \underset{\sim}{\vec{e}}$
strain	:	$\boldsymbol{\varepsilon} = \frac{1}{2}\left\{\left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right)\right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$
compatibility	:	$\nabla^2\{\mathrm{tr}(\boldsymbol{\varepsilon})\} - \vec{\nabla}\cdot(\vec{\nabla}\cdot\boldsymbol{\varepsilon})^T = 0$
stress	:	$\boldsymbol{\sigma} = \underset{\sim}{\vec{e}}^T \underline{\sigma}\, \underset{\sim}{\vec{e}}$
balance laws	:	$\vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
material law	:	$\boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\varepsilon} \quad ; \quad \boldsymbol{\varepsilon} = {}^4\mathbf{C}^{-1} : \boldsymbol{\sigma} = {}^4\mathbf{S} : \boldsymbol{\sigma}$
th.mech. mat. law	:	$\boldsymbol{\sigma} = {}^4\mathbf{C} : (\boldsymbol{\varepsilon} - \alpha\Delta T\mathbf{I}) \quad : \quad \boldsymbol{\varepsilon} = {}^4\mathbf{S} : \boldsymbol{\sigma} + \alpha\Delta T\mathbf{I}$

Cartesian components





Cartesian components, 3D

$$\underset{\sim}{x}^T = \left[\begin{array}{ccc} x & y & z \end{array}\right] \quad ; \quad \underset{\sim}{\nabla}^T = \left[\begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array}\right] \quad ; \quad \underset{\sim}{u}^T = \left[\begin{array}{ccc} u_x & u_y & u_z \end{array}\right]$$

$$\underline{\varepsilon} = \frac{1}{2}\left[\begin{array}{ccc} 2u_{x,x} & u_{x,y}+u_{y,x} & u_{x,z}+u_{z,x} \\ \cdots & 2u_{y,y} & u_{y,z}+u_{z,y} \\ \cdots & \cdots & 2u_{z,z} \end{array}\right]$$

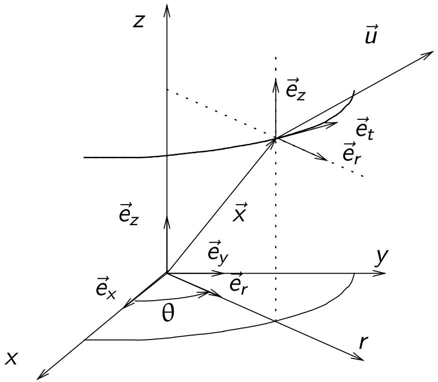
$$2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$
$$\varepsilon_{xx,yz} + \varepsilon_{yz,xx} - \varepsilon_{zx,xy} - \varepsilon_{xy,xz} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$

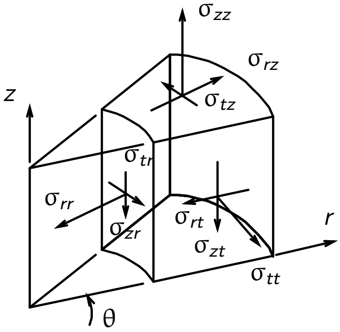
$$\underset{\approx}{\varepsilon}^T = \underset{\sim}{\varepsilon}^T = \left[\begin{array}{cccccc} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \varepsilon_{xy} & \varepsilon_{yz} & \varepsilon_{zx} \end{array}\right]$$
$$\underset{\approx}{\sigma}^T = \underset{\sim}{\sigma}^T = \left[\begin{array}{cccccc} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{array}\right]$$

$$\begin{array}{ll} \sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x = \rho \ddot{u}_x & \qquad (\sigma_{xy} = \sigma_{yx}) \\ \sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y = \rho \ddot{u}_y & \qquad (\sigma_{yz} = \sigma_{zy}) \\ \sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z = \rho \ddot{u}_z & \qquad (\sigma_{zx} = \sigma_{xz}) \end{array}$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon}\underset{\sim}{\sigma} = \underline{\underline{C}}\,\underset{\sim}{\varepsilon} \qquad ; \qquad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}\underset{\sim}{\varepsilon} = \underline{\underline{S}}\,\underset{\sim}{\sigma}$$

Cylindrical components





Cylindrical components, 3D

$$\underset{\sim}{x}^T = \begin{bmatrix} r & \theta & z \end{bmatrix} \quad ; \quad \underset{\sim}{\nabla}^T = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \end{bmatrix} \quad ; \quad \underset{\sim}{u}^T = \begin{bmatrix} u_r & u_t & u_z \end{bmatrix}$$

$$\underline{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \cdots & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ \cdots & \cdots & 2u_{z,z} \end{bmatrix}$$

$$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0 \quad \rightarrow \quad \text{cyc. } 2\text{x}$$
$$\varepsilon_{rr,tz} + \varepsilon_{tz,rr} - \varepsilon_{zr,rt} - \varepsilon_{rt,rz} = 0 \quad \rightarrow \quad \text{cyc. } 2\text{x}$$

$$\underset{\approx}{\varepsilon}^T = \underset{\sim}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{tt} & \varepsilon_{zz} & \varepsilon_{rt} & \varepsilon_{tz} & \varepsilon_{zr} \end{bmatrix}$$
$$\underset{\approx}{\sigma}^T = \underset{\sim}{\sigma}^T = \begin{bmatrix} \sigma_{rr} & \sigma_{tt} & \sigma_{zz} & \sigma_{rt} & \sigma_{tz} & \sigma_{zr} \end{bmatrix}$$

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = \rho \ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$
$$\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t = \rho \ddot{u}_t \qquad (\sigma_{tz} = \sigma_{zt})$$
$$\sigma_{zr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}\sigma_{zr} + \sigma_{rz,z} + \rho q_z = \rho \ddot{u}_z \qquad (\sigma_{zr} = \sigma_{rz})$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon}\underset{\sim}{\sigma} = \underline{\underline{C}}\,\underset{\sim}{\varepsilon} \quad ; \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}\underset{\sim}{\varepsilon} = \underline{\underline{S}}\,\underset{\sim}{\sigma}$$

Material law, 3D (No ΔT)

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2K & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L & 0 \\ 0 & 0 & 0 & 0 & 0 & 2M \end{bmatrix} \rightarrow \underline{\underline{S}} = \underline{\underline{C}}^{-1} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}k & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}l & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}m \end{bmatrix}$$

quadratic	$B = A\ ;\ S = R\ ;\ M = L;$
transversal isotropic	$B = A\ ;\ S = R\ ;\ M = L\ ;\ K = \frac{1}{2}(A - Q)$
cubic	$C = B = A\ ;\ S = R = Q\ ;\ M = L = K \neq \frac{1}{2}(A - Q)$
isotropic	$C = B = A\ ;\ S = R = Q\ ;\ M = L = K = \frac{1}{2}(A - Q)$

3D isotropic ▷
3D transversal isotropic ▷
3D orthotropic ▷
3D general orthotropic ▷
plane strain / plane stress / planar ▷ ▷ ▷

Planar deformation : Cartesian

$$\left.\begin{array}{lll} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_x = u_x(x,y) \\ u_y = u_y(x,y) \end{array}\right.$$

$$\underset{\approx}{\varepsilon}^T = \underset{\sim}{\varepsilon}^T = \left[\begin{array}{ccc} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{xy} \end{array}\right] = \left[\begin{array}{ccc} u_{x,x} & u_{y,y} & \frac{1}{2}(u_{x,y} + u_{y,x}) \end{array}\right]$$

$$2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0$$

$$\underset{\approx}{\sigma}^T = \underset{\sim}{\sigma}^T = \left[\begin{array}{ccc} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{array}\right]$$

$$\begin{array}{l} \sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \\ \sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y \end{array} \qquad (\sigma_{xy} = \sigma_{yx})$$

$$\underline{\underline{C}}_p = \left[\begin{array}{ccc} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{array}\right] \quad ; \quad \underline{\underline{S}}_p = \left[\begin{array}{ccc} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{array}\right]$$

orthotr. pe/ps ▷ ▷ transv.iso. pe/ps ▷ ▷ isotropic pe/ps ▷ ▷

Planar deformation : cylindrical

$$\left.\begin{array}{lll} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta) \end{array}\right.$$

$$\underset{\approx}{\varepsilon}^T = \underset{\sim}{\varepsilon}^T = \left[\begin{array}{ccc} \varepsilon_{rr} & \varepsilon_{tt} & \varepsilon_{rt} \end{array}\right] = \left[\begin{array}{ccc} u_{r,r} & \frac{1}{r}(u_r + u_{t,t}) & \frac{1}{2}\left(\frac{1}{r}(u_{r,t} - u_t) + u_{t,r}\right) \end{array}\right]$$

$$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0$$

$$\underset{\approx}{\sigma}^T = \underset{\sim}{\sigma}^T = \left[\begin{array}{ccc} \sigma_{rr} & \sigma_{tt} & \sigma_{rt} \end{array}\right]$$

$$\sigma_{rr,r} + \frac{1}{r}\,\sigma_{rt,t} + \frac{1}{r}\,(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$

$$\sigma_{tr,r} + \frac{1}{r}\,\sigma_{tt,t} + \frac{1}{r}\,(\sigma_{tr} + \sigma_{rt}) + \rho q_t = \rho \ddot{u}_t$$

$$\underline{\underline{C}}_p = \left[\begin{array}{ccc} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{array}\right] \quad ; \quad \underline{\underline{S}}_p = \left[\begin{array}{ccc} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{array}\right]$$

orthotr. pe/ps ▷ ▷ transv.iso. pe/ps ▷ ▷ isotropic pe/ps ▷ ▷

Axi-symmetric + $u_t = 0$

$$\begin{array}{lcl} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array} \Bigg\} \qquad \begin{cases} u_r = u_r(r) \\ u_t = 0 \end{cases}$$

$$\underset{\approx}{\varepsilon}^T = \underset{\sim}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{tt} \end{bmatrix} = \begin{bmatrix} u_{r,r} & \frac{1}{r}(u_r) \end{bmatrix}$$

$$\varepsilon_{rr} = u_{r,r} = (r\varepsilon_{tt})_{,r} = \varepsilon_{tt} + r\varepsilon_{tt,r}$$

$$\underset{\approx}{\sigma}^T = \underset{\sim}{\sigma}^T = \begin{bmatrix} \sigma_{rr} & \sigma_{tt} \end{bmatrix}$$

$$\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r$$

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p \\ Q_p & B_p \end{bmatrix} \qquad ; \qquad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p \\ q_p & b_p \end{bmatrix}$$

orthotr. pe/ps ▷ ▷ transv.iso. pe/ps ▷ ▷ isotropic pe/ps ▷ ▷

SOLUTION STRATEGIES

back to index

Governing equations

unknown variables

displacements	:	$\vec{u} = \vec{u}(\vec{x}) \quad \rightarrow \quad \mathbf{F} = \left(\vec{\nabla}_0 \vec{x}\right)^T \quad \rightarrow \quad \mathbf{E}\ ,\ \boldsymbol{\varepsilon}$
stresses	:	$\boldsymbol{\sigma} \quad \rightarrow \quad g(\boldsymbol{\sigma}) = g(\sigma_1, \sigma_2, \sigma_3) = g_t$

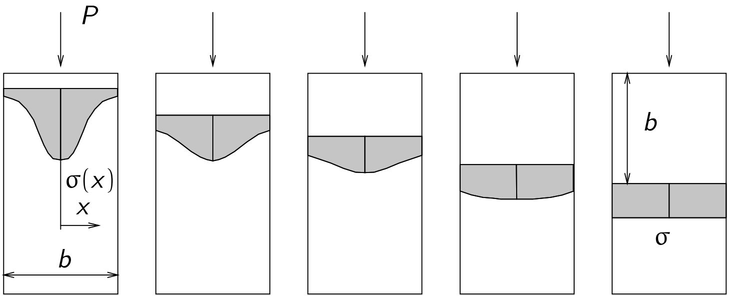
equations

compatibility	:	$\nabla^2 \{\mathrm{tr}(\boldsymbol{\varepsilon})\} - \vec{\nabla} \cdot (\vec{\nabla} \cdot \boldsymbol{\varepsilon})^T = 0$
equilibrium	:	$\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \rho \ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
material law	:	$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{F}) \quad \rightarrow \quad \boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\varepsilon} \quad \rightarrow \quad \boldsymbol{\varepsilon} = {}^4\mathbf{S} : \boldsymbol{\sigma}$

boundary conditions

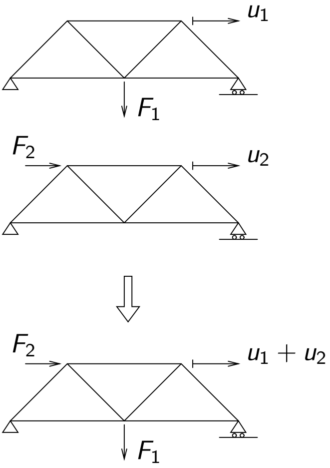
displacement	:	$\vec{u} = \vec{u}_p \quad \forall \quad \vec{x} \in A_u$
edge load	:	$\vec{p} = \vec{n} \cdot \boldsymbol{\sigma} = \vec{p}_p \quad \forall \quad \vec{x} \in A_p$

Saint-Venant's principle



$$P = \int_A \sigma(x)\, dA = \sigma\, A \qquad ; \qquad A = b * t$$

Superposition



Solution : displacement method

$$\left.\begin{aligned} &\vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \\ &\boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\varepsilon} \end{aligned}\right\} \quad \rightarrow \quad \left.\begin{aligned} &\vec{\nabla}\cdot\left({}^4\mathbf{C} : \boldsymbol{\varepsilon}\right)^T + \rho\vec{q} = \rho\ddot{\vec{u}} \\ &\boldsymbol{\varepsilon} = \tfrac{1}{2}\left\{\left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right)\right\} \end{aligned}\right\} \quad \rightarrow$$

$$\vec{\nabla}\cdot\left\{{}^4\mathbf{C} : \left(\vec{\nabla}\vec{u}\right)\right\}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \qquad \rightarrow$$

$$\vec{u} \quad \rightarrow \quad \boldsymbol{\varepsilon} \quad \rightarrow \quad \boldsymbol{\sigma}$$

Planar, Cartesian : Navier equations

$$\left.\begin{array}{l}\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \qquad ; \qquad \sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y \\[2ex] \sigma_{xx} = A_p \varepsilon_{xx} + Q_p \varepsilon_{yy} \\ \sigma_{yy} = Q_p \varepsilon_{xx} + B_p \varepsilon_{yy} \\ \sigma_{xy} = 2K \varepsilon_{xy}\end{array}\right\}$$

$$\left.\begin{array}{l}A_p \varepsilon_{xx,x} + Q_p \varepsilon_{yy,x} + 2K \varepsilon_{xy,y} + \rho q_x = \rho \ddot{u}_x \\ 2K \varepsilon_{xy,x} + Q_p \varepsilon_{xx,y} + B_p \varepsilon_{yy,y} + \rho q_y = \rho \ddot{u}_y\end{array}\right\}$$

$$\left.\begin{array}{l}A_p u_{x,xx} + K u_{x,yy} + (Q_p + K) u_{y,yx} + \rho q_x = \rho \ddot{u}_x \\ K u_{y,xx} + B_p u_{y,yy} + (Q_p + K) u_{x,xy} + \rho q_y = \rho \ddot{u}_y\end{array}\right\}$$

Planar, axi-symmetric with $u_t = 0$, isotropic

$$\left.\begin{array}{l} \sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \\ \\ \left.\begin{array}{l} \sigma_{rr} = A_p \varepsilon_{rr} + Q_p \varepsilon_{tt} - \Theta_{p1}\alpha\Delta T \\ \sigma_{tt} = Q_p \varepsilon_{rr} + A_p \varepsilon_{tt} - \Theta_{p1}\alpha\Delta T \end{array}\right\} \end{array}\right\}$$

$$\left.\begin{array}{l} A_p \varepsilon_{rr,r} + Q_p \varepsilon_{tt,r} - \Theta_{p1}\alpha(\Delta T)_r + \\ \qquad \frac{1}{r}\{(A_p - Q_p)\varepsilon_{rr} + (Q_p - A_p)\varepsilon_{tt}\} + \rho q_r = \rho \ddot{u}_r \\ \varepsilon_{rr} = u_{r,r} \quad ; \quad \varepsilon_{tt} = \frac{1}{r} u_r \end{array}\right\}$$

$$u_{r,rr} + \frac{1}{r} u_{r,r} - \frac{1}{r^2} u_r = \frac{\rho}{A_p}\,(\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p}\,\alpha(\Delta T)_{,r} = f(r)$$

WEIGHTED RESIDUAL FORMULATION

back to index

Weighted residual formulation for 3D deformation

equilibrium equation $\qquad \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \vec{0} \qquad \forall\ \vec{x} \in V$

approximation $\rightarrow$ residual $\qquad \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \vec{\Delta}(\vec{x}) \neq \vec{0} \qquad \forall\ \vec{x} \in V$

weighted residual

$$\int_V \vec{w}(\vec{x}) \cdot \vec{\Delta}(\vec{x})\, dV = \int_V \vec{w} \cdot \left[\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} \right] dV$$

equivalent problem formulation

$$\int_V \vec{w} \cdot \left[\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} \right] dV = 0 \qquad \forall \quad \vec{w}(\vec{x}) \quad \leftrightarrow \quad \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \vec{0} \qquad \forall\ \vec{x} \in V$$

Weak formulation

$$\left.\begin{array}{l}\displaystyle\int_V \vec{w}\cdot\left[\vec{\nabla}\cdot\boldsymbol{\sigma}^T+\rho\vec{q}\right]dV=0 \\ \vec{\nabla}\cdot(\boldsymbol{\sigma}^T\cdot\vec{w})=(\vec{\nabla}\vec{w})^T:\boldsymbol{\sigma}^T+\vec{w}\cdot(\vec{\nabla}\cdot\boldsymbol{\sigma}^T)\end{array}\right\}\quad\rightarrow$$

$$\left.\begin{array}{l}\displaystyle\int_V\left[\vec{\nabla}\cdot(\boldsymbol{\sigma}^T\cdot\vec{w})-(\vec{\nabla}\vec{w})^T:\boldsymbol{\sigma}^T+\vec{w}\cdot\rho\vec{q}\right]dV=0 \qquad \forall\ \vec{w} \\ \text{Gauss / Stokes}\ :\ \displaystyle\int_V\vec{\nabla}\cdot(\boldsymbol{\sigma}^T\cdot\vec{w})=\int_V\vec{n}\cdot\boldsymbol{\sigma}^T\cdot\vec{w}\,dA=\int_A\vec{w}\cdot\vec{p}\,dA\end{array}\right\}\quad\rightarrow$$

$$\int_V(\vec{\nabla}\vec{w})^T:\boldsymbol{\sigma}\,dV=\int_V\vec{w}\cdot\rho\vec{q}\,dV+\int_A\vec{w}\cdot\vec{p}\,dA \qquad \forall\ \vec{w}$$

$$\int_V(\vec{\nabla}\vec{w})^T:\boldsymbol{\sigma}\,dV=f_e(\vec{w}) \qquad \forall\ \vec{w}$$

Linear elastic formulation

$$\int_{V_0} (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma} \, dV_0 = \int_{V_0} \vec{w} \cdot \rho \vec{q} \, dV_0 + \int_{A_0} \vec{w} \cdot \vec{p} \, dA_0 = f_{e0}(\vec{w}) \qquad \forall \ \vec{w}$$

$$\begin{aligned} \boldsymbol{\sigma} &= {}^4\mathbf{C} : \boldsymbol{\varepsilon} \\ &= {}^4\mathbf{C} : \tfrac{1}{2}\left\{(\vec{\nabla}_0\vec{u}) + (\vec{\nabla}_0\vec{u})^T\right\} = {}^4\mathbf{C} : (\vec{\nabla}_0\vec{u}) \end{aligned}$$

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^T : {}^4\mathbf{C} : (\vec{\nabla}_0\vec{u}) \, dV_0 = \int_{V_0} \vec{w} \cdot \rho \vec{q} \, dV_0 + \int_{A_0} \vec{w} \cdot \vec{p} \, dA_0 = f_{e0}(\vec{w}) \qquad \forall \ \vec{w}$$

Matrix/column notation

$$\int_{V_0} \left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{C}} \left(\underset{\approx}{L}_{0u} \right)_t \, dV_0 = f_{e0}(\underset{\sim}{w}) \qquad \forall \; \underset{\sim}{w}$$

Total Lagrange formulation

$$\int_V (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma}\, dV = f_e(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

transformation to undeformed configuration t_0

$$\vec{\nabla} = \mathbf{F}^{-c} \cdot \vec{\nabla}_0 \quad \rightarrow \quad (\vec{\nabla}\vec{w})^c = (\vec{\nabla}_0\vec{w})^c \cdot \mathbf{F}^{-1}$$
$$dV = \det(\mathbf{F})dV_0 = J dV_0$$

weighted residual integral

$$\left.\begin{array}{l} \displaystyle\int_{V_0} (\vec{\nabla}_0\vec{w})^c \cdot \mathbf{F}^{-1} : \boldsymbol{\sigma} J\, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \\ \\ \mathbf{P} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-c} \end{array}\right\} \quad \rightarrow$$

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^c : (\mathbf{P} \cdot \mathbf{F}^c)\, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

Iterative solution process

$$\left.\begin{array}{l}
\displaystyle\int_{V_0} (\vec{\nabla}_0 \vec{w})^c : (\mathbf{P} \cdot \mathbf{F}^c)\, dV_0 = f_{e0}(\vec{w}) \qquad \forall\ \vec{w}(\vec{x}) \\[2ex]
\mathbf{F} = (\vec{\nabla}_0 \vec{x})^c = \{\vec{\nabla}_0(\vec{x}^* + \delta\vec{x})\}^c = (\vec{\nabla}_0 \vec{x}^*)^c + (\vec{\nabla}_0 \delta\vec{x})^c \\[2ex]
\qquad\qquad = \mathbf{F}^* + \delta\mathbf{F} = \mathbf{F}^* + \mathbf{L}_{0u} \\[2ex]
\mathbf{P} = \mathbf{P}^* + \delta\mathbf{P}
\end{array}\right\} \quad \rightarrow$$

$$\int_{V_0} (\vec{\nabla}_0 \vec{w})^c : (\mathbf{P}^* + \delta\mathbf{P}) \cdot (\mathbf{F}^* + \mathbf{L}_{0u})^c\, dV_0 = f_{e0}(\vec{w}) \qquad \forall\ \vec{w}(\vec{x})$$

Linearisation

$$\int_{V_0} \mathbf{L}_{0w} : (\mathbf{P}^* + \delta\mathbf{P}) \cdot (\mathbf{F}^* + \mathbf{L}_{0u})^c \, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

$$\int_{V_0} \mathbf{L}_{0w} : (\mathbf{P}^* \cdot \mathbf{F}^{*c} + \mathbf{P}^* \cdot \mathbf{L}_{0u}^c + \delta\mathbf{P} \cdot \mathbf{F}^{*c}) \; dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

$$\int_{V_0} \mathbf{L}_{0w} : (\mathbf{P}^* \cdot \mathbf{L}_{0u}^c + \delta\mathbf{P} \cdot \mathbf{F}^{*c}) \; dV_0 =$$

$$f_{e0}(\vec{w}) - \int_{V_0} \mathbf{L}_{0w} : (\mathbf{P}^* \cdot \mathbf{F}^{*c}) \, dV_0 = r^* \qquad \forall \quad \vec{w}(\vec{x})$$

Material model

$$\delta \mathbf{P} = {}^4\mathbf{M} : \mathbf{L}_{0u} \quad \rightarrow$$

$$\int_{V_0} \mathbf{L}_{0w} : \left(\mathbf{P}^* \cdot \mathbf{L}_{0u}^c + ({}^4\mathbf{M} : \mathbf{L}_{0u}) \cdot \mathbf{F}^{*c} \right) \, dV_0 =$$

$$f_{e0}(\vec{w}) - \int_{V_0} \mathbf{L}_{0w} : (\mathbf{P}^* \cdot \mathbf{F}^{*c}) \, dV_0 \qquad \forall \quad \vec{w}(\vec{x})$$

$$\int_{V_0} \left[\mathbf{L}_{0w} : (\mathbf{P}^* \cdot \mathbf{L}_{0u}^c) + \mathbf{L}_{0w} : (\mathbf{F}^* \cdot {}^4\mathbf{M}^{lrc}) : \mathbf{L}_{0u}^c \right] \, dV_0 =$$

$$f_{e0}(\vec{w}) - \int_{V_0} \mathbf{L}_{0w} : (\mathbf{P}^* \cdot \mathbf{F}^{*c}) \, dV_0 \qquad \forall \quad \vec{w}(\vec{x})$$

Matrix/column notation

$$\int_{V_0} \left[\left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{P}} \left(\underset{\approx}{L}_{0u} \right)_t + \left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{F}}_{cr} \, \underline{\underline{M}}_{0c} \left(\underset{\approx}{L}_{0u} \right)_t \right] dV_0 =$$

$$f_{e0}(\underset{\sim}{w}) - \int_{V_0} \left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{F}}_{cr} \, \underset{\approx}{P} \, dV_0 = f_{e0}(\underset{\sim}{w}) - f_{i0}(\underset{\sim}{w})$$

$$\int_{V_0} \left(\underset{\approx}{L}_{0w} \right)_t^T \left[\underline{\underline{P}} + \underline{\underline{F}}_{cr} \, \underline{\underline{M}}_{0c} \right] \left(\underset{\approx}{L}_{0u} \right)_t dV_0 = f_{e0}(\underset{\sim}{w}) - f_{i0}(\underset{\sim}{w})$$

Updated Lagrange formulation

$$\int_V (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma}\, dV = f_e(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

transformation to begin increment configuration t_n

$$\vec{\nabla} = \mathbf{F}_n^{-c} \cdot \vec{\nabla}_n \quad \longrightarrow \quad (\vec{\nabla}\vec{w})^c = (\vec{\nabla}_n\vec{w})^c \cdot \mathbf{F}_n^{-1}$$
$$dV = \det(\mathbf{F}_n) dV_n$$

weighted residual integral

$$\int_{V_n} (\vec{\nabla}_n\vec{w})^c \cdot \mathbf{F}_n^{-1} : \boldsymbol{\sigma} \det(\mathbf{F}_n)\, dV_n = f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \quad \longrightarrow$$

$$\int_{V_n} (\vec{\nabla}_n\vec{w})^c : (\mathbf{F}_n^{-1} \cdot \boldsymbol{\sigma}) \det(\mathbf{F}_n)\, dV_n = f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

Iterative solution process

$$\left.\begin{array}{l}
\displaystyle\int_{V_n} (\vec{\nabla}_n \vec{w})^c : (\mathbf{F}_n^{-1} \cdot \boldsymbol{\sigma}) \det(\mathbf{F}_n)\, dV_n = f_{en}(\vec{w}) \qquad \forall\ \vec{w}(\vec{x}) \\[4em]
\mathbf{F}_n = (\vec{\nabla}_n \vec{x})^c = \{\vec{\nabla}_n(\vec{x}^* + \delta\vec{x})\}^c = (\vec{\nabla}_n \vec{x}^*)^c + (\vec{\nabla}_n \delta\vec{x})^c \\[1em]
\qquad = \mathbf{F}_n^* + \delta\mathbf{F}_n = \mathbf{F}_n^* + (\vec{\nabla}^* \delta\vec{x})^c \cdot (\vec{\nabla}_n \vec{x}^*)^c = \mathbf{F}_n^* + \mathbf{L}_u^* \cdot \mathbf{F}_n^* = (\mathbf{I} + \mathbf{L}_u^*) \cdot \mathbf{F}_n^* \\[1em]
\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}
\end{array}\right\} \rightarrow$$

$$\int_{V_n} (\vec{\nabla}_n \vec{w})^c : \left[(\mathbf{F}_n^*)^{-1} \cdot (\mathbf{I} + \mathbf{L}_u^*)^{-1} \cdot (\boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma})\ \det\{(\mathbf{I} + \mathbf{L}_u^*) \cdot \mathbf{F}_n^*\}\right]\ dV_n$$

$$= f_{en}(\vec{w}) \qquad \forall\ \vec{w}(\vec{x})$$

Linearisation

$$(\mathbf{I}+\mathbf{L}_u^*)^{-1} \approx \mathbf{I}-\mathbf{L}_u^*$$

$$\det\{(\mathbf{I}+\mathbf{L}_u^*)\cdot\mathbf{F}_n^*\} = \det(\mathbf{I}+\mathbf{L}_u^*)\det(\mathbf{F}_n^*) \approx \mathrm{tr}(\mathbf{I}+\mathbf{L}_u^*)\det(\mathbf{F}_n^*) = (1+\mathbf{I}:\mathbf{L}_u^*)\det(\mathbf{F}_n^*)$$

weighted residual integral

$$\begin{aligned}\int_{V_n} &(\vec{\nabla}_n\vec{w})^c : \\ &\left[(\mathbf{F}_n^*)^{-1}\cdot(\mathbf{I}-\mathbf{L}_u^*)\cdot(\boldsymbol{\sigma}^*+\delta\boldsymbol{\sigma})(1+\mathbf{I}:\mathbf{L}_u^*)\det(\mathbf{F}_n^*)\right]\, dV_n \\ &= f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})\end{aligned}$$

further linearisation

$$\int_{V^*} [\mathbf{L}_w^* : \boldsymbol{\sigma}^*\mathbf{I}:\mathbf{L}_u^{*c} + \mathbf{L}_w^* : \delta\boldsymbol{\sigma} - \mathbf{L}_w^* : (\boldsymbol{\sigma}^{*c}\cdot\mathbf{L}_u^{*c})^c]\, dV^* =$$

$$f_e^*(\vec{w}) - \int_{V^*} \mathbf{L}_w^* : \boldsymbol{\sigma}^*\, dV^* =$$

$$r^* \qquad \forall \quad \vec{w}(\vec{x})$$

Material model

$$\delta\boldsymbol{\sigma} = {}^4\mathbf{M} : \mathbf{L}_u^* \quad \rightarrow$$

$$\int_{V^*} \left[\mathbf{L}_w^* : \boldsymbol{\sigma}^* \mathbf{I} : \mathbf{L}_u^{*c} + \mathbf{L}_w^* : {}^4\mathbf{M} : \mathbf{L}_u^* - \mathbf{L}_w^* : (\boldsymbol{\sigma}^{*c} \cdot \mathbf{L}_u^{*c})^c \right] dV^* =$$

$$f_e^*(\vec{w}) - \int_{V^*} \mathbf{L}_w^* : \boldsymbol{\sigma}^* \, dV^* \qquad \forall \quad \vec{w}(\vec{x})$$

Matrix/column notation

$$\int_{V^*} \left[\left(\underset{\approx}{L}_w \right)_t^T \underset{\approx}{\sigma} \underset{\approx}{I}^T \left(\underset{\approx}{L}_u \right)_t + \left(\underset{\approx}{L}_w \right)_t^T \underline{\underline{M}} \left(\underset{\approx}{L}_u \right)_t - \left(\underset{\approx}{L}_w \right)_t^T \underline{\underline{\sigma}}_{tr} \left(\underset{\approx}{L}_u \right)_t \right] dV^* =$$

$$f_e(\underset{\sim}{w}) - \int_{V^*} \left(\underset{\approx}{L}_w \right)_t^T \underset{\approx}{\sigma}\, dV^* =$$

$$f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$

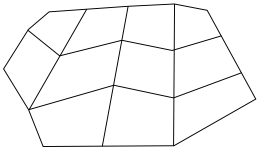
$$\int_{V^*} \left(\underset{\approx}{L}_w \right)_t^T \left[\underset{\approx}{\sigma} \underset{\approx}{I}^T - \underline{\underline{\sigma}}_{tr} + \underline{\underline{M}} \right] \left(\underset{\approx}{L}_u \right)_t dV^* = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$

$$\int_{V^*} \left(\underset{\approx}{L}_w \right)_t^T \left[\underline{\underline{\Sigma}} + \underline{\underline{M}} \right] \left(\underset{\approx}{L}_u \right)_t dV^* = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$

FINITE ELEMENT METHOD

back to index

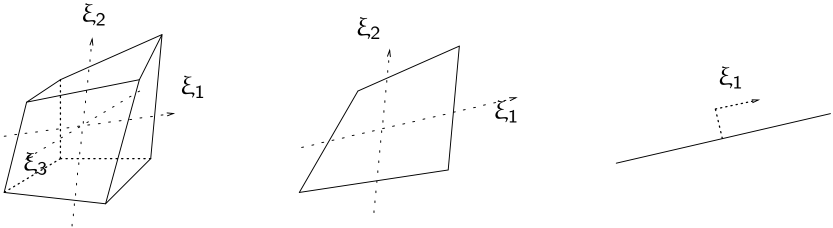
Discretisation



$$\int_V \left(\underset{\approx}{\underline{L}}_w\right)_t^T \left[\underline{\underline{W}}\right] \left(\underset{\approx}{\underline{L}}_u\right)_t \, dV = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$

$$\sum_e \int_{V^e} \left(\underset{\approx}{\underline{L}}_w\right)_t^T \left[\underline{\underline{W}}\right] \left(\underset{\approx}{\underline{L}}_u\right)_t \, dV^e = \sum_e f_e^e(\underset{\sim}{w}) - \sum_e f_i^e(\underset{\sim}{w}) \qquad \forall \; \underset{\sim}{w}$$

Isoparametric elements



isoparametric (local) coordinates

$$(\xi_1, \xi_2, \xi_3) \quad ; \quad -1 \leq \xi_i \leq 1 \quad i = 1, 2, 3$$

Jacobian matrix

$$\underline{J} = \left(\underset{\sim}{\nabla}_{\xi} \underset{\sim}{x}^T \right)^T \; dV^e = \det(\underline{J}) \, d\xi_1 d\xi_2 d\xi_3$$

Interpolation

$$\vec{a} = N^1 \vec{a}^1 + N^2 \vec{a}^2 + \cdots + N^{nep} \vec{a}^{nep} = \utilde{N}^T \utilde{\vec{a}}^e \quad \longrightarrow$$

$$a_i = N^1 a_i^1 + N^2 a_i^2 + \cdots + N^{nep} a_i^{nep} = \utilde{N}^T \utilde{a}_i^e \quad \longrightarrow \quad \utilde{a} = \underline{N}\, \utilde{a}^e$$

Gradient

$$\vec{a} = N^1\vec{a}^1 + N^2\vec{a}^2 + \cdots + N^{nep}\vec{a}^{nep} = \underset{\sim}{N}^T \underset{\sim}{\vec{a}}^e \quad \rightarrow$$

$$a_i = N^1 a_i^1 + N^2 a_i^2 + \cdots + N^{nep} a_i^{nep} = \underset{\sim}{N}^T \underset{\sim}{a}_i^e \quad \rightarrow \quad \underset{\sim}{a} = \underline{N}\,\underset{\sim}{a}^e$$

$$\mathbf{L}^c = \vec{\nabla}\vec{a} \quad \rightarrow \quad \underset{\approx}{L}_t = \underline{B}\,\underset{\sim}{a}^e$$

Integration

$$\vec{a} = N^1 \vec{a}^1 + N^2 \vec{a}^2 + \cdots + N^{nep} \vec{a}^{nep} = \underset{\sim}{N}^T \underset{\sim}{\vec{a}}^e \quad \rightarrow$$

$$a_i = N^1 a_i^1 + N^2 a_i^2 + \cdots + N^{nep} a_i^{nep} = \underset{\sim}{N}^T \underset{\sim}{a}_i^e \quad \rightarrow \quad \underset{\sim}{a} = \underline{N}\, \underset{\sim}{a}^e$$

$$\mathbf{L}^c = \vec{\nabla} \vec{a} \quad \rightarrow \quad \underset{\approx}{L}_t = \underline{B}\, \underset{\sim}{a}^e$$

$$\underset{\sim}{w}^{eT} \left[\int_{V^e} \underline{B}^T \, \underline{\underline{W}} \, \underline{B} \, dV^e \right] \delta \underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}_e^e - \underset{\sim}{w}^{eT} \underset{\sim}{f}_i^e$$

$$\underset{\sim}{w}^{eT} \left[\int_{\xi_1=-1}^{\xi_1=1} \int_{\xi_2=-1}^{\xi_2=1} \int_{\xi_3=-1}^{\xi_3=1} \underline{B}^T \left[\underline{\underline{W}} \right] \underline{B} \, \det(\underline{J}) \, d\xi_1 d\xi_2 d\xi_3 \right] \delta \underset{\sim}{u}^e =$$

$$\underset{\sim}{w}^{eT} \underset{\sim}{f}_e^e - \underset{\sim}{w}^{eT} \underset{\sim}{f}_i^e$$

$$\underset{\sim}{w}^{eT} \, \underline{K}^e \, \delta \underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \, \underset{\sim}{f}_e^e - \underset{\sim}{w}^{eT} \, \underset{\sim}{f}_i^e$$

Integration

$$\int_{V^e} g(x_1, x_2, x_3)\, dV^e = \int_{\xi_1=-1}^{1} \int_{\xi_2=-1}^{1} \int_{\xi_3=-1}^{1} f(\xi_1, \xi_2, \xi_3)\, d\xi_1 d\xi_2 d\xi_3$$

$$= \sum_{ip=1}^{nip} c^{ip}\, f(\xi_1^{ip}, \xi_2^{ip}, \xi_3^{ip})$$

Assemblation

$$\sum_e \underset{\sim}{w}^{eT} \underline{K}^e \, \delta \underset{\sim}{u}^e = \sum_e \underset{\sim}{w}^{eT} \underset{\sim}{f}_e^e - \sum_e \underset{\sim}{w}^{eT} \underset{\sim}{f}_i^e \quad \rightarrow$$

$$\underset{\sim}{w}^T \, \underline{K} \, \delta \underset{\sim}{u} = \underset{\sim}{w}^T \, \underset{\sim}{f}_e - \underset{\sim}{w}^T \, \underset{\sim}{f}_i = \underset{\sim}{w}^T \, \underset{\sim}{r} \qquad \forall \; \underset{\sim}{w} \qquad \rightarrow$$

$$\underline{K} \, \delta \underset{\sim}{u} = \underset{\sim}{f}_e - \underset{\sim}{f}_i = \underset{\sim}{r}$$

Solution

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots \\ k_{21} & k_{22} & k_{23} & \cdots \\ k_{31} & k_{32} & k_{33} & \cdots \\ \vdots & \vdots & \vdots & . \end{bmatrix} \begin{bmatrix} a \\ a \\ a \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \rightarrow \quad \underline{K} = \text{ singular} \quad \rightarrow \quad \det \underline{K} = 0$$

prevent rigid body movement with BC's
other BC's : prescribed displacements / loads / temperature

$$\delta \underset{\sim}{u} = \underline{K}^{-1} \underset{\sim}{r}$$

Program structure

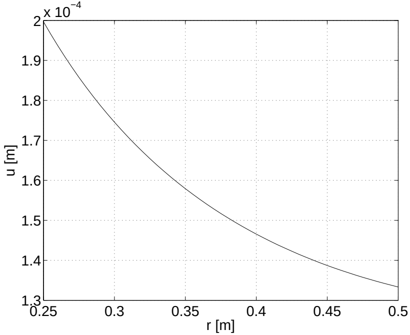
```
read input data from input file
calculate additional variables from input data
initialize values and arrays

while load increments to be done

   for all elements
      for all integration points
         calculate contribution to initial element stiffness matrix
      end integration point loop
      assemble global stiffness matrix
   end element loop

   determine external incremental load from input

   while non-converged iteration step

      take tyings into account
      take boundary conditions into account

      calculate iterative nodal displacements
      calculate total deformation

      for all elements
         for all integration points
            calculate stresses from material behavior
            calculate material stiffness from material behavior
            calculate contribution to element internal nodal forces
            calculate contribution to element stiffness matrix
         end ntegration point loop
         assemble global stiffness matrix
         assemble global internal load column
      end element loop

      calculate residual load column
      calculate convergence norm

   end iteration step

   store data for post-processing
end load increment
```

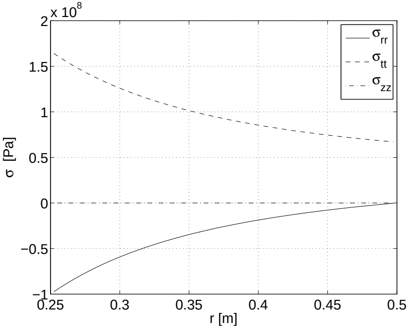
NUMERICAL SOLUTIONS

back to index

Example

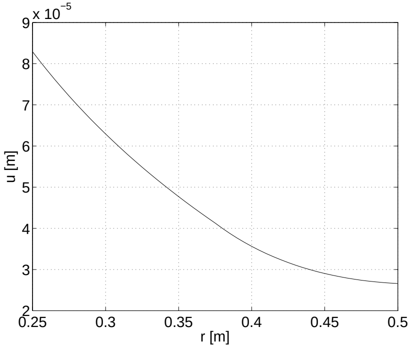
isotropic	plane stress	$p_i = 100$ MPa	$p_e = 0$ MPa
$a = 0.25$ m	$b = 0.5$ m	$h = 0.5$ m	$E = 250$ GPa | $\nu = 0.33$

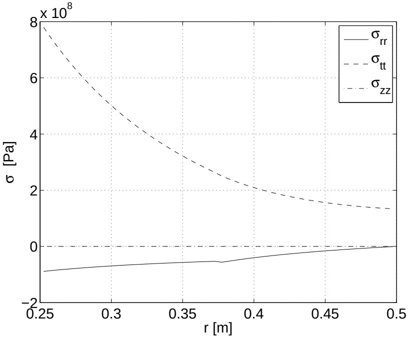




Compound thick-walled pressurized cylinder

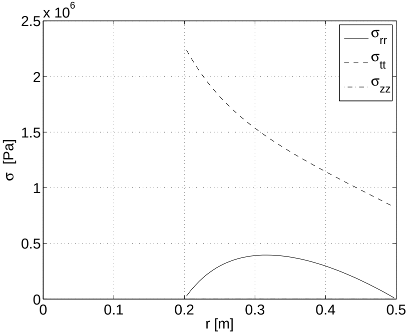
isotropic	plane stress	$p_i = 100$ MPa	$p_e = 0$ MPa
$a_1 = 0.25$ m	$a_2 = 0.375$ m	$E = 250$ GPa	$\nu = 0.33$
$a_2 = 0.375$ m	$b = 0.5$ m	$E1 = E$ GPa	$E2 = 10E$ GPa
$\nu_{12} = \nu/10$	$\nu_{32} = \nu/10$		

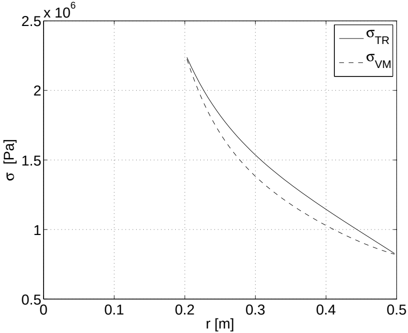




Rotating disc

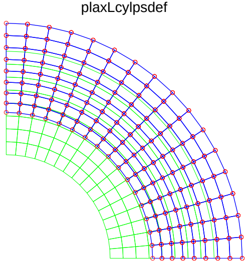
orthotropic	plane stress	$\omega = 6$ c/s		
$a = 0.2$ m	$b = 0.5$ m	$E = 200$ GPa	$\nu = 0.3$	$Gr = 7500$ kg/m^3

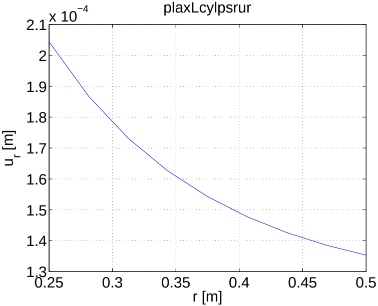




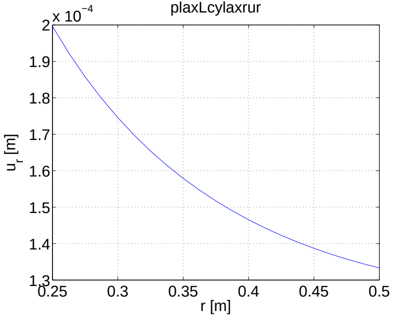
Thick-walled pressurized cylinder: plane stress

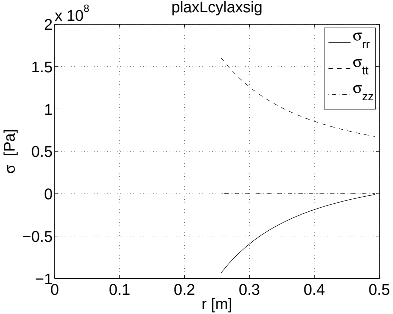
$$| \, a = 0.25 \text{ m} \quad | \, b = 0.5 \text{ m} \quad | \, h = 0.5 \text{ m} \, | \, E = 250 \text{ GPa} \, | \, \nu = 0.33 \, |$$
$$| \, p_i = 100 \text{ MPa} \, | \, p_e = 0 \text{ MPa} \, |$$



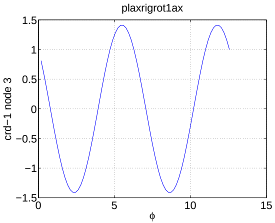


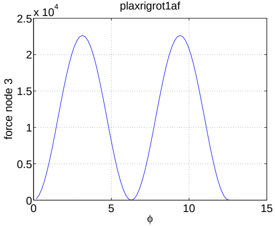
Thick-walled pressurized cylinder: axi-symmetric

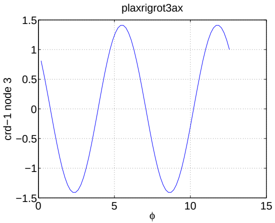


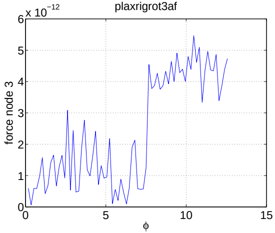


Rigid rotation









THREE-DIMENSIONAL MATERIAL MODELS

back to index

ELASTIC

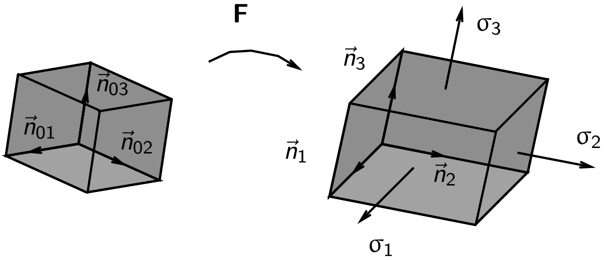
back to index

Elastic material behavior

$$\mathbf{P} = \mathbf{G}(\mathbf{E}) \qquad \text{with} \qquad \mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I}) = \tfrac{1}{2}(\mathbf{F}^c \cdot \mathbf{F} - \mathbf{I})$$

$$\begin{aligned}
\boldsymbol{\sigma} &= J^{-1}\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}^c \\
&= J^{-1}\mathbf{F} \cdot \mathbf{G}(\mathbf{E}) \cdot \mathbf{F}^c && \text{with} \qquad J = \det(\mathbf{F}) \\
&= \mathbf{K}(\mathbf{A}) && \text{with} \qquad \mathbf{A} = \tfrac{1}{2}(\mathbf{B} - \mathbf{I}) = \tfrac{1}{2}(\mathbf{F} \cdot \mathbf{F}^c - \mathbf{I})
\end{aligned}$$

Isotropic elastic material models



$$
\begin{aligned}
\mathbf{U} &= \lambda_1 \vec{n}_{01}\vec{n}_{01} + \lambda_2 \vec{n}_{02}\vec{n}_{02} + \lambda_3 \vec{n}_{03}\vec{n}_{03} \\
\mathbf{R} &= \vec{n}_1\vec{n}_{01} + \vec{n}_2\vec{n}_{02} + \vec{n}_3\vec{n}_{03} \\
\mathbf{F} &= \lambda_1 \vec{n}_1\vec{n}_{01} + \lambda_2 \vec{n}_2\vec{n}_{02} + \lambda_3 \vec{n}_3\vec{n}_{03} \\
\mathbf{P} &= J\mathbf{F}^{-1} \cdot \left(\sigma_1 \vec{n}_1\vec{n}_1 + \sigma_2 \vec{n}_2\vec{n}_2 + \sigma_3 \vec{n}_3\vec{n}_3\right) \cdot \mathbf{F}^{-c} \\
&= J\left\{\sigma_1\lambda_1^{-2}\vec{n}_{01}\vec{n}_{01} + \sigma_2\lambda_2^{-2}\vec{n}_{02}\vec{n}_{02} + \sigma_3\lambda_3^{-2}\vec{n}_{03}\vec{n}_{03}\right\} \\
&= s_1\vec{n}_{01}\vec{n}_{01} + s_2\vec{n}_{01}\vec{n}_{01} + s_3\vec{n}_{01}\vec{n}_{01}
\end{aligned}
$$

$\mathbf{P} - \mathbf{E}$ model

$$\mathbf{P} = s_1 \vec{n}_{01} \vec{n}_{01} + s_2 \vec{n}_{02} \vec{n}_{02} + s_3 \vec{n}_{03} \vec{n}_{03}$$
$$\mathbf{E} = \varepsilon_1 \vec{n}_{01} \vec{n}_{01} + \varepsilon_2 \vec{n}_{02} \vec{n}_{02} + \varepsilon_3 \vec{n}_{03} \vec{n}_{03}$$

$$\begin{aligned} \mathbf{P} &= \sum s_i \vec{n}_{0i} \vec{n}_{0i} \\ &= \mathbf{G}(\mathbf{E}) = \sum G(\varepsilon_i) \vec{n}_{0i} \vec{n}_{0i} \\ &= a_0 \mathbf{I} + a_1 \mathbf{E} + a_2 \mathbf{E}^2 + a_3 \mathbf{E}^3 + \ldots \end{aligned}$$

Cayley-Hamilton's theorem $\qquad \mathbf{E}^3 = J_1(\mathbf{E})\mathbf{E}^2 - J_2(\mathbf{E})\mathbf{E} + J_3(\mathbf{E})\mathbf{I}$

$\mathbf{P} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{E} + \alpha_2 \mathbf{E}^2$ with $\alpha_i = \alpha_i \{J_1(\mathbf{E}), J_2(\mathbf{E}), J_3(\mathbf{E})\}$

$\boldsymbol{\sigma} - \mathbf{A}$ model

$$\boldsymbol{\sigma} = \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3$$
$$\mathbf{A} = A_1 \vec{n}_1 \vec{n}_1 + A_2 \vec{n}_2 \vec{n}_2 + A_3 \vec{n}_3 \vec{n}_3$$

$$\begin{aligned}\boldsymbol{\sigma} &= \sum \sigma_i \vec{n}_i \vec{n}_i \\ &= \mathbf{K}(\mathbf{A}) = \sum K(A_i) \vec{n}_i \vec{n}_i \\ &= b_0 \mathbf{I} + b_1 \mathbf{A} + b_2 \mathbf{A}^2 + b_3 \mathbf{A}^3 + \ldots\end{aligned}$$

Cayley-Hamilton's theorem $\quad \mathbf{A}^3 = J_1(\mathbf{A})\mathbf{A}^2 - J_2(\mathbf{A})\mathbf{A} + J_3(\mathbf{A})\mathbf{I}$

$\boldsymbol{\sigma} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \beta_2 \mathbf{A}^2$ with $\beta_i = \beta_i \{J_1(\mathbf{A}), J_2(\mathbf{A}), J_3(\mathbf{A})\}$

Isotropic elastic material : $\boldsymbol{\sigma} - \mathbf{A}$

$$
\begin{aligned}
\boldsymbol{\sigma} &= J^{-1}\mathbf{F}\cdot\mathbf{P}\cdot\mathbf{F}^c \\
&= J^{-1}\mathbf{F}\cdot\left[\alpha_0\mathbf{I}+\alpha_1\mathbf{E}+\alpha_2\mathbf{E}^2\right]\cdot\mathbf{F}^c \\
&= J^{-1}\mathbf{F}\cdot\left[(\alpha_0-\tfrac{1}{2}\alpha_1+\alpha_2)\mathbf{I}+(\tfrac{1}{2}\alpha_1-\tfrac{1}{2}\alpha_2)\mathbf{C}+\tfrac{1}{4}\alpha_2\mathbf{C}^2\right]\cdot\mathbf{F}^c \\
&= \{J_3(\mathbf{B})\}^{-1/2}\left[(\alpha_0-\tfrac{1}{2}\alpha_1+\alpha_2)\mathbf{B}+(\tfrac{1}{2}\alpha_1-\tfrac{1}{2}\alpha_2)\mathbf{B}^2+\tfrac{1}{4}\alpha_2\mathbf{B}^3\right]
\end{aligned}
$$

$$
\mathbf{B}^3 = J_1(\mathbf{B})\mathbf{B}^2 - J_2(\mathbf{B})\mathbf{B} + J_3(\mathbf{B})\mathbf{I}
$$

$$
\begin{aligned}
&= J_3^{-1/2}\left[(\tfrac{1}{2}\alpha_1-\tfrac{1}{2}\alpha_2+\tfrac{1}{4}\alpha_2J_1)\mathbf{B}^2+\right. \\
&\qquad\qquad \left.(\alpha_0-\tfrac{1}{2}\alpha_1+\alpha_2-\tfrac{1}{4}\alpha_2J_2)\mathbf{B}+\tfrac{1}{4}\alpha_2J_3\mathbf{I}\right]
\end{aligned}
$$

$$
\mathbf{A} = \tfrac{1}{2}(\mathbf{B}-\mathbf{I}) \quad\rightarrow\quad \mathbf{B} = 2\mathbf{A}+\mathbf{I}
$$

$$
\mathbf{A}^2 = \tfrac{1}{4}\mathbf{B}^2 - \tfrac{1}{2}\mathbf{B} + \tfrac{1}{4}\mathbf{I} \quad\rightarrow\quad \mathbf{B}^2 = 4\mathbf{A}^2 + 2\mathbf{B} - \mathbf{I}
$$

$$
\begin{aligned}
&= J_3^{-1/2}\left[(2\alpha_1-2\alpha_2+\alpha_2J_1)\mathbf{A}^2+(\alpha_0+\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2J_1-\tfrac{1}{4}\alpha_2J_2)\mathbf{A}+\right. \\
&\qquad\qquad \left.(\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2+\tfrac{3}{4}\alpha_2J_1-\tfrac{1}{4}\alpha_2J_2+\tfrac{1}{4}\alpha_2J_3)\mathbf{I}\right] \\
&= \beta_2\mathbf{A}^2+\beta_1\mathbf{A}+\beta_0\mathbf{I}
\end{aligned}
$$

Linear $\mathbf{P} - \mathbf{E}$ model

$$\mathbf{P} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{E} + \alpha_2 \mathbf{E}^2 \qquad \text{with} \qquad \alpha_i = \alpha_i \{ J_1(\mathbf{E}), J_2(\mathbf{E}), J_3(\mathbf{E}) \}$$

linear $\longrightarrow$

1. $\alpha_2 = 0$
2. $\alpha_1 = \text{constant} = c_1$
3. $\alpha_0 = \text{linear in } \mathbf{E} = c_0 \text{tr}(\mathbf{E})$

$$\mathbf{P} = c_0 \text{tr}(\mathbf{E}) \mathbf{I} + c_1 \mathbf{E}$$

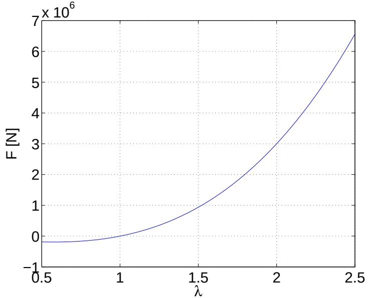
Tensile test

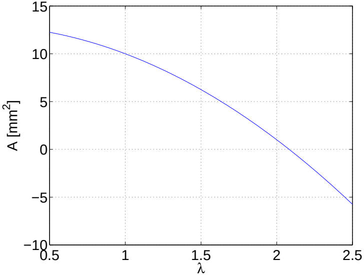
$$\left.\begin{aligned} P &= c_0 \tfrac{1}{2}(\lambda^2 - 1) + 2c_0 \tfrac{1}{2}(\mu^2 - 1) + c_1 \tfrac{1}{2}(\lambda^2 - 1) \\ 0 &= c_0 \tfrac{1}{2}(\lambda^2 - 1) + 2c_0 \tfrac{1}{2}(\mu^2 - 1) + c_1 \tfrac{1}{2}(\mu^2 - 1) \end{aligned}\right\} \quad \rightarrow$$

$$\tfrac{1}{2}(\mu^2 - 1) = -\frac{c_0}{2c_0 + c_1}\,\tfrac{1}{2}(\lambda^2 - 1) = -\nu\,\tfrac{1}{2}(\lambda^2 - 1)$$

$$P = \frac{c_1(3c_0 + c_1)}{2c_0 + c_1}\,\tfrac{1}{2}(\lambda^2 - 1) = E\,\tfrac{1}{2}(\lambda^2 - 1)$$

$$F = \sigma A = \frac{\lambda}{\mu^2}\,P\mu^2 A_0 = \lambda P A_0 = \tfrac{1}{2}\lambda(\lambda^2 - 1)\,E A_0$$

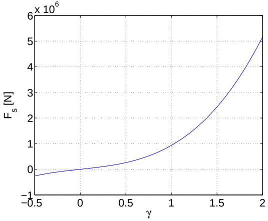


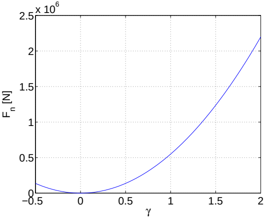


Simple shear test : plane stress

$$\sigma_{33} = P_{33} = 0 \quad \rightarrow \quad c_0(E_{11} + E_{22} + E_{33}) + c_1 E_{33} = 0 \quad \rightarrow \quad E_{33} = -\frac{c_0}{c_0 + c_1}(E_{11} + E_{22})$$

$$\begin{aligned}
\mathbf{F} &= \mathbf{I} + (F_{33} - 1)\vec{e}_3\vec{e}_3 + \gamma\vec{e}_1\vec{e}_2 \\
\mathbf{E} &= \tfrac{1}{2}(\mathbf{F}^c \cdot \mathbf{F} - \mathbf{I}) = \tfrac{1}{2}\left[\gamma^2\vec{e}_2\vec{e}_2 + \gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + \left\{2(F_{33} - 1) + (F_{33} - 1)^2\right\}\vec{e}_3\vec{e}_3\right] \\
F_{33} &= \sqrt{2E_{33} + 1} \quad \rightarrow \quad J = \det(\mathbf{F}) = F_{33} = \sqrt{2E_{33} + 1} \\
\mathbf{P} &= \frac{c_0 c_1}{c_0 + c_1}(E_{11} + E_{22}) + c_1\mathbf{E} \\
&= \frac{c_0 c_1}{c_0 + c_1}\tfrac{1}{2}\gamma^2\mathbf{I} + c_1\tfrac{1}{2}\gamma^2\vec{e}_2\vec{e}_2 + c_1\tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
\boldsymbol{\sigma} &= J^{-1}\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}^c = J^{-1}\left[\mathbf{P} + (\gamma P_{12} + \gamma P_{21} + \gamma^2 P_{22})\vec{e}_1\vec{e}_1 + \gamma P_{22}(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)\right] \\
p_n &= \vec{e}_2 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \sigma_{22} \quad ; \quad p_s = \vec{e}_1 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \sigma_{12} \\
F_n &= p_n d w_0 = p_n F_{33} d_0 w_0 \quad ; \quad F_s = p_s d w_0 = p_s F_{33} d_0 w_0
\end{aligned}$$





Linear $\boldsymbol{\sigma} - \mathbf{A}$ model

$$\boldsymbol{\sigma} = \beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \beta_2 \mathbf{A}^2 \qquad \text{with} \qquad \beta_i = \beta_i \{J_1(\mathbf{A}), J_2(\mathbf{A}), J_3(\mathbf{A})\}$$

linear $\rightarrow$

1. $\beta_2 = 0$
2. $\beta_1 = \text{constant} = c_1$
3. $\beta_0 = \text{linear in } \mathbf{A} = c_0 \text{tr}(\mathbf{A})$

$$\boldsymbol{\sigma} = c_0 \text{tr}(\mathbf{A}) \mathbf{I} + c_1 \mathbf{A}$$

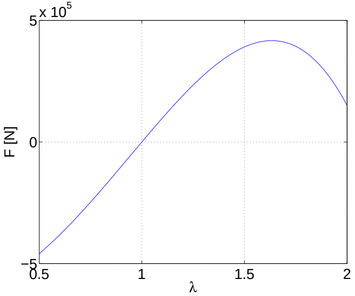
Tensile test

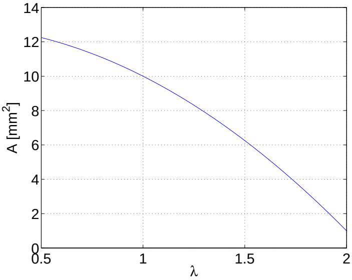
$$\left.\begin{aligned} \sigma &= c_0\,\tfrac{1}{2}(\lambda^2-1) + 2c_0\,\tfrac{1}{2}(\mu^2-1) + c_1\,\tfrac{1}{2}(\lambda^2-1) \\ 0 &= c_0\,\tfrac{1}{2}(\lambda^2-1) + 2c_0\,\tfrac{1}{2}(\mu^2-1) + c_1\,\tfrac{1}{2}(\mu^2-1) \end{aligned}\right\} \quad \rightarrow$$

$$\tfrac{1}{2}(\mu^2-1) = -\frac{c_0}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = -\nu\,\tfrac{1}{2}(\lambda^2-1)$$

$$\sigma = \frac{c_0(3c_0+c_1)}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = E\,\tfrac{1}{2}(\lambda^2-1)$$

$$F = \sigma A = \sigma\mu^2 A_0 = \tfrac{1}{2}(\lambda^2-1)\{1-\nu(\lambda^2-1)\}\,EA_0$$

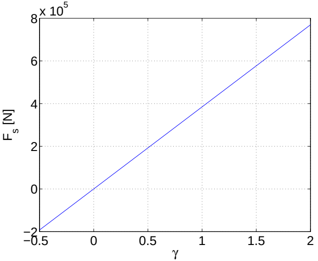


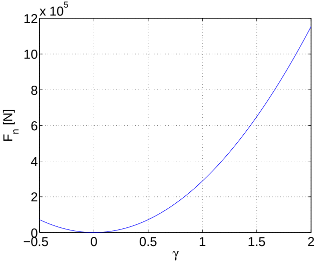


Simple shear test : plane strain

$$\mathbf{F} = \mathbf{I} + \gamma\, \vec{e}_1 \vec{e}_2$$
$$\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^c = \mathbf{I} + \gamma^2 \vec{e}_1 \vec{e}_1 + \gamma(\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_1)$$
$$\mathbf{A} = \tfrac{1}{2}(\mathbf{B} - \mathbf{I}) = \tfrac{1}{2}\gamma^2 \vec{e}_1 \vec{e}_1 + \tfrac{1}{2}\gamma(\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_1)$$

$$\boldsymbol{\sigma} = c_0 \tfrac{1}{2}\gamma^2 \mathbf{I} + c_1 \tfrac{1}{2}\gamma^2 \vec{e}_1 \vec{e}_1 + c_1 \tfrac{1}{2}\gamma(\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_1)$$
$$\sigma_{33} = c_0 \tfrac{1}{2}\gamma^2$$
$$p_n = \vec{e}_2 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = c_0 \tfrac{1}{2}\gamma^2 \quad ; \quad p_s = \vec{e}_1 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = c_1 \tfrac{1}{2}\gamma$$
$$F_n = p_n d_0 w_0 \quad ; \quad F_s = p_s d_0 w_0$$

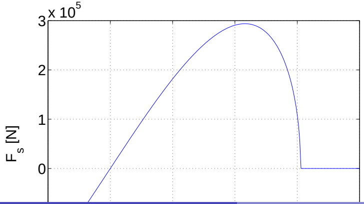


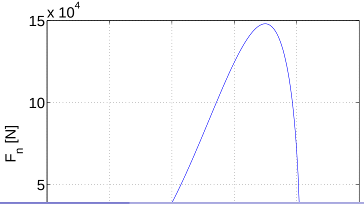


Simple shear test : plane stress

$$\sigma_{33} = c_0(A_{11} + A_{22} + A_{33}) + c_1 A_{33} = 0 \quad \rightarrow$$
$$A_{33} = -\frac{c_0}{c_0 + c_1}(A_{11} + A_{22}) \quad \rightarrow \quad F_{33} = \sqrt{2A_{33} + 1}$$
$$\boldsymbol{\sigma} = \frac{c_0 c_1}{c_0 + c_1}(A_{11} + A_{22})\mathbf{I} + c_1 \mathbf{A}$$

$$\mathbf{A} = \tfrac{1}{2}\gamma^2 \vec{e}_1\vec{e}_1 + \tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)$$
$$\boldsymbol{\sigma} = \frac{c_0 c_1}{c_0 + c_1}\tfrac{1}{2}\gamma^2\mathbf{I} + c_1\,\tfrac{1}{2}\gamma^2\vec{e}_1\vec{e}_1 + c_1\,\tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)$$
$$p_n = \vec{e}_2 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \frac{c_0 c_1}{c_0 + c_1}\tfrac{1}{2}\gamma^2 \quad ; \quad p_s = \vec{e}_1 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = c_1\,\tfrac{1}{2}\gamma$$
$$F_n = p_n d w_0 = p_n F_{33} d_0 w_0 \quad ; \quad F_s = p_s d w_0 = p_s F_{33} d_0 w_0$$





Hyper-elastic material models

$$\phi = \phi(\mathbf{E}) \quad \rightarrow \quad W = W(\mathbf{C}) \qquad \rightarrow$$

$$\mathbf{P} = \frac{d\phi(d\mathbf{E})}{\mathbf{E}} = \frac{dW(\mathbf{C})}{d\mathbf{C}} : \frac{d\mathbf{C}}{d\mathbf{E}} = 2\frac{dW(\mathbf{C})}{d\mathbf{C}} = \mathbf{G}(\mathbf{E})$$

$$\sigma = \frac{1}{J}\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}^c = \frac{2}{J}\mathbf{F} \cdot \frac{dW(\mathbf{C})}{d\mathbf{C}} \cdot \mathbf{F}^c$$

Isotropic hyper-elastic model : $\mathbf{P} - \mathbf{E}$

$$\phi = \phi(\mathbf{E}) = \phi\{J_1(\mathbf{E}), J_2(\mathbf{E}), J_3(\mathbf{E})\} \quad \rightarrow$$

$$\mathbf{P} = \frac{\partial \phi}{\partial J_1}\frac{dJ_1}{d\mathbf{E}} + \frac{\partial \phi}{\partial J_2}\frac{dJ_2}{d\mathbf{E}} + \frac{\partial \phi}{\partial J_3}\frac{dJ_3}{d\mathbf{E}}$$

$$\frac{dJ_1}{d\mathbf{E}} = \mathbf{I} \qquad ; \qquad \frac{dJ_2}{d\mathbf{E}} = J_1\mathbf{I} - \mathbf{E} \qquad ; \qquad \frac{dJ_3}{d\mathbf{E}} = J_2\mathbf{I} - J_1\mathbf{E} + \mathbf{E}^2 \qquad \rightarrow$$

$$\begin{aligned}\mathbf{P} &= \left(\frac{\partial \phi}{\partial J_1} + \frac{\partial \phi}{\partial J_2}J_1 + \frac{\partial \phi}{\partial J_3}J_2\right)\mathbf{I} + \left(-\frac{\partial \phi}{\partial J_2} - \frac{\partial \phi}{\partial J_3}J_1\right)\mathbf{E} + \frac{\partial \phi}{\partial J_3}\mathbf{E}^2 \\ &= \alpha_0\mathbf{I} + \alpha_1\mathbf{E} + \alpha_2\mathbf{E}^2\end{aligned}$$

Isotropic hyper-elastic model : $\mathbf{P} - \mathbf{C}$

$$W = W(\mathbf{C}) = W\{J_1(\mathbf{C}), J_2(\mathbf{C}), J_3(\mathbf{C})\} \quad \rightarrow$$

$$\mathbf{P} = 2\left(\frac{\partial W}{\partial J_1}\frac{dJ_1}{d\mathbf{C}} + \frac{\partial W}{\partial J_2}\frac{dJ_2}{d\mathbf{C}} + \frac{\partial W}{\partial J_3}\frac{dJ_3}{d\mathbf{C}}\right)$$

$$\frac{dJ_1}{d\mathbf{C}} = \mathbf{I} \qquad ; \qquad \frac{dJ_2}{d\mathbf{C}} = J_1\mathbf{I} - \mathbf{C} \qquad ; \qquad \frac{dJ_3}{d\mathbf{C}} = J_2\mathbf{I} - J_1\mathbf{C} + \mathbf{C}^2 \qquad \rightarrow$$

$$\mathbf{P} = 2\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1 + \frac{\partial W}{\partial J_3}J_2\right)\mathbf{I} + 2\left(-\frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3}J_1\right)\mathbf{C} + 2\frac{\partial W}{\partial J_3}\mathbf{C}^2$$

$$= \bar{\alpha}_0\mathbf{I} + \bar{\alpha}_1\mathbf{E} + \bar{\alpha}_2\mathbf{E}^2$$

Isotropic hyper-elastic model : $\boldsymbol{\sigma} - \mathbf{A}$ model

$$\begin{aligned}
\boldsymbol{\sigma} &= \frac{1}{J}\,\mathbf{F}\cdot\mathbf{P}\cdot\mathbf{F}^c = \frac{2}{J}\,\mathbf{F}\cdot\frac{dW(\mathbf{C})}{d\mathbf{C}}\cdot\mathbf{F}^c \\
&= \frac{2}{\sqrt{J_3}}\,\mathbf{F}\cdot\left(\frac{\partial W}{\partial J_1}\frac{dJ_1}{d\mathbf{C}} + \frac{\partial W}{\partial J_2}\frac{dJ_2}{d\mathbf{C}} + \frac{\partial W}{\partial J_3}\frac{dJ_3}{d\mathbf{C}}\right)\cdot\mathbf{F}^c \\
&= \frac{2}{\sqrt{J_3}}\,\mathbf{F}\cdot\left\{\left(\frac{\partial W}{\partial J_1} + J_1\frac{\partial W}{\partial J_2} + J_2\frac{\partial W}{\partial J_3}\right)\mathbf{I} + \left(-\frac{\partial W}{\partial J_2} - J_1\frac{\partial W}{\partial J_3}\right)\mathbf{C} + \left(\frac{\partial W}{\partial J_3}\right)\mathbf{C}^2\right\} \\
&= \frac{2}{\sqrt{J_3}}\,\mathbf{F}\cdot\left(\gamma_0\mathbf{I} + \gamma_1\mathbf{C} + \gamma_2\mathbf{C}^2\right)\cdot\mathbf{F}^c = \frac{2}{\sqrt{J_3}}\left(\gamma_0\mathbf{B} + \gamma_1\mathbf{B}^2 + \gamma_2\mathbf{B}^3\right)
\end{aligned}$$

$$\mathbf{B}^3 = J_1\mathbf{B}^2 - J_2\mathbf{B} + J_3\mathbf{I}$$

$$= \frac{2}{\sqrt{J_3}}\left[(\gamma_1 + \gamma_2 J_1)\mathbf{B}^2 + (\gamma_0 - \gamma_2 J_2)\mathbf{B} + (\gamma_2 J_3)\mathbf{I}\right]$$

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{B} - \mathbf{I}) \quad\rightarrow\quad \mathbf{B} = 2\mathbf{A} + \mathbf{I} \quad\rightarrow\quad \mathbf{B}^2 = 4\mathbf{A}^2 + 2\mathbf{B} - \mathbf{I}$$

$$\begin{aligned}
&= \frac{2}{\sqrt{J_3}}\left[(4\gamma_1 + 4\gamma_2 J_1)\mathbf{A}^2 + (\gamma_0 + 2\gamma_1 + 2\gamma_2 J_1 - \gamma_2 J_2)\mathbf{A} + \right. \\
&\qquad\qquad \left.(\gamma_0 - \gamma_1 + \gamma_2 J_1 - \gamma_2 J_2 + \gamma_2 J_3)\mathbf{I}\right] \\
&= \beta_2\mathbf{A}^2 + \beta_1\mathbf{A} + \beta_0\mathbf{I}
\end{aligned}$$

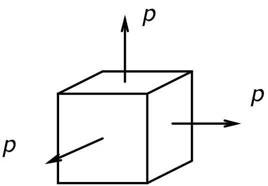
Incompressibility

$$J = \det(\mathbf{F}) = 1 \quad \rightarrow \quad \det(\mathbf{C}) = J_3(\mathbf{C}) = 1 \quad \rightarrow \quad W(\mathbf{C}) = W\{J_1(\mathbf{C}), J_2(\mathbf{C})\}$$

$$\mathbf{P} = 2\left(\frac{\partial W}{\partial J_1}\frac{dJ_1}{d\mathbf{C}} + \frac{\partial W}{\partial J_2}\frac{dJ_2}{d\mathbf{C}}\right) = 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\mathbf{I} - \frac{\partial W}{\partial J_2}\mathbf{C}\right\}$$

$$\boldsymbol{\sigma} = \mathbf{F}\cdot\mathbf{P}\cdot\mathbf{F}^c = 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\mathbf{B} - \frac{\partial W}{\partial J_2}\mathbf{B}^2\right\}$$

Incompressibility



$$\begin{aligned}\boldsymbol{\sigma} &= -p\mathbf{I} + \mathbf{F}\cdot\mathbf{P}\cdot\mathbf{F}^c \\ &= -p\mathbf{I} + 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\mathbf{B} - \frac{\partial W}{\partial J_2}\mathbf{B}^2\right\} \\ &= -p\mathbf{I} + \boldsymbol{\tau}\end{aligned}$$

hydrostatic pressure	:	p	:	extra unknown
incompressibility condition	:	$\det(\mathbf{F}) = J = 1$	:	extra equation

Rivlin models

$$W(\mathbf{C}) = \sum_{i=0}^{m} \sum_{j=0}^{n} C_{ij} \{J_1(\mathbf{C}) - 3\}^i \{J_2(\mathbf{C}) - 3\}^j \qquad \text{with} \quad C_{00} = 0$$

$$\begin{aligned}
J_1 &= \operatorname{tr}(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
J_2 &= \tfrac{1}{2} \left\{ \operatorname{tr}^2(\mathbf{C}) - \operatorname{tr}(\mathbf{C}^2) \right\} \\
&= \tfrac{1}{2} \left\{ \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right)^2 - \left(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 \right) \right\} \\
&= \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 \\
J_3 &= \det(\mathbf{C}) = \lambda_1^2 \lambda_2^2 \lambda_3^2 = 1
\end{aligned}$$

$$W(\mathbf{C}) = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \right)^i \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3 \right)^j$$

Neo-Hookean model

$$W = C_{10}(J_1 - 3)$$

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2C_{10}\mathbf{B}$$

Tensile test

$$\mathbf{B} = \lambda^2 \vec{e}_1 \vec{e}_1 + \mu^2 (\vec{e}_2 \vec{e}_2 + \vec{e}_3 \vec{e}_3) = \lambda^2 \vec{e}_1 \vec{e}_1 + \frac{1}{\lambda} (\vec{e}_2 \vec{e}_2 + \vec{e}_3 \vec{e}_3)$$

$$\sigma = -p\mathbf{I} + 2C_{10}\lambda^2 \vec{e}_1 \vec{e}_1 + 2C_{10}\frac{1}{\lambda}(\vec{e}_2 \vec{e}_2 + \vec{e}_3 \vec{e}_3)$$

$$\left.\begin{aligned} \sigma &= -p + 2C_{10}\lambda^2 \\ 0 &= -p + 2C_{10}\frac{1}{\lambda} \end{aligned}\right\} \quad \rightarrow$$

$$\sigma = 2C_{10}(\lambda^2 - \frac{1}{\lambda})$$

$$F = \sigma A = \sigma \mu^2 A_0 = \sigma \frac{1}{\lambda} A_0 = 2C_{10} A_0 (\lambda - \frac{1}{\lambda^2})$$

Mooney-Rivlin material model

$$W = C_{10}(J_1 - 3) + C_{01}(J_2 - 3)$$

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\{C_{10} + C_{01}\mathrm{tr}(\mathbf{B})\}\mathbf{B} - 2C_{01}\mathbf{B}^2$$

Tensile test

$$\mathbf{B} = \lambda^2 \vec{e}_1\vec{e}_1 + \mu^2(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) = \lambda^2\vec{e}_1\vec{e}_1 + \frac{1}{\lambda}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)$$

$$\mathrm{tr}(\mathbf{B}) = \lambda^2 + \frac{2}{\lambda}$$

$$\mathbf{B}^2 = \lambda^4\vec{e}_1\vec{e}_1 + \frac{1}{\lambda^2}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)$$

$$\begin{aligned}\sigma = -p\mathbf{I} + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\{\lambda^2\vec{e}_1\vec{e}_1 + \frac{1}{\lambda}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)\} \\ - 2C_{01}\{\lambda^4\vec{e}_1\vec{e}_1 + \frac{1}{\lambda^2}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)\}\end{aligned}$$

$$\left.\begin{aligned}\sigma &= -p + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\lambda^2 - 2C_{01}\lambda^4 \\ 0 &= -p + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\frac{1}{\lambda} - 2C_{01}\frac{1}{\lambda^2}\end{aligned}\right\} \quad \rightarrow$$

$$\sigma = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) + 2C_{01}(\lambda - \frac{1}{\lambda^2})$$

$$F = \sigma A = \sigma\mu^2 A_0 = \sigma\frac{1}{\lambda}A_0 = 2A_0\{C_{10}(\lambda - \frac{1}{\lambda^2}) + C_{01}(1 - \frac{1}{\lambda^3})\}$$

Other energy functions

3-term Mooney-Rivlin $\quad W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3)$

Signiorini $\quad W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{20}(J_1 - 3)^2$

Yeoh $\quad W = c_{10}(J_1 - 3) + c_{20}(J_1 - 3)^2 + c_{30}(J_1 - 3)^3$

2nd-order invariant model

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3) + c_{20}(J_1 - 3)^2$$

Kloaner-Segal

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{20}(J_1 - 3)^2 + c_{03}(J_2 - 3)^3$$

James, Green, Simpson (3rd-order deformation model)

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3) + c_{20}(J_1 - 3)^2 + c_{30}(J_1 - 3)^3$$

Ogden models

$$W = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n} J^{\frac{-\alpha_n}{3}} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right) + 4.5K \left(1 - J^{\frac{1}{3}}\right)^2$$

μ_n	:	moduli
α_n	:	exponents
K	:	bulk modulus
J	:	volume ratio $= \det(\mathbf{F})$

foam model

$$W = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right) + \sum_{n=1}^{N} \frac{\mu_n}{\beta_n} \left(1 - J^{\beta_n}\right)$$

Linear $\mathbf{P} - \mathbf{E}$ model

- stress update
- consistent material stiffness tensor for $\delta \mathbf{P}$ $\quad \longrightarrow$ Total Lagrange formulation
- consistent material stiffness tensor for $\delta \boldsymbol{\sigma}$ $\quad \longrightarrow$ Updated Lagrange formulation

Stress update

$$\mathbf{P} = c_0 \operatorname{tr}(\mathbf{E})\mathbf{I} + c_1 \mathbf{E} \qquad \text{with} \qquad \mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I})$$

$$= \tfrac{1}{2} c_0 \mathbf{C} : \mathbf{II} + \tfrac{1}{2} c_1 \mathbf{C} - \tfrac{1}{2}(3c_0 + c_1)\mathbf{I} \qquad \text{with} \qquad \mathbf{C} = \mathbf{F}^c \cdot \mathbf{F}$$

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}^c = J^{-1}\mathbf{F} \cdot (\mathbf{P} \cdot \mathbf{F}^c) = J^{-1}\mathbf{F} \cdot (\mathbf{F} \cdot \mathbf{P}^c)^c$$

Stiffness

$$
\begin{aligned}
\delta\mathbf{P} &= \tfrac{1}{2}c_0\delta\mathbf{C} : \mathbf{II} + \tfrac{1}{2}c_1\delta\mathbf{C} \\
&\qquad \mathbf{C} = \mathbf{F}^c \cdot \mathbf{F} \qquad \rightarrow \qquad \delta\mathbf{C} = \delta\mathbf{F}^c \cdot \mathbf{F} + \mathbf{F}^c \cdot \delta\mathbf{F} \\
&= \tfrac{1}{2}c_0\left(\delta\mathbf{F}^c \cdot \mathbf{F} + \mathbf{F}^c \cdot \delta\mathbf{F}\right) : \mathbf{II} + \tfrac{1}{2}c_1\left(\delta\mathbf{F}^c \cdot \mathbf{F} + \mathbf{F}^c \cdot \delta\mathbf{F}\right) \\
&= c_0(\mathbf{F}^c \cdot \delta\mathbf{F}) : \mathbf{II} + \tfrac{1}{2}c_1\left(\delta\mathbf{F}^c \cdot \mathbf{F} + \mathbf{F}^c \cdot \delta\mathbf{F}\right) \\
&= c_0\mathbf{I}(\mathbf{F}^c : \delta\mathbf{F}) + \tfrac{1}{2}c_1\left\{(\mathbf{F}^c \cdot \delta\mathbf{F})^c + (\mathbf{F}^c \cdot \delta\mathbf{F})\right\}
\end{aligned}
$$

$$
\boldsymbol{\sigma} = J^{-1}\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}^c \quad \rightarrow
$$

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= J^{-1}\left[-\delta J\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}^c + \delta\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}^c + \mathbf{F} \cdot \delta\mathbf{P} \cdot \mathbf{F}^c + \mathbf{F} \cdot \mathbf{P} \cdot \delta\mathbf{F}^c\right] \\
&\qquad \delta J = J\mathrm{tr}(\mathbf{L}) = J\mathbf{L} : \mathbf{I} \quad ; \quad \delta\mathbf{F} = \mathbf{L} \cdot \mathbf{F} \\
&= J^{-1}\left[-(\mathbf{L} : \mathbf{I})\mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}^c + (\mathbf{L} \cdot \mathbf{F}) \cdot \mathbf{P} \cdot \mathbf{F}^c + \right. \\
&\qquad\qquad \left. \mathbf{F} \cdot \delta\mathbf{P} \cdot \mathbf{F}^c + \mathbf{F} \cdot \mathbf{P} \cdot (\mathbf{F}^c \cdot \mathbf{L}^c)\right] \\
&= -(\mathbf{L} : \mathbf{I})\boldsymbol{\sigma} + \mathbf{L} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{L}^c + J^{-1}\mathbf{F} \cdot \delta\mathbf{P} \cdot \mathbf{F}^c \\
&= -\boldsymbol{\sigma}(\mathbf{I} : \mathbf{L}) + (\boldsymbol{\sigma}^c \cdot \mathbf{L}^c)^c + \boldsymbol{\sigma} \cdot \mathbf{L}^c + J^{-1}\mathbf{F} \cdot (\mathbf{F} \cdot \delta\mathbf{P}^c)^c
\end{aligned}
$$

Matrix/column notation

$$\mathbf{P} = \tfrac{1}{2} c_0 \mathbf{C} : \mathbf{II} + \tfrac{1}{2} c_1 \mathbf{C} - \tfrac{1}{2}(3c_0 + c_1)\mathbf{I} \qquad \text{with} \qquad \mathbf{C} = \mathbf{F}^c \cdot \mathbf{F}$$

$$\underset{\approx}{P} = \tfrac{1}{2} c_0 \underset{\approx}{C}^T \underset{\approx}{I}_t \underset{\approx}{I} + \tfrac{1}{2} c_1 \underset{\approx}{C} - \tfrac{1}{2}(3c_0 + c_1)\underset{\approx}{I} \qquad \text{with} \qquad \underset{\approx}{C} = \underline{\underline{F}}_t \underset{\approx}{F}$$

$$\boldsymbol{\sigma} = J^{-1} \mathbf{F} \cdot (\mathbf{F} \cdot \mathbf{P}^c)^c$$

$$\underset{\approx}{\sigma} = J^{-1} \underline{\underline{F}} \left(\underline{\underline{F}}_r \underset{\approx}{P}_t \right) = J^{-1} \underline{\underline{F}} \, \underline{\underline{F}}_r \underset{\approx}{P}_t$$

Matrix/column notation

$$
\begin{aligned}
\delta \mathbf{P} &= c_0 \mathbf{I}(\mathbf{F}^c : \delta \mathbf{F}) + \tfrac{1}{2} c_1 \left\{ (\mathbf{F}^c \cdot \delta \mathbf{F})^c + (\mathbf{F}^c \cdot \delta \mathbf{F}) \right\} \\
\delta \underset{\approx}{P} &= c_0 \underset{\approx}{I}\, \underset{\approx}{F}_t^T \delta \underset{\approx}{F}_t + \tfrac{1}{2} c_1 \left\{ (\underline{\underline{F}}_t \delta \underset{\approx}{F})_r + (\underline{\underline{F}}_t \delta \underset{\approx}{F}) \right\} \\
&= c_0 \underset{\approx}{I}\, \underset{\approx}{F}_{tc}^T \delta \underset{\approx}{F} + \tfrac{1}{2} c_1 \left(\underline{\underline{F}}_{tr} \delta \underset{\approx}{F} + \underline{\underline{F}}_t \delta \underset{\approx}{F} \right) \\
&= \underline{\underline{M}}_0 \delta \underset{\approx}{F} = \underline{\underline{M}}_0 \left(\underset{\approx}{L}_0 \right)_t = \underline{\underline{M}}_0 \underline{\underline{F}}_{tr} \underset{\approx}{L}_t = \underline{\underline{M}}_1 \underset{\approx}{L}_t
\end{aligned}
$$

$$
\begin{aligned}
\delta \boldsymbol{\sigma} &= -\boldsymbol{\sigma}(\mathbf{I} : \mathbf{L}) + (\boldsymbol{\sigma}^c \cdot \mathbf{L}^c)^c + \boldsymbol{\sigma} \cdot \mathbf{L}^c + J^{-1} \mathbf{F} \cdot (\mathbf{F} \cdot \delta \mathbf{P}^c)^c \\
\delta \underset{\approx}{\sigma} &= -\underset{\approx}{\sigma}\, \underset{\approx}{I}^T \underset{\approx}{L}_t + \underline{\underline{\sigma}}_{tr} \underset{\approx}{L}_t + \underline{\underline{\sigma}} \underset{\approx}{L}_t + J^{-1} \underline{\underline{F}}\, \underline{\underline{F}}_r \delta \underset{\approx}{P}_t \\
&= -\underset{\approx}{\sigma}\, \underset{\approx}{I}^T \underset{\approx}{L}_t + \underline{\underline{\sigma}}_{tr} \underset{\approx}{L}_t + \underline{\underline{\sigma}} \underset{\approx}{L}_t + J^{-1} \underline{\underline{F}}\, \underline{\underline{F}}_{rc} \delta \underset{\approx}{P} \\
&= \left[-\underset{\approx}{\sigma}\, \underset{\approx}{I}^T + \underline{\underline{\sigma}}_{tr} + \underline{\underline{\sigma}} + J^{-1} \underline{\underline{F}}\, \underline{\underline{F}}_{rc} \underline{\underline{M}}_1 \right] \underset{\approx}{L}_t = \underline{\underline{M}}\, \underset{\approx}{L}_t
\end{aligned}
$$

$\boldsymbol{\sigma} - \mathbf{A}$ model

- stress update
- consistent material stiffness tensor for $\delta\boldsymbol{\sigma}$ $\quad \rightarrow$ Updated Lagrange formulation

Stress update

$$\boldsymbol{\sigma} = c_0 \mathrm{tr}(\mathbf{A})\mathbf{I} + c_1 \mathbf{A} \qquad \text{with} \qquad \mathbf{A} = \tfrac{1}{2}(\mathbf{B} - \mathbf{I})$$

$$= \tfrac{1}{2} c_0 \mathbf{B} : \mathbf{II} + \tfrac{1}{2} c_1 \mathbf{B} - \tfrac{1}{2}(3c_0 + c_1)\mathbf{I} \qquad \text{with} \qquad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^c$$

Stiffness

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= \tfrac{1}{2}c_0\delta\mathbf{B} : \mathbf{II} + \tfrac{1}{2}c_1\delta\mathbf{B} \\
&= \tfrac{1}{2}c_0\left\{(\mathbf{F}\cdot\delta\mathbf{F}^c)^c + \mathbf{F}\cdot\delta\mathbf{F}^c\right\} : \mathbf{II} + \tfrac{1}{2}c_1\left\{(\mathbf{F}\cdot\delta\mathbf{F}^c)^c + \mathbf{F}\cdot\delta\mathbf{F}^c\right\} \\
&= c_0(\mathbf{F}\cdot\delta\mathbf{F}^c) : \mathbf{II} + \tfrac{1}{2}c_1\left\{(\mathbf{F}\cdot\delta\mathbf{F}^c)^c + \mathbf{F}\cdot\delta\mathbf{F}^c\right\} \\
&= c_0\mathbf{IF} : \delta\mathbf{F}^c + \tfrac{1}{2}c_1\left\{(\mathbf{F}\cdot\delta\mathbf{F}^c)^c + \mathbf{F}\cdot\delta\mathbf{F}^c\right\}
\end{aligned}
$$

with $\qquad \delta\mathbf{F} = \mathbf{L}\cdot\mathbf{F} = (\mathbf{F}^c\cdot\mathbf{L}^c)^c \qquad$ and $\quad \mathbf{L}^c = \vec{\nabla}\vec{u}$

Matrix/column notation

$$\boldsymbol{\sigma} = \tfrac{1}{2}c_0\mathbf{B}:\mathbf{II} + \tfrac{1}{2}c_1\mathbf{B} - \tfrac{1}{2}(3c_0+c_1)\mathbf{I} \qquad \text{with} \qquad \mathbf{B} = \mathbf{F}\cdot\mathbf{F}^c$$

$$\underset{\approx}{\sigma} = \tfrac{1}{2}c_0\underset{\approx}{B}^T\underset{\approx t}{I}\underset{\approx}{I} + \tfrac{1}{2}c_1\underset{\approx}{B} - \tfrac{1}{2}(3c_0+c_1)\underset{\approx}{I} \qquad \text{with} \qquad \underset{\approx}{B} = \underline{\underline{F}}\,\underset{\approx}{F}_t$$

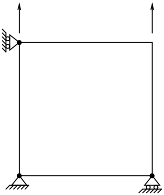
$$\delta\boldsymbol{\sigma} = c_0\mathbf{IF}:\delta\mathbf{F}^c + \tfrac{1}{2}c_1\{(\mathbf{F}\cdot\delta\mathbf{F}^c)^c + \mathbf{F}\cdot\delta\mathbf{F}^c\} \qquad \text{with} \qquad \delta\mathbf{F} = \mathbf{L}\cdot\mathbf{F} = (\mathbf{F}^c\cdot\mathbf{L}^c)^c$$

$$\begin{aligned}
\delta\underset{\approx}{\sigma} &= c_0\underset{\approx}{I}\underset{\approx}{F}^T\delta\underset{\approx}{F} + \tfrac{1}{2}c_1\left\{\underline{\underline{F}}_r\delta\underset{\approx}{F}_t + \underline{\underline{F}}\delta\underset{\approx}{F}_t\right\} \\
&= \left[c_0\underset{\approx}{I}\underset{\approx}{F}^T + \tfrac{1}{2}c_1\left\{\underline{\underline{F}}_{rc} + \underline{\underline{F}}_c\right\}\right]\delta\underset{\approx}{F} \qquad \text{with} \qquad \delta\underset{\approx}{F} = \left(\underline{\underline{F}}_t\underset{\approx}{L}_t\right)_r = \underline{\underline{F}}_{tr}\,\underset{\approx}{L}_t \\
&= \left[c_0\underset{\approx}{I}\underset{\approx}{F}^T\underline{\underline{F}}_{tr} + \tfrac{1}{2}c_1\left(\underline{\underline{F}}_{rc}\underline{\underline{F}}_{tr} + \underline{\underline{F}}_c\underline{\underline{F}}_{tr}\right)\right]\underset{\approx}{L}_t = \underline{\underline{M}}\,\underset{\approx}{L}_t
\end{aligned}$$

Examples

- Tensile test
- Shear test

Tensile test



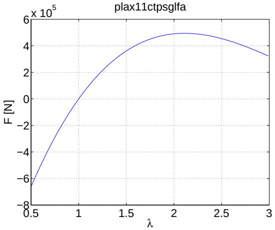
Cartesian			
initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

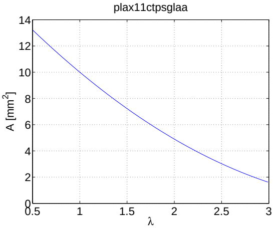
cylindrical			
initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

modulus	C	100000	MPa
Poisson ratio	ν	0.3	-

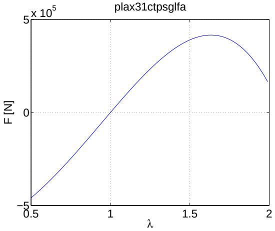
Elastic models in tensile test

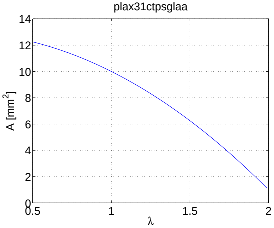
plane stress; $\boldsymbol{\sigma} \sim \boldsymbol{\varepsilon}$





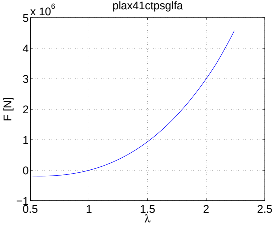
plane stress; $\boldsymbol{\sigma} \sim \mathbf{A}$

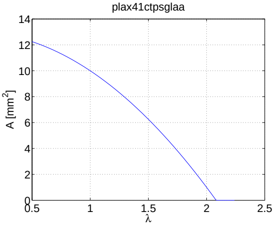




Elastic models in tensile test

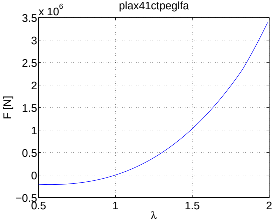
plane stress; $\mathbf{P} \sim \mathbf{E}$

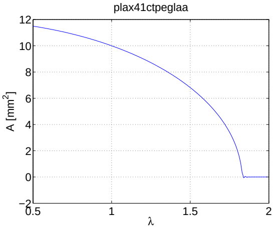




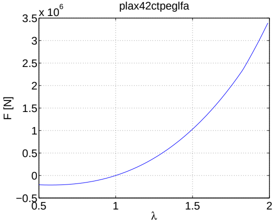
Total Lagrange formulation

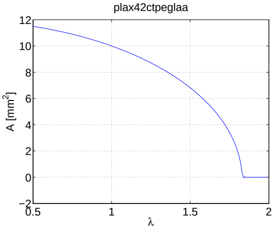
plane strain; $\mathbf{P} \sim \mathbf{E}$; Upd.Lag.



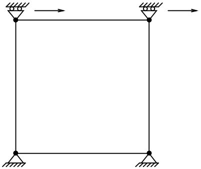


plane strain; $\mathbf{P} \sim \mathbf{E}$; Tot.Lag.





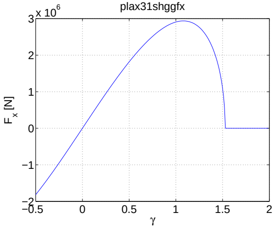
Shear test

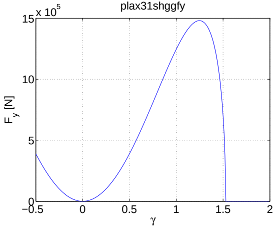


initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

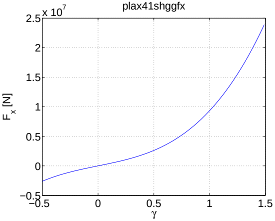
Elastic models in shear test

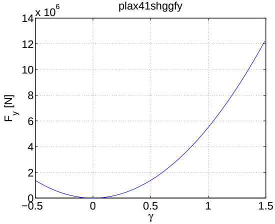
plane stress; $\boldsymbol{\sigma} \sim \mathbf{A}$





plane stress; $\mathbf{P} \sim \mathbf{E}$

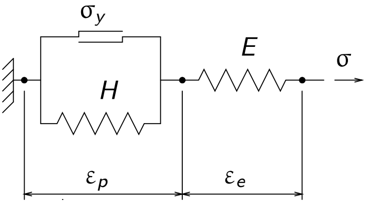




ELASTOPLASTIC

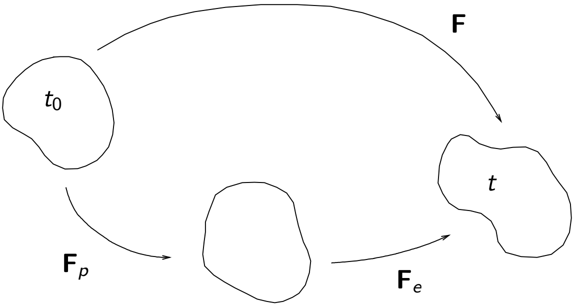
back to index

Elastoplastic material behavior



$$
\begin{aligned}
\sigma < \sigma_y &\quad \rightarrow \quad \varepsilon = \varepsilon_e \\
\sigma = \sigma_y &\quad \rightarrow \quad \dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p
\end{aligned}
$$

Kinematics



$$\mathbf{F} = (\vec{\nabla}_0 \vec{x})^c = \mathbf{F}_e \cdot \mathbf{F}_p$$

$$\mathbf{C} = \mathbf{F}^c \cdot \mathbf{F} \qquad ; \qquad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^c \qquad ; \qquad \mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I})$$

$$\begin{aligned} \mathbf{L} &= \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = (\vec{\nabla} \vec{v})^c \\ &= \mathbf{L}_e + \mathbf{L}_p = (\mathbf{D}_e + \mathbf{\Omega}_e) + (\mathbf{D}_p + \mathbf{\Omega}_p) = (\mathbf{D}_e + \mathbf{\Omega}_e) + \mathbf{D}_p \end{aligned}$$

Elastic deformation

metal alloys $\quad\rightarrow\quad$ small elastic strains $\quad\rightarrow\quad$ hypo-elastic model

$$\left.\begin{array}{l} \boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\Lambda}_e \\[2ex] {}^4\mathbf{C} = c_0 \mathbf{II} + \frac{1}{2} c_1 ({}^4\mathbf{I} + {}^4\mathbf{I}^{rc}) = K\mathbf{II} + 2G \left({}^4\mathbf{I} - \frac{1}{3}\mathbf{II} \right) \end{array} \right\}$$

invariant tensors

$$\boldsymbol{\sigma}_A = \mathbf{A} \cdot \boldsymbol{\sigma} \cdot \mathbf{A}^c = \boldsymbol{\sigma}_A^* \qquad \text{with} \quad \mathbf{A}^* = \mathbf{A} \cdot \mathbf{Q}^c \quad \forall \quad \mathbf{Q}$$

$$\dot{\boldsymbol{\sigma}}_A = \mathbf{A} \cdot \left\{ (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\mathbf{A}^{-1} \cdot \dot{\mathbf{A}})^c + \dot{\boldsymbol{\sigma}} \right\} \cdot \mathbf{A}^c = \mathbf{A} \cdot \overset{\odot}{\boldsymbol{\sigma}}_A \cdot \mathbf{A}^c = \dot{\boldsymbol{\sigma}}_A^*$$

objective elastic law $\qquad \overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\mathbf{C} : \mathbf{D}_e$

Yield criterion and hardening

yield criterion $F = \bar{\sigma}^2 - \sigma_y^2(\bar{\varepsilon}_p)$

effective plastic strain $\bar{\varepsilon}_p = \int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p \, d\tau$

hardening law $\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p) \quad \text{with} \quad \dfrac{\partial \sigma_y}{\partial \bar{\varepsilon}_p} = H(\bar{\varepsilon}_p)$

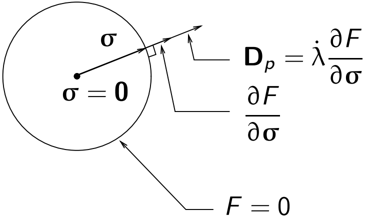
Kuhn-Tucker relations

$$\{(F < 0) \vee (F = 0 \wedge \dot{F} < 0)\} \quad \rightarrow \quad \text{elastic}$$
$$\{(F = 0) \wedge (\dot{F} = 0)\} \quad \rightarrow \quad \text{elastoplastic}$$

Von Mises plasticity

$$
\begin{aligned}
\bar{\sigma} &= \sqrt{\tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} \\
\dot{\bar{\varepsilon}}_p &= \sqrt{\tfrac{2}{3}\mathbf{D}_p : \mathbf{D}_p} \\
F &= \tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d - \sigma_y^2(\bar{\varepsilon}_p) \\
\dot{F} &= 2\bar{\sigma}\dot{\bar{\sigma}} - 2\sigma_y\dot{\sigma}_y = 2\bar{\sigma}\dot{\bar{\sigma}} - 2\sigma_y H\dot{\bar{\varepsilon}}_p \\
&= 3\boldsymbol{\sigma}^d : \dot{\boldsymbol{\sigma}} - 2\sigma_y H\dot{\bar{\varepsilon}}_p = 3\boldsymbol{\sigma}_A^d : \dot{\boldsymbol{\sigma}}_A - 2\sigma_y H\dot{\bar{\varepsilon}}_p = 0
\end{aligned}
$$

Elastoplastic deformation



$$\mathbf{D}_p = \dot{\lambda}\,\frac{\partial F}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\mathbf{a}$$

$$\mathbf{a} = \frac{\partial F}{\partial \boldsymbol{\sigma}^d} : \frac{\partial \boldsymbol{\sigma}^d}{\partial \boldsymbol{\sigma}} = \left[3\boldsymbol{\sigma}^d : {}^4\mathbf{I}\right] : \frac{\partial}{\partial \boldsymbol{\sigma}}\left\{\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}\right\} = 3\boldsymbol{\sigma}^d : \left({}^4\mathbf{I} - \tfrac{1}{3}\mathbf{I}\mathbf{I}\right) = 3\boldsymbol{\sigma}^d$$

$$\dot{\bar{\varepsilon}}_p = \dot{\lambda}\sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}}$$

Constitutive model

$$\{(F<0)\vee(F=0\wedge\dot{F}<0)\} \quad\rightarrow\quad \mathbf{D}=\mathbf{D}_e \quad\rightarrow\quad \dot{\bar{\varepsilon}}_p=0$$

$$\overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\mathbf{C}:\mathbf{D} \quad\rightarrow\quad \dot{\boldsymbol{\sigma}}_A = {}^4\mathbf{C}_A:\mathbf{D}_A$$

$$\{(F=0)\wedge(\dot{F}=0)\} \quad\rightarrow\quad \mathbf{D}=\mathbf{D}_e+\mathbf{D}_p$$

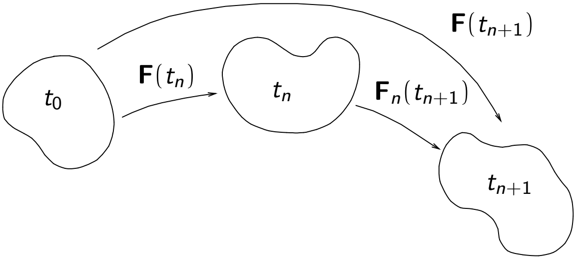
$$\left.\begin{array}{l}\overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\mathbf{C}:(\mathbf{D}-\dot{\lambda}\mathbf{a}) \\ 2\bar{\sigma}\dot{\bar{\sigma}}-2\sigma_y H\dot{\bar{\varepsilon}}_p=0\end{array}\right\} \quad\rightarrow$$

$$\left.\begin{array}{l}\dot{\boldsymbol{\sigma}}_A = {}^4\mathbf{C}_A:\left(\mathbf{D}_A-\dot{\lambda}\mathbf{a}_A\right) \\ 3\boldsymbol{\sigma}_A^d:\dot{\boldsymbol{\sigma}}_A-2\sigma_y H\dot{\lambda}\sqrt{\frac{2}{3}\mathbf{a}_A:\mathbf{a}_A}=0\end{array}\right\}$$

$$\left.\begin{array}{l}\dot{\boldsymbol{\sigma}}_A = {}^4\mathbf{C}_A:\left(\mathbf{D}_A-\dot{\lambda}\mathbf{a}_A\right) \\ 3\boldsymbol{\sigma}_A^d:{}^4\mathbf{C}_A:\mathbf{D}_A-\dot{\lambda}\left(3\boldsymbol{\sigma}_A^d:{}^4\mathbf{C}_A:\mathbf{a}_A+2\sigma_v H\sqrt{\frac{2}{3}\mathbf{a}_A:\mathbf{a}_A}\right)=0\end{array}\right\}$$

$$\sigma_y=\sigma_y(\sigma_{y0},\bar{\varepsilon}_p)$$

Incremental analysis



$$\mathbf{F}(\tau) = \mathbf{F}_n(\tau) \cdot \mathbf{F}(t_n) \quad \rightarrow \quad \mathbf{F}_n(\tau) = (\vec{\nabla}_n \vec{x})^c = \mathbf{F}(\tau) \cdot \mathbf{F}^{-1}(t_n)$$

$$\mathbf{D} = \frac{1}{2}\left(\dot{\mathbf{F}}_n \cdot \mathbf{F}_n^{-1} + \mathbf{F}_n^{-c} \cdot \dot{\mathbf{F}}_n^c\right) = \frac{1}{2}\mathbf{R}_n \cdot \left(\dot{\mathbf{U}}_n \cdot \mathbf{U}_n^{-1} + \mathbf{U}_n^{-1} \cdot \dot{\mathbf{U}}_n\right) \cdot \mathbf{R}_n^c$$

$$\mathbf{\Omega} = \frac{1}{2}\left\{\dot{\mathbf{F}}_n \cdot \mathbf{F}_n^{-1} - \mathbf{F}_n^{-c} \cdot \dot{\mathbf{F}}_n^c\right\} = \dot{\mathbf{R}}_n \cdot \mathbf{R}_n^c + \frac{1}{2}\mathbf{R}_n \cdot \left(\dot{\mathbf{U}}_n \cdot \mathbf{U}_n^{-1} - \mathbf{U}_n^{-1} \cdot \dot{\mathbf{U}}_n\right) \cdot \mathbf{R}_n^c$$

$$\mathbf{U}_n = \sum_{i=1}^{3} \lambda_{ni} \vec{n}_{ni} \vec{n}_{ni} \quad ; \quad \mathbf{\Lambda}_n = \sum_{i=1}^{3} \ln(\lambda_{ni}) \vec{n}_{ni} \vec{n}_{ni}$$

Elastic stress predictor

elastic trial stress $\quad \boldsymbol{\sigma}_e = \boldsymbol{\sigma}(t_n) + {}^4\mathbf{C} : (\boldsymbol{\Lambda} - \boldsymbol{\Lambda}(t_n))$

yield criterion $\quad F = \frac{3}{2}\boldsymbol{\sigma}_e^d : \boldsymbol{\sigma}_e^d - \sigma_y^2(\sigma_{y0}, \bar{\varepsilon}_p(t_n))$

$$
\begin{array}{lll}
F \leq 0 & \rightarrow & \text{elastic increment} \\
F > 0 & \rightarrow & \text{elastoplastic increment}
\end{array}
$$

matrix/column notation

$$
\begin{aligned}
&\underline{\underline{C}} = K \underset{\approx}{I}\underset{\approx}{I}^T + 2G\left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}^T\right) \\
&\underline{\Lambda}_n \quad \rightarrow \quad \underset{\approx}{\Lambda}_n \\
&\underset{\approx}{\sigma}_{D_e} = \underset{\approx}{\sigma}(t_n) + \underline{\underline{C}}_c \, \underset{\approx}{\Lambda}_n \quad \rightarrow \quad \underline{\sigma}_{D_e} \quad \rightarrow \\
&\underline{\sigma}_e = \underline{R}_n \, \underline{\sigma}_{D_e} \, \underline{R}_n^T \\
&F = \tfrac{3}{2}\left(\underset{\approx}{\sigma}_{D_{tr}}\right)^T \left(\underset{\approx}{\sigma}_{D_{tr}}\right) - \sigma_y^2(\bar{\varepsilon}_p)
\end{aligned}
$$

Elastic increment

$$
\begin{aligned}
\boldsymbol{\sigma}(t_{n+1}) &= \boldsymbol{\sigma}_e \\
\Delta\lambda &= 0 \\
\bar{\varepsilon}_p(t_{n+1}) &= \bar{\varepsilon}_p(t_n) \\
\sigma_y(t_{n+1}) &= \sigma_y(t_n)
\end{aligned}
$$

Elastoplastic increment

$$\left.\begin{array}{l}\dot{\boldsymbol{\sigma}}_A = {}^4\mathbf{C}_A : \left(\mathbf{D}_A - \dot{\lambda}\mathbf{a}_A\right) \\ 3\boldsymbol{\sigma}_A^d : {}^4\mathbf{C}_A : \mathbf{D}_A - \dot{\lambda}\left(3\boldsymbol{\sigma}_A^d : {}^4\mathbf{C}_A : \mathbf{a}_A + 2\sigma_v H\sqrt{\frac{2}{3}\mathbf{a}_A : \mathbf{a}_A}\right) = 0\end{array}\right\}$$

Dienes tensor and Dienes derivative

$$\left.\begin{array}{l}\boldsymbol{\sigma}_D = \mathbf{R}_n^c \cdot \boldsymbol{\sigma} \cdot \mathbf{R}_n \quad \rightarrow \quad \dot{\boldsymbol{\sigma}}_D = \mathbf{R}_n^c \cdot \overset{\odot}{\boldsymbol{\sigma}}_D \cdot \mathbf{R}_n \\ \mathbf{D}_D = \mathbf{R}_n^c \cdot \mathbf{D} \cdot \mathbf{R}_n = \frac{1}{2}\left(\dot{\mathbf{U}}_n \cdot \mathbf{U}_n^{-1} + \mathbf{U}_n^{-1} \cdot \dot{\mathbf{U}}_n\right)\end{array}\right\}$$

$$\left.\begin{array}{l}\dot{\boldsymbol{\sigma}}_D = {}^4\mathbf{C}_D : \left(\mathbf{D}_D - \dot{\lambda}\mathbf{a}_D\right) \\ 3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C}_D : \mathbf{D}_D - \dot{\lambda}\left(3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C}_D : \mathbf{a}_D + 2\sigma_y H\sqrt{\frac{2}{3}\mathbf{a}_D : \mathbf{a}_D}\right) = 0\end{array}\right\}$$

Rotation neutralized elastoplastic increment

- incremental rotation neutralized

$$\begin{array}{lcl} t_n \leq \tau < t_{n+1} & : & \mathbf{R}_n = I \quad ; \quad \mathbf{D}_D = \mathbf{D} \quad ; \quad \mathbf{a}_D = \mathbf{a} \quad ; \quad {}^4\mathbf{C}_D = {}^4\mathbf{C} \\ \tau = t_{n+1} & : & \mathbf{R}_n(t_{n+1}) = \mathbf{F}(t_{n+1}) \cdot \mathbf{U}^{-1}(t_{n+1}) \end{array}$$

- incremental principal strain directions constant $\quad \vec{n}_{ni}(\tau) = \vec{n}_{ni}(t_n)$

$$\mathbf{U}_n(\tau) = \sum_{i=1}^{3} \lambda_{ni}(\tau)\vec{n}_{ni}(t_n)\vec{n}_{ni}(t_n)$$

$$\mathbf{D} = \dot{\mathbf{U}}_n \cdot \mathbf{U}_n^{-1} = \sum_{i=1}^{3} \left(\frac{\dot{\lambda}_{ni}(\tau)}{\lambda_{ni}(\tau)} \right) \vec{n}_{ni}(t_n)\vec{n}_{ni}(t_n) = \dot{\mathbf{\Lambda}}_n$$

constitutive equations

$$\left. \begin{array}{l} \dot{\boldsymbol{\sigma}}_D = {}^4\mathbf{C} : \left\{ \dot{\mathbf{\Lambda}}_n - \dot{\lambda}\mathbf{a} \right\} \\ 3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \dot{\mathbf{\Lambda}}_n - \dot{\lambda} \left(3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \mathbf{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}} \right) = 0 \end{array} \right\}$$

Rotation neutralized stress update

$$\left.\begin{aligned}
&\dot{\boldsymbol{\sigma}}_D = {}^4\mathbf{C} : \left\{ \dot{\boldsymbol{\Lambda}}_n - \dot{\lambda}\mathbf{a} \right\} \\
&3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \dot{\boldsymbol{\Lambda}}_n - \dot{\lambda}\left(3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \mathbf{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}} \right) = 0
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\boldsymbol{\sigma}_D = \boldsymbol{\sigma}_D(t_n) + {}^4\mathbf{C} : (\boldsymbol{\Lambda}_n - \Delta\lambda\mathbf{a}) \\
&3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \boldsymbol{\Lambda}_n - \Delta\lambda\left(3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \mathbf{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}} \right) = 0
\end{aligned}\right\}$$

Iterative stress update

$$\left.\begin{aligned} {}^4\mathbf{R} : \delta\boldsymbol{\sigma}_D + \mathbf{t}\delta\lambda &= -\mathbf{s}_1 \\ \mathbf{u} : \delta\boldsymbol{\sigma}_D + v\delta\lambda &= -s_2 \end{aligned}\right\}$$

$$\begin{aligned}
{}^4\mathbf{R} &= {}^4\mathbf{I} + 3\Delta\lambda\, {}^4\mathbf{C} : {}^4\mathbf{I} \\
\mathbf{t} &= {}^4\mathbf{C} : \mathbf{a} \\
\mathbf{u} &= \left(3\,{}^4\mathbf{C} - \mathbf{II} : {}^4\mathbf{C}\right) : \boldsymbol{\Lambda}_n - \Delta\lambda \left\{ \left(3\,{}^4\mathbf{C} - \mathbf{II} : {}^4\mathbf{C}\right) : \mathbf{a} + 4\sigma_y H \left(\tfrac{2}{3}\mathbf{a} : \mathbf{a}\right)^{-\frac{1}{2}} \mathbf{a} : {}^4\mathbf{I} \right\} \\
v &= 3\,{}^4\mathbf{C} : \mathbf{a} : \boldsymbol{\sigma}_D^d + 2\sigma_y H \sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}} \\
\mathbf{s}_1 &= \boldsymbol{\sigma}_D - \boldsymbol{\sigma}_D(t_n) - {}^4\mathbf{C} : \boldsymbol{\Lambda}_n + \Delta\lambda\, {}^4\mathbf{C} : \mathbf{a} \\
s_2 &= 3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \boldsymbol{\Lambda}_n - \Delta\lambda \left(3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \mathbf{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}} \right)
\end{aligned}$$

Stiffness

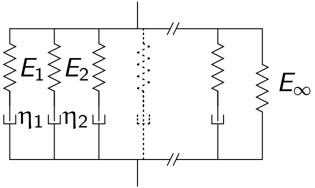
$$\left.\begin{aligned}
&\boldsymbol{\sigma}_D - \boldsymbol{\sigma}_D(t_n) - {}^4\mathbf{C} : \boldsymbol{\Lambda}_n + \Delta\lambda\,{}^4\mathbf{C} : \mathbf{a} = 0 \\
&3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \boldsymbol{\Lambda}_n - \Delta\lambda\left(3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \mathbf{a} + 2\sigma_y H\sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}}\right) = 0
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\delta\boldsymbol{\sigma}_D = \boldsymbol{\sigma}_D(t_n) + {}^4\mathbf{C} : \delta\boldsymbol{\Lambda}_n - \delta\lambda\,{}^4\mathbf{C} : \mathbf{a} - \Delta\lambda\,{}^4\mathbf{C} : \delta\mathbf{a} = 0 \\
&3\delta\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \boldsymbol{\Lambda}_n + 3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \delta\boldsymbol{\Lambda}_n - \\
&\qquad \delta\lambda\left(3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \mathbf{a} + 2\sigma_y H\sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}}\right) - \\
&\qquad \Delta\lambda\left(3\delta\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \mathbf{a} + 3\boldsymbol{\sigma}_D^d : {}^4\mathbf{C} : \delta\mathbf{a} + \right. \\
&\qquad\qquad \left. 2\delta\sigma_y H\sqrt{\tfrac{2}{3}\mathbf{a} : \mathbf{a}} + 2\sigma_y H\tfrac{1}{2}[\tfrac{2}{3}\mathbf{a} : \mathbf{a}]^{-1/2}\tfrac{4}{3}\mathbf{a} : \delta\mathbf{a}\right) = 0
\end{aligned}\right\}$$

LINEAR VISCOELASTIC

back to index

Linear viscoelastic material behavior



$$\boldsymbol{\sigma}(t) = \int\limits_{\tau=0}^{t} {}^4\mathbf{C}(t-\tau) : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau$$

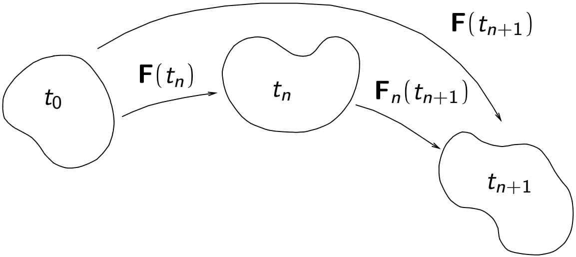
$${}^4\mathbf{C}(t) = {}^4\mathbf{C}_\infty + \sum_{i=1}^{N} {}^4\mathbf{C}_i e^{-\frac{t}{\tau_i}}$$

Constitutive model

$$
\left.\begin{aligned}
\boldsymbol{\sigma}(t) &= \int\limits_{\tau=0}^{t} {}^4\mathbf{C}(t-\tau) : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\
{}^4\mathbf{C}(t) &= {}^4\mathbf{C}_\infty + \sum_{i=1}^{N} {}^4\mathbf{C}_i \; e^{-\frac{t}{\tau_i}}
\end{aligned}\right\} \quad \rightarrow
$$

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= \int\limits_{\tau=0}^{t} \left[{}^4\mathbf{C}_\infty + \sum_{i=1}^{N} {}^4\mathbf{C}_i e^{-\frac{t-\tau}{\tau_i}} \right] : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\
&= {}^4\mathbf{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} {}^4\mathbf{C}_i : \int\limits_{\tau=0}^{t} e^{-\frac{t-\tau}{\tau_i}} \; \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\
&= {}^4\mathbf{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t)
\end{aligned}
$$

Incremental analysis



$$[0, t] \quad \rightarrow \quad [t_1 = 0, t_2, t_3, .., t_n, t_{n+1} = t]$$

$$\Delta t = t_{i+1} - t_i \qquad ; \quad i = 1, ..., n$$

$$\varepsilon(\tau) = \varepsilon(t_n) + (\tau - t_n)\,\frac{\Delta\varepsilon}{\Delta t} \quad \rightarrow \quad \dot{\varepsilon}(\tau) = \frac{\Delta\varepsilon}{\Delta t}$$

Stress update

$$
\begin{aligned}
\boldsymbol{\sigma}_i(t) &= {}^4\mathbf{C}_i : \int\limits_{\tau=0}^{t} e^{-\frac{t-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\
&= {}^4\mathbf{C}_i : \left[\int\limits_{\tau=0}^{t_n} e^{-\frac{t-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau + \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \right] \\
&= {}^4\mathbf{C}_i : \left[e^{-\frac{\Delta t}{\tau_i}} \int\limits_{\tau=0}^{t_n} e^{-\frac{t_n-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau + \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \right] \\
&= e^{-\frac{\Delta t}{\tau_i}}\ {}^4\mathbf{C}_i : \int\limits_{\tau=0}^{t_n} e^{-\frac{t_n-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau + {}^4\mathbf{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\
&= e^{-\frac{\Delta t}{\tau_i}}\ \boldsymbol{\sigma}_i(t_n) + {}^4\mathbf{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau
\end{aligned}
$$

Stress update

$$\sigma_i(t) = e^{-\frac{\Delta t}{\tau_i}}\, \boldsymbol{\sigma}_i(t_n) + {}^4\mathbf{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, \frac{\Delta \varepsilon}{\Delta t}\, d\tau = e^{-\frac{\Delta t}{\tau_i}}\, \boldsymbol{\sigma}_i(t_n) + {}^4\mathbf{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, d\tau$$

$$= e^{-\frac{\Delta t}{\tau_i}}\boldsymbol{\sigma}_i(t_n) + {}^4\mathbf{C}_i : \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta \varepsilon}{\Delta t}$$

$$\begin{aligned}\boldsymbol{\sigma}(t) &= {}^4\mathbf{C}_\infty : \varepsilon(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t) \\ &= {}^4\mathbf{C}_\infty : \varepsilon(t) + \\ &\qquad \sum_{i=1}^{N}\left[e^{-\frac{\Delta t}{\tau_i}}\boldsymbol{\sigma}_i(t_n) + {}^4\mathbf{C}_i : \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta \varepsilon}{\Delta t}\right]\end{aligned}$$

Stiffness

$$\sigma(t) = {}^4\mathbf{C}_\infty : \varepsilon(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \sigma_i(t_n) + {}^4\mathbf{C}_i : \frac{\Delta\varepsilon}{\Delta t}\, \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}} \right) \right]$$

$$\begin{aligned} \delta\sigma &= \left[{}^4\mathbf{C}_\infty + \sum_{i=1}^{N} {}^4\mathbf{C}_i \, \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}} \right) \right] : \delta\varepsilon \\ &= {}^4\mathbf{M} : \delta\varepsilon \end{aligned}$$

Isotropic material

$$
\begin{aligned}
\boldsymbol{\sigma} &= {}^4\mathbf{C} : \boldsymbol{\varepsilon} \\
&= \left[\lambda \mathbf{I}\mathbf{I} + 2\mu\, {}^4\mathbf{I}^s\right] : \boldsymbol{\varepsilon} = \left[\lambda \mathbf{I}\mathbf{I} + \mu \left({}^4\mathbf{I} + {}^4\mathbf{I}^{rc} \right)\right] : \boldsymbol{\varepsilon} = \lambda \mathbf{I}\, \mathrm{tr}(\boldsymbol{\varepsilon}) + 2\mu\, \boldsymbol{\varepsilon} \\
&= (3\lambda + 2\mu)\, \tfrac{1}{3}\, \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\, \boldsymbol{\varepsilon}^d = (3\lambda + 2\mu)\, \boldsymbol{\varepsilon}^h + 2\mu\, \boldsymbol{\varepsilon}^d \\
&= 3K\, \boldsymbol{\varepsilon}^h + 2G\, \boldsymbol{\varepsilon}^d \\
&= \boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d
\end{aligned}
$$

$$
\begin{aligned}
K &= \tfrac{1}{3}\,(3\lambda + 2\mu) = \frac{E}{3(1-2\nu)} \\
\mu &= G = \frac{E}{2(1+\nu)} \\
\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)}
\end{aligned}
$$

Isotropic viscoelastic material

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= \boldsymbol{\sigma}^h(t) + \boldsymbol{\sigma}^d(t) \\
&= 3 \int\limits_{\tau=0}^{t} K(t-\tau)\, \frac{d}{d\tau} \left\{ \varepsilon^h(\tau) \right\}\, d\tau + 2 \int\limits_{\tau=0}^{t} G(t-\tau)\, \frac{d}{d\tau} \left\{ \varepsilon^d(\tau) \right\}\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
K(t) &= K_\infty + \sum_{i=1}^{n} K_i\, e^{-\frac{t}{\tau_i}} = \frac{1}{3(1-2\nu)} \left[E_\infty + \sum_{i=1}^{n} E_i\, e^{-\frac{t}{\tau_i}} \right] \\
G(t) &= G_\infty + \sum_{i=1}^{n} G_i\, e^{-\frac{t}{\tau_i}} = \frac{1}{2(1+\nu)} \left[E_\infty + \sum_{i=1}^{n} E_i\, e^{-\frac{t}{\tau_i}} \right]
\end{aligned}
$$

Stress update

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= {}^4\mathbf{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t) \\
&= {}^4\mathbf{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + {}^4\mathbf{C}_i : \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}} \right) \frac{\Delta \boldsymbol{\varepsilon}}{\Delta t} \right] \\
&= 3K_\infty \, \Delta \boldsymbol{\varepsilon}^h + 2G_\infty \, \Delta \boldsymbol{\varepsilon}^d + \\
&\qquad \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}} \right) \left\{ 3K_i \Delta \boldsymbol{\varepsilon}^h + 2G_i \Delta \boldsymbol{\varepsilon}^d \right\} \right]
\end{aligned}
$$

Stiffness

$$\delta\boldsymbol{\sigma} = 3K_\infty \delta\varepsilon^h + 2G_\infty \delta\varepsilon^d + \\ \sum_{i=1}^{N} \frac{\tau_i}{\Delta t}\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\left\{3K_i \delta\varepsilon^h + 2G_i \delta\varepsilon^d\right\}$$

Matrix/column notation

$$\underset{\sim}{\sigma}(t) = \left(3K_\infty \underline{\underline{A}}^h + 2G_\infty \underline{\underline{A}}^d\right) \Delta\underset{\sim}{\varepsilon} + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \underset{\sim}{\sigma}_i(t_n) + \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \left\{3K_i \underline{\underline{A}}^h + 2G_i \underline{\underline{A}}^d\right\}\right] \Delta\underset{\sim}{\varepsilon}$$

$$\delta\underset{\sim}{\sigma}(t) = \left[\left(3K_\infty \underline{\underline{A}}^h + 2G_\infty \underline{\underline{A}}^d\right) \delta\underset{\sim}{\varepsilon} + \sum_{i=1}^{N} \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \left(3K_i \underline{\underline{A}}^h + 2G_i \underline{\underline{A}}^d\right)\right] \delta\underset{\sim}{\varepsilon}$$

Initial stiffness formulation

$$\Delta\boldsymbol{\sigma}(t) = 3K_0\Delta\boldsymbol{\varepsilon}^h + 2G_0\Delta\boldsymbol{\varepsilon}^d - \sum_{i=1}^{N}\left[1-\left(1-e^{-\frac{\Delta t}{\tau_i}}\right)\frac{\tau_i}{\Delta t}\right]\left\{3K_i\Delta\boldsymbol{\varepsilon}^h + 2G_i\Delta\boldsymbol{\varepsilon}^d\right\} -$$

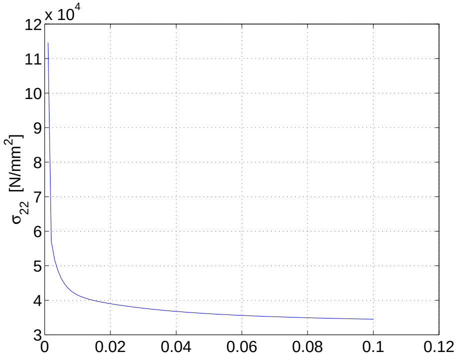
$$\sum_{i=1}^{N}\left(1-e^{-\frac{\Delta t}{\tau_i}}\right)\left\{\boldsymbol{\sigma}_i^h(t_n) + \boldsymbol{\sigma}_i^d(t_n)\right\}$$

$$\delta\boldsymbol{\sigma} = 3K_0\delta\boldsymbol{\varepsilon}^h + 2G_0\delta\boldsymbol{\varepsilon}^d -$$

$$\sum_{i=1}^{N}\left[1-\left(1-e^{-\frac{\Delta t}{\tau_i}}\right)\frac{\tau_i}{\Delta t}\right]\left\{3K_i\delta\boldsymbol{\varepsilon}^h + 2G_i\delta\boldsymbol{\varepsilon}^d\right\}$$

Tensile test

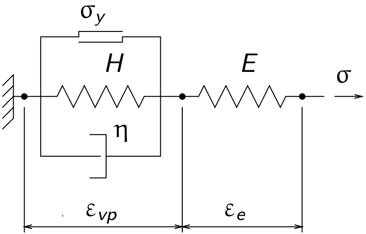
	E [MPa]	τ [s]		E [MPa]	τ [s]
1	3.0e6	3.1e-8	2	1.4e6	3.0e-7
3	3.9e6	3.0e-6	4	5.4e6	2.9e-5
5	1.3e6	2.8e-4	6	2.3e5	2.7e-3
7	7.6e4	2.6e-2	8	3.7e4	2.5e-1
9	3.3e4	2.5e+0	10	1.7e4	2.4e+1
11	8.0e3	2.3e+2	12	1.2e4	2.2e+3



VISCOPLASTIC

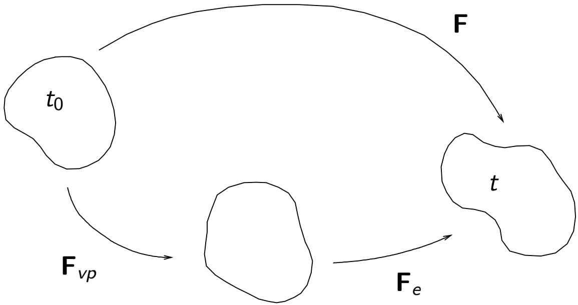
back to index

Viscoplastic material behavior



$$\sigma < \sigma_y \quad \rightarrow \quad \varepsilon = \varepsilon_e$$
$$\sigma \geq \sigma_y \quad \rightarrow \quad \dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_{vp}$$

Kinematics



$$\mathbf{F} = (\vec{\nabla}_0 \vec{x})^c = \mathbf{F}_e \cdot \mathbf{F}_{vp}$$

$$\mathbf{C} = \mathbf{F}^c \cdot \mathbf{F} \qquad ; \qquad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^c \qquad ; \qquad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = (\vec{\nabla} \vec{v})^c$$

$$= \mathbf{L}_e + \mathbf{L}_{vp} = (\mathbf{D}_e + \mathbf{\Omega}_e) + (\mathbf{D}_{vp} + \mathbf{\Omega}_{vp}) = (\mathbf{D}_e + \mathbf{\Omega}_e) + \mathbf{D}_{vp}$$

Elastic deformation

polymers $\quad\rightarrow\quad$ large elastic strains $\quad\rightarrow\quad$ hyper-elastic model

$$\left.\begin{aligned}
&\mathbf{P} = \frac{\partial W(\mathbf{E}_e)}{\partial \mathbf{E}_e} = 2\,\frac{\partial W}{\partial \mathbf{C}_e} = \mathbf{F}^{-1}\cdot\boldsymbol{\tau}\cdot\mathbf{F}^{-c} \quad\rightarrow\quad \dot{\mathbf{P}} = 2\frac{\partial^2 W}{\partial \mathbf{C}^2} : \dot{\mathbf{C}} \\[2em]
&W(\lambda_1,\lambda_2,\lambda_3) = \tfrac{1}{2}\,\mu\left\{\lambda_1^2+\lambda_2^2+\lambda_3^2-3-2\ln(J)\right\} + \tfrac{1}{2}\,\lambda\left\{\ln(J)\right\}^2 \\
&\text{with} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad ; \qquad \mu = \frac{E}{2(1+\nu)}
\end{aligned}\right\}$$

Yield criterion and hardening

yield criterion $\quad F = \bar{\tau} - \tau_y(\bar{\varepsilon}_{vp})$

effective viscoplastic strain $\quad \bar{\varepsilon}_{vp} = \int\limits_{\tau=0}^{t} \dot{\bar{\varepsilon}}_{vp} \, d\tau$

hardening law $\quad \tau_y = \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp}) \quad$ with $\quad \dfrac{\partial \tau_y}{\partial \bar{\varepsilon}_p} = H(\bar{\varepsilon}_p)$

Kuhn-Tucker relations

$F < 0 \quad \rightarrow \quad$ elastic deformation

$F \geq 0 \quad \rightarrow \quad$ viscoplastic deformation

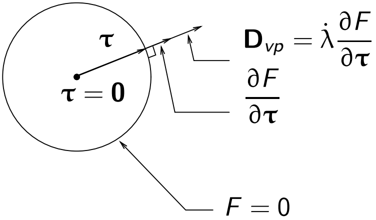
Von Mises plasticity

$$\bar{\tau} = \sqrt{\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d}$$

$$\dot{\bar{\varepsilon}}_{vp} = \sqrt{\tfrac{2}{3}\,\mathbf{D}_{vp} : \mathbf{D}_{vp}}$$

$$F = \sqrt{\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d} - \tau_y(\bar{\varepsilon}_{vp})$$

Viscoplastic deformation



$$\mathbf{D}_{vp} = \dot{\lambda}\,\frac{\partial F}{\partial \boldsymbol{\tau}} = \dot{\lambda}\mathbf{a} \quad \rightarrow \quad \dot{\mathbf{C}}_{vp} = 2\,\mathbf{F}^c \cdot \mathbf{D}_{vp} \cdot \mathbf{F} = 2\dot{\lambda}\,\mathbf{F}^c \cdot \mathbf{a} \cdot \mathbf{F}$$

$$\begin{aligned}
\mathbf{a} &= \frac{\partial F}{\partial \boldsymbol{\tau}^d} : \frac{\partial \boldsymbol{\tau}^d}{\partial \boldsymbol{\tau}} = \left[\tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2}\boldsymbol{\tau}^d : {}^4\mathbf{I}\right] : \left[\frac{\partial}{\partial \boldsymbol{\tau}}\left\{\boldsymbol{\tau} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}\right\}\right] \\
&= \tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2}\boldsymbol{\tau}^d : \left({}^4\mathbf{I} - \tfrac{1}{3}\mathbf{II}\right) = \tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2}\boldsymbol{\tau}^d \\
&= \tfrac{3}{2}\,\frac{1}{\bar{\tau}}\,\boldsymbol{\tau}^d \\
\dot{\bar{\varepsilon}}_{vp} &= \dot{\lambda}\sqrt{\tfrac{2}{3}\,\mathbf{a} : \mathbf{a}}
\end{aligned}$$

Constitutive model

$$F < 0 \quad \rightarrow \quad \dot{\mathbf{C}} = \dot{\mathbf{C}}_e$$

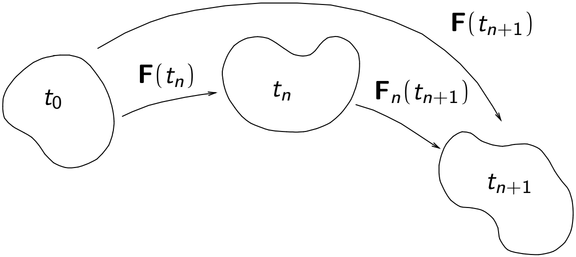
$$\dot{\mathbf{P}} = 2\,\frac{\partial^2 W}{\partial \mathbf{C}^2} : \dot{\mathbf{C}} \qquad ; \qquad \dot{\mathbf{C}}_{vp} = \mathbf{O} \qquad ; \qquad \dot{\bar{\varepsilon}}_{vp} = 0$$

$$F \geq 0 \quad \rightarrow \quad \dot{\mathbf{C}} = \dot{\mathbf{C}}_e + \dot{\mathbf{C}}_{vp} \quad \rightarrow$$

$$\left.\begin{array}{l} \dot{\mathbf{P}} = 2\,\dfrac{\partial^2 W}{\partial \mathbf{C}^2} : \left(\dot{\mathbf{C}} - 2\,\mathbf{F}^c \cdot \dot{\lambda}\mathbf{a} \cdot \mathbf{F}\right) \\ \dot{\lambda} = \gamma\phi(F) = \gamma\left(\dfrac{F}{\tau_{y0}}\right)^N \end{array}\right\}$$

$$\tau_y = \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp}) \qquad ; \qquad \dot{\bar{\varepsilon}}_{vp} = \dot{\lambda}\sqrt{\tfrac{2}{3}\,\mathbf{a} : \mathbf{a}}$$

Incremental analysis



$$\mathbf{F}(\tau) = \mathbf{F}_n(\tau) \cdot \mathbf{F}(t_n) \quad \rightarrow \quad \mathbf{F}_n(\tau) = \mathbf{F}(\tau) \cdot \mathbf{F}^{-1}(t_n)$$

$$\mathbf{F}_n = (\vec{\nabla}_n \vec{x})^c = \mathbf{R}_n \cdot \mathbf{U}_n \quad ; \quad J_n = \det(\mathbf{F}_n) \quad ; \quad \vec{\nabla} = \mathbf{F}_n^{-c} \cdot \vec{\nabla}_n$$

$$\mathbf{D} = \frac{1}{2} \left\{ (\vec{\nabla}\vec{v})^c + (\vec{\nabla}\vec{v}) \right\} = \frac{1}{2} \left(\dot{\mathbf{F}}_n \cdot \mathbf{F}_n^{-1} + \mathbf{F}_n^{-c} \cdot \dot{\mathbf{F}}_n^c \right)$$

Elastic stress predictor

elastic trial stress $\quad \mathbf{P}_e = \mathbf{P}_n + 2\,\dfrac{\partial^2 W}{\partial \mathbf{G}^2} : (\mathbf{C} - \mathbf{C}(t_n)) \quad \rightarrow \quad \boldsymbol{\tau}_e = \mathbf{F} \cdot \mathbf{P}_e \cdot \mathbf{F}^c$

yield criterion $\quad F = \sqrt{\tfrac{3}{2}\,(\boldsymbol{\tau}_e)^d : (\boldsymbol{\tau}_e)^d} - \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp}(t_n))$

$$
\begin{array}{lll}
F < 0 & \rightarrow & \text{elastic increment} \\
F \geq 0 & \rightarrow & \text{elastoviscoplastic increment}
\end{array}
$$

matrix/column notation

$$
\underset{\approx}{\tau}_e = \underset{\approx}{A} + \underline{\underline{H}}_c\, \underset{\approx}{e}_n
$$

$$
F = \sqrt{\tfrac{3}{2}\left(\underset{\approx}{\tau}_e\right)^T \left(\underset{\approx}{\tau}_e\right)_t} - \zeta(\kappa)
$$

$$
\text{with} \quad \left\{
\begin{array}{l}
\underline{\underline{H}} = 2\{\mu - \lambda \ln(J)\}\, \underline{\underline{I}} + \lambda \underset{\approx}{I}\underset{\approx}{I}^T \\
\underline{e}_n = \frac{1}{2}\left(\underline{I} - \underline{F}_n^{-T} \underline{F}_n^{-1}\right) \rightarrow \underset{\approx}{e}_n \\
\underline{A} = \underline{F}_n\, \underline{\tau}(t_n) \underline{F}_n^T \rightarrow \underset{\approx}{A}
\end{array}
\right.
$$

Elastic increment

$$
\begin{aligned}
&\boldsymbol{\tau}(t_{n+1}) = \boldsymbol{\tau}_e \\
&\Delta\lambda = 0 \\
&\bar{\varepsilon}_{vp}(t_{n+1}) = \bar{\varepsilon}_{vp}(t_n) \\
&\tau_y(t_{n+1}) = \tau_y(t_n)
\end{aligned}
$$

Viscoplastic increment

$$\left.\begin{aligned}
\dot{\mathbf{P}} &= 2\,\frac{\partial^2 W}{\partial \mathbf{C}^2} : (\dot{\mathbf{C}} - 2\,\mathbf{F}^c \cdot \dot{\lambda}\mathbf{a} \cdot \mathbf{F}) \\
\dot{\lambda} &= \gamma \phi(F) = \gamma \left(\frac{F}{\tau_{y0}}\right)^N
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\mathbf{P} &= \mathbf{P}(t_n) + 2\,\frac{\partial^2 W}{\partial \mathbf{C}^2} : \{\mathbf{C} - \mathbf{C}(t_n) - 2\,\mathbf{F}^c \cdot \Delta\lambda\,\mathbf{a} \cdot \mathbf{F}\} \\
\Delta\lambda &= \Delta t \gamma \phi(F)
\end{aligned}\right\}$$

Viscoplastic increment

$$\left.\begin{aligned}
&\mathbf{F}^{-1}\cdot\boldsymbol{\tau}\cdot\mathbf{F}^{-c} = \mathbf{F}^{-1}(t_n)\cdot\boldsymbol{\tau}(t_n)\cdot\mathbf{F}^{-c}(t_n) + 2\frac{\partial^2 W}{\partial \mathbf{C}^2} : \{\mathbf{C} - \mathbf{C}(t_n) - 2\,\mathbf{F}^c\cdot\Delta\lambda\,\mathbf{a}\cdot\mathbf{F}\}\\
&\Delta\lambda = \Delta t\,\gamma\phi(F)
\end{aligned}\right\}$$

$$\mathbf{F}_n = \mathbf{F}\cdot\mathbf{F}^{-1}(t_n) \quad\rightarrow\quad \mathbf{C} - \mathbf{C}(t_n) = \mathbf{F}^c\cdot(\mathbf{I} - \mathbf{F}_n^{-c}\cdot\mathbf{F}_n^{-1})\cdot\mathbf{F} = 2\,\mathbf{F}^c\cdot\mathbf{e}_n\cdot\mathbf{F}$$

$$\left.\begin{aligned}
&\boldsymbol{\tau} = \mathbf{F}_n\cdot\boldsymbol{\tau}(t_n)\cdot\mathbf{F}_n^c + 4\mathbf{F}\cdot\frac{\partial^2 W}{\partial \mathbf{C}^2} : \mathbf{F}^c\cdot\,(\mathbf{e}_n - \Delta\lambda\,\mathbf{a})\,\cdot\mathbf{F}\cdot\mathbf{F}^c\\
&\Delta\lambda = \Delta t\,\gamma\phi(F)
\end{aligned}\right\}$$

$${}^4\mathbf{H} = 4\,\mathbf{F}\cdot\left(\mathbf{F}\cdot\frac{\partial^2 W}{\partial \mathbf{C}^2}\cdot\mathbf{F}^c\right)^{lc,rc}\cdot\mathbf{F}^c = 2\{\mu - \lambda\ln(J)\}\,{}^4\mathbf{I}^{rc} + \lambda\mathbf{II}$$

$$\left.\begin{aligned}
&\boldsymbol{\tau} = \mathbf{F}_n\cdot\boldsymbol{\tau}(t_n)\cdot\mathbf{F}_n^c + {}^4\mathbf{H} : (\mathbf{e}_n - \Delta\lambda\,\mathbf{a}) = \boldsymbol{\tau}_e - \Delta\lambda\,{}^4\mathbf{H} : \mathbf{a}\\
&\Delta\lambda = \Delta t\,\gamma\phi(F)
\end{aligned}\right\}$$

Iterative stress update

$$\left.\begin{array}{l} \boldsymbol{\tau} - \boldsymbol{\tau}_e + \Delta\lambda\,{}^4\mathbf{H} : \mathbf{a} = \mathbf{0} \\ \Delta\lambda - \Delta t\,\gamma\phi(F) = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} \delta\boldsymbol{\tau} - \delta\boldsymbol{\tau}_e + \,{}^4\mathbf{H} : \mathbf{a}\,\delta\lambda + \Delta\lambda\,\delta{}^4\mathbf{H} : \mathbf{a} + \Delta\lambda\,\,{}^4\mathbf{H} : \delta\mathbf{a} = -\mathbf{s}_1 \\ \delta\lambda - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\mathbf{a} : \delta\boldsymbol{\tau} - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\left(\dfrac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right)\delta\lambda = -s_2 \end{array}\right\}$$

$$\text{with} \quad \begin{cases} \delta\boldsymbol{\tau}_e = \mathbf{M}_1\delta\lambda + \,{}^4\mathbf{M}_2 : \delta\boldsymbol{\tau} \\ \delta{}^4\mathbf{H} = \left(\dfrac{\partial\,{}^4\mathbf{H}}{\partial J}\right)\,\delta J = \,{}^4\mathbf{c}\,\delta J \\ \delta\mathbf{a} = \left(\dfrac{\partial\mathbf{a}}{\partial\boldsymbol{\tau}}\right) : \delta\boldsymbol{\tau} = \,{}^4\mathbf{b} : \delta\boldsymbol{\tau} \\ \delta J = J_1\delta\lambda + \mathbf{J}_2 : \delta\boldsymbol{\tau} \end{cases}$$

plane strain $\qquad \delta\boldsymbol{\tau}_{tr} = \mathbf{0} \quad ; \quad \delta J = 0$

Iterative stress update

$$\left.\begin{aligned} {}^4\mathbf{R} : \delta\boldsymbol{\tau} + \mathbf{t}\,\delta\lambda &= -\mathbf{s}_1 \\ \mathbf{u} : \delta\boldsymbol{\tau} + v\,\delta\lambda &= -s_2 \end{aligned}\right\}$$

$$\begin{aligned}
{}^4\mathbf{R} &= {}^4\mathbf{I} + \Delta\lambda\,{}^4\mathbf{H} : {}^4\mathbf{b} + \Delta\lambda\,{}^4\mathbf{c} : \mathbf{a}\mathbf{J}_2 - {}^4\mathbf{M}_2 \\
\mathbf{t} &= {}^4\mathbf{H} : \mathbf{a} + \Delta\lambda\,{}^4\mathbf{c} : \mathbf{a}J_1 - \mathbf{M}_1 \\
\mathbf{u} &= -\Delta t\,\gamma\left(\frac{\partial\phi}{\partial F}\right)\mathbf{a} \\
v &= 1 - \Delta t\,\gamma\left(\frac{\partial\phi}{\partial F}\right)\left(\frac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right) \\
\mathbf{s}_1 &= \boldsymbol{\tau} - \boldsymbol{\tau}_e + \Delta\lambda\,{}^4\mathbf{H} : \mathbf{a} \\
s_2 &= \Delta\lambda - \Delta t\,\gamma\,\phi(F)
\end{aligned}$$

Stiffness

$$\left.\begin{aligned} &\boldsymbol{\tau} = \mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \mathbf{F}_n^c + {}^4\mathbf{H} : \mathbf{e}_n - \Delta\lambda\, {}^4\mathbf{H} : \mathbf{a} \\ &\Delta\lambda = \Delta t\, \gamma\, \phi(F) \end{aligned}\right\} \quad \rightarrow$$

$$\left.\begin{aligned} \delta\boldsymbol{\tau} =& \delta\mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \mathbf{F}_n^c + \mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\mathbf{F}_n^c + \delta\, {}^4\mathbf{H} : (\mathbf{e}_n - \Delta\lambda\, \mathbf{a}) + \\ & {}^4\mathbf{H} : \delta\mathbf{e}_n - {}^4\mathbf{H} : \mathbf{a}\, \delta\lambda - \Delta\lambda\, {}^4\mathbf{H} : \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\tau}}\right) : \delta\boldsymbol{\tau} \\ \delta\lambda =& \left[\left\{\Delta t\, \gamma \left(\frac{\partial\phi}{\partial F}\right)\right\} / \left\{1 - \Delta t\, \gamma \left(\frac{\partial\phi}{\partial F}\right)\left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}}\right)\right\}\right] \mathbf{a} : \delta\boldsymbol{\tau} = c_1 \mathbf{a} : \delta\boldsymbol{\tau} \end{aligned}\right\}$$

$$\begin{aligned} &\left\{ {}^4\mathbf{I} + \Delta\lambda\, {}^4\mathbf{H} : \left(\frac{\partial \mathbf{a}}{\partial \boldsymbol{\tau}}\right) + c_1\, {}^4\mathbf{H} : \mathbf{a}\mathbf{a} \right\} : \delta\boldsymbol{\tau} = \\ &\qquad \delta\mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \mathbf{F}_n^c + \mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\mathbf{F}_n^c + \delta\, {}^4\mathbf{H} : (\mathbf{e}_n - \Delta\lambda\, \mathbf{a}) + {}^4\mathbf{H} : \delta\mathbf{e}_n \end{aligned}$$

$${}^4\mathbf{V} : \delta\boldsymbol{\tau} = {}^4\mathbf{E} : \delta\mathbf{F}_n \quad \rightarrow \quad \delta\boldsymbol{\tau} = {}^4\mathbf{V}^{-1} : {}^4\mathbf{E} : \delta\mathbf{F}_n$$

Stiffness

$$\delta \mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \mathbf{F}_n^c + \mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta \mathbf{F}_n^c = {}^4\mathbf{T} : \delta \mathbf{F}_n$$

$$J = \det(\mathbf{F}_n) = \det(\mathbf{F}_n + \delta \mathbf{F}_n) = J(1 + \mathbf{F}_n^{-1} : \delta \mathbf{F}_n) \quad \rightarrow \quad \delta J = J\, \mathbf{F}_n^{-1} : \delta \mathbf{F}_n$$

$$\delta\, {}^4\mathbf{H} = \left(\frac{\partial\, {}^4\mathbf{H}}{\partial J} \right) \delta J = \left(\frac{\partial\, {}^4\mathbf{H}}{\partial J} \right) \left(J \mathbf{F}_n^{-1} : \delta \mathbf{F}_n \right)$$

$$\left. \begin{array}{l} \delta \mathbf{e}_n = -\frac{1}{2} \delta \mathbf{F}_n^{-c} \cdot \mathbf{F}_n^{-1} - \frac{1}{2} \mathbf{F}_n^{-c} \cdot \delta \mathbf{F}_n^{-1} = -\,{}^4\mathbf{A}_1 : \delta \mathbf{F}_n^{-1} \\ \delta \mathbf{F}_n^{-1} = -\mathbf{F}_n^{-1} \cdot \delta \mathbf{F}_n \cdot \mathbf{F}_n^{-1} = -\,{}^4\mathbf{A}_2 : \delta \mathbf{F}_n \end{array} \right\} \quad \rightarrow$$

$$\delta \mathbf{e}_n = (\,{}^4\mathbf{A}_1 :\, {}^4\mathbf{A}_2) : \delta \mathbf{F}_n = {}^4\mathbf{P} : \delta \mathbf{F}_n$$

Consistent material stiffness tensor

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \quad \rightarrow \quad \boldsymbol{\sigma} = \frac{1}{J}\,\boldsymbol{\tau} \quad \rightarrow$$

$$\begin{aligned}
\delta\boldsymbol{\sigma} &= \frac{1}{J}\left(\delta\boldsymbol{\tau} - \boldsymbol{\sigma}\delta J\right) \\
&= \frac{1}{J}\left\{\,{}^4\mathbf{V}^{-1} : {}^4\mathbf{E} - \boldsymbol{\sigma} J\mathbf{F}_n^{-1}\right\} : \delta\mathbf{F}_n \\
&= {}^4\mathbf{C} : \delta\mathbf{F}_n \\
&= {}^4\mathbf{M} : \mathbf{L}_u
\end{aligned}$$

Matrix/column notation

$$\delta\boldsymbol{\sigma} = {}^4\mathbf{C} : \delta\mathbf{F}_n \quad \rightarrow \quad \delta\underset{\approx}{\sigma} = \underline{\underline{C}}\,\delta\underset{\approx}{F_n}{}_t$$

$$\delta\mathbf{F}_n = \left(\mathbf{F}^{-c}(t_n)\cdot\delta\mathbf{F}^c\right)^c \quad \rightarrow \quad \delta\underset{\approx}{F_n} = \left(\underline{\underline{F}}_t^{-1}(t_n)\delta\underset{\approx}{F}_t\right)_t \rightarrow \delta\underset{\approx}{F_n}{}_t = \underline{\underline{F}}_t^{-1}(t_n)\delta\underset{\approx}{F}_t$$

$$\delta\mathbf{F}^c = \mathbf{F}^c\cdot\mathbf{L}_u^c \quad \rightarrow \quad \delta\underset{\approx}{F}_t = \underline{\underline{F}}_t\underset{\approx}{L_u}{}_t$$

$$\delta\underset{\approx}{\sigma} = \left[\underline{\underline{C}}\,\underline{\underline{F}}_t^{-1}(t_n)\underline{\underline{F}}_t\right]\underset{\approx}{L_u}{}_t = \underline{\underline{M}}\underset{\approx}{L_u}{}_t$$

$$\underline{\underline{M}} = \underline{\underline{C}}\,\underline{\underline{F}}_t^{-1}(t_n)\underline{\underline{F}}_t$$

$$\underline{\underline{C}} = \frac{1}{J}\left(\underline{\underline{V}}^{-1}\,\underline{\underline{E}}_r - J\underset{\approx}{\sigma}\underset{\approx}{F_n}{}^{-T}\right)$$

$$\underline{\underline{V}} = \underline{\underline{I}} + \Delta\lambda\,\underline{\underline{H}}_c\underline{\underline{b}} + c_1\underline{\underline{H}}_c\underset{\approx}{a}\underset{\approx}{a}^T$$

$$\underline{\underline{E}} = \underline{\underline{T}} - 2\lambda\underline{\underline{I}}\left(\underset{\approx}{e_n} - \Delta\lambda\,\underset{\approx}{a}\right)\left(\underset{\approx}{F_n}{}^{-1}\right)^T + \underline{\underline{H}}_c\underline{\underline{P}}$$

Plane strain

$$\delta J = J_1 \delta\lambda + \mathbf{J}_2 : \delta\boldsymbol{\tau} = 0$$

$$\delta\boldsymbol{\tau}_{tr} = \mathbf{M}_1 \delta\lambda + {}^4\mathbf{M}_2 : \delta\boldsymbol{\tau} = \mathbf{O}$$

Iterative stress update

$$\left.\begin{aligned} {}^4\mathbf{R} : \delta\boldsymbol{\tau} + \mathbf{t}\delta\lambda &= -\mathbf{s}_1 \\ \mathbf{u} : \delta\boldsymbol{\tau} + v\delta\lambda &= -s_2 \end{aligned}\right\}$$

$$\begin{aligned} {}^4\mathbf{R} &= {}^4\mathbf{I} + \Delta\lambda\, {}^4\mathbf{H} : {}^4\mathbf{b} \\ \mathbf{t} &= {}^4\mathbf{H} : \mathbf{a} \\ \mathbf{u} &= -\Delta t\,\gamma \left(\frac{\partial\phi}{\partial F}\right)\mathbf{a} \\ v &= 1 - \Delta t\,\gamma \left(\frac{\partial\phi}{\partial F}\right)\left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}}\right) \\ \mathbf{s}_1 &= \boldsymbol{\tau} - \boldsymbol{\tau}_{tr} + \Delta\lambda\, {}^4\mathbf{H} : \mathbf{a} \\ s_2 &= \Delta\lambda - \Delta t\,\gamma\,\phi(F) \end{aligned}$$

Matrix/column notation

$$\begin{bmatrix} \underline{\underline{R}}_c & \underset{\approx}{t} \\ \underset{\approx}{u}_t^T & v \end{bmatrix} \begin{bmatrix} \delta\underset{\approx}{\tau} \\ \delta\lambda \end{bmatrix} = - \begin{bmatrix} \underset{\approx}{s}_1 \\ s_2 \end{bmatrix}$$

$$\begin{aligned}
&\underline{\underline{R}} = \underline{\underline{I}} + \Delta\lambda\, \underline{\underline{H}}\, \underline{\underline{b}}_t \\
&\underset{\approx}{t} = \underline{\underline{H}}\, \underset{\approx}{a}_t \\
&\underset{\approx}{u} = -\Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \underset{\approx}{a} \\
&v = 1 - \Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}} \right) \\
&\underset{\approx}{s}_1 = \underset{\approx}{\tau} - \underset{\approx}{\tau}_{tr} + \Delta\lambda\, \underline{\underline{H}}\underset{\approx}{a}_t \\
&s_2 = \Delta\lambda - \Delta t\, \gamma \phi(F)
\end{aligned}$$

Stiffness

$$\delta\boldsymbol{\sigma} = {}^4\mathbf{C} : \delta\mathbf{F}_n = \frac{1}{J}\left\{ {}^4\mathbf{V}^{-1} : {}^4\mathbf{E} - \boldsymbol{\sigma} J \mathbf{F}_n^{-1} \right\} : \delta\mathbf{F}_n$$

$${}^4\mathbf{V} = \left\{ {}^4\mathbf{I} + \Delta\lambda\, {}^4\mathbf{H} : {}^4\mathbf{b} + c_1\, {}^4\mathbf{H} : \mathbf{aa} \right\}$$

$${}^4\mathbf{E} = \left\{ {}^4\mathbf{T} + {}^4\mathbf{c} : (\mathbf{e}_n - \Delta\lambda\, \mathbf{a}) J \mathbf{F}_n^{-1} + {}^4\mathbf{H} : {}^4\mathbf{P} \right\}$$

$$\delta\mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \mathbf{F}_n^c + \mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\mathbf{F}_n^c = {}^4\mathbf{T} : \delta\mathbf{F}_n$$

$$\delta\mathbf{e}_n = {}^4\mathbf{P} : \delta\mathbf{F}_n$$

Matrix/column notation

$$\delta \underset{\approx}{\sigma} = \underline{\underline{C}} \left(\delta \underset{\approx}{F}_n \right)_t = \left[\frac{1}{J} \left\{ \underline{\underline{V}}^{-1} \underline{\underline{E}}_r - \underset{\approx}{\sigma} J \underset{\approx}{F}_n^{-T} \right\} \right] \left(\delta \underset{\approx}{F}_n \right)_t$$

$$\underline{\underline{V}} = \underline{\underline{I}} + \Delta\lambda \, \underline{\underline{H}}_c \underline{\underline{b}} + c_1 \underline{\underline{H}}_c \underset{\approx}{a} \underset{\approx}{a}^T$$

$$\underline{\underline{E}} = \underline{\underline{T}} + 2\lambda \underline{\underline{I}} \left(\underset{\approx}{e} - \Delta\lambda \underset{\approx}{a} \right) J \underset{\approx}{F}_n^{-T} + \underline{\underline{H}}_c \underline{\underline{P}}$$

Plane stress

Iterative stress update

$$\left.\begin{array}{l} {}^4\mathbf{R} : \delta\boldsymbol{\tau} + \mathbf{t}\delta\lambda = -\mathbf{s}_1 \\[2ex] \mathbf{u} : \delta\boldsymbol{\tau} + v\delta\lambda = -s_2 \end{array}\right\}$$

$$\begin{aligned}
{}^4\mathbf{R} &= {}^4\mathbf{I} - {}^4\mathbf{M}_2 + \Delta\lambda\, {}^4\mathbf{C} : \mathbf{a}\mathbf{J}_2 + \Delta\lambda\, {}^4\mathbf{H} : {}^4\mathbf{b} \\
\mathbf{t} &= -\mathbf{M}_1 + \Delta\lambda\, {}^4\mathbf{C} : \mathbf{a}J_1 + {}^4\mathbf{H} : \mathbf{a} \\
\mathbf{u} &= -\Delta t\, \gamma \left(\frac{\partial\phi}{\partial F}\right) \mathbf{a} \\
v &= 1 - \Delta t\, \gamma \left(\frac{\partial\phi}{\partial F}\right) \left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}}\right) \\
\mathbf{s}_1 &= \boldsymbol{\tau} - \boldsymbol{\tau}_{trial} + \Delta\lambda\, {}^4\mathbf{H} : \mathbf{a} \\
s_2 &= \Delta\lambda - \Delta t\, \gamma\, \phi(F)
\end{aligned}$$

Matrix/column notation

$$
\begin{bmatrix} \underline{\underline{R}}_c & \underset{\approx}{t} \\ \underset{\approx}{u}_t^T & v \end{bmatrix}
\begin{bmatrix} \delta \underset{\approx}{\tau} \\ \delta\lambda \end{bmatrix}
= -
\begin{bmatrix} \underset{\approx}{s}_1 \\ s_2 \end{bmatrix}
$$

$$
\begin{aligned}
\underline{\underline{R}} &= \underline{\underline{I}} - \underline{\underline{M}}_2 + \Delta\lambda \underline{\underline{C}}\, \underset{\approx}{a}_r \underset{\approx}{J}_2^T + \Delta\lambda \underline{\underline{H}}\, \underline{\underline{b}}_r \\
\underset{\approx}{t} &= -\underline{\underline{M}}_1 + \Delta\lambda \underline{\underline{C}} \underset{\approx}{a}_t J_1 + \underline{\underline{H}} \underset{\approx}{a}_t \\
\underset{\approx}{u} &= -\Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \underset{\approx}{a} \\
v &= 1 - \Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}} \right) \\
\underset{\approx}{s}_1 &= \underset{\approx}{\tau} - \underset{\approx}{\tau}_{tr} + \Delta\lambda \underline{\underline{H}} \underset{\approx}{a}_t \\
s_2 &= \Delta\lambda - \Delta t \gamma \phi(F)
\end{aligned}
$$

Stiffness

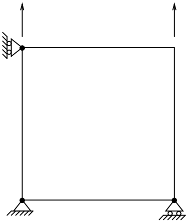
$$\delta\boldsymbol{\sigma} = {}^4\mathbf{C} : \delta\mathbf{F}_n = \frac{1}{J}\left\{ {}^4\mathbf{V}^{-1} : {}^4\mathbf{E} - \boldsymbol{\sigma} J \mathbf{F}_n^{-1} \right\} : \delta\mathbf{F}_n$$

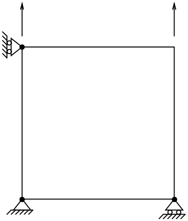
$$\begin{aligned}
{}^4\mathbf{V} &= \left\{ {}^4\mathbf{I} + \Delta\lambda\, {}^4\mathbf{H} : \left(\frac{\partial\mathbf{a}}{\partial\boldsymbol{\tau}}\right) + c_1\, {}^4\mathbf{H} : \mathbf{a}\mathbf{a} \right\} \\
{}^4\mathbf{E} &= \left\{ {}^4\mathbf{T} + \left(\frac{\partial\, {}^4\mathbf{H}}{\partial J}\right) : (\mathbf{e} - \Delta\lambda\, \mathbf{a}) J \mathbf{F}_n^{-1} + {}^4\mathbf{H} : {}^4\mathbf{P} \right\}
\end{aligned}$$

$$\delta\mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \mathbf{F}_n^c + \mathbf{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\mathbf{F}_n^c = {}^4\mathbf{T} : \delta\mathbf{F}_n$$

Examples

Tensile test





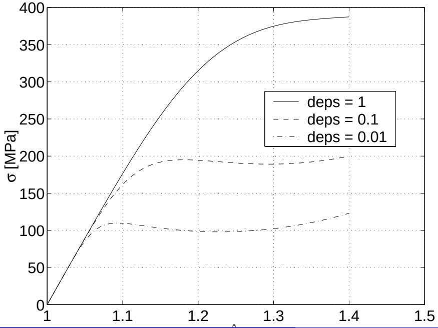
initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

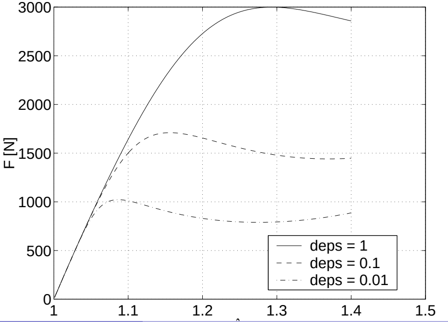
initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

Tensile test at various strain rates

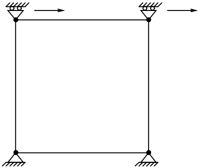
E	1800	MPa	ν	0.37	-
σ_{y0}	37	MPa	H	-200	MPa
γ	0.001	1/s	N	3	-
a	500	MPa	b	700	MPa
c	800	MPa	d	30000	MPa

elongation rate $$\frac{\dot{\Delta l}}{h_0} = \{0.01, 0.1, 1\}\ \mathrm{s}^{-1}$$



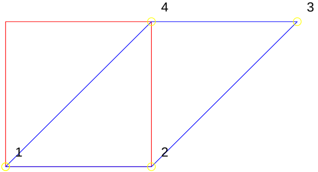


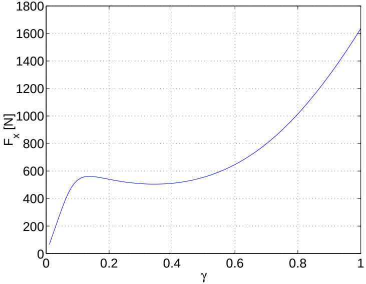
Shear test



initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

strain rate $$\dot{\gamma} = \frac{\dot{u}}{h_0} = 0.01\ \mathrm{s}^{-1}$$

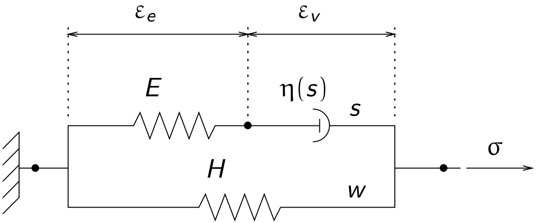




NONLINEAR VISCOELASTIC

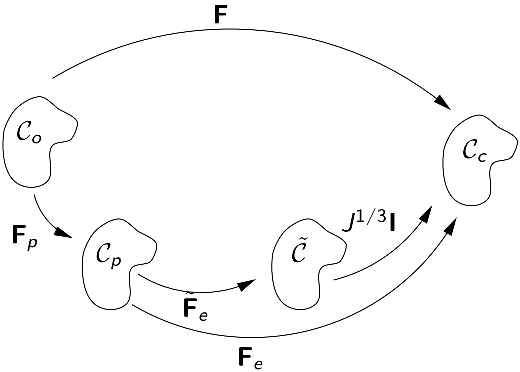
back to index

Nonlinear viscoelastic material behavior



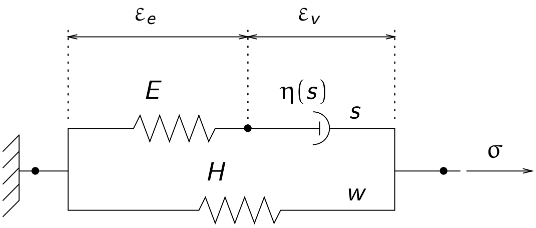
$$\sigma = s + w$$

Kinematics



$$
\begin{aligned}
\mathbf{F} &= (\vec{\nabla}_0\vec{x})^c = \mathbf{F}_e \cdot \mathbf{F}_p = J^{1/3}\mathbf{I} \cdot \tilde{\mathbf{F}}_e \cdot \mathbf{F}_p \\
\mathbf{C} &= \mathbf{F}^c \cdot \mathbf{F} \qquad ; \qquad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^c \quad \rightarrow \quad \tilde{\mathbf{B}}_e = \tilde{\mathbf{F}}_e \cdot \tilde{\mathbf{F}}_e^c \\
\mathbf{L} &= \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = (\vec{\nabla}\vec{v})^c \\
&= \mathbf{L}_e + \mathbf{L}_p = (\mathbf{D}_e + \mathbf{\Omega}_e) + (\mathbf{D}_p + \mathbf{\Omega}_p) = (\mathbf{D}_e + \mathbf{\Omega}_e) + \mathbf{D}_p
\end{aligned}
$$

Stress decomposition



$$\boldsymbol{\sigma} = \mathbf{s} + \mathbf{w} = \mathbf{s}^d + \mathbf{s}^h + \mathbf{w}$$

$$\mathbf{s} = G\tilde{\mathbf{B}}_e^d + \kappa(J-1)\mathbf{I} \quad ; \quad \mathbf{w} = H\tilde{\mathbf{B}}^d$$

Elastic deformation

$$\tilde{\mathbf{B}}_e = \tilde{\mathbf{F}}_e \cdot \tilde{\mathbf{F}}_e^c \quad \rightarrow \quad \dot{\tilde{\mathbf{B}}}_e = \dot{\tilde{\mathbf{F}}}_e \cdot \tilde{\mathbf{F}}_e^c + \tilde{\mathbf{F}}_e \cdot \dot{\tilde{\mathbf{F}}}_e^c$$

$$\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_e \cdot \mathbf{F}_p \quad \rightarrow \quad \tilde{\mathbf{F}}_e = \tilde{\mathbf{F}} \cdot \mathbf{F}_p^{-1} \quad \rightarrow \quad \dot{\tilde{\mathbf{F}}}_e = \dot{\tilde{\mathbf{F}}} \cdot \mathbf{F}_p^{-1} + \tilde{\mathbf{F}} \cdot \dot{\mathbf{F}}_p^{-1}$$

$$\begin{aligned}
\dot{\tilde{\mathbf{B}}}_e &= \left(\dot{\tilde{\mathbf{F}}} \cdot \mathbf{F}_p^{-1} + \tilde{\mathbf{F}} \cdot \dot{\mathbf{F}}_p^{-1}\right) \cdot \tilde{\mathbf{F}}_e^c + \tilde{\mathbf{F}}_e \cdot \left(\mathbf{F}_p^{-c} \cdot \dot{\tilde{\mathbf{F}}}^c + \dot{\mathbf{F}}_p^{-c} \cdot \tilde{\mathbf{F}}^c\right) \\
&= \left(\dot{\tilde{\mathbf{F}}} \cdot \mathbf{F}_p^{-1} \cdot \tilde{\mathbf{F}}_e^{-1} + \tilde{\mathbf{F}} \cdot \dot{\mathbf{F}}_p^{-1} \cdot \tilde{\mathbf{F}}_e^{-1}\right) \cdot \tilde{\mathbf{B}}_e + \\
&\qquad \tilde{\mathbf{B}}_e \cdot \left(\tilde{\mathbf{F}}_e^{-c} \cdot \mathbf{F}_p^{-c} \cdot \dot{\tilde{\mathbf{F}}}^c + \tilde{\mathbf{F}}_e^{-c} \cdot \dot{\mathbf{F}}_p^{-c} \cdot \tilde{\mathbf{F}}^c\right) \\
&= \left(\tilde{\mathbf{L}} + \tilde{\mathbf{F}}_e \cdot \mathbf{F}_p \cdot \dot{\mathbf{F}}_p^{-1} \tilde{\mathbf{F}}_e^{-1}\right) \cdot \tilde{\mathbf{B}}_e + \tilde{\mathbf{B}}_e \cdot \left(\tilde{\mathbf{L}}^c + \tilde{\mathbf{F}}_e^{-c} \cdot \dot{\mathbf{F}}_p^{-c} \cdot \mathbf{F}_p^c \cdot \tilde{\mathbf{F}}_e^c\right) \\
&\qquad \mathbf{F}_p \cdot \mathbf{F}_p^{-1} = \mathbf{I} \quad \rightarrow \quad \mathbf{F}_p \cdot \dot{\mathbf{F}}_p^{-1} = -\dot{\mathbf{F}}_p \cdot \mathbf{F}_p^{-1} \quad \rightarrow \\
&= (\tilde{\mathbf{L}} - \mathbf{D}_p) \cdot \tilde{\mathbf{B}}_e + \tilde{\mathbf{B}}_e \cdot (\tilde{\mathbf{L}}^c - \mathbf{D}_p)
\end{aligned}$$

Viscoplastic deformation

$$\mathbf{D}_p = \frac{1}{2\eta}\,\mathbf{s}^d$$

$$\begin{aligned}
\eta &= \eta(\bar{s}, p, T, D) \\
\bar{s} &= \sqrt{\tfrac{3}{2}\mathbf{s}^d : \mathbf{s}^d} \\
p &= \kappa(J-1)\mathbf{I}
\end{aligned}$$

Eyring viscosity

plastic deformation rate $$\mathbf{D}_p = \frac{1}{2\eta(\bar{s}, p, T, D)}\,\mathbf{s}^d$$

$$\eta = \frac{A\bar{s}}{\sqrt{3}\sinh\left(\dfrac{\bar{s}}{\sqrt{3}\tau_0}\right)}$$

$$\bar{s} = \sqrt{\tfrac{3}{2}\mathbf{s}^d : \mathbf{s}^d}$$

$$A = A_0 \exp\left[\frac{\Delta H}{RT} + \frac{\mu p}{\tau_0} - D\right]$$

$$\tau_0 = \frac{RT}{V} \qquad ; \qquad p = -\tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})$$

$$\dot{D} = h\left(1 - \frac{D}{D_\infty}\right)\,\frac{\bar{s}}{\sqrt{6}\,\eta} \qquad ; \qquad D \in [0, D_\infty]$$

Bodner-Partom viscosity

plastic deformation rate $$\mathbf{D}_p = \frac{1}{2\eta(\bar{s}, \mathbf{D}_p)}\,\mathbf{s}^d$$

$$\eta = \frac{\bar{s}}{\sqrt{12\Gamma_0}}\,\exp\left[\tfrac{1}{2}\left(\frac{Z}{\bar{\sigma}}\right)^{2n}\right]$$

$$\bar{s} = \sqrt{\tfrac{3}{2}\mathbf{s}^d : \mathbf{s}^d}$$

$$Z = Z_1 + (Z_0 - Z_1)e^{-m\bar{\varepsilon}_p}$$

$$\dot{\bar{\varepsilon}}_p = \sqrt{\tfrac{2}{3}\mathbf{D}_p : \mathbf{D}_p} \qquad \rightarrow \qquad \bar{\varepsilon}_p$$

Plastic strain rate

$$\left.\left.\begin{array}{l}\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_e \cdot \mathbf{F}_p \quad \rightarrow \quad \mathbf{C}_p = \mathbf{F}_p^c \cdot \mathbf{F}_p = \tilde{\mathbf{F}}^c \cdot \tilde{\mathbf{B}}_e^{-1} \cdot \tilde{\mathbf{F}} \quad \rightarrow \\ \dot{\mathbf{C}}_p = \tilde{\mathbf{F}}^c \cdot \tilde{\mathbf{B}}_e^{-1} \cdot \left[\tilde{\mathbf{B}}_e \cdot \tilde{\mathbf{L}}^c + \tilde{\mathbf{B}}_e \cdot \dot{\tilde{\mathbf{B}}}_e^{-1} \cdot \tilde{\mathbf{B}}_e + \tilde{\mathbf{L}} \cdot \tilde{\mathbf{B}}_e\right] \cdot \tilde{\mathbf{B}}_e^{-1} \cdot \tilde{\mathbf{F}}\end{array}\right\} \\ \left.\begin{array}{l}\dot{\tilde{\mathbf{B}}}_e = (\tilde{\mathbf{L}} - \mathbf{D}_p) \cdot \tilde{\mathbf{B}}_e + \tilde{\mathbf{B}}_e \cdot (\tilde{\mathbf{L}}^c - \mathbf{D}_p) \quad \rightarrow \\ \tilde{\mathbf{B}}_e \cdot \dot{\tilde{\mathbf{B}}}_e^{-1} = -\tilde{\mathbf{L}} - \tilde{\mathbf{B}}_e \cdot \tilde{\mathbf{L}}^c \cdot \tilde{\mathbf{B}}_e^{-1} + \mathbf{D}_p + \tilde{\mathbf{B}}_e \cdot \mathbf{D}_p \cdot \tilde{\mathbf{B}}_e^{-1}\end{array}\right\}\right\} \rightarrow$$

$$\dot{\mathbf{C}}_p = \tilde{\mathbf{F}}^c \cdot \tilde{\mathbf{B}}_e^{-1} \cdot \left[\mathbf{D}_p \cdot \tilde{\mathbf{B}}_e + \tilde{\mathbf{B}}_e \cdot \mathbf{D}_p\right] \cdot \tilde{\mathbf{B}}_e^{-1} \cdot \tilde{\mathbf{F}}$$

$$\text{with} \quad \mathbf{D}_p = \frac{1}{2\eta}\,\mathbf{s}^d = \frac{G}{2\eta}\,\tilde{\mathbf{B}}_e^d \quad \rightarrow$$

$$= \frac{G}{\eta}\left(\tilde{\mathbf{C}} - \tfrac{1}{3}\,\mathrm{tr}(\tilde{\mathbf{B}}_e)\mathbf{C}_p\right) = \Gamma\left(\tilde{\mathbf{C}} - \frac{1}{\alpha}\,\mathbf{C}_p\right) = \Gamma\,\mathbf{A}$$

Constitutive model

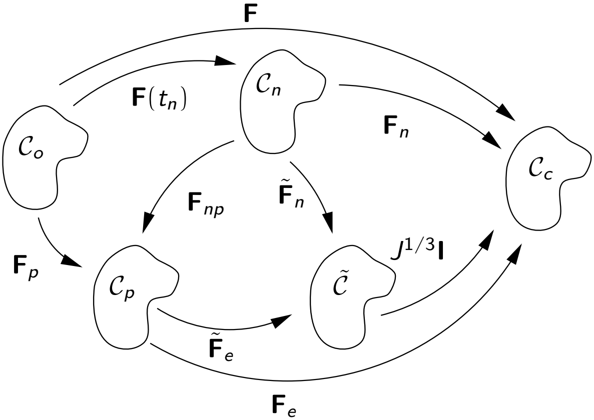
$$J = \det(\mathbf{F}) \quad \rightarrow \quad \tilde{\mathbf{F}} = J^{-1/3}\mathbf{F} \quad \rightarrow \quad \tilde{\mathbf{B}} = \tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}^c \quad \rightarrow \quad \mathbf{w} = H\tilde{\mathbf{B}}^d$$

$$p = \kappa(J-1) \quad \rightarrow \quad \mathbf{s}^h = p\mathbf{I}$$

$$\left.\begin{array}{l} \dot{\mathbf{C}}_p = \dfrac{G}{\eta}\left(\tilde{\mathbf{C}} - \frac{1}{3}\mathrm{tr}(\tilde{\mathbf{B}}_e)\,\mathbf{C}_p\right) \\ \tilde{\mathbf{B}}_e = \tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-1} \cdot \tilde{\mathbf{F}}^c \end{array}\right\} \quad \rightarrow \quad \mathbf{s}^d = G\tilde{\mathbf{B}}_e^d \rightarrow \bar{s} = \sqrt{\tfrac{3}{2}\mathbf{s}^d : \mathbf{s}^d}$$

$$\boldsymbol{\sigma} = \mathbf{s}^d + \mathbf{s}^h + \mathbf{w}$$

Incremental analysis



$$\mathbf{F}(\tau) = \mathbf{F}_n(\tau) \cdot \mathbf{F}(t_n) \quad \longrightarrow \quad \mathbf{F}_n(\tau) = \mathbf{F}(\tau) \cdot \mathbf{F}^{-1}(t_n)$$

$$\tilde{\mathbf{F}}(\tau) = \tilde{\mathbf{F}}_n(\tau) \cdot \tilde{\mathbf{F}}(t_n)$$

$$\mathbf{F}_n = \left(\vec{\nabla}_n \vec{x}\right)^c = \mathbf{R}_n \cdot \mathbf{U}_n$$

Incremental plastic strain

$$
\begin{aligned}
\mathbf{C}_p(\tau) &= \mathbf{F}_p^c(\tau) \cdot \mathbf{F}_p(\tau) = \tilde{\mathbf{F}}^c(\tau) \cdot \tilde{\mathbf{B}}_e^{-1}(\tau) \cdot \tilde{\mathbf{F}}(\tau) \\
& \qquad\qquad \text{with} \quad \tilde{\mathbf{F}}(\tau) = \tilde{\mathbf{F}}_n(\tau) \cdot \tilde{\mathbf{F}}(t_n) \quad \rightarrow \\
&= \tilde{\mathbf{F}}^c(t_n) \cdot \left[\tilde{\mathbf{F}}_n^c(\tau) \cdot \tilde{\mathbf{B}}_e^{-1}(\tau) \cdot \tilde{\mathbf{F}}_n(\tau) \right] \cdot \tilde{\mathbf{F}}(t_n) \\
&= \tilde{\mathbf{F}}^c(t_n) \cdot \mathbf{C}_{p_n}(\tau) \cdot \tilde{\mathbf{F}}(t_n)
\end{aligned}
$$

incremental rotation neutralized plastic strain

$$
\begin{aligned}
\mathbf{C}_{p_n}(\tau) &= \tilde{\mathbf{F}}_n^c(\tau) \cdot \tilde{\mathbf{B}}_e^{-1}(\tau) \cdot \tilde{\mathbf{F}}_n(\tau) \\
&= \tilde{\mathbf{U}}_n(\tau) \cdot \left[\mathbf{R}_n^c(\tau) \cdot \tilde{\mathbf{B}}_e^{-1}(\tau) \cdot \mathbf{R}_n(\tau) \right] \cdot \tilde{\mathbf{U}}_n(\tau) \\
&= \tilde{\mathbf{U}}_n(\tau) \cdot \bar{\tilde{\mathbf{B}}}_{e_n}^{-1}(\tau) \cdot \tilde{\mathbf{U}}_n(\tau)
\end{aligned}
$$

Constitutive equations

$$J = \det(\mathbf{F}) \quad \rightarrow \quad \tilde{\mathbf{F}} = J^{-1/3}\mathbf{F} \quad \rightarrow \quad \tilde{\mathbf{B}} = \tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}^c \quad \rightarrow \quad \mathbf{w} = H\tilde{\mathbf{B}}^d$$

$$p = \kappa(J-1) \quad \rightarrow \quad \mathbf{s}^h = p\mathbf{I}$$

$$\left.\begin{array}{l}
\dot{\mathbf{C}}_{p_n} = \dfrac{G}{\eta}\left(\tilde{\mathbf{C}}_n - \frac{1}{3}\mathrm{tr}\left(\bar{\tilde{\bar{\mathbf{B}}}}_{e_n}\right)\mathbf{C}_{p_n}\right) \\
\bar{\tilde{\bar{\mathbf{B}}}}_{e_n} = \tilde{\mathbf{U}}_n \cdot \mathbf{C}_{p_n}^{-1} \cdot \tilde{\mathbf{U}}_n^c \rightarrow \tilde{\mathbf{B}}_e = \mathbf{R}_n \cdot \bar{\tilde{\bar{\mathbf{B}}}}_{e_n} \cdot \mathbf{R}_n^c \\
\dot{D} = h\left(1 - \dfrac{D}{D_\infty}\right)\dfrac{\bar{s}}{\sqrt{6}\,\eta} \\
\eta = \eta(\bar{s}, p, T, D)
\end{array}\right\} \rightarrow \mathbf{s}^d = G\tilde{\mathbf{B}}_e^d \rightarrow \bar{s} = \sqrt{\tfrac{3}{2}\mathbf{s}^d : \mathbf{s}^d}$$

$$\sigma = \mathbf{s}^d + \mathbf{s}^h + \mathbf{w}$$

Stress update

$$\dot{\mathbf{C}}_{p_n}(\tau) = \Gamma(\tau)\left[\tilde{\mathbf{C}}_n(\tau) - \frac{1}{\bar{\alpha}_n(\tau)}\,\mathbf{C}_{p_n}(\tau)\right] \quad ; \quad \frac{1}{\bar{\alpha}_n} = \tfrac{1}{3}\mathrm{tr}\left(\bar{\tilde{\mathbf{B}}}_{e_n}\right)$$

$$\frac{1}{\Delta t}\left[\mathbf{C}_{p_n} - \mathbf{C}_{p_n}(t_n)\right] = \Gamma\left[\tilde{\mathbf{C}}_n - \frac{1}{\bar{\alpha}_n}\,\mathbf{C}_{p_n}\right] \quad \rightarrow$$

$$\mathbf{C}_{p_n} = \frac{\bar{\alpha}_n\,\Delta t\,\Gamma}{\bar{\alpha}_n + \Delta t\,\Gamma}\,\tilde{\mathbf{C}}_n + \frac{\bar{\alpha}_n}{\bar{\alpha}_n + \Delta t\,\Gamma}\,\mathbf{C}_{p_n}(t_n) \quad \rightarrow$$

$$\mathbf{C}_{p_n} = \bar{\alpha}_n(1-\lambda)\tilde{\mathbf{C}}_n + \lambda\mathbf{C}_{p_n}(t_n) \quad ; \quad \lambda = \frac{\bar{\alpha}_n}{\bar{\alpha}_n + \Delta t\,\Gamma} = \text{elasticity parameter}$$

$$\tilde{\mathbf{B}}_e = \mathbf{R}_n \cdot \bar{\tilde{\mathbf{B}}}_{e_n} \cdot \mathbf{R}_n^c = \tilde{\mathbf{F}}_n \cdot \mathbf{C}_{p_n}^{-1} \cdot \tilde{\mathbf{F}}_n^c$$

Sub-incremental plastic strain update

$$\dot{\mathbf{C}}_{p_n}(\tau) = \Gamma(\tau)\left[\tilde{\mathbf{C}}_n(\tau) - \frac{1}{\bar{\alpha}_n(\tau)}\,\mathbf{C}_{p_n}(\tau)\right] \quad ; \quad \frac{1}{\bar{\alpha}_n} = \tfrac{1}{3}\mathrm{tr}\left(\bar{\tilde{\mathbf{B}}}_{e_n}\right)$$

$$\left.\begin{array}{lll} \text{sub-incremental deformation :} & j = 1\cdots ns+1 & \\ j=1 \quad : \quad \tau = t_n & ; \quad j = ns+1 \quad : \quad \tau = t_{n+1} & \\ \delta t = \Delta t/ns \quad ; & \delta\tilde{\mathbf{C}}_n = \left\{\tilde{\mathbf{C}}_n\right\}^{1/ns} \quad ; & \tilde{\mathbf{C}}_n^j = \left\{\delta\tilde{\mathbf{C}}_n\right\}^j \end{array}\right\}$$

$$\frac{1}{\delta t}\left[\mathbf{C}_{p_n}^j - \mathbf{C}_{p_n}^{j-1}\right] = \Gamma^j\left[\tilde{\mathbf{C}}_n^j - \frac{1}{\bar{\alpha}_n^j}\,\mathbf{C}_{p_n}^j\right] \quad \rightarrow$$

$$\mathbf{C}_{p_n}^j = \frac{\bar{\alpha}_n^j\,\delta t\,\Gamma^j}{\bar{\alpha}_n^j + \delta t\,\Gamma^j}\,\tilde{\mathbf{C}}_n^j + \frac{\bar{\alpha}_n^j}{\bar{\alpha}_n^j + \delta t\,\Gamma^j}\,\mathbf{C}_{p_n}^{j-1} \quad \rightarrow$$

$$\mathbf{C}_{p_n}^j = \bar{\alpha}_n^j(1-\lambda^j)\tilde{\mathbf{C}}_n^j + \lambda^j\mathbf{C}_{p_n}^{j-1} \quad ; \quad \lambda^j = \frac{\bar{\alpha}_n^j}{\bar{\alpha}_n^j + \delta t\,\Gamma^j}$$

incremental plastic strain $\quad \mathbf{C}_{p_n} = \mathbf{C}_{p_n}(t_{n+1}) = \mathbf{C}_{p_n}^{ns+1}$

total isochoric elastic strain $\quad \tilde{\mathbf{B}}_e = \mathbf{R}_n \cdot \bar{\tilde{\mathbf{B}}}_{e_n} \cdot \mathbf{R}_n^c = \tilde{\mathbf{F}}_n \cdot \mathbf{C}_{p_n}^{-1} \cdot \tilde{\mathbf{F}}_n^c$

Scalar variable update

$$\lambda = 1/(1 + \Delta t \Gamma) \qquad \rightarrow \qquad f(\lambda, D) = \lambda(1 + \Delta t \Gamma) = 1$$

$$\frac{1}{\Delta t}\{D - D(t_n)\} = \dot{D} \qquad \rightarrow \qquad g(\lambda, D) = D - \Delta t \dot{D} = D(t_n)$$

Newton-Raphson iterative solution procedure

$$\begin{bmatrix} \dfrac{\partial f}{\partial \lambda} & \dfrac{\partial f}{\partial D} \\ \dfrac{\partial g}{\partial \lambda} & \dfrac{\partial g}{\partial D} \end{bmatrix} \begin{bmatrix} \delta\lambda \\ \delta D \end{bmatrix} = \begin{bmatrix} 1 - f^* \\ D(t_n) - g^* \end{bmatrix} = \begin{bmatrix} r_\lambda^* \\ r_D^* \end{bmatrix}$$

Partial derivatives

$$
\begin{aligned}
\frac{\partial f}{\partial \lambda} &= 1 + \Delta t \Gamma + \lambda \Delta t \, \frac{\partial \Gamma}{\partial \lambda} = 1 + \Delta t \Gamma - \lambda \Delta t \frac{G}{\eta^2} \, \frac{\partial \eta}{\partial \bar{\sigma}} \, \frac{\partial \bar{\sigma}}{\partial \lambda} \\
&= 1 + \Delta t \Gamma - \lambda \Delta t \frac{G}{\eta^2} \left[\eta \left(\frac{1}{\bar{\sigma}} - \frac{1}{\sqrt{3}\tau_0} \right) \right] \bar{\sigma} \\
\frac{\partial f}{\partial D} &= \lambda \Delta t \, \frac{\partial \Gamma}{\partial D} = -\lambda \Delta t \frac{G}{\eta^2} \, \frac{\partial \eta}{\partial D} = \lambda \Delta t \frac{G}{\eta^2} \, \eta = \lambda \Delta t \Gamma \\
\frac{\partial g}{\partial \lambda} &= -\Delta t \, \frac{\partial \dot{D}}{\partial \lambda} = -\Delta t \, \frac{\partial \dot{D}}{\partial \bar{\sigma}} \, \frac{\partial \bar{\sigma}}{\partial \lambda} = -\Delta t \, \left[\frac{\dot{D}}{\sqrt{3}\tau_0} \right] \bar{\sigma} \\
\frac{\partial g}{\partial D} &= 1 - \Delta t \, \frac{\partial \dot{D}}{\partial D} = 1 - \Delta t \, \left[\dot{D} - \frac{h \bar{\sigma}}{\sqrt{6} D_\infty \eta} \right]
\end{aligned}
$$

Matrix/column notation

$$J = \det(\underline{F}) \quad \rightarrow \quad \tilde{\underline{F}} = J^{-1/3}\underline{F} \quad \rightarrow \quad \tilde{\underline{B}} = \tilde{\underline{F}}\,\tilde{\underline{F}}^T \quad \rightarrow \quad \underline{w} = H\tilde{\underline{B}}^d$$

$$p = \kappa(J-1) \quad \rightarrow \quad \underline{s}^h = p\underline{I}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \lambda = 1/(1+\Delta t\Gamma) \\ \frac{1}{\Delta t}\{D - D(t_n)\} = \dot{D} \end{array}\right\} \quad \rightarrow \quad \lambda, D \\ \underline{C}_{p_n} = (1-\lambda)\tilde{\underline{C}}_n + \lambda\underline{C}_{p_n}(t_n) \\ \bar{\tilde{\underline{B}}}_{e_n} = \tilde{\underline{U}}_n\,\underline{C}_{p_n}^{-1}\,\tilde{\underline{U}}_n^T \\ \bar{\underline{s}}^d = G\bar{\tilde{\underline{B}}}_{e_n} \quad \rightarrow \quad \bar{s} = \sqrt{\frac{3}{2}\mathrm{tr}(\bar{\underline{s}}^d\,\bar{\underline{s}}^d)} \\ \eta = \eta(\bar{s}, p, T, D) \end{array}\right\} \rightarrow \bar{\tilde{\underline{B}}}_{e_n} \rightarrow \left\{\begin{array}{l} \tilde{\underline{B}}_e = \underline{R}_n\,\bar{\tilde{\underline{B}}}_{e_n}\,\underline{R}_n^T \rightarrow \\ \underline{s}^d = G\tilde{\underline{B}}_e^d \end{array}\right.$$

$$\underline{\sigma} = \underline{s}^d + \underline{s}^h + \underline{w}$$

Stiffness

$$\left.\begin{array}{l}\sigma = \mathbf{s}^d + \mathbf{s}^h + \mathbf{w} = G\tilde{\mathbf{B}}_e^d + \kappa\mathbf{I}(J-1) + H\tilde{\mathbf{B}}^d \\ \\ \quad\left.\begin{array}{l}\tilde{\mathbf{B}}_e = \tilde{\mathbf{F}}\cdot\mathbf{C}_p^{-1}\cdot\tilde{\mathbf{F}}^c \\ \mathbf{C}_p = (1-\lambda)\tilde{\mathbf{C}} + \lambda\mathbf{C}_p(t_n) \\ \tilde{\mathbf{F}} = J^{-1/3}\mathbf{F}\end{array}\right\}\end{array}\right\}$$

$$\begin{aligned}\delta\sigma &= \delta\mathbf{s}^d + \delta\mathbf{s}^h + \delta\mathbf{w} \\ &= G\,\delta\tilde{\mathbf{B}}_e^d + \kappa\,\mathbf{I}\delta J + H\,\delta\tilde{\mathbf{B}}^d = \left({}^4\mathbf{S}_d + {}^4\mathbf{S}_h + {}^4\mathbf{H}\right) : \delta\mathbf{F} \\ &= {}^4\mathbf{S} : \delta\mathbf{F} = {}^4\mathbf{S}^{rc} : \delta\mathbf{F}^c \quad \text{with} \qquad \delta\mathbf{F}^c = \vec{\nabla}_0\vec{u} = \mathbf{F}^c\cdot\vec{\nabla}\vec{u} = \mathbf{F}^c\cdot\mathbf{L}_u^c \\ &= {}^4\mathbf{S}^{rc} : (\mathbf{F}^c\cdot\mathbf{L}_u^c) \\ &= {}^4\mathbf{M} : \mathbf{L}_u^c\end{aligned}$$

Elastic strain variation

$$
\begin{aligned}
\tilde{\mathbf{B}}_e &= \tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-1} \cdot \tilde{\mathbf{F}}^c \\
\delta\tilde{\mathbf{B}}_e &= \delta\tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-1} \cdot \tilde{\mathbf{F}}^c - \tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-1} \cdot \delta\mathbf{C}_p \cdot \mathbf{C}_p^{-1} \cdot \tilde{\mathbf{F}}^c + \tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-1} \cdot \delta\tilde{\mathbf{F}}^c \\
&= \left(\tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-c} \cdot \delta\tilde{\mathbf{F}}^c\right)^c - \tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-1} \cdot \left(\tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-c} \cdot \delta\mathbf{C}_p^c\right)^c + \tilde{\mathbf{F}} \cdot \mathbf{C}_p^{-1} \cdot \delta\tilde{\mathbf{F}}^c \\
&= \left(\mathbf{M}^{(1)} \cdot \delta\tilde{\mathbf{F}}^c\right)^c - \mathbf{M}^{(2)} \cdot \left(\mathbf{M}^{(1)} \cdot \delta\mathbf{C}_p^c\right)^c + \mathbf{M}^{(2)} \cdot \delta\tilde{\mathbf{F}}^c
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathbf{B}}_e^d &= \tilde{\mathbf{B}}_e - \tfrac{1}{3}\mathrm{tr}(\tilde{\mathbf{B}}_e)\mathbf{I} = \left({}^4\mathbf{I} - \tfrac{1}{3}\mathbf{II}\right) : \tilde{\mathbf{B}}_e \\
\delta\tilde{\mathbf{B}}_e^d &= \left({}^4\mathbf{I} - \tfrac{1}{3}\mathbf{II}\right) : \delta\tilde{\mathbf{B}}_e
\end{aligned}
$$

Plastic strain variation

$$
\begin{aligned}
\mathbf{C}_p &= (1-\lambda)\tilde{\mathbf{C}} + \lambda \mathbf{C}_p(t_n) \\
\delta\mathbf{C}_p &= (1-\lambda)\delta\tilde{\mathbf{C}} + \left(\mathbf{C}_p(t_n) - \tilde{\mathbf{C}}\right)\delta\lambda \\
&= (1-\lambda)\left(\delta\tilde{\mathbf{F}}^c \cdot \tilde{\mathbf{F}} + \tilde{\mathbf{F}}^c \cdot \delta\tilde{\mathbf{F}}\right) + \left(\mathbf{C}_p(t_n) - \tilde{\mathbf{C}}\right)\delta\lambda \\
&= (1-\lambda)\left[\left(\tilde{\mathbf{F}}^c \cdot \delta\tilde{\mathbf{F}}\right)^c + \tilde{\mathbf{F}}^c \cdot \delta\tilde{\mathbf{F}}\right] + \left(\mathbf{C}_p(t_n) - \tilde{\mathbf{C}}\right)\delta\lambda
\end{aligned}
$$

Deformation tensor variation

$$
\begin{aligned}
\tilde{\mathbf{F}} &= J^{-1/3}\mathbf{F} \\
\delta\tilde{\mathbf{F}} &= -\tfrac{1}{6}J^{-1/3}\mathbf{F}\mathbf{I} : \left(\delta\mathbf{F}\cdot\mathbf{F}^{-1} + \mathbf{F}^{-c}\cdot\delta\mathbf{F}^{c}\right) + J^{-1/3}\delta\mathbf{F} \\
&= -\tfrac{1}{3}J^{-1/3}\mathbf{F}\left(\mathbf{F}^{-c} : \delta\mathbf{F}^{c}\right) + J^{-1/3}\delta\mathbf{F}
\end{aligned}
$$

Elasticity scalar variation

$$\delta\lambda = \frac{\lambda \Delta t \Gamma}{G\Delta t + \eta}\,\delta\eta = l_1 \tilde{\mathbf{B}}_e^d : \delta\tilde{\mathbf{B}}_e + l_2 \mathbf{I} : \delta\mathbf{F}$$

$$l_1 = \frac{\lambda \Delta t \,\Gamma\, h_1}{\Delta t\, G + \eta} \quad ; \quad l_2 = \frac{l_1 h_2}{h_1}$$

$$h_1 = \frac{3G^2}{2\bar{\sigma}}\left(\frac{\partial\eta}{\partial\bar{\sigma}} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial\bar{\sigma}}\right) \quad ; \quad h_2 = -\kappa J\left(\frac{\partial\eta}{\partial p} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial p}\right)$$

$$\frac{\partial\eta}{\partial\bar{\sigma}} = \eta\left(\frac{1}{\bar{\sigma}} - \frac{1}{\sqrt{3}\tau_0}\right) \quad ; \quad \frac{\partial\eta}{\partial p} = \frac{\eta\mu}{\tau_0} \quad ; \quad \frac{\partial\eta}{\partial D} = -\eta$$

$$\frac{\partial D}{\partial\bar{\sigma}} = \frac{\Delta t\,\frac{\partial\dot{D}}{\partial\bar{\sigma}}}{1 - \Delta t\,\frac{\partial\dot{D}}{\partial D}} \quad ; \quad \frac{\partial D}{\partial p} = \frac{\Delta t\,\frac{\partial\dot{D}}{\partial p}}{1 - \Delta t\,\frac{\partial\dot{D}}{\partial D}}$$

$$\frac{\partial\dot{D}}{\partial\bar{\sigma}} = \frac{\dot{D}}{\sqrt{3}\tau_0} \quad ; \quad \frac{\partial\dot{D}}{\partial p} = -\frac{\dot{D}\mu}{\tau_0} \quad ; \quad \frac{\partial\dot{D}}{\partial D} = \dot{D} - \frac{h\bar{\sigma}}{\sqrt{6}\,D_\infty\,\eta}$$

with $\dot{D} = h\left(1 - \frac{D}{D_\infty}\right)\frac{\bar{\sigma}}{\sqrt{6}\,\eta}$

Deviatoric stress variation

$$\delta \mathbf{s}^d = G\, \delta \tilde{\mathbf{B}}_e^d = {}^4\mathbf{S}_d : \delta \mathbf{F}$$

Hydrostatic stress variation

$$\delta \mathbf{s}^h = \kappa \, \mathbf{I} \delta J = {}^4\mathbf{S}_h : \delta \mathbf{F}$$

$$\begin{aligned}
\dot{J} &= J \, \mathrm{tr}(\mathbf{D}) = J \, \tfrac{1}{2} \, \mathrm{tr} \left\{ \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} + \left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \right)^c \right\} \quad \rightarrow \\
\delta J &= \tfrac{1}{2} J \, \mathrm{tr} \left(\delta \mathbf{F} \cdot \mathbf{F}^{-1} + \mathbf{F}^{-c} \cdot \delta \mathbf{F}^c \right) \\
&= \tfrac{1}{2} J \left(\mathbf{F}^{-c} : \delta \mathbf{F}^c \right) + \tfrac{1}{2} J \left(\mathbf{F}^{-c} : \delta \mathbf{F}^c \right) \\
&= J \, \mathbf{F}^{-c} : \delta \mathbf{F}^c = J \, \mathbf{F}^{-1} : \delta \mathbf{F}
\end{aligned}$$

Hardening stress variation

$$\delta \mathbf{w} = H\,\delta \tilde{\mathbf{B}}^d = {}^4\mathbf{H} : \delta \mathbf{F}$$

$$\begin{aligned}
\tilde{\mathbf{B}} &= \tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}^c \\
\delta \tilde{\mathbf{B}} &= \delta \tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}^c + \tilde{\mathbf{F}} \cdot \delta \tilde{\mathbf{F}}^c
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{B}}^d &= \tilde{\mathbf{B}} - \tfrac{1}{3}\mathrm{tr}(\tilde{\mathbf{B}})\mathbf{I} = \left({}^4\mathbf{I} - \tfrac{1}{3}\mathbf{II}\right) : \tilde{\mathbf{B}} \\
\delta \tilde{\mathbf{B}}^d &= \left({}^4\mathbf{I} - \tfrac{1}{3}\mathbf{II}\right) : \left\{ \left(\tilde{\mathbf{F}} \cdot \delta \tilde{\mathbf{F}}^c\right)^c + \tilde{\mathbf{F}} \cdot \delta \tilde{\mathbf{F}}^c \right\}
\end{aligned}$$

Consistent material stiffness tensor

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= \delta\mathbf{s}^d + \delta\mathbf{s}^h + \delta\mathbf{w} \\
&= \left({}^4\mathbf{S}_d + {}^4\mathbf{S}_h + {}^4\mathbf{H} \right) : \delta\mathbf{F} = {}^4\mathbf{S} : \delta\mathbf{F} = {}^4\mathbf{S}^{rc} : \delta\mathbf{F}^c \\
&\qquad\qquad\qquad \text{with} \quad \delta\mathbf{F}^c = \vec{\nabla}_0 \vec{u} = \mathbf{F}^c \cdot \vec{\nabla}\vec{u} = \mathbf{F}^c \cdot \mathbf{L}_u^c \quad \rightarrow \\
&= {}^4\mathbf{S}^{rc} : (\mathbf{F}^c \cdot \mathbf{L}_u^c) = {}^4\mathbf{M} : \mathbf{L}_u^c
\end{aligned}
$$

Matrix/column notation

$$\delta \tilde{\underset{\approx}{B}}_e = \left(\underline{\underline{M}}_{cr}^{(1)} + \underline{\underline{M}}_{c}^{(2)} \right) \delta \tilde{\underset{\approx}{F}} - \underline{\underline{M}}_{c}^{(2)} \underline{\underline{M}}_{c}^{(1)} \delta \underset{\approx}{C}_p \quad ; \quad \underline{M}^{(1)} = \underline{\tilde{F}}\, \underline{C}_p^{-T} \; ; \; \underline{M}^{(2)} = \underline{\tilde{F}}\, \underline{C}_p^{-1}$$

$$= \underline{\underline{A}}^{(1)} \delta \tilde{\underset{\approx}{F}} + \underline{\underline{A}}^{(2)} \delta \underset{\approx}{C}_p$$

$$\delta \tilde{\underset{\approx}{B}}_e^d = \left(\underline{\underline{I}} - \tfrac{1}{3} \underset{\approx}{I} \underset{\approx}{I}_t^T \right) \delta \tilde{\underset{\approx}{B}}_e$$

$$\delta \underset{\approx}{C}_p = \left[(1-\lambda) \left(\tilde{\underline{\underline{F}}}_{tr} + \tilde{\underline{\underline{F}}}_t \right) \right] \delta \tilde{\underset{\approx}{F}} + \left(\underset{\approx}{C}_p(t_n) - \tilde{\underset{\approx}{C}} \right) \delta\lambda = \underline{\underline{C}}^{(1)} \delta \tilde{\underset{\approx}{F}} + \underset{\approx}{C}^{(2)} \delta\lambda$$

$$\delta \tilde{\underset{\approx}{F}} = \left[-\tfrac{1}{3} J^{-1/3} \underset{\approx}{F} \left(\underset{\approx}{F}^{-1} \right)_t^T + J^{-1/3} \underline{\underline{I}} \right] \delta \underset{\approx}{F} = \underline{\underline{F}}\, \delta \underset{\approx}{F}$$

$$\delta\lambda = = l_1 \left(\tilde{\underset{\approx}{B}}_e^d \right)_t^T \delta \tilde{\underset{\approx}{B}}_e + l_2 \underset{\approx}{I}_t^T \delta \underset{\approx}{F}$$

Matrix/column notation

$$
\begin{aligned}
\delta\tilde{\underset{\approx}{B}}_e &= \underline{\underline{A}}^{(1)}\delta\tilde{\underset{\approx}{F}} + \underline{\underline{A}}^{(2)}\delta\underset{\approx}{C}_p \\
&= \left(\underline{\underline{A}}^{(1)} + \underline{\underline{A}}^{(2)}\underline{\underline{C}}^{(1)}\right)\delta\tilde{\underset{\approx}{F}} + \underline{\underline{A}}^{(2)}\underset{\approx}{C}^{(2)}\delta\lambda = \underline{\underline{B}}^{(1)}\delta\tilde{\underset{\approx}{F}} + \underset{\approx}{B}^{(2)}\delta\lambda \\
&= \underline{\underline{B}}^{(1)}\underline{\underline{F}}\delta\underset{\approx}{F} + l_1\underset{\approx}{B}^{(2)}\left(\tilde{\underset{\approx}{B}}{}_e^d\right)_t^T\delta\tilde{\underset{\approx}{B}}_e + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}_t^T\delta\underset{\approx}{F} \\
\delta\tilde{\underset{\approx}{B}}_e &= \left[\underline{\underline{I}} - l_1\underset{\approx}{B}^{(2)}\left(\tilde{\underset{\approx}{B}}{}_e^d\right)_t^T\right]^{-1}\left[\underline{\underline{B}}^{(1)}\underline{\underline{F}} + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}_t^T\right]\delta\underset{\approx}{F} \\
\delta\tilde{\underset{\approx}{B}}{}_e^d &= \left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\right)\delta\tilde{\underset{\approx}{B}}_e \\
&= \left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\right)\left[\underline{\underline{I}} - l_1\underset{\approx}{B}^{(2)}\left(\tilde{\underset{\approx}{B}}{}_e^d\right)_t^T\right]^{-1}\left[\underline{\underline{B}}^{(1)}\underline{\underline{F}} + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}_t^T\right]\delta\underset{\approx}{F} = \underline{\underline{B}}^{(3)}\delta\underset{\approx}{F} \\
\delta\underset{\approx}{B}^d &= \left(\underline{\underline{I}}_c - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\right)\left(\tilde{\underline{\underline{F}}}_{cr} + \tilde{\underline{\underline{F}}}_c\right)\delta\underset{\approx}{F} \\
&= \left(\tilde{\underline{\underline{F}}}_{cr} + \tilde{\underline{\underline{F}}}_c - \tfrac{2}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\tilde{\underline{\underline{F}}}_c\right)\underline{\underline{F}}\delta\underset{\approx}{F} = \underline{\underline{B}}^{(4)}\delta\underset{\approx}{F}
\end{aligned}
$$

Matrix/column notation

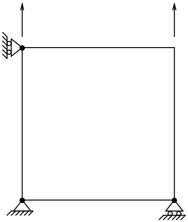
$$\delta \underset{\approx}{s}^d = G \delta \underset{\approx}{\tilde{B}}_e^d = G \underline{\underline{B}}^{(3)} \delta \underset{\approx}{F} = \underline{\underline{S}}_d \delta \underset{\approx}{F}$$

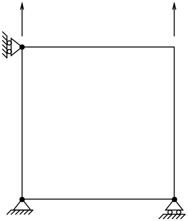
$$\delta \underset{\approx}{s}^h = \kappa \underset{\approx}{I} \delta J = \kappa J \underset{\approx}{I} \left(\underset{\approx}{F}^{-1} \right)_t^T \delta \underset{\approx}{F} = \underline{\underline{S}}_h \delta \underset{\approx}{F}$$

$$\delta \underset{\approx}{w} = H \, \delta \underset{\approx}{\tilde{B}}^d = H \, \underline{\underline{B}}^{(4)} \delta \underset{\approx}{F} = \underline{\underline{H}} \delta \underset{\approx}{F}$$

$$\begin{aligned}
\delta \underset{\approx}{\sigma} &= \delta \underset{\approx}{s}^d + \delta \underset{\approx}{s}^h + \delta \underset{\approx}{w} \\
&= \left(\underline{\underline{S}}_d + \underline{\underline{S}}_h + \underline{\underline{H}} \right) \delta \underset{\approx}{F} = \underline{\underline{S}} \delta \underset{\approx}{F} = \underline{\underline{S}}_c \delta \underset{\approx}{F}_t \\
&\qquad\qquad \text{with} \quad \delta \underset{\approx}{F}_t = \underline{\underline{F}}_t \left(\underset{\approx}{L}_u \right)_t \\
&= \underline{\underline{S}}_c \underline{\underline{F}}_t \left(\underset{\approx}{L}_u \right)_t \\
&= \underline{\underline{M}} \left(\underset{\approx}{L}_u \right)_t
\end{aligned}$$

Tensile test





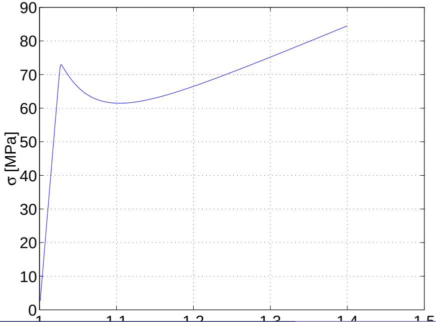
initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

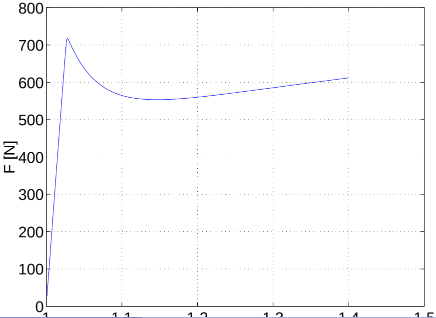
initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

Viscoelastic model in tensile test

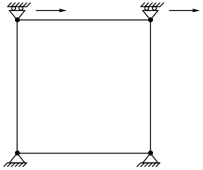
E	2305	MPa	ν	0.37	-
H	29	MPa	h	270	-
D_∞	19	-	A_0	9.7573E-27	s
ΔH	2.9E5	J/mol	μ	0.06984	-
τ_0	0.72	MPa			

elongation rate $$\frac{\dot{\Delta l}}{h_0} = 0.01\ \mathrm{s}^{-1}$$





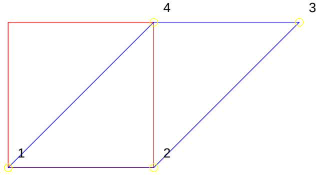
Shear test

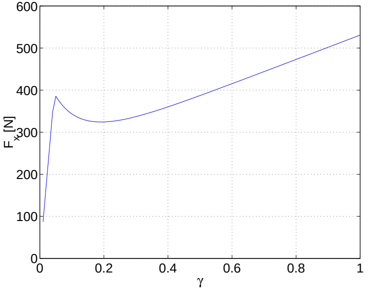


initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

strain rate

$$\dot{\gamma} = \frac{\dot{u}}{h_0} = 0.01 \text{ s}^{-1}$$





Bathe, K.-J. *Finite Element Procedures*. Prentice Hall, New Jersey, 1996.

Boyle, J.T.; Spence, J. *Stress analysis for creep*. Butterwort, 1983, pp 283.

Crisfield, M. *Non-linear Finite Element Analysis of Solids and Structures, Vol. 1: Essentials*. John Wiley and Sons Ltd., West Sussex, England...

Crisfield, M. *Non-linear Finite Element Analysis of Solids and Structures, Vol. 2: Advanced Topics*. John Wiley and Sons Ltd., West Sussex, England...

Gordon, J.E. *The new science of strong materials*. Penguin Books, 1976.

Gordon, J.E. *Structures*. Penguin Books, 1978.

Hughes, T. *Numerical implementation of constitutive models: rate-independent deviatoric plasticity*. In: Theoretical Foundation for Large-scale Computations for Nonlinear Material Behaviour, Ed: Nemat-Nasser, R. Asaro, G. Hegemier Martinus Nijhoff Publishers, Dordrecht, The Netherlands.., pp 29-57.

Hunter, S.C. *Mechanics of continuous media, 2nd edition*. Ellis Horwood Limited, 1983.

Simo, J.C.; Hughes, T. *Computational Inelasticity. Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1998.

Skrzypek, J. *Plasticity and Creep - Theory, Examples and Problems*. CRC Press, Inc., Florida, USA, 1993.

Timoshenko, Stephen P. *History of strength of materials: with a brief account of the history of elasticity and theory of structures*. London : McGraww-Hill, 1953, pp 452.

Tschoegl, N.W. *The phenomenological theory of linear viscoelastic behaviour; An introduction*. Springer-Verlag, 1989.

Zienkiewicz, O.; Taylor, R. *The Finite Element Method, Vol. 1, Basic Formulation and Linear Problems*. McGraw-Hill, London, UK, 1989.

Zienkiewicz, O.; Taylor, R. *The Finite Element Method, Vol. 2, Solid and Fluid Mechanics, Dynamics and Non-linearity*. McGraw-Hill, London, UK, 1989.

APPENDICES

back to index

Utilities m2cc.m and m2mm.m

back to index

m2cc

```
%**********************************************************************
function [C] = m2cc(m,s);

C = zeros(s,1);

if s==9
  C = [m(1,1); m(2,2); m(3,3);
       m(1,2); m(2,1); m(2,3); m(3,2); m(3,1); m(1,3)];
elseif s==5
  C = [m(1,1); m(2,2); m(3,3); m(1,2); m(2,1)];
elseif s==4
  C = [m(1,1); m(2,2); m(1,2); m(2,1)];
end;
%**********************************************************************
```

m2mm

```
%*****************************************************************
function [M] = m2mm(m,s);

M = zeros(s);

if s==9
   M = [ m(1,1) 0      0      0      m(1,2) 0      0      m(1,3) 0
         0      m(2,2) 0      m(2,1) 0      0      m(2,3) 0      0
         0      0      m(3,3) 0      0      m(3,2) 0      0      m(3,1)
         0      m(1,2) 0      m(1,1) 0      0      m(1,3) 0      0
         m(2,1) 0      0      0      m(2,2) 0      0      m(2,3) 0
         0      0      m(2,3) 0      0      m(2,2) 0      0      m(2,1)
         0      m(3,2) 0      m(3,1) 0      0      m(3,3) 0      0
         m(3,1) 0      0      0      m(3,2) 0      0      m(3,3) 0
         0      0      m(1,3) 0      0      m(1,2) 0      0      m(1,1) ];
elseif s==5
   M = [ m(1,1) 0      0      0      m(1,2)
         0      m(2,2) 0      m(2,1) 0
         0      0      m(3,3) 0      0
         0      m(1,2) 0      m(1,1) 0
         m(2,1) 0      0      0      m(2,2) ];
elseif s==4
   M = [ m(1,1) 0      0      m(1,2)
         0      m(2,2) m(2,1) 0
         0      m(1,2) m(1,1) 0
         m(2,1) 0      0      m(2,2) ];
end;
%*****************************************************************
```

Stiffness and compliance matrices

back to index

General orthotropic stiffness matrix

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \underline{\underline{C}}\, \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \underline{\underline{C}}^{-1}\, \underset{\sim}{\sigma} = \underline{\underline{S}}\, \underset{\sim}{\sigma}$$

General orthotropic compliance matrix

$$\underline{\underline{C}}^{-1} = \frac{1}{\Delta_c} \begin{bmatrix} BC - S^2 & -QC + RS & QS - BR & 0 & 0 & 0 \\ -QC + RS & AC - R^2 & -AS + QR & 0 & 0 & 0 \\ QS - BR & -AS + QR & AB - Q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_c(1/K) & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_c(1/L) & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_c(1/M) \end{bmatrix}$$

with $\quad \Delta_c = ABC - AS^2 - BR^2 - CQ^2 + 2QRS$

$$\underline{\underline{S}}^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & qs - br & 0 & 0 & 0 \\ -qc + rs & ac - r^2 & -as + qr & 0 & 0 & 0 \\ qs - br & -as + qr & ab - q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_s(1/k) & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_s(1/l) & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_s(1/m) \end{bmatrix}$$

with $\quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$

Material symmetry

quadratic	$B = A \; ; \; S = R \; ; \; M = L;$
transversal isotropic	$B = A \; ; \; S = R \; ; \; M = L \; ; \; K = \frac{1}{2}(A - Q)$
cubic	$C = B = A \; ; \; S = R = Q \; ; \; M = L = K \neq \frac{1}{2}(A - Q)$
isotropic	$C = B = A \; ; \; S = R = Q \; ; \; M = L = K = \frac{1}{2}(A - Q)$

Orthotropic thermo-elasticity

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} A+Q+R \\ Q+B+S \\ R+S+C \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Plane strain

$$\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0 \quad \longrightarrow \quad \sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\varepsilon \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{AB - Q^2} \begin{bmatrix} B & -Q & 0 \\ -Q & A & 0 \\ 0 & 0 & \dfrac{AB - Q^2}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$= \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\varepsilon \underset{\sim}{\sigma}$$

Plane strain

$$\varepsilon_{33} = 0 = r\sigma_{11} + s\sigma_{22} + c\sigma_{33} \quad \rightarrow \quad \sigma_{33} = -\frac{r}{c}\sigma_{11} - \frac{s}{c}\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} - \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \begin{bmatrix} \frac{r}{c} & \frac{s}{c} & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} ac - r^2 & qc - rs & 0 \\ qc - sr & bc - s^2 & 0 \\ 0 & 0 & kc \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\varepsilon \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & 0 \\ -qc + rs & ac - r^2 & 0 \\ 0 & 0 & \frac{\Delta_s}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$$

$$= \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \underline{\underline{C}}_\varepsilon \underset{\sim}{\sigma}$$

$$\sigma_{33} = -\frac{1}{\Delta_s}\left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22}\right]$$

Plane stress

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \quad \longrightarrow \quad \varepsilon_{33} = r\sigma_{11} + s\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{ab - q^2} \begin{bmatrix} b & -q & 0 \\ -q & a & 0 \\ 0 & 0 & \dfrac{ab - q^2}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

$$\varepsilon_{33} = \frac{1}{ab - q^2} \left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

Plane stress

$$\sigma_{33} = 0 = R\varepsilon_{11} + S\varepsilon_{22} + C\varepsilon_{33} \quad \rightarrow \quad \varepsilon_{33} = -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \begin{bmatrix} R \\ S \\ 0 \end{bmatrix} \begin{bmatrix} \frac{R}{C} & \frac{S}{C} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \frac{1}{C} \begin{bmatrix} AC - R^2 & QC - RS & 0 \\ QC - SR & BC - S^2 & 0 \\ 0 & 0 & KC \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

General planar material laws

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} - \alpha \Delta T \begin{bmatrix} \Theta_{p1} \\ \Theta_{p2} \\ 0 \end{bmatrix}$$

$$\underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix} + \sigma \Delta T \begin{bmatrix} \theta_{p1} \\ \theta_{p2} \\ 0 \end{bmatrix}$$

plane strain : $(\ )_p = (\ )_\varepsilon$
plane stress : $(\ )_p = (\ )_\sigma$

Linear elastic orthotropic, 3D

$$\underline{\underline{S}} = \begin{bmatrix} E_1^{-1} & -\nu_{21}E_2^{-1} & -\nu_{31}E_3^{-1} & 0 & 0 & 0 \\ -\nu_{12}E_1^{-1} & E_2^{-1} & -\nu_{32}E_3^{-1} & 0 & 0 & 0 \\ -\nu_{13}E_1^{-1} & -\nu_{23}E_2^{-1} & E_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{23}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{31}^{-1} \end{bmatrix}$$

$$\text{with} \qquad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \quad ; \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3} \quad ; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}$$

$$\underline{\underline{C}} = \frac{1}{\Delta_s} \begin{bmatrix} \frac{1-\nu_{32}\nu_{23}}{E_2E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2E_3} & \frac{\nu_{21}\nu_{32}+\nu_{31}}{E_2E_3} & 0 & 0 & 0 \\ \frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1E_3} & \frac{\nu_{12}\nu_{31}+\nu_{32}}{E_1E_3} & 0 & 0 & 0 \\ \frac{\nu_{12}\nu_{23}+\nu_{13}}{E_1E_2} & \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1E_2} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_s G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_s G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_s G_{31} \end{bmatrix}$$

$$\text{with} \qquad \Delta_s = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

Voigt notation

$$\underset{\sim}{\sigma}^T = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{31}] = [\sigma_1\ \sigma_2\ \sigma_3\ \sigma_6\ \sigma_4\ \sigma_5]$$
$$\underset{\sim}{\varepsilon}^T = [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \gamma_{12}\ \gamma_{23}\ \gamma_{31}] = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \varepsilon_6\ \varepsilon_4\ \varepsilon_5]$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

Linear elastic orthotropic, plane strain

$$\sigma_{33} = \nu_{13}\,\frac{E_3}{E_1}\,\sigma_{11} + \nu_{23}\,\frac{E_3}{E_2}\,\sigma_{22}$$

$$\underline{\underline{S}}_{\varepsilon} = \begin{bmatrix} \frac{1-\nu_{31}\nu_{13}}{E_1} & -\frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1} & \frac{1-\nu_{32}\nu_{23}}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

$$\underline{\underline{C}}_{\varepsilon} = \underline{\underline{S}}_{\varepsilon}^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} \frac{1-\nu_{32}\nu_{23}}{E_2 E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2 E_3} & 0 \\ \frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1 E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1 E_3} & 0 \\ 0 & 0 & \Delta_s G_{12} \end{bmatrix}$$

with $$\Delta_s = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3}$$

$$\sigma_{33} = \frac{1}{\Delta_s}\left\{\frac{\nu_{12}\nu_{32}+\nu_{13}}{E_1 E_2}\,\varepsilon_{11} + \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1 E_2}\,\varepsilon_{22}\right\}$$

Linear elastic orthotropic, plane stress

$$\varepsilon_{33} = -\nu_{13} E_1^{-1} \sigma_{11} - \nu_{23} E_2^{-1} \sigma_{22}$$

$$\underline{\underline{S}}_\sigma = \begin{bmatrix} E_1^{-1} & -\nu_{21} E_2^{-1} & 0 \\ -\nu_{12} E_1^{-1} & E_2^{-1} & 0 \\ 0 & 0 & G_{12}^{-1} \end{bmatrix}$$

$$\underline{\underline{C}}_\sigma = \underline{\underline{S}}_\sigma^{-1} = \frac{1}{1 - \nu_{21}\nu_{12}} \begin{bmatrix} E_1 & \nu_{21} E_1 & 0 \\ \nu_{12} E_2 & E_2 & 0 \\ 0 & 0 & (1 - \nu_{21}\nu_{12}) G_{12} \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{1}{1 - \nu_{12}\nu_{21}} \left\{ (\nu_{12}\nu_{23} + \nu_{13}) \varepsilon_{11} + (\nu_{21}\nu_{13} + \nu_{23}) \varepsilon_{22} \right\}$$

Linear elastic transversal isotropic, 3D

$$\underline{\underline{S}} = \begin{bmatrix} E_p^{-1} & -\nu_p E_p^{-1} & -\nu_{3p} E_3^{-1} & 0 & 0 & 0 \\ -\nu_p E_p^{-1} & E_p^{-1} & -\nu_{3p} E_3^{-1} & 0 & 0 & 0 \\ -\nu_{p3} E_p^{-1} & -\nu_{p3} E_p^{-1} & E_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_p^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{p3}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{3p}^{-1} \end{bmatrix}$$

with $\quad \dfrac{\nu_{p3}}{E_p} = \dfrac{\nu_{3p}}{E_3}$

$$\underline{\underline{C}} = \underline{\underline{S}}^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_p E_3} & \frac{\nu_{3p}\nu_{p3}+\nu_p}{E_p E_3} & \frac{\nu_p\nu_{3p}+\nu_{3p}}{E_p E_3} & 0 & 0 & 0 \\ \frac{\nu_{p3}\nu_{3p}+\nu_p}{E_p E_3} & \frac{1-\nu_{3p}\nu_{p3}}{E_p E_3} & \frac{\nu_p\nu_{3p}+\nu_{3p}}{E_p E_3} & 0 & 0 & 0 \\ \frac{\nu_p\nu_{p3}+\nu_{p3}}{E_p E_p} & \frac{\nu_p\nu_{p3}+\nu_{p3}}{E_p E_p} & \frac{1-\nu_p\nu_p}{E_p E_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_s G_p & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_s G_{p3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_s G_{3p} \end{bmatrix}$$

with $\quad \Delta_s = \dfrac{1-\nu_p\nu_p-\nu_{p3}\nu_{3p}-\nu_{3p}\nu_{p3}-\nu_p\nu_{p3}\nu_{3p}-\nu_p\nu_{3p}\nu_{p3}}{E_p E_p E_3}$

Linear elastic transversal isotropic, plane strain

$$\sigma_{33} = \frac{E_3 \nu_{p3}}{E_p} \left(\sigma_{11} + \sigma_{22} \right) = \nu_{3p} (\sigma_{11} + \sigma_{22})$$

$$\underline{\underline{S}}_{\varepsilon} = \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_p} & -\frac{\nu_{3p}\nu_{p3}+\nu_p}{E_p} & 0 \\ -\frac{\nu_{p3}\nu_{3p}+\nu_p}{E_p} & \frac{1-\nu_{3p}\nu_{p3}}{E_p} & 0 \\ 0 & 0 & \frac{1}{G_p} \end{bmatrix}$$

$$\underline{\underline{C}}_{\varepsilon} = \underline{\underline{S}}_{\varepsilon}^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_p E_3} & \frac{\nu_{3p}\nu_{p3}+\nu_p}{E_p E_3} & 0 \\ \frac{\nu_{p3}\nu_{3p}+\nu_p}{E_p E_3} & \frac{1-\nu_{3p}\nu_{p3}}{E_p E_3} & 0 \\ 0 & 0 & \Delta_s G_p \end{bmatrix}$$

with $$\Delta_s = \frac{1 - \nu_p \nu_p - \nu_{p3}\nu_{3p} - \nu_{3p}\nu_{p3} - \nu_p \nu_{p3}\nu_{3p} - \nu_p \nu_{3p}\nu_{p3}}{E_p E_p E_3}$$

$$\sigma_{33} = \frac{1}{\Delta_s} \frac{\nu_{p3}(\nu_p + 1)}{E_p^2} \left(\varepsilon_{11} + \varepsilon_{22} \right)$$

Linear elastic transversal isotropic, plane stress

$$\varepsilon_{33} = -\frac{\nu_{p3}}{E_p}(\sigma_{11} + \sigma_{22})$$

$$\underline{\underline{S}}_\sigma = \begin{bmatrix} E_p^{-1} & -\nu_p E_p^{-1} & 0 \\ -\nu_p E_p^{-1} & E_p^{-1} & 0 \\ 0 & 0 & G_p^{-1} \end{bmatrix}$$

$$\underline{\underline{C}}_\sigma = \underline{\underline{S}}_\sigma^{-1} = \frac{1}{1-\nu_p\nu_p}\begin{bmatrix} E_p & \nu_p E_p & 0 \\ \nu_p E_p & E_p & 0 \\ 0 & 0 & (1-\nu_p\nu_p)G_p \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu_{p3}}{1-\nu_p}\left(\varepsilon_{11} + \varepsilon_{22}\right)$$

Linear elastic isotropic, 3D

$$\underline{\underline{S}} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{S}}^{-1} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

Linear elastic isotropic, plane strain

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

$$\underline{\underline{S}}_{\varepsilon} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underline{\underline{C}}_{\varepsilon} = \underline{\underline{S}}_{\varepsilon}^{-1} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)} \nu(\varepsilon_{11} + \varepsilon_{22})$$

Linear elastic isotropic, plane stress

$$\varepsilon_{33} = -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right)$$

$$\underline{\underline{S}}_{\sigma} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$\underline{\underline{C}}_{\sigma} = \underline{\underline{S}}_{\sigma}^{-1} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}\left(\varepsilon_{11} + \varepsilon_{22}\right)$$

WR for axi-symmetric deformation

back to index

Weighted residual formulation for axi-symmetric deformation

$$\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r = 0 \quad \forall \quad r \quad \leftrightarrow$$

$$\int_V w\{\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r\}\, dV = 0 \qquad \forall \quad w(r)$$

$$2\pi t \int_{R_i}^{R_o} w\{\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r\} r\, dr = 0 \qquad \forall \quad w(r)$$

Weak formulation

$$w\sigma_{rr,r}r = w\frac{d\sigma_{rr}}{dr}r = \frac{d}{dr}(w\sigma_{rr}r) - \frac{dw}{dr}\sigma_{rr}r - w\sigma_{rr} \qquad \rightarrow$$

$$\int_{R_i}^{R_o}(w_{,r}\sigma_{rr}r + w\sigma_{tt})\,dr = \int_{R_i}^{R_o} wq_r r\,dr + [w\sigma_{rr}t]_{R_i}^{R_o} = f_e$$

Linear elastic deformation

$$\left.\begin{aligned}\sigma_{rr} &= A_p\varepsilon_{rr} + Q_p\varepsilon_{tt} = A_p u_{r,r} + Q_p\frac{u_r}{r} \\ \sigma_{tt} &= Q_p\varepsilon_{rr} + B_p\varepsilon_{tt} = Q_p u_{r,r} + B_p\frac{u_r}{r}\end{aligned}\right\} \quad \rightarrow$$

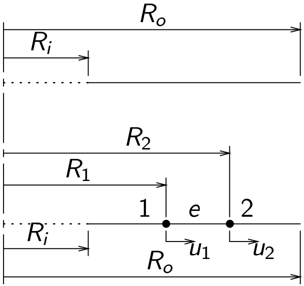
$$\int_{R_i}^{R_o} \left\{ w_{,r}\left(A_p u_{r,r} + Q_p\frac{u_r}{r}\right) r + w\left(Q_p u_{r,r} + B_p\frac{u_r}{r}\right)\right\} dr = f_e$$

FEM for axi-symmerric deformation

back to index

Finite element method for an axi-symmetric ring

Discretisation



$$\sum_{e=1}^{ne} \int_{R_1}^{R_2} \left[A_p w_{,r} u_{r,r} r + Q_p w_{,r} u_r + Q_p w u_{r,r} + B_p w \frac{1}{r} u_r \right] dr = \sum_{e=1}^{ne} f_e^e$$

Interpolation

$$u_r = \psi_1 u_1 + \psi_2 u_2$$

$$\text{Galerkin} \quad \rightarrow \quad w = \psi_1 w_1 + \psi_2 w_2$$

Substitution

$$\left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \int_{R_1}^{R_2} \left\{ A_p \left[\begin{array}{c} \psi_{1,r} \\ \psi_{2,r} \end{array}\right] \left[\begin{array}{cc} \psi_{1,r} & \psi_{2,r} \end{array}\right] r + Q_p \left[\begin{array}{c} \psi_{1,r} \\ \psi_{2,r} \end{array}\right] \left[\begin{array}{cc} \psi_1 & \psi_2 \end{array}\right] + \right.$$

$$\left. Q_p \left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right] \left[\begin{array}{cc} \psi_{1,r} & \psi_{2,r} \end{array}\right] + B_p \left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right] \left[\begin{array}{cc} \psi_1 & \psi_2 \end{array}\right] \frac{1}{r} \right\} \left[\begin{array}{c} u_{r1} \\ u_{r2} \end{array}\right] = \left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \underset{\sim}{f}_e^e$$

$$\left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \left\{ A_p \left[\begin{array}{cc} \psi_{1,r}\psi_{1,r} & \psi_{1,r}\psi_{2,r} \\ \psi_{2,r}\psi_{1,r} & \psi_{2,r}\psi_{2,r} \end{array}\right] r + Q_p \left[\begin{array}{cc} \psi_{1,r}\psi_1 & \psi_{1,r}\psi_2 \\ \psi_{2,r}\psi_1 & \psi_{2,r}\psi_2 \end{array}\right] + \right.$$

$$\left. Q_p \left[\begin{array}{cc} \psi_1\psi_{1,r} & \psi_1\psi_{2,r} \\ \psi_2\psi_{1,r} & \psi_2\psi_{2,r} \end{array}\right] + B_p \left[\begin{array}{cc} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{array}\right] \frac{1}{r} \right\} dr \left[\begin{array}{c} u_{r1} \\ u_{r2} \end{array}\right] = \left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \underset{\sim}{f}_e^e$$

$$\underset{\sim}{w}^{eT} \underline{K}^e \underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}_e^e$$

Integration

$$K_{11}^e = \int_{R_i}^{R_o} \left[A_p \psi_{1,r} \psi_{1,r} r + Q_p \psi_{1,r} \psi_1 + Q_p \psi_1 \psi_{1,r} + B_p \psi_1 \psi_1 \frac{1}{r} \right] \, dr$$

$$K_{12}^e = \int_{R_1}^{R_2} \left[A_p \psi_{1,r} \psi_{2,r} r + Q_p \psi_{1,r} \psi_2 + Q_p \psi_1 \psi_{2,r} + B_p \psi_1 \psi_2 \frac{1}{r} \right] \, dr$$

$$K_{21}^e = \int_{R_1}^{R_2} \left[A_p \psi_{2,r} \psi_{1,r} r + Q_p \psi_{2,r} \psi_1 + Q_p \psi_2 \psi_{1,r} + B_p \psi_2 \psi_1 \frac{1}{r} \right] \, dr$$

$$K_{22}^e = \int_{R_1}^{R_2} \left[A_p \psi_{2,r} \psi_{2,r} r + Q_p \psi_{2,r} \psi_2 + Q_p \psi_2 \psi_{2,r} + B_p \psi_2 \psi_2 \frac{1}{r} \right] \, dr$$

External load

$$f_e = \int_{R_i}^{R_o} wq_r r \, dr + [w\sigma_{rr} r]_{R_i}^{R_o} = \sum_{e=1}^{ne} \int_{R_1}^{R_2} wq_r r \, dr + [w\sigma_{rr} r]_{R_i}^{R_o} = \sum_{e=1}^{ne} q_e^e + [w\sigma_{rr} r]_{R_i}^{R_o}$$

Volume load = centrifugal load

$$q_e^e = \rho\omega^2 \int_{R_1}^{R_2} w r^2 \, dr = \left[\begin{array}{cc} w_1 & w_2 \end{array} \right] \rho\omega^2 \int_{R_1}^{R_2} \left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right] r^2 \, dr = \underset{\sim}{w}^{eT} \underset{\sim}{q}^e$$

$$f_e^e = \underset{\sim}{w}^{eT} \underset{\sim}{q}^e + w_o \sigma_{rr}(r = R_o) R_o - w_i \sigma_{rr}(r = R_i) R_i$$

Assembling

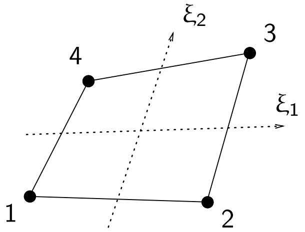
$$\underset{\sim}{w}^T \, \underline{K} \, \underset{\sim}{u} = \underset{\sim}{w}^T \, \underset{\sim}{f}_e \quad \forall \quad \underset{\sim}{w} \qquad \Rightarrow \quad \underline{K} \, \underset{\sim}{u} = \underset{\sim}{f}_e$$

Boundary conditions

FEM for planar deformation

back to index

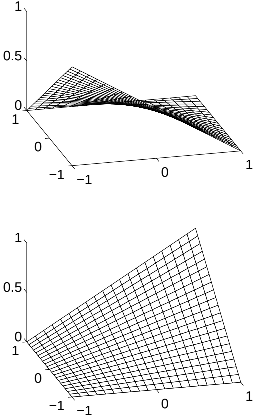
Four-node quadrilateral element



interpolation (shape) functions bi-linear in (ξ_1, ξ_2)

$$N^1 = \frac{1}{4}(1-\xi_1)(1-\xi_2) \quad ; \quad N^2 = \frac{1}{4}(1+\xi_1)(1-\xi_2)$$
$$N^3 = \frac{1}{4}(1+\xi_1)(1+\xi_2) \quad ; \quad N^4 = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

Shape functions



Cartesian coordinate system

displacement

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 & 0 \\ 0 & N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 \end{bmatrix} \begin{bmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \\ u_x^3 \\ u_y^3 \\ u_x^4 \\ u_y^4 \end{bmatrix} \longrightarrow \underset{\sim}{u} = \underline{N}\, \underset{\sim}{u}^e$$

element shape $\quad \underset{\sim}{x} = \underline{N}\, \underset{\sim}{x}^e \quad ; \quad \underset{\sim}{x}_0 = \underline{N}\, \underset{\sim}{x}_0^e$

weighting function $\quad \underset{\sim}{w} = \underline{N}\, \underset{\sim}{w}^e$

Derivatives

$$\begin{bmatrix} u_{x,x} \\ u_{y,y} \\ u_{y,x} \\ u_{x,y} \end{bmatrix} = \begin{bmatrix} N^1_{,x} & 0 & N^2_{,x} & 0 & N^3_{,x} & 0 & N^4_{,x} & 0 \\ 0 & N^1_{,y} & 0 & N^2_{,y} & 0 & N^3_{,y} & 0 & N^4_{,y} \\ 0 & N^1_{,x} & 0 & N^2_{,x} & 0 & N^3_{,x} & 0 & N^4_{,x} \\ N^1_{,y} & 0 & N^2_{,y} & 0 & N^3_{,y} & 0 & N^4_{,y} & 0 \end{bmatrix} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \\ u^3_x \\ u^3_y \\ u^4_x \\ u^4_y \end{bmatrix} \rightarrow \left(\underset{\approx}{L}_u \right)_t = \underline{B}\, \underset{\sim}{u}^e$$

$$\begin{bmatrix} N^1_{,x} & N^1_{,y} \\ N^2_{,x} & N^2_{,y} \\ N^3_{,x} & N^3_{,y} \\ N^4_{,x} & N^4_{,y} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^1_{,2} \\ N^2_{,1} & N^2_{,2} \\ N^3_{,1} & N^3_{,2} \\ N^4_{,1} & N^4_{,2} \end{bmatrix} \begin{bmatrix} \xi_{1,x} & \xi_{1,y} \\ \xi_{2,x} & \xi_{2,y} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^1_{,2} \\ N^2_{,1} & N^2_{,2} \\ N^3_{,1} & N^3_{,2} \\ N^4_{,1} & N^4_{,2} \end{bmatrix} \underline{J}^{-T}$$

$$\underline{J} = \begin{bmatrix} x_{,1} & y_{,1} \\ x_{,2} & y_{,2} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^2_{,1} & N^3_{,1} & N^4_{,1} \\ N^1_{,2} & N^2_{,2} & N^3_{,2} & N^4_{,2} \end{bmatrix} \begin{bmatrix} x^1_e & y^1_e \\ x^2_e & y^2_e \\ x^3_e & y^3_e \\ x^4_e & y^4_e \end{bmatrix}$$

Deformation matrix

$$\underline{F} = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & 0 \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & 0 \\ 0 & 0 & F_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} \end{bmatrix} = \begin{bmatrix} N^1_{,x0} & N^2_{,x0} & N^3_{,x0} & N^4_{,x0} \\ N^1_{,y0} & N^2_{,y0} & N^3_{,y0} & N^4_{,y0} \end{bmatrix} \begin{bmatrix} x_e^1 & y_e^1 \\ x_e^2 & y_e^2 \\ x_e^3 & y_e^3 \\ x_e^4 & y_e^4 \end{bmatrix}$$

$$= \begin{bmatrix} \xi_{1,x0} & \xi_{2,x0} \\ \xi_{1,y0} & \xi_{2,y0} \end{bmatrix} \begin{bmatrix} N^1_{,1} & N^2_{,1} & N^3_{,1} & N^4_{,1} \\ N^1_{,2} & N^2_{,2} & N^3_{,2} & N^4_{,2} \end{bmatrix} \begin{bmatrix} x_e^1 & y_e^1 \\ x_e^2 & y_e^2 \\ x_e^3 & y_e^3 \\ x_e^4 & y_e^4 \end{bmatrix}$$

$$= \underline{J}_0^{-1} \underline{J}$$

Cylindrical coordinate system

displacement

$$\left[\begin{array}{c} u_r \\ u_z \end{array}\right] = \left[\begin{array}{cccccccc} N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 & 0 \\ 0 & N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 \end{array}\right] \left[\begin{array}{c} u_r^1 \\ u_z^1 \\ u_r^2 \\ u_z^2 \\ u_r^3 \\ u_z^3 \\ u_r^4 \\ u_z^4 \end{array}\right] \longrightarrow \underset{\sim}{u} = \underline{N}\, \underset{\sim}{u}_e$$

element shape

$$r = \underset{\sim}{N}^T \underset{\sim}{r} \quad ; \quad z = \underset{\sim}{N}^T \underset{\sim}{z}$$

$$r_0 = \underset{\sim}{N}^T \underset{\sim}{r}_0 \quad ; \quad z_0 = \underset{\sim}{N}^T \underset{\sim}{z}_0$$

weighting function

$$\underset{\sim}{w} = \underline{N}\, \underset{\sim}{w}^e$$

Derivatives

$$\begin{bmatrix} u_{r,r} \\ u_{z,z} \\ \frac{1}{r}u_r \\ u_{z,r} \\ u_{r,z} \end{bmatrix} = \begin{bmatrix} N^1_{,r} & 0 & N^2_{,r} & 0 & N^3_{,r} & 0 & N^4_{,r} & 0 \\ 0 & N^1_{,z} & 0 & N^2_{,z} & 0 & N^3_{,z} & 0 & N^4_{,z} \\ \frac{1}{r}N^1 & 0 & \frac{1}{r}N^2 & 0 & \frac{1}{r}N^3 & 0 & \frac{1}{r}N^4 & 0 \\ 0 & N^1_{,r} & 0 & N^2_{,r} & 0 & N^3_{,r} & 0 & N^4_{,r} \\ N^1_{,z} & 0 & N^2_{,z} & 0 & N^3_{,z} & 0 & N^4_{,z} & 0 \end{bmatrix} \begin{bmatrix} u^1_r \\ u^1_z \\ u^2_r \\ u^2_z \\ u^3_r \\ u^3_z \\ u^4_r \\ u^4_z \end{bmatrix} \rightarrow \left(\underset{\approx}{L}\,u\right)_t = \underline{B}\,\underset{\sim}{u}^e$$

$$\begin{bmatrix} N^1_{,r} & N^1_{,z} \\ N^2_{,r} & N^2_{,z} \\ N^3_{,r} & N^3_{,z} \\ N^4_{,r} & N^4_{,z} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^1_{,2} \\ N^2_{,1} & N^2_{,2} \\ N^3_{,1} & N^3_{,2} \\ N^4_{,1} & N^4_{,2} \end{bmatrix} \begin{bmatrix} \xi_{1,r} & \xi_{1,z} \\ \xi_{2,r} & \xi_{2,z} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^1_{,2} \\ N^2_{,1} & N^2_{,2} \\ N^3_{,1} & N^3_{,2} \\ N^4_{,1} & N^4_{,2} \end{bmatrix} \underline{J}^{-T}$$

$$\underline{J} = \begin{bmatrix} r_{,1} & z_{,1} \\ r_{,2} & z_{,2} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^2_{,1} & N^3_{,1} & N^4_{,1} \\ N^1_{,2} & N^2_{,2} & N^3_{,2} & N^4_{,2} \end{bmatrix} \begin{bmatrix} r^1_e & z^1_e \\ r^2_e & z^2_e \\ r^3_e & z^3_e \\ r^4_e & z^4_e \end{bmatrix}$$

Deformation matrix

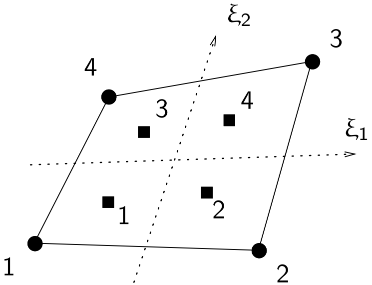
$$\underline{F} = \begin{bmatrix} \frac{\partial r}{\partial r_0} & \frac{\partial r}{\partial z_0} & 0 \\ \frac{\partial z}{\partial r_0} & \frac{\partial z}{\partial z_0} & 0 \\ 0 & 0 & \frac{r}{r_0} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial r}{\partial r_0} & \frac{\partial z}{\partial r_0} \\ \frac{\partial r}{\partial z_0} & \frac{\partial z}{\partial z_0} \end{bmatrix} = \begin{bmatrix} N^1_{,r0} & N^2_{,r0} & N^3_{,r0} & N^4_{,r0} \\ N^1_{,z0} & N^2_{,z0} & N^3_{,z0} & N^4_{,z0} \end{bmatrix} \begin{bmatrix} r_e^1 & z_e^1 \\ r_e^2 & z_e^2 \\ r_e^3 & z_e^3 \\ r_e^4 & z_e^4 \end{bmatrix}$$

$$= \begin{bmatrix} \xi_{1,r0} & \xi_{2,r0} \\ \xi_{1,z0} & \xi_{2,z0} \end{bmatrix} \begin{bmatrix} N^1_{,1} & N^2_{,1} & N^3_{,1} & N^4_{,1} \\ N^1_{,2} & N^2_{,2} & N^3_{,2} & N^4_{,2} \end{bmatrix} \begin{bmatrix} r_e^1 & z_e^1 \\ r_e^2 & z_e^2 \\ r_e^3 & z_e^3 \\ r_e^4 & z_e^4 \end{bmatrix}$$

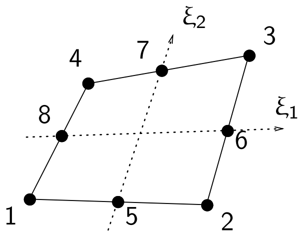
$$= \underline{J}_0^{-1} \underline{J}$$

Numerical integration



ip	ξ_1	ξ_2	ζ
1	$-\frac{1}{3}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	1
2	$\frac{1}{3}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	1
3	$-\frac{1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	1
4	$\frac{1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	1

Eight-node quadrilateral element

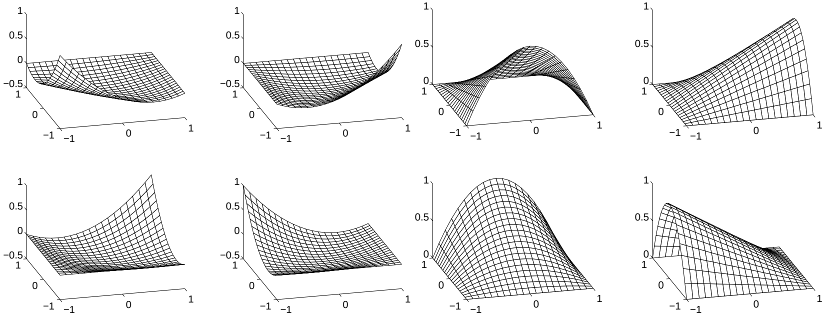


interpolation functions quadratic in (ξ_1, ξ_2)

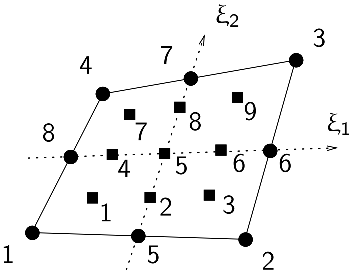
$$N^1 = \tfrac{1}{4}(\xi_1 - 1)(\xi_2 - 1)(-\xi_1 - \xi_2 - 1)$$
$$N^2 = \tfrac{1}{4}(\xi_1 + 1)(\xi_2 - 1)(-\xi_1 + \xi_2 + 1)$$
$$N^3 = \tfrac{1}{4}(\xi_1 + 1)(\xi_2 + 1)(\xi_1 + \xi_2 - 1)$$
$$N^4 = \tfrac{1}{4}(\xi_1 - 1)(\xi_2 + 1)(\xi_1 - \xi_2 + 1)$$

$$N^5 = \tfrac{1}{2}(\xi_1^2 - 1)(\xi_2 - 1)$$
$$N^6 = \tfrac{1}{2}(-\xi_1 - 1)(\xi_2^2 - 1)$$
$$N^7 = \tfrac{1}{2}(\xi_1^2 - 1)(-\xi_2 - 1)$$
$$N^8 = \tfrac{1}{2}(\xi_1 - 1)(\xi_2^2 - 1)$$

Shape functions



Numerical integration



$$a = 0.77459\,; \quad p = 0.55556\,; \quad q = 0.88889$$

ip	ξ_1	ξ_2	ζ
1	$-a$	$-a$	$p \times p$
2	0	$-a$	$p \times q$
3	a	$-a$	$p \times p$
4	$-a$	0	$p \times q$
5	0	0	$q \times q$
6	a	0	$p \times q$
7	$-a$	a	$p \times p$
8	0	a	$p \times q$
9	a	a	$p \times p$