MATERIAL MODELS

Piet Schreurs

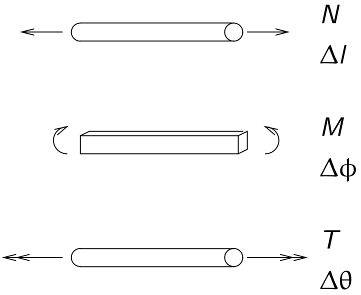
Department of Mechanical Engineering
Eindhoven University of Technology

http://www.mate.tue.nl/~piet

2012/2013

STRUCTURAL ELEMENTS

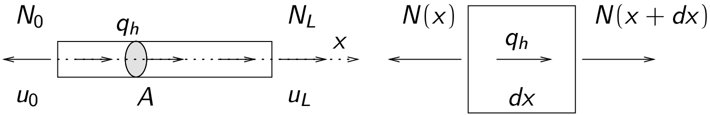
structural elements



- equilibrium equation(s)
- material behavior
- differential equation
- boundary conditions

material model needed for statistically undetermined problems

Tension

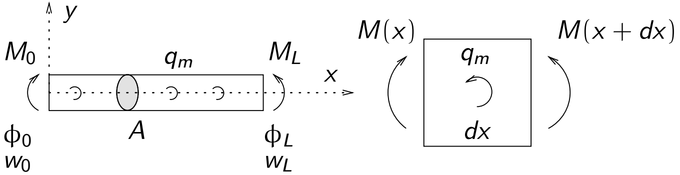


$$\frac{dN}{dx} + q_h = 0 \quad ; \quad N = \int_A \sigma \, dA$$

Hooke: $\sigma = E\varepsilon = E\,\dfrac{du}{dx} \quad \rightarrow \quad N = EA\,\dfrac{du}{dx}$

(E, A uniform) $\qquad EA\,\dfrac{d^2u}{dx^2} + q_h = 0$

Pure bending

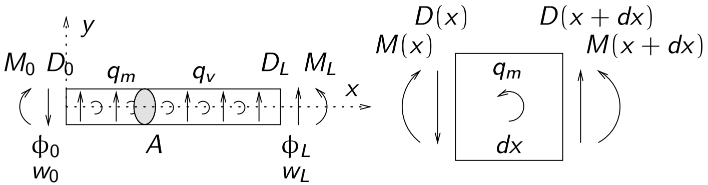


$$\frac{dM}{dx} + q_m = 0 \quad ; \quad M = -\int_A \sigma y \, dA$$

$$\sigma = E\varepsilon = E\left(-y\,\frac{d\phi}{dx}\right) \rightarrow M = EI\,\frac{d\phi}{dx} = EI\,\frac{d^2 w}{dx^2}$$

$$(E,\ I \text{ uniform}) \qquad EI\,\frac{d^2\phi}{dx^2} + q_m = 0$$

Bending



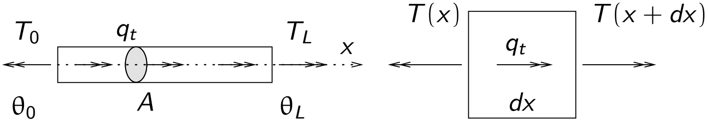
$$\frac{dM}{dx} + q_m + D = 0 \qquad ; \qquad \frac{dD}{dx} + q_v = 0$$

$$M = -\int_A \sigma y \, dA \quad ; \quad D = \int_A \tau \, dA$$

$$\sigma = E\varepsilon = E\left(-y\,\frac{d\phi}{dx}\right) \rightarrow M = EI\,\frac{d\phi}{dx} = EI\,\frac{d^2 w}{dx^2}$$

$$(E,\ I \text{ uniform}) \qquad EI\,\frac{d^3\phi}{dx^3} + \frac{dq_m}{dx} - q_v = 0$$

Torsion



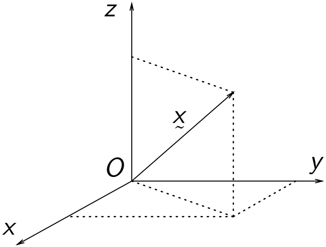
$$\frac{dT}{dx} + q_t = 0 \quad ; \quad T = \int_A \tau r \, dA$$

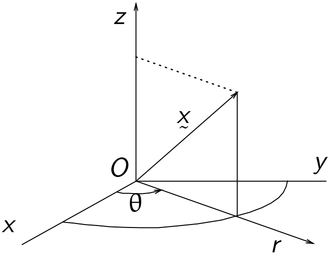
$$\tau = G\gamma = G\,r\,\frac{d\theta}{dx} \quad \rightarrow \quad T = GK\,\frac{d\theta}{dx}$$

$$(G,\ K \text{ uniform}) \qquad GK\,\frac{d^2\theta}{dx^2} + q_t = 0$$

3D DEFORMATION

Coordinate systems



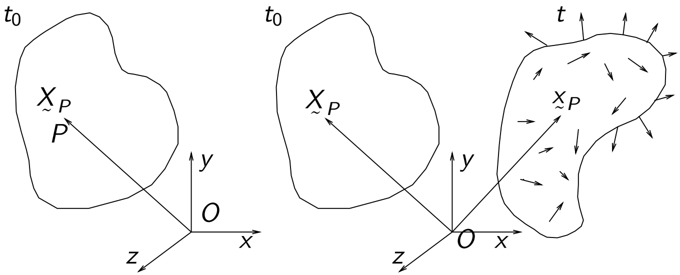


$$\underset{\sim}{x}^T = \begin{bmatrix} x & y & z \end{bmatrix} \qquad \underset{\sim}{x}^T = \begin{bmatrix} r & \theta & z \end{bmatrix}$$

$$x = r\cos(\theta) \quad ; \quad y = r\sin(\theta) \quad ; \quad z = z$$

$$r = \sqrt{x^2 + y^2} \quad ; \quad \theta = \arctan\left(\frac{y}{x}\right) \quad ; \quad z = z$$

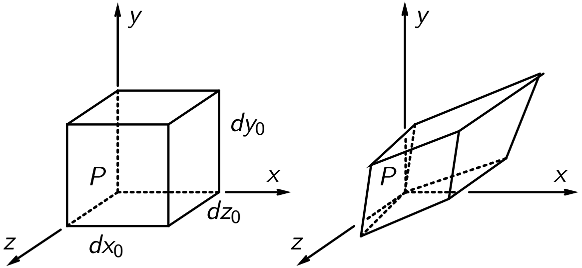
Global deformation



$$\underset{\sim}{X}_P = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}^T \qquad ; \qquad \underset{\sim}{x}_P = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$\begin{aligned} \underset{\sim}{u}_P &= \underset{\sim}{x}_P - \underset{\sim}{X}_P \\ &= \begin{bmatrix} (x - x_0) & (y - y_0) & (z - z_0) \end{bmatrix}^T \\ &= \begin{bmatrix} u & v & w \end{bmatrix}^T \end{aligned}$$

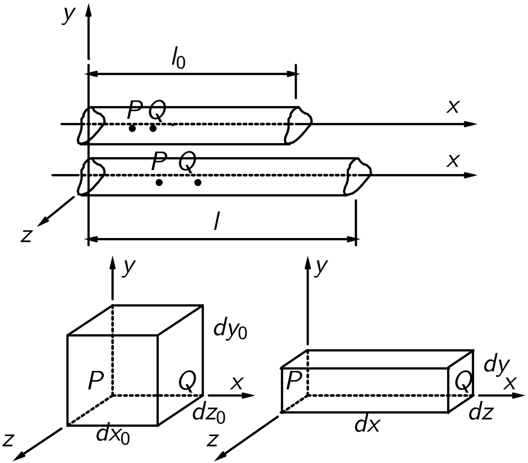
Local deformation



total deformation $=$

1. elongation $\rightarrow$ stretch
2. rotation $\rightarrow$ shear

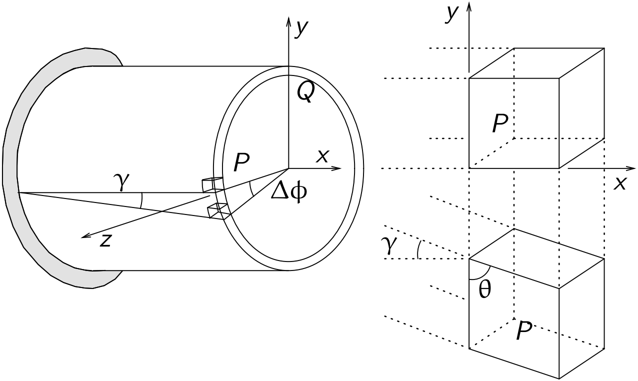
Elongation $\rightarrow$ stretch



$$\lambda_{xx} = \frac{dx}{dx_0}$$
$$\lambda_{yy} = \frac{dy}{dy_0}$$
$$\lambda_{zz} = \frac{dz}{dz_0}$$

$$\lambda > 0$$
$$\lambda < 1 \quad \vee \quad \lambda > 1$$

Rotation $\rightarrow$ shear



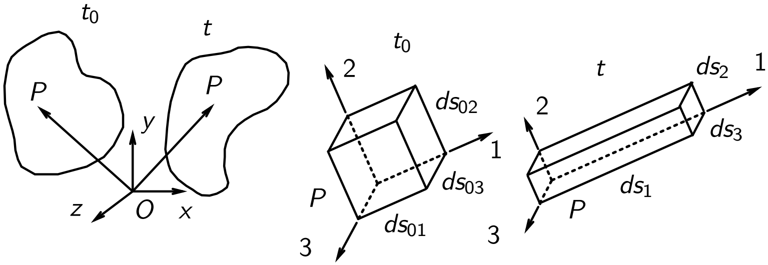
$$\gamma_{xy} = \sin\left(\frac{\pi}{2} - \theta_{xy}\right) \quad (\Uparrow) \quad ; \quad \gamma_{yz} = \sin\left(\frac{\pi}{2} - \theta_{yz}\right) \quad ; \quad \gamma_{zx} = \sin\left(\frac{\pi}{2} - \theta_{zx}\right)$$

$$\gamma_{yx} = \gamma_{xy} \quad ; \quad \gamma_{zy} = \gamma_{yz} \quad ; \quad \gamma_{xz} = \gamma_{zx}$$

Total deformation

$$\underset{\sim}{U} = \left[\begin{array}{cccccc} \lambda_{xx} & \lambda_{yy} & \lambda_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{array} \right]^T$$

Principal directions of deformation



$$\lambda_1 = \frac{ds_1}{ds_{01}} \quad ; \quad \lambda_2 = \frac{ds_2}{ds_{02}} \quad ; \quad \lambda_3 = \frac{ds_3}{ds_{03}} \qquad ; \qquad \gamma_{12} = \gamma_{23} = \gamma_{31} = 0$$

$$\underset{\sim}{U} = \left[\begin{array}{cccccc} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \end{array} \right]^T$$

volume change $\quad : \quad \dfrac{dV}{dV_0} = \dfrac{ds_1\, ds_2\, ds_3}{ds_{01}\, ds_{02}\, ds_{03}} = \lambda_1 \lambda_2 \lambda_3 = J$

incompressible $\quad \rightarrow \quad \lambda_1 \lambda_2 \lambda_3 = 1$

Strains

strain definitions $\qquad \varepsilon = f(\lambda)$

$$\text{with} \quad : \quad f(\lambda = 1) = 0 \quad ; \quad \lim_{\lambda \longrightarrow 1} f(\lambda) = \lambda - 1$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{\ell_1} & \varepsilon_{\ell_2} & \varepsilon_{\ell_3} & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} \lambda_1 - 1 & \lambda_2 - 1 & \lambda_3 - 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\underset{\sim}{\Lambda} = \begin{bmatrix} \varepsilon_{ln_1} & \varepsilon_{ln_2} & \varepsilon_{ln_3} & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} \ln(\lambda_1) & \ln(\lambda_2) & \ln(\lambda_3) & 0 & 0 & 0 \end{bmatrix}^T$$

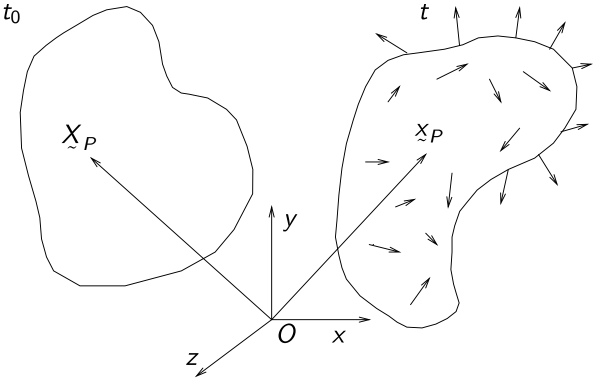
$$\underset{\sim}{E} = \begin{bmatrix} \varepsilon_{gl_1} & \varepsilon_{gl_2} & \varepsilon_{gl_3} & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2}(\lambda_1^2 - 1) & \frac{1}{2}(\lambda_2^2 - 1) & \frac{1}{2}(\lambda_3^2 - 1) & 0 & 0 & 0 \end{bmatrix}^T$$

no deformation $\quad : \quad \underset{\sim}{\mathcal{E}} = \underset{\sim}{\Lambda} = \underset{\sim}{E} = \underset{\sim}{O}$

small deformations $\quad : \quad \lambda_1 = \lambda_2 = \lambda_3 \approx 1 \quad \rightarrow \quad \underset{\sim}{\mathcal{E}} \approx \underset{\sim}{\Lambda} \approx \underset{\sim}{E}$

STRESS

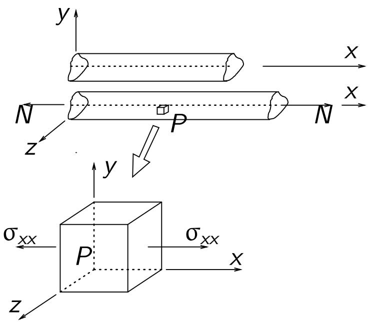
Forces



forces : - volume forces

- boundary forces

Axial stress

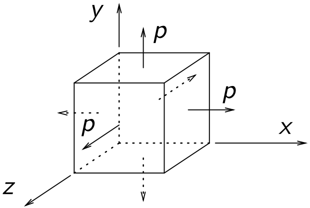


axial tensile force $$N = \int_A \sigma \, dA = \sigma A$$

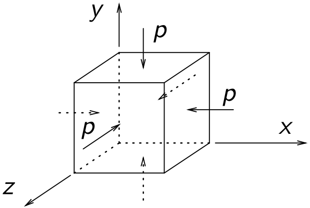
true or Cauchy stress $$\sigma = \frac{N}{A} = \sigma_{xx}$$

engineering stress $$\sigma_n = \frac{N}{A_0}$$

Hydrostatic stress

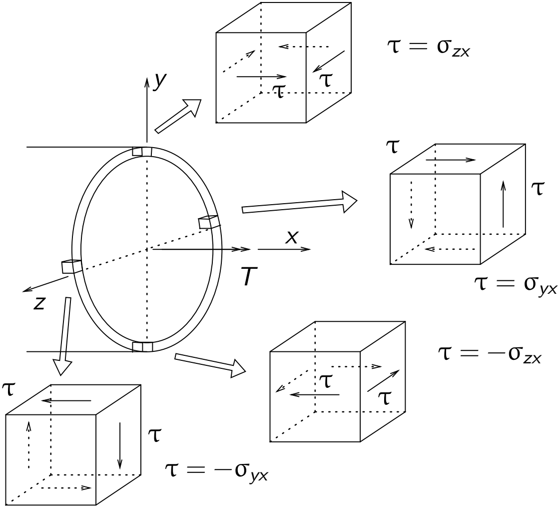


$$\sigma_{xx} = p$$
$$\sigma_{yy} = p$$
$$\sigma_{zz} = p$$



$$\sigma_{xx} = -p$$
$$\sigma_{yy} = -p$$
$$\sigma_{zz} = -p$$

Shear stress



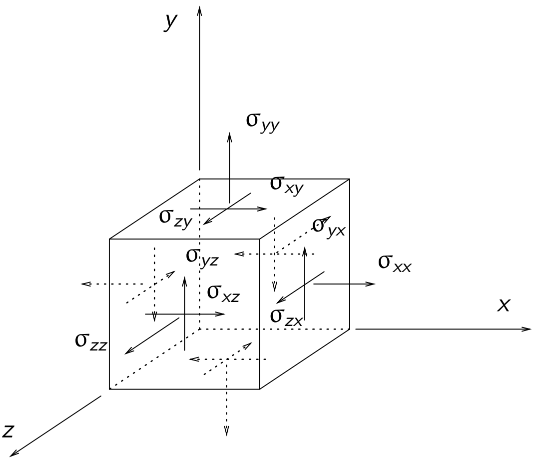
torsion moment

$$T = \tau\, R\, 2\pi R t$$

shear stress

$$\tau = \frac{T}{2\pi R^2 t}$$

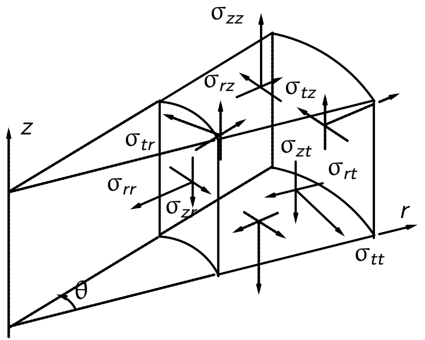
Stress cube : Cartesian



$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{bmatrix}^T$$

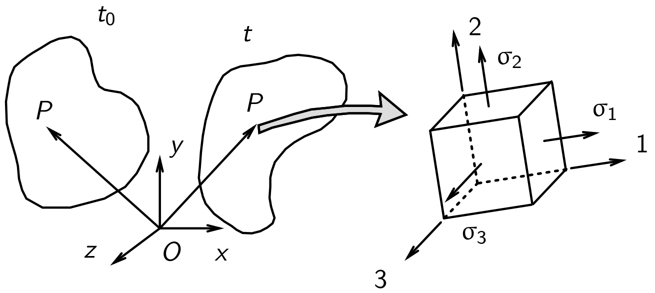
Stress cube : cylindrical



$$\underline{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{rt} & \sigma_{rz} \\ \sigma_{tr} & \sigma_{tt} & \sigma_{tz} \\ \sigma_{zr} & \sigma_{zt} & \sigma_{zz} \end{bmatrix}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{tt} & \sigma_{zz} & \sigma_{rt} & \sigma_{tz} & \sigma_{zr} \end{bmatrix}^T$$

Principal stresses and directions



$$\underline{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \qquad ; \qquad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & 0 & 0 & 0 \end{bmatrix}^T$$

CONSTITUTIVE EQUATION(S)

Constitutive equation(s)

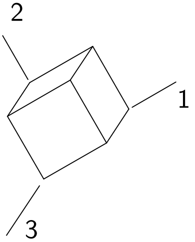
constitutive equation(s)	:	$\underset{\sim}{\sigma}(t) = \underset{\sim}{f}\left(\underset{\sim}{\sigma}(\tau), \underset{\sim}{\varepsilon}(\tau), \dot{\underset{\sim}{\varepsilon}}(\tau) \quad \middle| \quad \tau \leq t\right)$
linear behavior	:	$\underset{\sim}{\sigma} = \underline{C}\,\underset{\sim}{\varepsilon} \qquad \rightarrow \qquad \underset{\sim}{\varepsilon} = \underline{C}^{-1}\underset{\sim}{\sigma} = \underline{S}\,\underset{\sim}{\sigma}$
stiffness matrix	:	$\underline{C}$
compliance matrix	:	$\underline{S}$

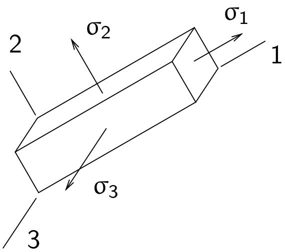
Isotropic linear material

$$\underline{C} = \begin{bmatrix} A & B & B & 0 & 0 & 0 \\ B & A & B & 0 & 0 & 0 \\ B & B & A & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{A-B}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{A-B}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{A-B}{2} \end{bmatrix}$$

$$\underline{S} = \begin{bmatrix} a & b & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 & 0 \\ b & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(a-b) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(a-b) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(a-b) \end{bmatrix}$$

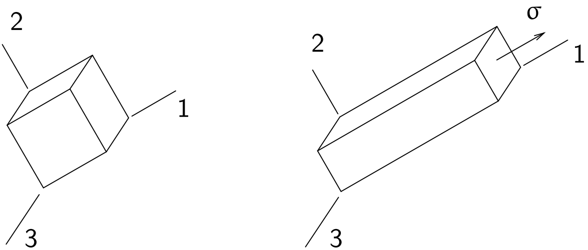
Isotropic material behavior in principal directions





$$\underset{\sim}{\sigma}(t) = \begin{bmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_3(t) \end{bmatrix} = \underset{\sim}{f}\left(\begin{bmatrix} \sigma_1(\tau) \\ \sigma_2(\tau) \\ \sigma_3(\tau) \end{bmatrix}, \begin{bmatrix} \varepsilon_1(\tau) \\ \varepsilon_2(\tau) \\ \varepsilon_3(\tau) \end{bmatrix}, \begin{bmatrix} \dot{\varepsilon}_1(\tau) \\ \dot{\varepsilon}_2(\tau) \\ \dot{\varepsilon}_3(\tau) \end{bmatrix} \quad \Big| \quad \tau \leq t \right)$$

One-dimensional material behavior

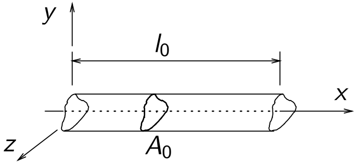


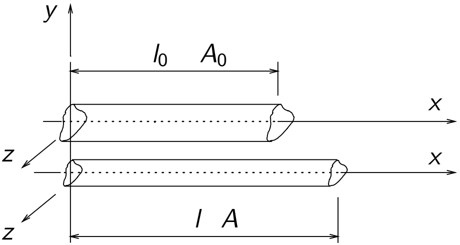
$$\sigma_2 = \sigma_3 = 0 \qquad ; \qquad \sigma_1 = \sigma$$
$$\varepsilon_2 = \varepsilon_3 = \varepsilon_d \qquad ; \qquad \varepsilon_1 = \varepsilon$$

$$\sigma(t) = f\left(\sigma(\tau), \varepsilon(\tau), \dot{\varepsilon}(\tau) \quad | \quad \tau \leq t\right)$$

HOMOGENEOUS TRUSS

Homogeneous truss

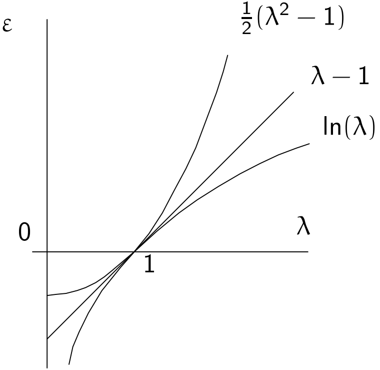




$$l_0 \quad \rightarrow \quad l \quad ; \quad A_0 \quad \rightarrow \quad A \quad ; \quad \lambda = \frac{l}{l_0} = \frac{l_0 + \Delta l}{l_0} = 1 + \frac{\Delta l}{l_0}$$

$$\mu = \sqrt{\frac{A}{A_0}} \quad \left(= \frac{d}{d_0} \right) \quad ; \quad J = \frac{lA}{l_0 A_0} = \lambda \mu^2$$

Strains



$$\varepsilon = \varepsilon_l = \lambda - 1$$
$$\varepsilon = \varepsilon_{ln} = \ln(\lambda)$$
$$\varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$$

Poisson's ratio / contraction strain / volume change

$$\varepsilon = f(\lambda) \quad \rightarrow \quad \varepsilon_d = f(\mu) = -\nu\varepsilon$$

$$J = (\varepsilon + 1)(-\nu\varepsilon + 1)^2 = \nu^2\varepsilon^3 + \nu(\nu - 2)\varepsilon^2 + (1 - 2\nu)\varepsilon + 1$$

Linear strain

$$\varepsilon = \varepsilon_l = \lambda - 1 = \frac{\Delta l}{l_0} \quad \varepsilon_d = \mu - 1 = -\nu\varepsilon_l$$

$$\mu = \sqrt{\tfrac{A}{A_0}} = 1 - \nu(\lambda - 1) \quad \rightarrow \quad A = A_0\{1 - \nu(\lambda - 1)\}^2$$

restriction of elongation

$$\mu > 0 \quad \rightarrow \quad \nu(\lambda - 1) < 1 \quad \rightarrow \quad \lambda - 1 < \tfrac{1}{\nu} \quad \rightarrow \quad \boxed{\lambda < \frac{1+\nu}{\nu}}$$

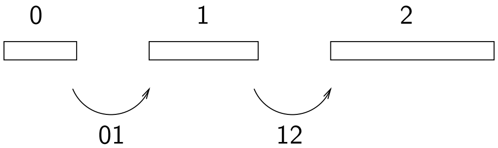
Logarithmic strain

$$\varepsilon = \varepsilon_{ln} = \ln(\lambda) \qquad \varepsilon_d = \ln(\mu) = -\nu\varepsilon_{ln} = -\nu \ln\lambda$$

$$\mu = \sqrt{\frac{A}{A_0}} = e^{-\nu\,\varepsilon_{ln}} = e^{-\nu \ln(\lambda)} = \left[e^{\ln(\lambda)}\right]^{-\nu} = \lambda^{-\nu} \qquad \rightarrow \qquad A = A_0\lambda^{-2\nu}$$

NO restriction of elongation

Addition of strains



$$l_0 \rightarrow l_1 \qquad \varepsilon_l(01) = \frac{l_1 - l_0}{l_0}$$

$$\varepsilon_{ln}(01) = \ln(\frac{l_1}{l_0})$$

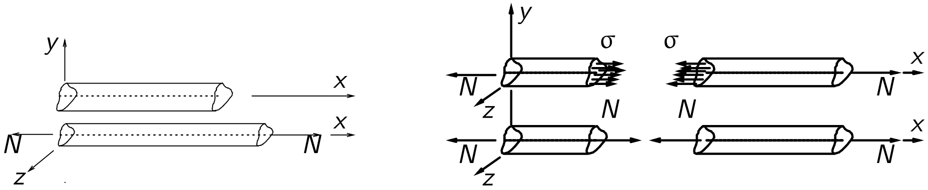
$$l_1 \rightarrow l_2 \qquad \varepsilon_l(12) = \frac{l_2 - l_1}{l_1}$$

$$\varepsilon_{ln}(12) = \ln(\frac{l_2}{l_1})$$

$$l_0 \rightarrow l_2 \qquad \varepsilon_l(02) = \frac{l_2 - l_0}{l_0} \neq \varepsilon_l(01) + \varepsilon_l(12)$$

$$\varepsilon_{ln}(02) = \ln(\frac{l_2}{l_0}) = \ln(\frac{l_2}{l_1}\frac{l_1}{l_0}) = \varepsilon_{ln}(01) + \varepsilon_{ln}(12)$$

Stresses



axial force $$N(x) = N = \int_A \sigma(y,z)\, dA = \int_A \sigma\, dA = \sigma A$$

engineering stress $$\sigma_n = \frac{N}{A_0}$$

true stress $$\sigma = \frac{N}{A} = \frac{A_0}{A}\,\frac{N}{A_0} = \frac{1}{\mu^2}\,\sigma_n$$

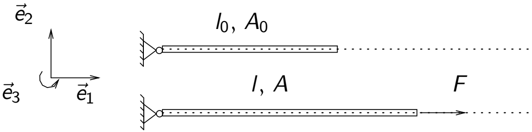
Stiffness

$$K = \lim_{\Delta l \longrightarrow 0} \frac{\Delta N}{\Delta l} = \lim_{\Delta \lambda \longrightarrow 1} \frac{\Delta N}{\Delta \lambda} \frac{1}{l_0} = \frac{\partial N}{\partial \lambda} \frac{1}{l_0}$$

$$= \frac{\partial \sigma}{\partial \lambda} A \frac{1}{l_0} + \sigma \frac{\partial A}{\partial \lambda} \frac{1}{l_0} = C_\lambda A \frac{1}{l_0} + \sigma \frac{\partial A}{\partial \lambda} \frac{1}{l_0}$$

$$= \frac{\partial \sigma}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \lambda} A \frac{1}{l_0} + \sigma \frac{\partial A}{\partial \lambda} \frac{1}{l_0} = C_\varepsilon \frac{\partial \varepsilon}{\partial \lambda} A \frac{1}{l_0} + \sigma \frac{\partial A}{\partial \lambda} \frac{1}{l_0}$$

Mechanical power



$$P = F\dot{l}$$

linear strain	$\varepsilon_l = \lambda - 1$	$\rightarrow$	$\dot{\varepsilon}_l = \dot{\lambda} = \frac{\dot{l}}{l_0}$
logarithmic strain	$\varepsilon_{ln} = \ln(\lambda)$	$\rightarrow$	$\dot{\varepsilon}_{ln} = \dot{\lambda}\lambda^{-1} = \frac{\dot{l}}{l}$
Green-Lagrange strain	$\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$	$\rightarrow$	$\dot{\varepsilon}_{gl} = \dot{\lambda}\lambda = \lambda\,\frac{\dot{l}}{l_0} = \lambda^2\,\frac{\dot{l}}{l}$

mechanical power using various strain rates

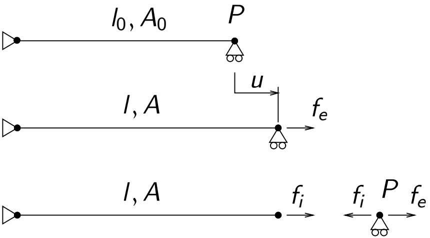
$$
\begin{array}{llllllllllllll}
P = F\dot{\ell} = & F\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A_0}A_0\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A_0}V_0\dot{\varepsilon}_l & = & & = & & = & V_0\sigma_n\dot{\varepsilon}_l \\
P = F\dot{\ell} = & F\ell\dot{\varepsilon}_{ln} & = & \dfrac{F}{A}A\ell\dot{\varepsilon}_{ln} & = & \dfrac{F}{A}V\dot{\varepsilon}_{ln} & = & V\sigma\dot{\varepsilon}_{ln} & = & V_0(J\sigma)\dot{\varepsilon}_{ln} & = & V_0\sigma_\kappa\dot{\varepsilon}_{ln} \\
P = F\dot{\ell} = & F\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A}A\ell\dfrac{\ell_0}{\ell}\dot{\varepsilon}_l & = & \dfrac{F}{A}V\lambda^{-1}\dot{\varepsilon}_l & = & V(\sigma\lambda^{-1})\dot{\varepsilon}_l & = & V_0(J\sigma\lambda^{-1})\dot{\varepsilon}_l & = & V_0\sigma_{p1}\dot{\varepsilon}_l \\
P = F\dot{\ell} = & F\ell\lambda^{-2}\dot{\varepsilon}_{gl} & = & \dfrac{F}{A}A\ell\lambda^{-2}\dot{\varepsilon}_{gl} & = & \dfrac{F}{A}V\lambda^{-2}\dot{\varepsilon}_{gl} & = & V(\sigma\lambda^{-2})\dot{\varepsilon}_{gl} & = & V_0(J\sigma\lambda^{-2})\dot{\varepsilon}_{gl} & = & V_0\sigma_{p2}\dot{\varepsilon}_{gl}
\end{array}
$$

Specific mechanical power

$$P = V_0 \dot{W}_0 = V \dot{W} \quad \rightarrow$$

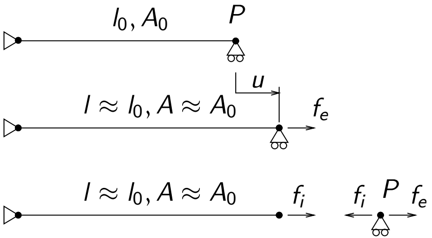
$$\begin{aligned} \dot{W}_0 &= \sigma_n \dot{\varepsilon}_l = \sigma_\kappa \dot{\varepsilon}_{ln} = \sigma_{p1} \dot{\varepsilon}_l = \sigma_{p2} \dot{\varepsilon}_{gl} \\ \dot{W} &= \qquad = \sigma \dot{\varepsilon}_{ln} = \sigma \lambda^{-1} \dot{\varepsilon}_l = \sigma \lambda^{-2} \dot{\varepsilon}_{gl} \end{aligned}$$

Equilibrium



external force	f_e
internal force	$f_i = f_i(u)$
equilibrium of point P	$f_i(u) = f_e$

Linear deformation

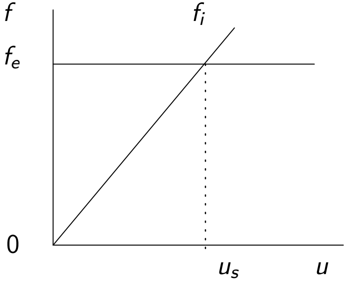


$$f_i = f_i(u) = \sigma_n A_0 \quad \rightarrow$$

$$E\,\varepsilon A_0 = \frac{EA_0}{l_0}\,u = Ku$$

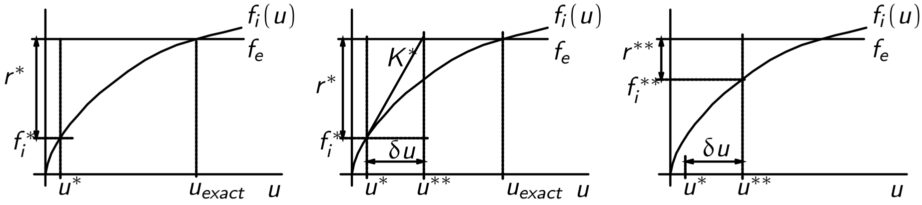
$$f_i = Ku = f_e \quad \rightarrow$$

$$u = u_s = \frac{f_e}{K} = \frac{l_0}{EA_0}\,f_e$$



proportionality & superposition

Nonlinear deformation



$f_i = \sigma A = f_i(u) = f_e$

$f_i(u)$ non-linear $\longrightarrow$ iterative solution process needed

exact solution : $u_{exact} \longrightarrow f_i(u_{exact}) = f_e \longrightarrow f_e - f_i(u_{exact}) = 0$

approximation : $u^* \longrightarrow f_e - f_i(u^*) = r(u^*) \neq 0$

residual $r^* = r(u^*)$

Newton-Raphson iteration procedure

known approximation u^* **unknown** error δu

$$\left.\begin{array}{l} f_i(u_{exact}) = f_e \\ u_{exact} = u^* + \delta u \end{array}\right\} \quad \rightarrow \quad f_i(u^* + \delta u) = f_i(u^*) + \left.\frac{df_i}{du}\right|_{u^*} \delta u = f_e$$

tangential stiffness $K^* = \left.\frac{df_i}{du}\right|_{u^*} \quad \rightarrow \quad f_i^* + K^* \delta u = f_e \rightarrow$

$K^* \, \delta u = f_e - f_i^* = r^* \quad \rightarrow \quad \delta u = \frac{1}{K^*} \, r^*$

new approximate solution $u^{**} = u^* + \delta u$; error $u_{exact} - u^{**}$

error smaller $\rightarrow$ convergence

convergence control : $|r^{**}| \leq c_r \quad \rightarrow$ stop iteration

convergence control : $|\delta u| \leq c_u \quad \rightarrow$ stop iteration

Tangential stiffness

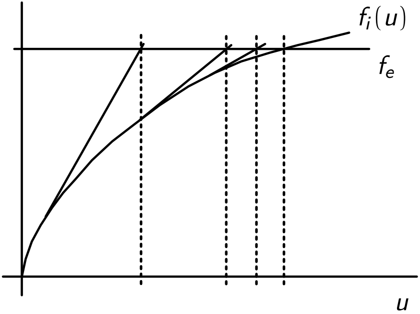
internal nodal force $\quad f_i^* = N(\lambda^*) = A^* \sigma^*$

tangential stiffness $\quad K^* = \left.\frac{\partial f_i}{\partial u}\right|_{u^*} = \left.\frac{\partial N(\lambda)}{\partial u}\right|_{u^*} = \left.\frac{dN}{d\lambda}\right|_{\lambda^*} \frac{\partial \lambda}{\partial u}$

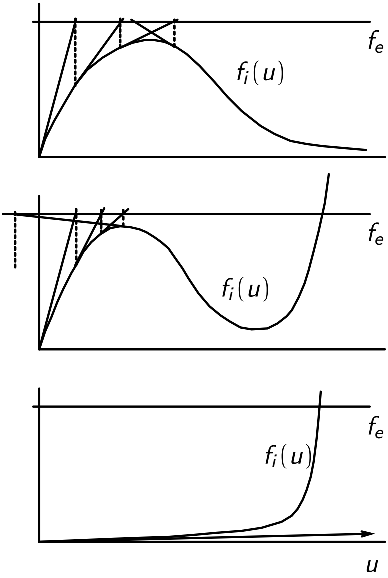
geometry $\quad \lambda = 1 + \frac{\Delta l}{l_0} = 1 + \frac{1}{l_0}\, u \quad \rightarrow \quad \frac{\partial \lambda}{\partial u} = \frac{1}{l_0}$

tangential stiffness
$$K^* = \left.\frac{dN}{d\lambda}\right|_{\lambda^*} \frac{\partial \lambda}{\partial u} = \left.\frac{dN}{d\lambda}\right|_{\lambda^*} \frac{1}{l_0} = \left.\frac{dN}{d\lambda}\right|^* \frac{1}{l_0}$$
$$= \frac{1}{l_0} \frac{d}{d\lambda}(\sigma A) = \frac{1}{l_0} \left.\frac{d\sigma}{d\lambda}\right|^* A^* + \frac{1}{l_0}\, \sigma^* \left.\frac{dA}{d\lambda}\right|^*$$

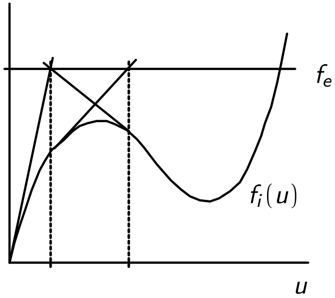
Converging iteration

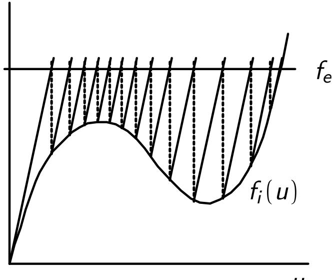


Non-converging iterations

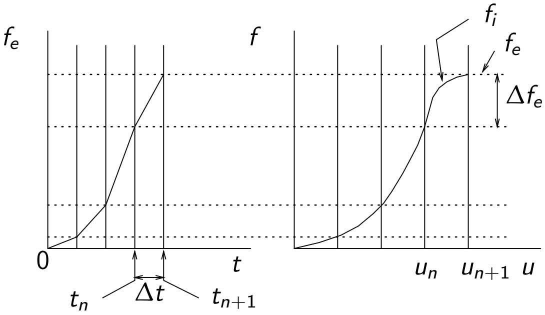


Modified Newton-Raphson



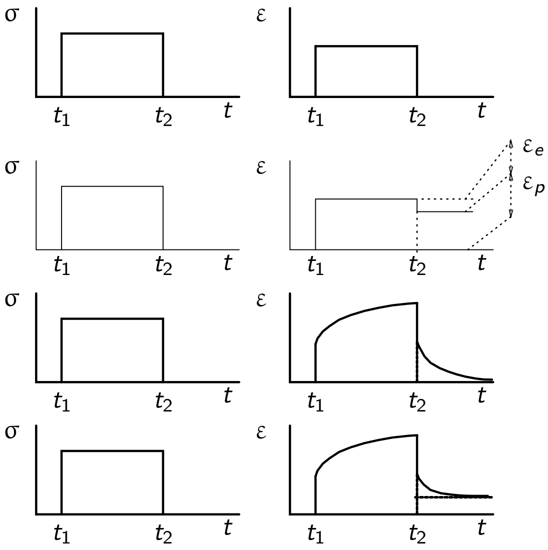


Incremental loading



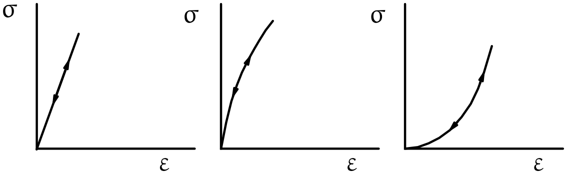
MATERIAL MODELS

Time-dependency

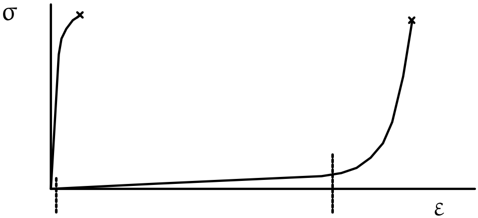


Elastic behavior

- Linear or non-linear

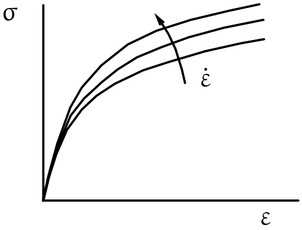


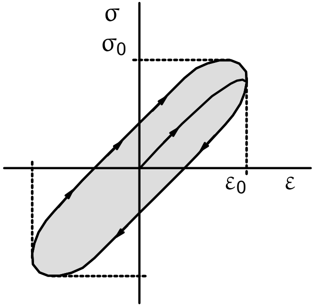
- metal versus polymer



Viscoelastic behavior

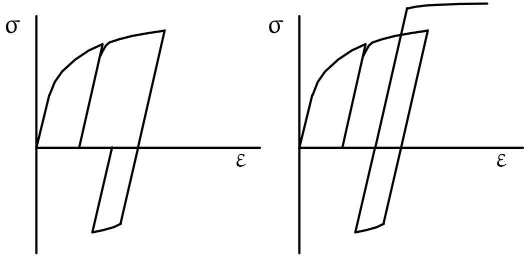
- time dependency
- strain rate dependency
- phase difference $\longrightarrow$ hysteresis





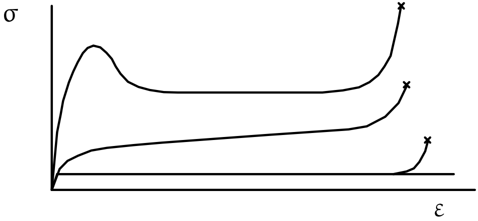
Elastoplastic behavior

- permanent deformation
- hardening



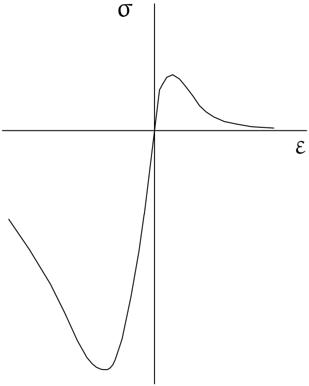
Viscoplastic behavior

- time & strain rate dependency
- permanent deformation
- softening & hardening



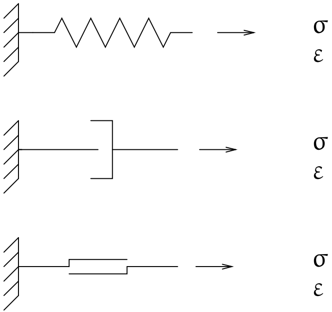
Damage

- softening
- different response for tensile and compression



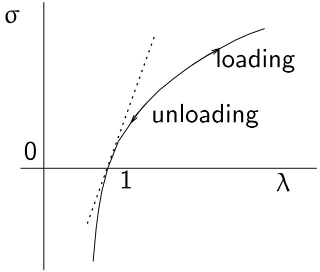
Discrete material models

- spring, dashpot, friction element



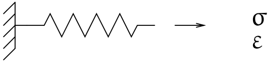
ELASTIC BEHAVIOR

Elastic behavior



- no permanent deformation after unloading
- no path- or time dependency
- no energy dissipation

discrete elastic model



constitutive eqn $\sigma = \sigma(\lambda)$

stiffness $C_\lambda = \dfrac{\partial \sigma}{\partial \lambda} = \dfrac{\partial \sigma}{\partial \varepsilon}\dfrac{\partial \varepsilon}{\partial \lambda} = C_\varepsilon \dfrac{\partial \varepsilon}{\partial \lambda}$

Small strain elastic behavior

$$\varepsilon = \varepsilon_{gl} = \varepsilon_{ln} = \varepsilon_{l} = \lambda - 1$$

$$\sigma = \frac{F}{A} = \frac{F}{A_0} = \sigma_n$$

$$\sigma = E\varepsilon = E(\lambda - 1)$$

$$E = \lim_{\lambda \longrightarrow 1} \frac{d\sigma}{d\lambda} = \lim_{\varepsilon \longrightarrow 0} \frac{d\sigma}{d\varepsilon}$$

Large strain elastic behavior

hypo-elastic models

- fit on experiments
- no thermodyn. basis
- exc. 3.8

hyper-elastic models

energy function	$W = W(I_1, I_2, I_3)$
invariants	$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$
	$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$
	$I_3 = \lambda_1\lambda_2\lambda_3$
specific elas. energy	$dW = \sigma_1 d\varepsilon_{ln_1} + \sigma_2 d\varepsilon_{ln_2} + \sigma_3 d\varepsilon_{ln_3}$
	uni-ax.: $dW = \sigma d\varepsilon_{ln}$ (exc. 3.1)
	bi-ax.: $dW = 2\sigma d\varepsilon_{ln}$ (exc. 3.17)

Mooney models

incompressible $\quad I_3 = \lambda_1 \lambda_2 \lambda_3 = 1 \quad \rightarrow \quad I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}$

energy function $\quad W = \sum_{i}^{n} \sum_{j}^{m} C_{ij} \left(I_1 - 3 \right)^i \left(I_2 - 3 \right)^j \qquad \text{with} \quad C_{00} = 0$

Neo-Hookean $\quad W = C_{10} \left(I_1 - 3 \right)$

Mooney-Rivlin $\quad W = C_{10} \left(I_1 - 3 \right) + C_{01} \left(I_2 - 3 \right)$

Ogden models

slightly compressible

$$W = \sum_{i=1}^{N} \frac{a_i}{b_i} \left[J^{-\frac{b_i}{3}} \left(\lambda_1^{b_i} + \lambda_2^{b_i} + \lambda_3^{b_i} \right) - 3 \right] + 4.5K \left(J^{\frac{1}{3}} - 1 \right)^2$$

highly compressible

$$W = \sum_{i=1}^{N} \frac{a_i}{b_i} \left(\lambda_1^{b_i} + \lambda_2^{b_i} + \lambda_3^{b_i} - 3 \right) + \sum_{i=1}^{N} \frac{a_i}{c_i} \left(1 - J^{c_i} \right)$$

Neo-Hookean one-dimensional models

$$
\begin{aligned}
W &= C_{10}\left(\lambda^2 + \frac{2}{\lambda} - 3\right) \\
\sigma &= C_{10}\left(2\lambda - \frac{2}{\lambda^2}\right)\lambda = 2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) \\
C_\lambda &= \frac{\partial\sigma}{\partial\lambda} = 2C_{10}\left(2\lambda + \frac{1}{\lambda^2}\right) \quad ; \\
E &= \lim_{\lambda\longrightarrow 1}\frac{\partial\sigma}{\partial\lambda} = 6C_{10} \\
F &= \sigma A = \sigma\frac{1}{\lambda}A_0 = 2C_{10}A_0\left(\lambda - \frac{1}{\lambda^2}\right)
\end{aligned}
$$

exc. 3.11

spherical balloon : exc. 3.17

Mooney-Rivlin one-dimensional

$$
\begin{aligned}
W &= C_{10}\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + C_{01}\left(\frac{1}{\lambda^2} + 2\lambda - 3\right) \\
\sigma &= 2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) + 2C_{01}\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{1}{\lambda} \\
C_\lambda &= \frac{\partial\sigma}{\partial\lambda} = 2C_{10}\left(2\lambda + \frac{1}{\lambda^2}\right) + 2C_{01}\left(1 + \frac{2}{\lambda^3}\right) \\
E &= \lim_{\lambda\longrightarrow 1}\frac{\partial\sigma}{\partial\lambda} = 6(C_{10} + C_{01}) \\
F &= \sigma A = \sigma\frac{1}{\lambda}A_0 = A_0\frac{1}{\lambda}\left[2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) + 2C_{01}\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{1}{\lambda}\right]
\end{aligned}
$$

exc. 3.12

ELASTOPLASTIC BEHAVIOR

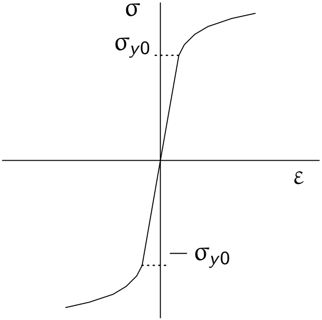
Uniaxial tensile/compression test with small strains

- linear strain

$$\varepsilon = \varepsilon_l = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}$$

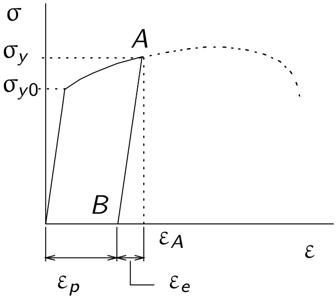
- engineering stress

$$\sigma = \sigma_n = \frac{F}{A_0}$$



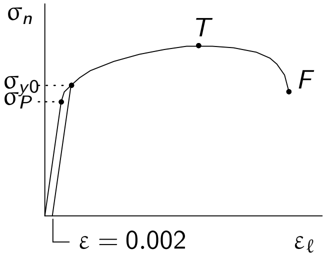
σ_P	proportional limit
$\sigma = \sigma_{y0}$	yield
σ_{y0}	initial yield stress
ε_{y0}	strain at σ_{y0} : $\varepsilon_{y0} = \sigma_{y0}/E$
$\varepsilon_{0.2}$	0.2-strain : $\varepsilon_p = 0.2\% = 0.002$
$f = \sigma^2 - \sigma_{y0}^2 = 0$	yield function

Interupted tensile test



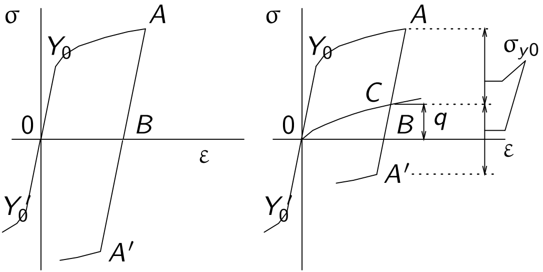
$\varepsilon = \varepsilon_A$	total strain		
ε_p	plastic strain		
ε_e	elastic strain (springback)		
$A - B$	elastic trajectory	elastic parameters constant	$\Delta\sigma = E\Delta\varepsilon = E\Delta\varepsilon_e$
$\sigma_y = \sigma_A$	current yield stress		
$\sigma_y > \sigma_{y0}$	hardening		
$\sigma_y \sim \varepsilon_p$	hardening model	ε_p : history parameter	

Total tensile test



σ_T	tensile strength
σ_F	fracture strength
ε_F	fracture strain ($\approx 5\% = 0.05$)
	(for metals : small strain)

Hardening



isotropic hardening

elastic area : increasing & symmetric w.r.t. $\sigma = 0$

$$\left.\begin{array}{ll} \text{tensile} & : \sigma = \sigma_y \\ \text{compression} & : \sigma = -\sigma_y \end{array}\right\} \rightarrow f = \sigma^2 - \sigma_y^2 = 0$$

kinematic hardening

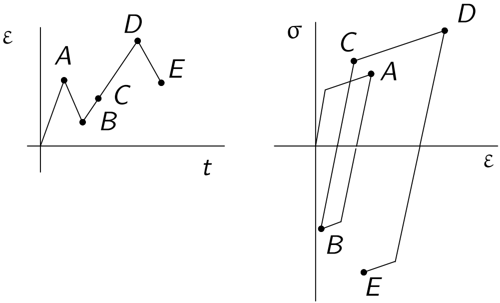
elastic area : constant & symmetric w.r.t. $\sigma = q$

shift stress : $q \rightarrow$ shift elastic area

Bauschinger effect

$$\left.\begin{array}{ll} \text{tens.} & : \sigma = q + \sigma_{y0} \\ \text{comp.} & : \sigma = q - \sigma_{y0} \end{array}\right\} \rightarrow f = (\sigma - q)^2 - \sigma_{y0}^2 = 0$$

Effective plastic strain

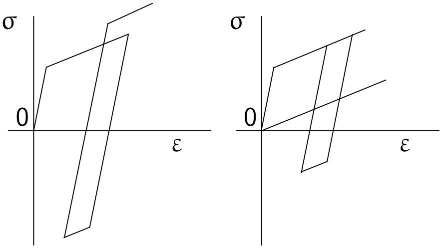


$\sigma_{yC} > \sigma_{yA} \quad ; \quad \varepsilon_{pC} < \varepsilon_{pA} \quad \rightarrow$ effective plastic strain : $\quad \bar{\varepsilon}_p = \sum_{\varepsilon} |\Delta\varepsilon_p|$

effective plastic strain rate

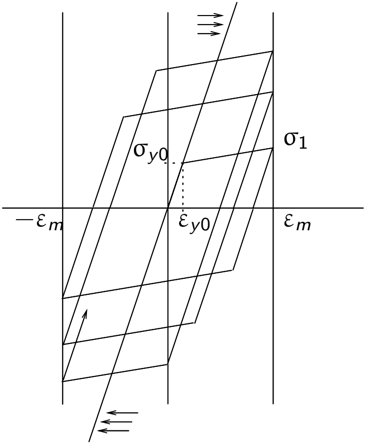
$$\bar{\varepsilon}_p = \sum_{\varepsilon} |\Delta\varepsilon_p| = \sum_{\tau=0}^{\tau=t} \frac{|\Delta\varepsilon_p|}{\Delta t}\,\Delta t$$
$$= \int_{\tau=0}^{t} |\dot{\varepsilon}_p|\, d\tau = \int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p\, d\tau$$

Linear hardening

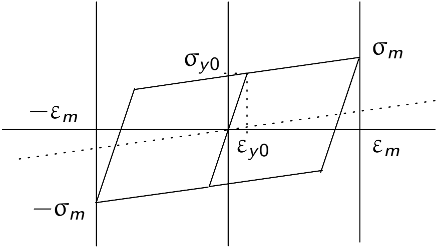


isotropic h.	σ_y can **not** decrease	$\longrightarrow$	effective plastic strain
	hardening		$\sigma_y = \sigma_{y0} + H\,\bar{\varepsilon}_p$
	yield criterion		$f = \sigma^2 - \sigma_y^2(\bar{\varepsilon}_p) = 0$
kinematic h.	q can decrease $\longrightarrow$		no *effective* plastic strain
	hardening		$q = K\,\varepsilon_p$
	yield criterion		$f = \{\sigma - q(\varepsilon_p)\}^2 - \sigma_{y0}^2 = 0$
isotr.-kinem. h.	hardening		$\sigma_y = \sigma_{y0} + H\,\bar{\varepsilon}_p \quad : \quad q = K\,\varepsilon_p$
	yield criterion		$f = \{\sigma - q(\varepsilon_p)\}^2 - \sigma_y^2(\bar{\varepsilon}_p) = 0$
other models			

cyclic load

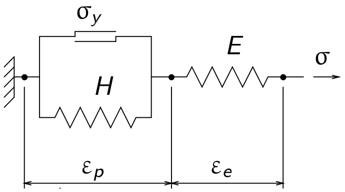


exc. 4.8



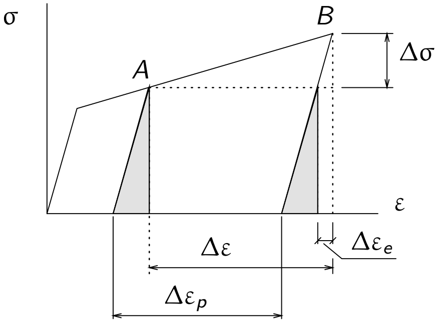
exc. 4.4, 4.9

Model



- $f = (\sigma - q)^2 - \sigma_y^2$
- $f < 0 \quad | \quad f = 0 \wedge \dot{f} < 0 \quad \rightarrow \quad$ elastic
  $f = 0 \wedge \dot{f} = 0 \quad \rightarrow \quad$ elastoplastic
- $\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p) \quad ; \quad q = q(\varepsilon_p)$
- $\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$
- $\sigma = E\varepsilon_e$
- $\bar{\varepsilon}_p = \sum_{\varepsilon} |\Delta\varepsilon_p|$

Monotonuous tensile test : isotropic hardening

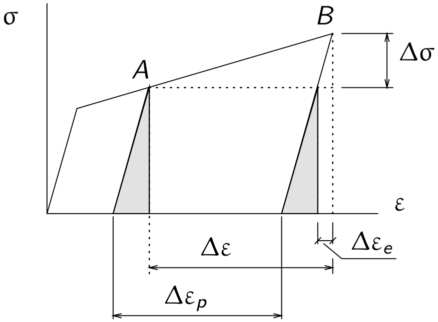


$$\Delta\sigma = E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) = E\left(\Delta\varepsilon - \frac{\Delta\sigma_y}{H}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{H}\right)$$

constitutive equation $\Delta\sigma = \frac{EH}{E+H}\Delta\varepsilon = S\Delta\varepsilon$

change plastic strain $\Delta\varepsilon_p = \frac{\Delta\sigma}{H} = \frac{E}{E+H}\Delta\varepsilon$

Monotonuous tensile test : kinematic hardening



$$\Delta\sigma = E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) = E\left(\Delta\varepsilon - \frac{\Delta q}{K}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{K}\right)$$

constitutive equation $\quad \Delta\sigma = \frac{EK}{E+K}\,\Delta\varepsilon = S\Delta\varepsilon$

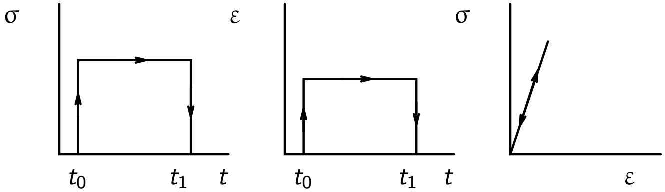
change plastic strain $\quad \Delta\varepsilon_p = \frac{\Delta\sigma}{K} = \frac{E}{E+K}\,\Delta\varepsilon$

Monotonuous tensile test

inhomogeneous deformation, residual stresses   exc. 4.19, 4.23

VISCOELASTIC BEHAVIOR

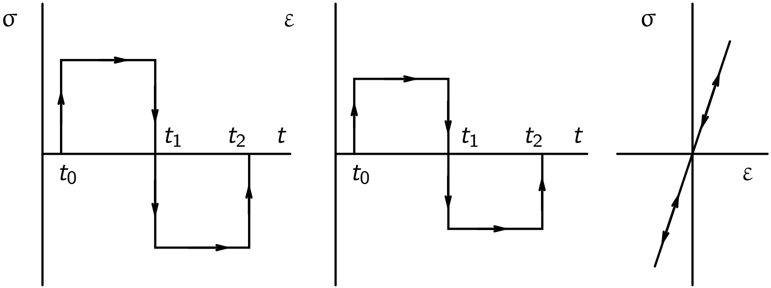
Linear elastic material behavior



$$\varepsilon = \frac{1}{E}\,\sigma \qquad \rightarrow \qquad \sigma = E\,\varepsilon \qquad \rightarrow \qquad N = \sigma A = EA\varepsilon = \frac{EA}{l}\,\Delta l = k\,\Delta l$$

- constant Young's modulus : Hooke's law E
- linear spring : spring stiffness $k = \frac{EA}{l}$

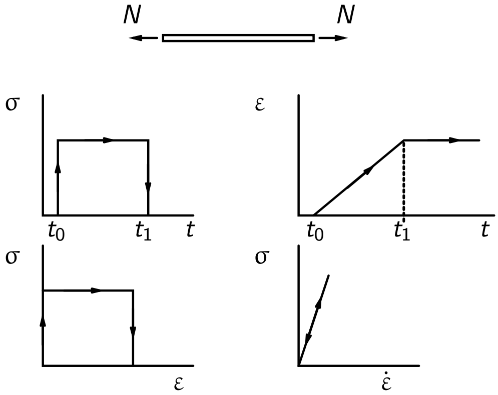
Linear elastic material behavior



- no dissipation : no area under (σ, ε)-curve

$$
\begin{aligned}
U_d &= \int_{t_0}^{t_1} \sigma \, d\varepsilon + \int_{t_1}^{t_2} \sigma \, d\varepsilon = \int_{t_0}^{t_1} E\varepsilon \, d\varepsilon + \int_{t_1}^{t_2} E\varepsilon \, d\varepsilon \\
&= \tfrac{1}{2} E[\varepsilon_1^2 - \varepsilon_0^2 + \varepsilon_2^2 - \varepsilon_1^2] \\
&= 0
\end{aligned}
$$

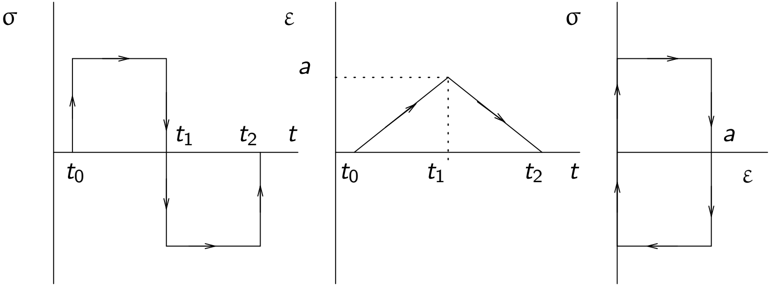
Linear viscous material behavior



$$\dot{\varepsilon} = \frac{1}{\eta}\,\sigma \qquad \rightarrow \qquad \sigma = \eta\dot{\varepsilon} \qquad \rightarrow \qquad N = \sigma A = \eta A\dot{\varepsilon} = \frac{\eta A}{l}\,\dot{\Delta l} = b\,\dot{\Delta l}$$

- constant viscosity : Newtonian fluid $\qquad \eta$
- linear dashpot : damping constant $\qquad b = \frac{\eta A}{l}$

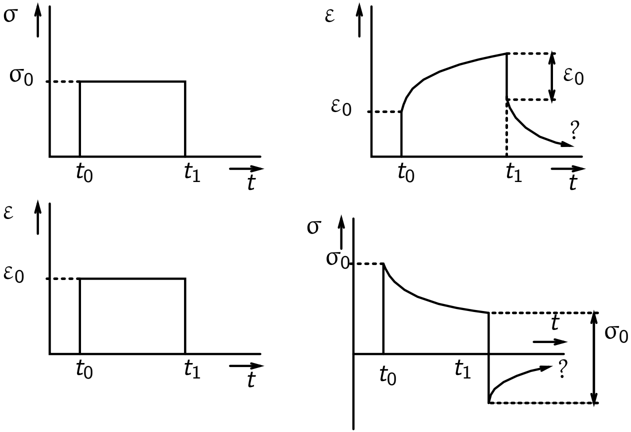
Linear viscous material behavior



- dissipated energy $\sim$ area

$$
\begin{aligned}
U_d &= \int_{t_0}^{t_1} \sigma \, d\varepsilon + \int_{t_1}^{t_2} \sigma \, d\varepsilon = \int_{t_0}^{t_1} \eta \dot{\varepsilon} \, d\varepsilon + \int_{t_1}^{t_2} \eta \dot{\varepsilon} \, d\varepsilon = \int_{t_0}^{t_1} \eta c \, d\varepsilon - \int_{t_1}^{t_2} \eta c \, d\varepsilon \\
&= \eta c [\varepsilon_1 - \varepsilon_0 - \varepsilon_2 + \varepsilon_1] \\
&= 2 \eta c a
\end{aligned}
$$

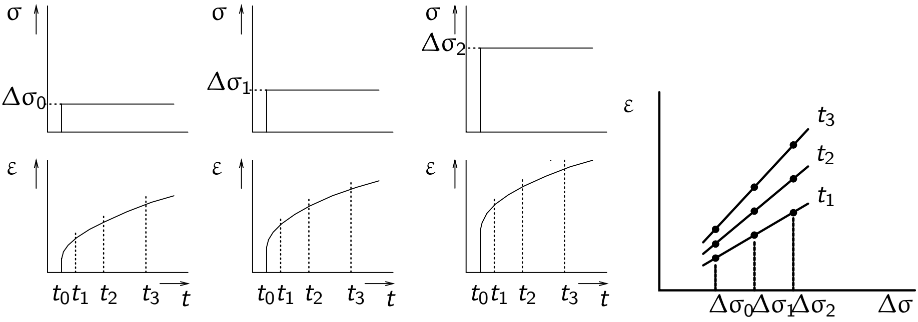
Viscoelastic material behavior



- small deformations
- spring-dashpot models

Proportionality

- uniaxial tensile stress step loading

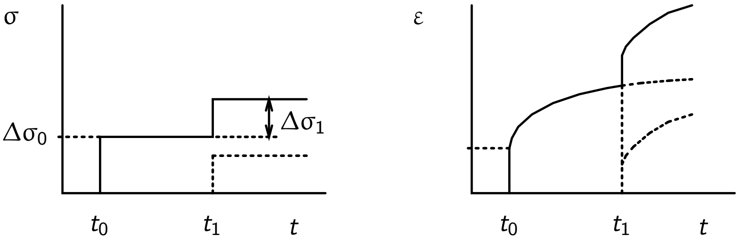


- linear **isochrones** $\rightarrow$ proportionality

$$\Rightarrow \quad \varepsilon(t) = \Delta\sigma D(t - t_0) \qquad \text{for} \qquad \forall \quad t \geq t_0$$

$D(t - t_0)$ is no function of the stresses

Superposition



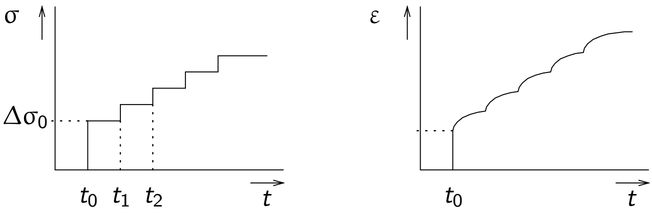
separate excitations

$$t_0 \; : \; \Delta\sigma_0 \rightarrow \varepsilon(t) = \Delta\sigma_0 D(t - t_0) \text{ for } t \geq t_0$$
$$t_1 \; : \; \Delta\sigma_1 \rightarrow \varepsilon(t) = \Delta\sigma_1 D(t - t_1) \text{ for } t \geq t_1$$

subsequent excitations

$$t_0 \; : \; \Delta\sigma_0 \rightarrow \varepsilon(t) = \Delta\sigma_0 D(t - t_0) \text{ for } t_0 \leq t < t_1$$
$$t_1 \; : \; \Delta\sigma = \Delta\sigma_0 + \Delta\sigma_1 \rightarrow \varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1) \text{ for } t \geq t_1$$

Boltzmann integral



$$\varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1) + \Delta\sigma_2 D(t - t_2) + .. = \sum_{i=1}^{n} \Delta\sigma_i D(t - t_i)$$

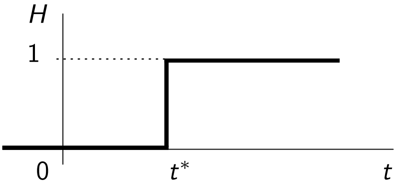
limit $n \to \infty$ $\qquad (t \to \tau)$

$$= \int_{\tau=t_0^-}^{t} D(t-\tau)\, d\sigma(\tau) = \int_{\tau=t_0^-}^{t} D(t-\tau)\frac{d\sigma(\tau)}{d\tau}\, d\tau = \int_{\tau=t_0^-}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau$$

idem : $\sigma(t) = \int_{\tau=t_0^-}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau$

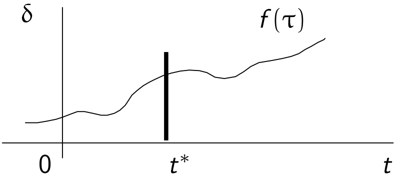
Step excitation

Heaviside function



$$H(t,t^*) \quad \left\{ \begin{array}{l} t < t^* \;:\; H(t,t^*) = 0 \\ t > t^* \;:\; H(t,t^*) = 1 \end{array} \right\}$$

Dirac function

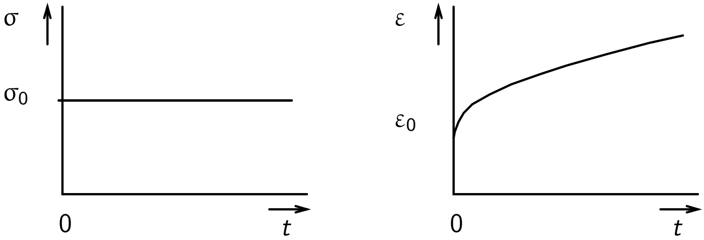


$$\delta(t,t^*) = \frac{d}{dt}\{H(t,t^*)\}$$

$$\int_{\tau=0}^{t>t^*} \delta(\tau,t^*)\; d\tau = 1$$

$$\int_{\tau=0}^{t>t^*} f(\tau)\delta(\tau,t^*)\; d\tau = f(t^*)$$

Creep (retardation)



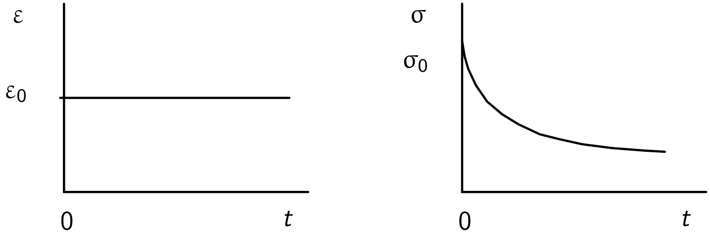
$$\sigma(t) = \sigma_0 H(t,0) \rightarrow \dot{\sigma}(t) = \sigma_0 \delta(t,0)$$

$$\varepsilon(t) = \int_{\tau=0^-}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau = \int_{\tau=0^-}^{t} D(t-\tau)\sigma_0\delta(\tau,0)\, d\tau = \sigma_0 D(t)$$

creep function : $D(t)$

$$\dot{D}(t) \geq 0 \quad \forall \quad t \geq 0 \quad ; \quad \ddot{D}(t) < 0 \quad \forall \quad t \geq 0$$

Relaxation

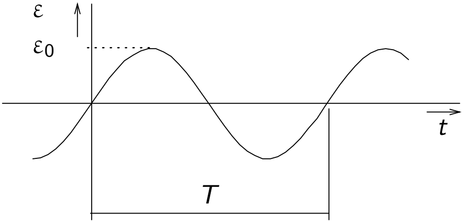


$$\varepsilon(t) = \varepsilon_0 H(t,0) \qquad \longrightarrow \qquad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0)$$
$$\sigma(t) = \int_{\tau=0^-}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau = \int_{\tau=0^-}^{t} E(t-\tau)\varepsilon_0\delta(\tau,0)\, d\tau = \varepsilon_0 E(t)$$

relaxation function $\quad : \quad E(t)$

$$\dot{E}(t) \leq 0 \qquad \forall \quad t \geq 0 \quad ; \quad \ddot{E}(t) > 0 \qquad \forall \quad t \geq 0$$
$$\int_{t=0}^{\infty} \dot{E}(t)\, dt \geq 0 \quad \longrightarrow \quad \lim_{t\longrightarrow\infty} \dot{E}(t) = 0$$

Harmonic strain excitation



$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \quad \longrightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$

amplitude	ε_0
angular frequency	ω [rad s^{-1}]
period and frequency	$T = \frac{2\pi}{\omega}$ [s^{-1}] ; $f = \frac{1}{T}$

Harmonic stress response to strain excitation

$$\sigma(t) = \varepsilon_0 \omega \int_{\xi=-\infty}^{t} E(t-\xi)\cos(\omega\xi)\,d\xi$$

$$t - \xi = s \quad \rightarrow \quad \xi = t - s \quad \rightarrow \quad d\xi = -ds$$

$$= \varepsilon_0 \omega \int_{s=0}^{\infty} E(s)\cos\{\omega(t-s)\}\,ds$$

$$\cos(\omega t - \omega s) = \cos(\omega t)\cos(\omega s) + \sin(\omega t)\sin(\omega s)$$

$$= \varepsilon_0 \left[\omega \int_{s=0}^{\infty} E(s)\sin(\omega s)\,ds\right]\sin(\omega t) + \varepsilon_0 \left[\omega \int_{s=0}^{\infty} E(s)\cos(\omega s)\,ds\right]\cos(\omega t)$$

$$= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)$$

storage modulus $\quad E'(\omega) \sim E(t)$

loss modulus $\quad E''(\omega) \sim E(t)$

Phase difference

- dissipated energy per unit of volume in one period $\quad 0 \leq t \leq \dfrac{2\pi}{\omega} = T = \dfrac{1}{f}$

$$U_d = \int_{\varepsilon(0)}^{\varepsilon(T)} \sigma \, d\varepsilon = \int_{t=0}^{T} \sigma \dot{\varepsilon} \, dt$$

$$= \int_{t=0}^{T} \{\varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)\}\{\varepsilon_0 \omega \cos(\omega t)\} \, dt$$

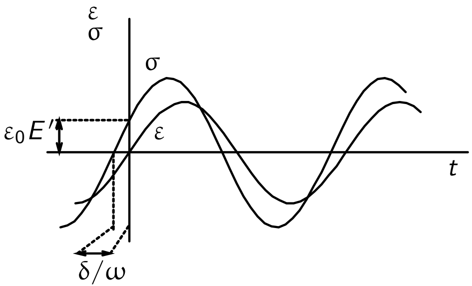
$$= \pi \varepsilon_0^2 E'' > 0 \quad \Rightarrow \quad E'' > 0 \quad \rightarrow$$

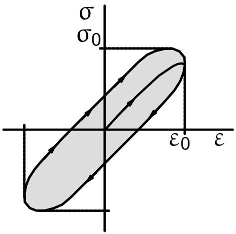
$$\sigma(t = 0) = \varepsilon_0 E'' > 0$$

- phase difference between stress and strain
- phase energy dissipation $\quad \rightarrow \quad$ heat

syl. p. 68

Phase difference





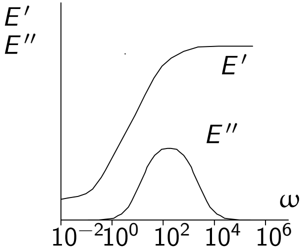
$$\sigma(t) = \sigma_0 \sin(\omega t + \delta) = \sigma_0 \cos(\delta) \sin(\omega t) + \sigma_0 \sin(\delta) \cos(\omega t)$$

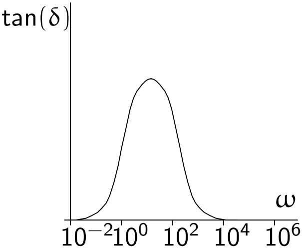
$$= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)$$

$$\left.\begin{aligned} E' &= \frac{\sigma_0}{\varepsilon_0}\cos(\delta) \\ E'' &= \frac{\sigma_0}{\varepsilon_0}\sin(\delta) \end{aligned}\right\} \rightarrow \left\{ \begin{aligned} \frac{E''}{E'} &= \tan(\delta) \quad \rightarrow \\ \delta &= \arctan\left(\frac{E''}{E'}\right) \end{aligned}\right.$$

amplitude $\quad \sigma_0 = \varepsilon_0 \sqrt{(E')^2 + (E'')^2}$

Measured E', E'' and $\tan(\delta)$





- measurement of E' and E'' can be done accurately
- $E'(\omega), E''(\omega) \quad \rightarrow \quad E(t)$ via fitting procedure
- range $\omega \quad \rightarrow \quad$ temperature $\quad \rightarrow \quad$ DMTA
- measurement of $E(t)$ in relaxation test is difficult $\rightarrow$ fit is inaccurate

spring : $\sigma(t) = E\varepsilon_0 \sin(\omega t) = \varepsilon_0 E' \sin(\omega t) \quad \rightarrow$

$$E' = E \; ; \; E'' = 0 \; ; \; \delta = 0$$

dashpot : $\sigma(t) = \eta\dot{\varepsilon}(t) = \eta\omega\varepsilon_o \cos(\omega t) = \varepsilon_0 E'' \cos(\omega t) \quad \rightarrow$

$$E' = 0 \; ; \; E'' = \eta\omega \; ; \; \delta = \frac{\pi}{2}$$

Harmonic stress excitation and strain response

$$\sigma(t) = \sigma_0 \sin(\omega t) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

$$\begin{aligned}
\varepsilon(t) &= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau = \int_{\tau=-\infty}^{t} D(t-\tau)\sigma_0 \omega \cos(\omega\tau)\, d\tau \\
&= \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s)\sin(\omega s)\, ds \right] \sin(\omega t) + \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s)\cos(\omega s)\, ds \right] \cos(\omega t) \\
&= \sigma_0\, D' \sin(\omega t) - \sigma_0\, D'' \cos(\omega t)
\end{aligned}$$

storage compliance $\qquad D'(\omega) \sim D(t)$

loss compliance $\qquad D''(\omega) \sim Dt$

Phase difference

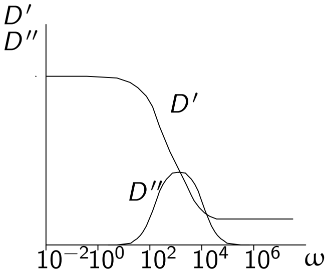
$$\varepsilon(t) = \varepsilon_0 \sin(\omega t - \delta) = \varepsilon_0 \cos(\delta) \sin(\omega t) - \varepsilon_0 \sin(\delta) \cos(\omega t)$$
$$= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)$$

storage and loss compliance

$$\left.\begin{array}{l} D' = \dfrac{\varepsilon_0}{\sigma_0} \cos(\delta) \\ D'' = \dfrac{\varepsilon_0}{\sigma_0} \sin(\delta) \end{array}\right\} \quad \rightarrow \quad \left\{ \quad \frac{D''}{D'} = \tan(\delta) \quad \rightarrow \quad \delta = \arctan\left(\frac{D''}{D'}\right)\right.$$

amplitude $\quad \varepsilon_0 = \sigma_0 \sqrt{(D')^2 + (D'')^2}$

Measured D' and D''



$$\left.\begin{aligned} \sigma_0 &= \varepsilon_0 \sqrt{(E')^2 + (E'')^2} \\ \varepsilon_0 &= \sigma_0 \sqrt{(D')^2 + (D'')^2} \end{aligned}\right\} \quad \rightarrow$$

$$\left[(E')^2 + (E'')^2\right]\left[(D')^2 + (D'')^2\right] = 1 \qquad (1)$$

$$\frac{D''}{D'} = \frac{E''}{E'} \quad \rightarrow \quad D'' = D' \frac{E''}{E'} \qquad (2)$$

(1) & (2) $\quad \rightarrow \quad D' = \dfrac{E'}{(E')^2 + (E'')^2} \quad ; \quad D'' = \dfrac{E''}{(E')^2 + (E'')^2}$

idem $\quad E' = \dfrac{D'}{(D')^2 + (D'')^2} \quad ; \quad E'' = \dfrac{D''}{(D')^2 + (D'')^2}$

Complex variables

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) = \varepsilon_0 \cos(\omega t - \tfrac{\pi}{2}) = Re\left[\varepsilon_0 e^{-i\frac{\pi}{2}} e^{i\omega t}\right] = Re\left[\varepsilon^* e^{i\omega t}\right]$$

$$\sigma(t) = \sigma_0 \sin(\omega t + \delta) = \sigma_0 \cos(\omega t - \tfrac{\pi}{2} + \delta) = Re\left[\sigma_0 e^{i(\delta - \frac{\pi}{2})} e^{i\omega t}\right] = Re\left[\sigma^* e^{i\omega t}\right]$$

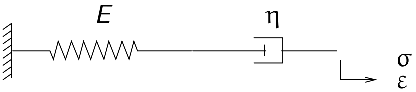
$$E^* = \frac{\sigma^*}{\varepsilon^*} = \frac{\sigma_0}{\varepsilon_0} e^{i\delta} = \frac{\sigma_0}{\varepsilon_0} \cos(\delta) + i \frac{\sigma_0}{\varepsilon_0} \sin(\delta) = E' + iE''$$

$$D^* = \frac{\varepsilon^*}{\sigma^*} = \frac{\varepsilon_0}{\sigma_0} e^{-i\delta} = \frac{\varepsilon_0}{\sigma_0} \cos(\delta) - i \frac{\varepsilon_0}{\sigma_0} \sin(\delta) = D' - iD''$$

$$E_d = |E^*| = \sqrt{(E')^2 + (E'')^2} = \frac{\sigma_0}{\varepsilon_0}$$

$$D_d = |D^*| = \sqrt{(D')^2 + (D'')^2} = \frac{\varepsilon_0}{\sigma_0}$$

Maxwell



$$\varepsilon = \varepsilon_E + \varepsilon_\eta \rightarrow$$

$$\dot{\varepsilon} = \dot{\varepsilon}_E + \dot{\varepsilon}_\eta = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

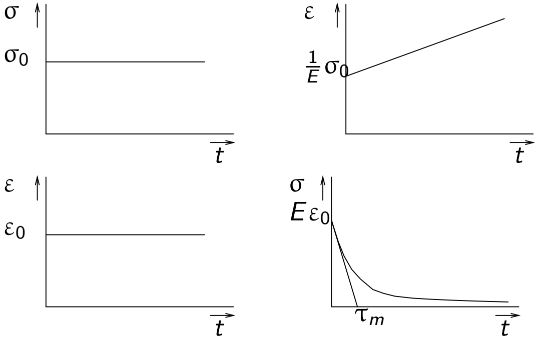
Step excitations

stress $\quad \sigma(t) = \sigma_0 H(t,0) \rightarrow \dot{\sigma}(t) = \sigma_0 \delta(t,0) \rightarrow \text{DV} \rightarrow$

$$\varepsilon(t) = \frac{\sigma_0}{E} H(t,0) + \frac{\sigma_0}{\eta}\, t = \sigma_0 \left[\frac{1}{\eta} \left(t + \frac{\eta}{E} \right) \right] = \sigma_0 D(t)$$

strain $\quad \varepsilon(t) = \varepsilon_0 H(t,0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0) \rightarrow \text{DV}$

$$\sigma(t) = \varepsilon_0 E\, e^{-\frac{E}{\eta}\, t} = \varepsilon_0 E\, e^{-\frac{t}{\tau_m}} = \varepsilon_0 E(t)$$

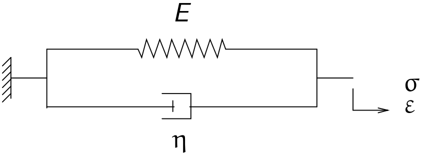


Boltzmann integrals

$$\varepsilon(t) = \int_{\tau=-\infty}^{t} \left[\frac{1}{\eta} \left\{ (t-\tau) + \frac{\eta}{E} \right\} \right] \dot{\sigma}(\tau)\, d\tau$$

$$\sigma(t) = \int_{\tau=-\infty}^{t} \left[E e^{-\frac{E}{\eta}(t-\tau)} \right] \dot{\varepsilon}(\tau)\, d\tau$$

Kelvin-Voigt



$$\sigma = \sigma_E + \sigma_\eta = E\varepsilon + \eta\dot{\varepsilon}$$

Step excitations

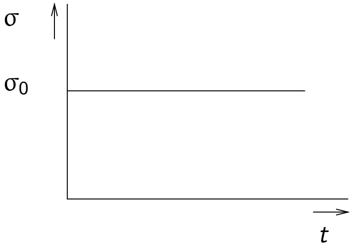
stress $\sigma(t) = \sigma_0 H(t,0) \rightarrow \text{DV} \rightarrow$

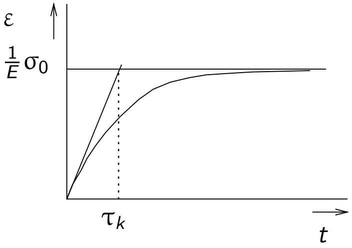
$$\eta\dot{\varepsilon}(t) + E\varepsilon(t) = \sigma_0 H(t,0) \rightarrow \varepsilon(t) = \varepsilon_H(t) + \varepsilon_P = C\,e^{-\frac{E}{\eta}t} + \frac{\sigma_0}{E}$$

$$\varepsilon(t=0) = 0 \rightarrow C = -\frac{\sigma_0}{E} \rightarrow \varepsilon(t) = \frac{\sigma_0}{E}\left[1 - e^{-\frac{E}{\eta}t}\right] = \sigma_0 D(t)$$

strain $\varepsilon(t) = \varepsilon_0 H(t,0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0) \quad \rightarrow \quad \text{DV} \rightarrow$

$$\sigma(t) = E\varepsilon_0 H(t,0) + \eta\varepsilon_0\delta(t,0) = \varepsilon_0\left[E + \eta\delta(t,0)\right] = \infty$$

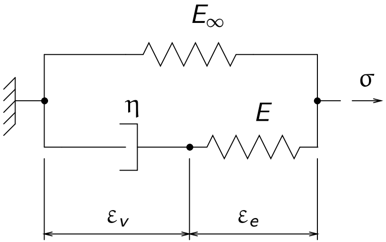




Boltzmann integral

$$\varepsilon(t) = \int\limits_{\tau=-\infty}^{t} \left[\frac{1}{E} \left\{ 1 - e^{-\frac{E}{\eta}(t-\tau)} \right\} \right] \dot{\sigma}(\tau)\, d\tau$$

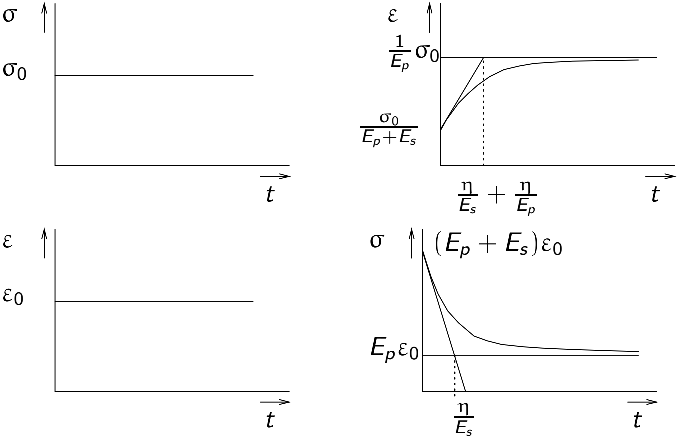
Standard Solid



$$
\begin{aligned}
\sigma = \sigma_\infty + \sigma_{ve} &= E_\infty\,\varepsilon + \eta\,\dot{\varepsilon}_v \\
&= E_\infty\,\varepsilon + \eta(\dot{\varepsilon} - \dot{\varepsilon}_e) = E_\infty\,\varepsilon + \eta\,\dot{\varepsilon} - \eta\,\frac{\dot{\sigma}_{ve}}{E} \\
&= E_\infty\,\varepsilon + \eta\,\dot{\varepsilon} - \frac{\eta}{E}\,(\dot{\sigma} - E_\infty\,\dot{\varepsilon}) \;\rightarrow
\end{aligned}
$$

$$
\sigma + \frac{\eta}{E}\,\dot{\sigma} = E_\infty\,\varepsilon + \frac{\eta(E + E_\infty)}{E}\,\dot{\varepsilon}
$$

Step excitations

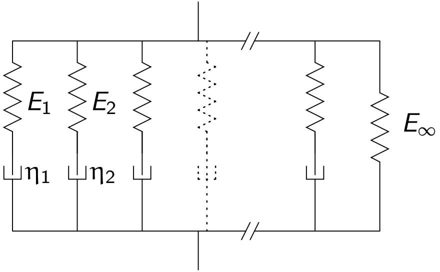


Boltzmann integrals

$$\varepsilon(t) = \int_{\tau=-\infty}^{t} \left[\frac{1}{E_\infty} - \frac{E}{E_\infty (E_\infty + E)} e^{-\frac{E_\infty E}{\eta (E_\infty + E)} (t-\tau)} \right] \dot{\sigma}(\tau)\, d\tau$$

$$\sigma(t) = \int_{\tau=-\infty}^{t} \left[E_\infty + E e^{-\frac{E}{\eta} (t-\tau)} \right] \dot{\varepsilon}(\tau)\, d\tau$$

Generalized Maxwell

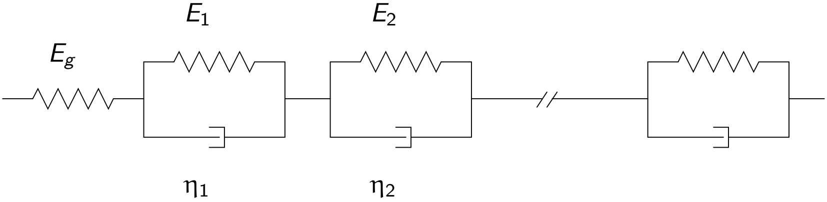


$$E(t) = E_\infty + \sum_i E_i e^{-\frac{t}{\tau_i}} \qquad ; \qquad \tau_i = \frac{\eta_i}{E_i}$$

equilibrium modulus $\quad E_\infty = \lim_{t \longrightarrow \infty} E(t)$

glass modulus $\quad E_g = \lim_{t \longrightarrow 0} E(t) = E_\infty + \sum_i E_i$

Generalized Kelvin



$$D(t) = \frac{1}{E_g} + \sum_i \frac{1}{E_i}(1 - e^{-\frac{t}{\tau_i}}) \qquad ; \qquad \tau_i = \frac{\eta_i}{E_i} = D_g + \sum_i D_i(1 - e^{-\frac{t}{\tau_i}})$$

glass compliance

$$D_g = \frac{1}{E_g} = \lim_{t \longrightarrow 0} D(t)$$

equilibrium compliance

$$D_\infty = \lim_{t \longrightarrow \infty} D(t) = D_g + \sum_i D_i$$

Examples

- harmonic excitation Maxwell : example
- Mentat/MARC

- exc. 5.12
- exc. 5.18
- exc. 5.20