Computational Material Models

Lecture notes - course 4K620

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Contents

Chapter 1

Introduction

For a good description of physical phenomena, proper constitutive equations, describing the material behaviour, are essential. Although many constitutive equations are used since long and mostly successful, they need continuous adaptation and extension due to both the use of new materials and the more extreme situations of their use. Experimental techniques to determine material parameters (-functions) need to be more and more sophisticated.

In the course CMM, attention goes to implementation of the constitutive equations in finite element software. When this is not done properly, the accuracy of the numerical solution and the efficiency of the solution process will be detrimental.

Subsequently attention is focussed on procedures, which are needed for various material behaviour 1) to calculate stresses and 2) to calculate the current material stiffness. Attention is given to material models for: hypo- and hyper-elastic behavior, elastoplastic behavior, viscoelastic behavior and viscoplastic behavior.

First, the material behavior is characterised and analysed with one-dimensional discrete mechanical models, made of springs, dashpots and friction elements. Calculation of stress response for a prescribed strain excitation is done with Matlab files.

The background of the finite element method (weighted residuals, interpolation, numerical integration, assemblage, partitioning) is summarized for truss, 2D (plane strain and plane stress) and axisymmetric elements. Implementation of the material models is done in both a truss element and in plane strain(/stress) and axisymmetric elements. The finite element software is available online as a set of Matlab command and function files. Input files for demo problems are also available. In the appendix of these lecture notes, some files are listed with material model procedures.

Chapter 2

Truss structures

A truss is a mechanical element whose dimension in one direction – the truss axis – is much larger than the dimensions in each direction perpendicular to the axis. A truss structure is an assembly of trusses, which are connected mutually and to the suroundings with hinges. The truss can transfer only axial forces along its axis, so bending is not possible, and the axis must be and remain straight.

In this chapter, we first consider small elongation and rotation of a truss. The material behaves linearly elastic and the resulting equilibrium equation is linear. The finite element method is used to model truss structures and to solve the resulting set of equilibrium equations.

Large elongations and rotations lead to a set of nonlinear equations. Moreover, the material behavior is likely to be nonlinear as well. Solution of the set of equations must be done iteratively. Implementation of various material models in the finite element software is the subject of the next chapter.

2.1 Homogeneous truss

We consider a truss to be oriented with its axis along the global x-axis. Its undeformed length is l_0. The undeformed cross-sectional area has a uniform value A_0. It is assumed that the material of the truss is isotropic and homogeneous.

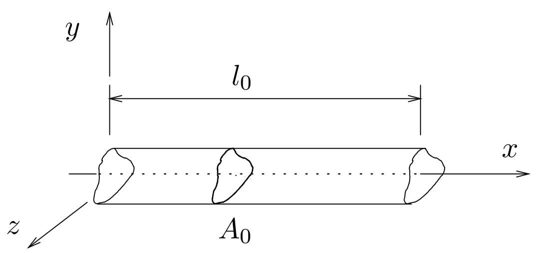


Fig. 2.1 : *Homogeneous truss*

2.1.1 Elongation and contraction

In the deformed state the length of the truss is l and its cross-sectional area is A. The elongation is described by the axial elongation factor λ. The change in cross-sectional area is described by the contraction μ. It is assumed that the load, which provokes the deformation, is such that the deformation is homogeneous. This means that λ and μ are the same in each point of the truss. The volume change is described by the volume ratio J.

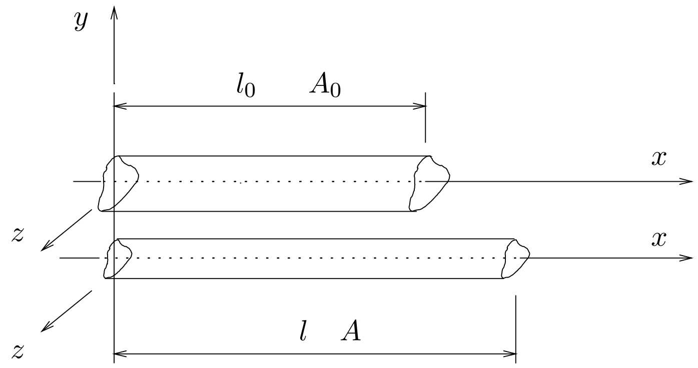


Fig. 2.2 : *Deformation of a homogeneous truss*

elongation factor $$\lambda = \frac{l}{l_0} = \frac{l_0 + \Delta l}{l_0} = 1 + \frac{\Delta l}{l_0}$$

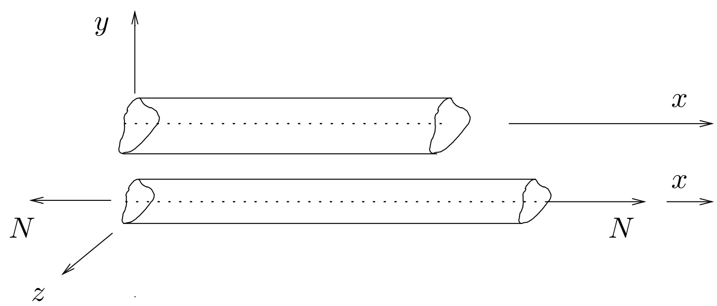
contraction $$\mu = \sqrt{\frac{A}{A_0}}$$

volume change $$J = \frac{lA}{l_0 A_0} = \lambda \mu^2$$

2.1.2 Stress

The deformation of the truss is caused by an external axial force N. In each cross-section (x) of the truss, an internal axial force $N(x)$ exists. With no volume load, the cross-sectional load will be the same in each cross-section. If a volume load is applied, this is not the case, but we will not consider such loading here.

The axial load is such that it causes only axial deformation and no bending. In the absence of a volume load the deformation will be homogeneous.



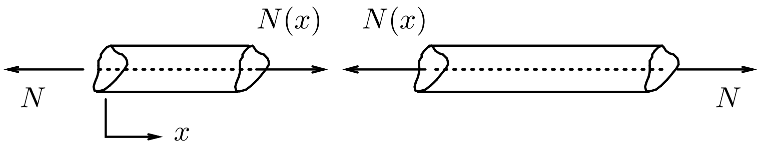


Fig. 2.3 : *Axial loading of a homogeneous truss*

The cross-sectional force is the result of the axial cross-sectional stress. For a homogeneous material with no volume loads, the stress is uniform over the cross-section. This leads to the definition of *true stress*, being the axial force divided by the deformed (= real) cross-sectional area. In many (engineering) applications the *engineering* or *nominal stress* is used, defined as the ratio of the axial force and the undeformed cross-sectional area. True stress and engineering stress, are related by the contraction μ.

In literature a truss is sometimes called a *tie* when it carries a tensile force and a *strut* when it is loaded in compression.

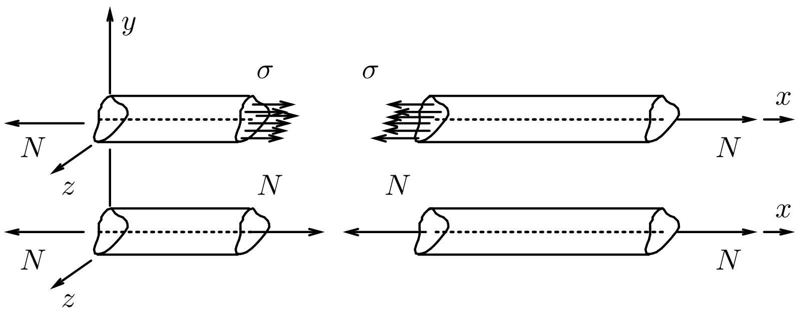


Fig. 2.4 : *Cross-sectional stress in an axially loaded homogeneous truss*

axial stress	$\sigma = \sigma(y,z)$
cross-sectional force	$N(x) = N = \int_A \sigma(y,z)\, dA$
stress uniform in cross-section	$N = \int_A \sigma\, dA = \sigma A$
true stress	$\sigma = \dfrac{N}{A}$
engineering stress	$\sigma_n = \dfrac{N}{A_0}$
relation	$\sigma = \dfrac{N}{A} = \dfrac{A_0}{A}\dfrac{N}{A_0} = \dfrac{1}{\mu^2}\,\sigma_n$

2.2 Linear deformation

When the elongation of the truss is very small, the contraction is even smaller so that the deformed cross-sectional area can be taken to be equal to the initial cross-sectional area. Consequently there is no difference between the true stress and the engineering stress.

2.2.1 Linear strain

Elongation is generally described by the strain ε. For small elongation and rotation, the linear strain is used. For the elongation, this strain is related to the elongation facor λ and for the contraction to the contraction μ.

linear strain	$\varepsilon = \varepsilon_l = \lambda - 1$
contractive linear strain	$\varepsilon_d = \mu - 1$

2.2.2 Linear elastic behavior

The linear elastic material behavior is characterized by two matarial constants: Young's modulus and Poisson's ratio. Young's modulus relates the axial stress to the axial strain. Poisson's ratio relates the contractive strain to the axial strain. For most materials Poisson's ratio is about 0.3. For small elongations this value is constant. For small deformation the volume change factor J can be expressed in the linear strain. For incompressible material $J = 1$ implying $\nu = \frac{1}{2}$.

axial stress $\sim$ strain	$\sigma = E\varepsilon$
contraction strain	$\varepsilon = \lambda - 1 \quad \rightarrow \quad \varepsilon_d = \mu - 1 = -\nu\varepsilon = -\nu(\lambda - 1)$
volume change	$J = (\varepsilon + 1)(-\nu\varepsilon + 1)^2 \approx \varepsilon(1 - 2\nu) + 1$

The table lists values of Young's modulus and Poisson's ratio for some materials.

material	E [GPa]	ν [-]	material	E [GPa]	ν [-]
Aluminum	69 - 79	0.31 - 0.34	Copper	105 - 150	0.33 - 0.35
Cast iron	105 - 150	0.21 - 0.30	Steel	200	0.33
Stainless steel	190 - 200	0.28	Lead	14	0.43
Magnesium	41 - 45	0.29 - 0.35	Nickel	180 - 215	0.31
Titanium	80 - 130	0.31 - 0.34	Tungsten	400	0.27
Diamond	820 - 1050	-	Graphite	240 - 390	-
Glass	70 - 80	0.24	Epoxy	3.5 - 17	0.34
Nylon	1.4 - 2.8	0.32 - 0.40	Rubber	0.01 - 0.1	0.5

2.2.3 Equilibrium

We consider a truss with length l_0 and cross-sectional area A_0 with its axis along the global x-axis. One end ($x = 0$) is fixed and the other ($x = l_0$) can be displaced in x-direction only. The elongation of the truss equals this displacement u. The displacement is caused by an *external axial force* f_e. In the deformed state the length of the truss is $l = l_0 + u$ and its cross-sectional area is A. The material of the truss is homogeneous.

When the external axial force f_e is prescribed, the elongation $\Delta l = u$ of the truss can be determined by solving the equilibrium equation in point P, which states that the internal force must be equal to the external force. The *internal force* f_i is a function of the elongation, a relation which is determined by the material behavior. It represents the resistance of the truss against elongation.

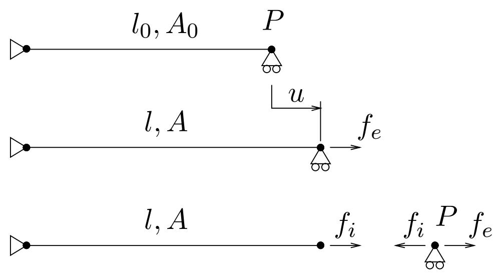


Fig. 2.5 : *Equilibrium of external and internal axial force*

external force	f_e
internal force	$f_i = f_i(u)$
equilibrium of point P	$f_i(u) = f_e$

When the deformation (= elongation) is very small, there is virtually no difference between the undeformed and the deformed geometry. Such deformation is referred to as being *geometrically linear*. The true axial stress $\sigma = N/A$ approximately equals the engineering stress $\sigma_n = N/A_0$, where N is the axial force.

When, moreover, the material behavior is not influenced by the deformation, as is the case for linear elastic behavior, – this is referred to as *physical linearity* – the total deformation is linear and the internal force f_i can be linearly related to the displacement u.

The equilibrium equation can be solved directly, yielding the displacement u.

2.2.4 Solution procedure

Because the relation between the external force f_e and the axial displacement u is linear, the latter can be solved directly from the equilibrium equation $f_i = f_e$, yielding the exact solution $u_{exact} = u_s$. The stiffness K of the truss depends on the Young's modulus E and on the initial geometry (A_0 and l_0).

$$f_i = \sigma_n A_0 = E\varepsilon A_0 = \frac{EA_0}{l_0} u = Ku$$

$$f_i = f_e \quad \rightarrow \quad Ku = f_e \quad \rightarrow \quad u = u_s = \frac{f_e}{K} = \frac{l_0}{EA_0} f_e$$

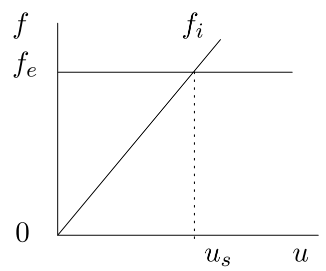


Fig. 2.6 : *Solution of linear equilibrium equation*

Proportionality and superposition

Two important characteristics hold for linear problems :

- the deformation is *proportional* to the load : when the external force f_e is multiplied by a factor, say α, the elongation u is also multiplied by α.
- *superposition* holds : when we determine the elongation u_1 and u_2 for two separate forces, f_{e1} and f_{e2} respectively, the elongation for the combined loading $f_{e1} + f_{e2}$ is the sum of the separate elongations : $u_1 + u_2$.

2.3 Finite element method for linear truss

When a truss structure is loaded by external forces or prescribed displacements, its deformation can sometimes be calculated analytically, especially when the structure is statically determinate. When the structure is statically indeterminate, this is only possible for very simple cases. Practical problems can be solved numerically, using the *finite element method.*

When the trusses in the structure show small elongation and rotation, and when moreover their material behavior is linearly elastic, the whole problem is linear and the finite element method can be explained rather straightforwardly.

In the following we restrict ourselves to two-dimensional structures.

Truss element

A truss element e with two nodal points is oriented with its axis in the 1-direction of a two-dimensional coordinate system. Both nodes move in this direction – being by definition positive – , leading to an elongation of the truss : its initial length l_0^e becomes l^e.

This elongation is resisted by the material of the truss, leading to reaction forces in both nodes : the *internal nodal point forces*, again defined to be positive in the positive 1-direction. In absence of distributed axial load the axial force N in the truss is constant and a function of the elongation.

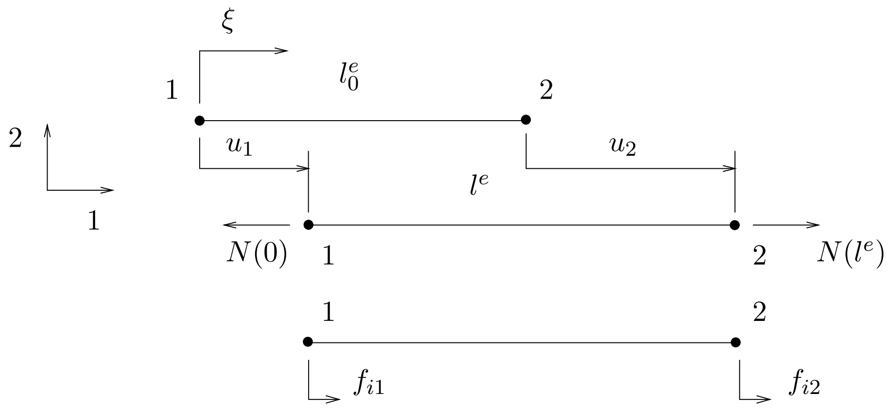


Fig. 2.7 : *One-dimensional truss element*

$$\underset{\sim}{u}^e = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u(0) \\ u(l^e) \end{bmatrix}$$

$$\underset{\sim}{f}_i^e = \begin{bmatrix} f_{i1} \\ f_{i2} \end{bmatrix} = \begin{bmatrix} -N(0) \\ N(l^e) \end{bmatrix} = \begin{bmatrix} -k(u_2 - u_1) \\ k(u_2 - u_1) \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

In the initial situation an angle α_0 may exist between the axis of a two-dimensional truss element and the 1-direction of the coordinate system. The displacements and forces of/in the nodal points have two components, and are defined positive in the positive coordinate directions. Due to the deformation, the current angle of the axis is α. For small deformations and rotations we have $\alpha \approx \alpha_0$.

The internal force components can be expressed in the axial force N and the *cosine* and *sine* of the angle α. For a linear element the nodal forces $\underset{\sim}{f}_i^e$ can be related to the elongation, expressed in the nodal displacements in the direction of the element axis, denoted as u_i^L, which are related to the displacement components of the nodal points $\underset{\sim}{u}^e$. This relation is expressed by the *element stiffness matrix* $\underline{K}^e$.

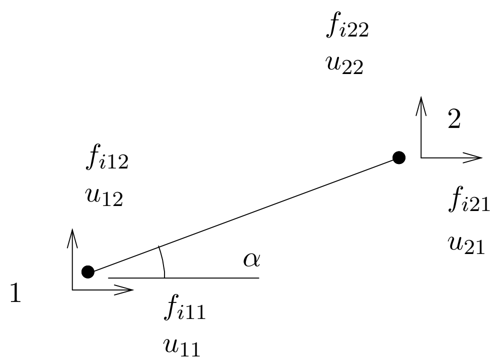


Fig. 2.8 : *Two-dimensional truss element*

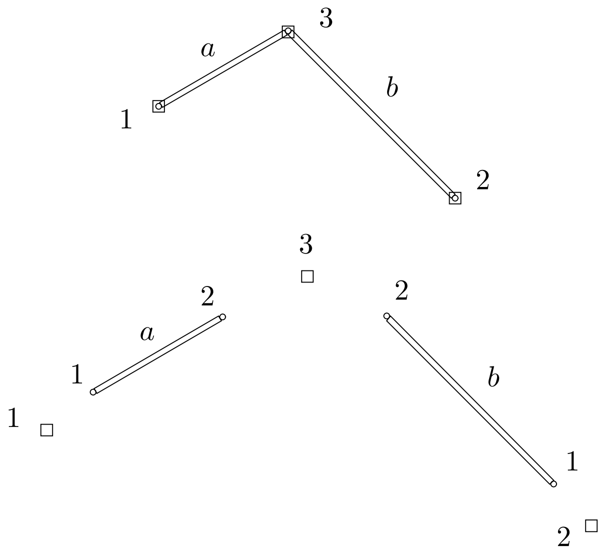
$$\begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \end{bmatrix} = \begin{bmatrix} cf_{i1}^L \\ sf_{i1}^L \\ cf_{i2}^L \\ sf_{i2}^L \end{bmatrix} = k(u_2^L - u_1^L) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix} = k(u_{21}c + u_{22}s - u_{11}c - u_{12}s) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}$$

$$= k \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} = \underline{K}^e \, \underset{\sim}{u}^e \qquad \begin{cases} c = \cos(\alpha) \\ s = \sin(\alpha) \end{cases}$$

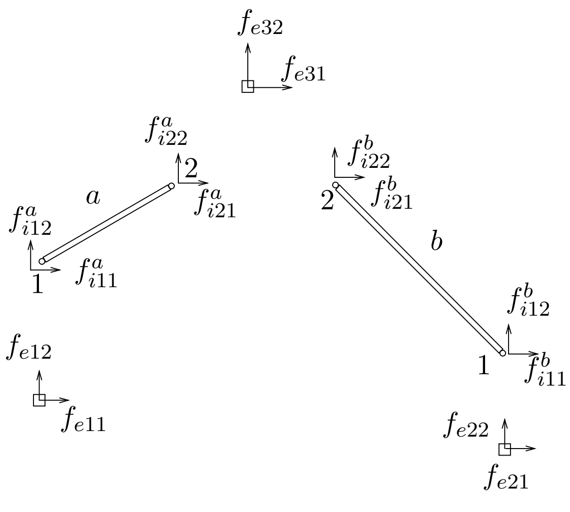
Assembling

For the truss structure, the internal nodal forces are stored in a column $\underset{\sim}{f}_i$ and are related to the nodal displacement components in the column $\underset{\sim}{u}$. This relation is expressed by the *structural* or *global stiffness matrix* $\underline{K}$. The contributions of the individual element stiffnesses to the structural stiffness are added in the assembling procedure.

The example shows two truss elements connected in one system node.



In each system node the internal nodal forces of all the connected elements must be added.



$$\begin{bmatrix} f_{e11} \\ f_{e12} \\ f_{e21} \\ f_{e22} \\ f_{e31} \\ f_{e32} \end{bmatrix} = \begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \\ f_{i31} \\ f_{i32} \end{bmatrix} = \begin{bmatrix} f^a_{i11} \\ f^a_{i12} \\ f^b_{i11} \\ f^b_{i12} \\ f^a_{i21} + f^b_{i21} \\ f^a_{i22} + f^b_{i22} \end{bmatrix} = \begin{bmatrix} f^a_{i11} \\ f^a_{i12} \\ 0 \\ 0 \\ f^a_{i21} \\ f^b_{i22} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f^b_{i11} \\ f^b_{i12} \\ f^b_{i21} \\ f^b_{i22} \end{bmatrix}$$

The displacements of element nodes which are connected to one and the same system node, must be equal to assure continuity.

$$\begin{bmatrix} f_{i11} \\ f_{i12} \\ f_{i21} \\ f_{i22} \\ f_{i31} \\ f_{i32} \end{bmatrix} = \begin{bmatrix} k^a c^2 & k^a cs & 0 & 0 & -k^a c^2 & -k^a cs \\ k^a cs & k^a s^2 & 0 & 0 & -k^a cs & -k^a s^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k^a c^2 & -k^a cs & 0 & 0 & k^a c^2 & k^a cs \\ -k^a cs & -k^a s^2 & 0 & 0 & k^a cs & k^a s^2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k^b c^2 & k^b cs & -k^b c^2 & -k^b cs \\ 0 & 0 & k^b cs & k^b s^2 & -k^b cs & -k^b s^2 \\ 0 & 0 & -k^b c^2 & -k^b cs & k^b c^2 & k^b cs \\ 0 & 0 & -k^b cs & -k^b s^2 & k^b cs & k^b s^2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{bmatrix} =$$

$$\begin{bmatrix} k^a c^2 & k^a cs & 0 & 0 & -k^a c^2 & -k^a cs \\ k^a cs & k^a s^2 & 0 & 0 & -k^a cs & -k^a s^2 \\ 0 & 0 & k^b c^2 & k^b cs & -k^b c^2 & -k^b cs \\ 0 & 0 & k^b cs & k^b s^2 & -k^b cs & -k^b s^2 \\ -k^a c^2 & -k^a cs & -k^b c^2 & -k^b cs & k^a c^2 + k^b c^2 & k^a cs + k^b cs \\ -k^a cs & -k^a s^2 & -k^b cs & -k^b s^2 & k^a cs + k^b cs & k^a s^2 + k^b s^2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \\ u_{32} \end{bmatrix}$$

Assembled system equations $\underset{\sim}{f}_i = \underline{K}\underset{\sim}{u}$

Equilibrium of the truss structure requires that the internal nodal point forces $\underset{\sim}{f}_i$ are equal to the external nodal point forces $\underset{\sim}{f}_e$. This leads to a linear system of equations for the nodal displacement components $\underset{\sim}{u}$. It is, however, not yet possible to solve this set of equations, because some essential boundary conditions have to be incorporated first.

$$\underset{\sim}{f}_i = \underset{\sim}{f}_e \qquad \rightarrow \qquad \underline{K}\underset{\sim}{u} = \underset{\sim}{f}_e = \underset{\sim}{f}$$

Boundary conditions

The equilibrium equations can only be solved uniquely, when proper boundary conditions are prescribed. These boundary conditions are suppressed displacements, prescribed displacements and prescribed forces.

It is always needed to prevent rigid body motions, because otherwise no (unique) solution can be determined. The algebraic system of equations $\underline{K}\underset{\sim}{u} = \underset{\sim}{f}$ has to be solved to determine the nodal displacements $\underset{\sim}{u}$. However, the stiffness matrix $\underline{K}$ is singular and cannot be inverted to solve $\underset{\sim}{u}$ when $\underset{\sim}{f}$ is known. This singularity is obvious as we must conclude that for a rigid body translation $\underset{\sim}{u} = \underset{\sim}{a} \neq \underset{\sim}{0}$ the nodal forces are zero.

rigid translation	$\underset{\sim}{u} = \underset{\sim}{a}$
no forces needed	$\underline{K}\,\underset{\sim}{a} = \underset{\sim}{0} \quad \text{with} \quad \underset{\sim}{a} \neq \underset{\sim}{0} \quad \rightarrow \quad \underline{K}$ singular

To get a non-singular matrix we have to suppress the rigid body movement of the construction, by prescribing enough nodal displacements. Besides boundary conditions to suppress rigid body motion, some more nodal displacements may be prescribed, as well as some nodal forces. When in a node a displacement component is prescribed, the associated external force component is unknown and vice versa.

Prescribed nodal displacement components are often denoted as *kinematic* boundary conditions and prescribed nodal forces as *dynamic* boundary conditions.

The *p*rescribed degrees of freedom $\underset{\sim}{u}_p$ are associated with unknown force components $\underset{\sim}{f}_p$. The *u*nknown degrees of freedom $\underset{\sim}{u}_u$ are associated with the known (prescribed or zero) force components $\underset{\sim}{f}_u$. The components of $\underset{\sim}{u}$ and $\underset{\sim}{f}$ are reorganized.

$$\text{reorganizing} \qquad \underset{\sim}{u} = \begin{bmatrix} \underset{\sim}{u}_u \\ \underset{\sim}{u}_p \end{bmatrix} \quad ; \quad \underset{\sim}{f} = \begin{bmatrix} \underset{\sim}{f}_u \\ \underset{\sim}{f}_p \end{bmatrix}$$

Reorganizing components of $\underset{\sim}{u}$ implies that *columns* of $\underline{K}$ have to be reorganized in the same way. Reorganizing components of $\underset{\sim}{f}$ implies that *rows* of $\underline{K}$ have to be reorganized in the same way. The components associated with the various parts of $\underset{\sim}{u}$ and $\underset{\sim}{f}$ can be placed in sub-matrices of the resulting matrix $\underline{K}$. The reorganization of columns and matrix described above is called *partitioning.*

As we can see, this partitioning leads to two sets of equations. Only the first set is relevant for the calculation of the unknown $\underset{\sim}{u}_u$. After determination of these unknowns, the second set is used to calculate the unknown reaction forces $\underset{\sim}{f}_p$.

$$\text{equilibrium} \qquad \underline{K}\underset{\sim}{u} = \underset{\sim}{f}$$

$$\text{partitioning} \qquad \begin{bmatrix} \underline{K}_{uu} & \underline{K}_{up} \\ \underline{K}_{pu} & \underline{K}_{pp} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}_u \\ \underset{\sim}{u}_p \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_u \\ \underset{\sim}{f}_p \end{bmatrix} \quad \rightarrow \quad \left.\begin{array}{l} \underline{K}_{uu}\underset{\sim}{u}_u + \underline{K}_{up}\underset{\sim}{u}_p = \underset{\sim}{f}_u \\ \underline{K}_{pu}\underset{\sim}{u}_u + \underline{K}_{pp}\underset{\sim}{u}_p = \underset{\sim}{f}_p \end{array}\right\}$$

$$\text{solving } \underset{\sim}{u}_u \qquad \underline{K}_{uu}\underset{\sim}{u}_u = \underset{\sim}{f}_u - \underline{K}_{up}\underset{\sim}{u}_p \quad \rightarrow \quad \underset{\sim}{u}_u = \underline{K}_{uu}^{-1}(\underset{\sim}{f}_u - \underline{K}_{up}\underset{\sim}{u}_p)$$

$$\text{calculating } \underset{\sim}{f}_p \qquad \underset{\sim}{f}_p = \underline{K}_{pu}\underset{\sim}{u}_u + \underline{K}_{pp}\underset{\sim}{u}_p$$

Links

Links (or tyings) are relations between some of the components of $\underset{\sim}{u}$. In these relations we make a difference between independent and dependent components. Dependent or *linked components* can be calculated from the independent ones after these have been solved. The linked components are removed from the equation system, as will be seen later. The independent components are not, so they are called *retained components.* Components which are not part of link relations are simply denoted as *free.* To identify the various components we use the indices l (linked), r (retained) and f (free).

Associated with the linked degrees of freedom are nodal forces, which ensure the relationship. They are calculated by requiring that the total virtual energy, associated with the links, is zero.

The column $\underset{\sim}{u}$ is reorganized such that free, retained and linked components are grouped in columns $\underset{\sim}{u}_f$, $\underset{\sim}{u}_r$ and $\underset{\sim}{u}_l$. The right-hand column $\underset{\sim}{f}$ is reorganized in the same way. The matrix $\underline{K}$ is adapted accordingly by moving rows and columns.

$$\begin{bmatrix} \underline{K}_{ff} & \underline{K}_{fr} & \underline{K}_{fl} \\ \underline{K}_{rf} & \underline{K}_{rr} & \underline{K}_{rl} \\ \underline{K}_{lf} & \underline{K}_{lr} & \underline{K}_{ll} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}_f \\ \underset{\sim}{u}_r \\ \underset{\sim}{u}_l \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_f \\ \underset{\sim}{f}_r + \bar{\underset{\sim}{f}}_r \\ \underset{\sim}{f}_l + \bar{\underset{\sim}{f}}_l \end{bmatrix} \quad \rightarrow \quad \begin{array}{l} \underline{K}_{ff}\underset{\sim}{u}_f + \underline{K}_{fr}\underset{\sim}{u}_r + \underline{K}_{fl}\underset{\sim}{u}_l = \underset{\sim}{f}_f \\ \underline{K}_{rf}\underset{\sim}{u}_f + \underline{K}_{rr}\underset{\sim}{u}_r + \underline{K}_{rl}\underset{\sim}{u}_l = \underset{\sim}{f}_r + \bar{\underset{\sim}{f}}_r \\ \underline{K}_{lf}\underset{\sim}{u}_f + \underline{K}_{lr}\underset{\sim}{u}_r + \underline{K}_{ll}\underset{\sim}{u}_l = \underset{\sim}{f}_l + \bar{\underset{\sim}{f}}_l \end{array}$$

The relation between $\underset{\sim}{u}_l$ and $\underset{\sim}{u}_r$ is denoted with a matrix $\underline{L}_{lr}$. Imposing the link relations will result in a change of the corresponding components of $\underset{\sim}{f}$. In a mechanical system $\bar{\underset{\sim}{f}}_r$ and $\bar{\underset{\sim}{f}}_l$ may be seen as forces which are needed to realize the links between $\underset{\sim}{u}_r$ and $\underset{\sim}{u}_l$. The resulting work of these forces at a random change in $\underset{\sim}{u}_r$ and $\underset{\sim}{u}_l$ must be zero, which results in a relation between $\bar{\underset{\sim}{f}}_r$ and $\bar{\underset{\sim}{f}}_l$.

$$\begin{array}{l} \underset{\sim}{u}_l = \underline{L}_{lr}\underset{\sim}{u}_r \\ \bar{\underset{\sim}{f}}_l^T \delta\underset{\sim}{u}_l + \bar{\underset{\sim}{f}}_r^T \delta\underset{\sim}{u}_r = 0 \quad \forall \quad \{\delta\underset{\sim}{u}_l, \delta\underset{\sim}{u}_r\} \qquad \rightarrow \\ \bar{\underset{\sim}{f}}_l^T \underline{L}_{lr} + \bar{\underset{\sim}{f}}_r^T = \underset{\sim}{0}^T \quad \rightarrow \quad \underline{L}_{lr}^T \bar{\underset{\sim}{f}}_l + \bar{\underset{\sim}{f}}_r = \underset{\sim}{0} \quad \rightarrow \quad \bar{\underset{\sim}{f}}_r = -\underline{L}_{lr}^T \bar{\underset{\sim}{f}}_l = -\underline{L}_{rl}\bar{\underset{\sim}{f}}_l \end{array}$$

Implementation of the link relations results in two systems of algebraic equations from which $\underset{\sim}{u}_r$ and $\underset{\sim}{u}_l$ can be solved.

$$\left.\begin{array}{l} \underline{K}_{ff}\underset{\sim}{u}_f + (\underline{K}_{fr} + \underline{K}_{fl}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_f \\ \underline{K}_{rf}\underset{\sim}{u}_f + (\underline{K}_{rr} + \underline{K}_{rl}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_r - \underline{L}_{rl}\bar{\underset{\sim}{f}}_l \\ \underline{K}_{lf}\underset{\sim}{u}_f + (\underline{K}_{lr} + \underline{K}_{ll}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_l + \bar{\underset{\sim}{f}}_l \\ \\ \text{elimination of } \bar{\underset{\sim}{f}}_l \end{array}\right\} \rightarrow$$

$$\left.\begin{array}{l} \underline{K}_{ff}\underset{\sim}{u}_f + (\underline{K}_{fr} + \underline{K}_{fl}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_f \\ (\underline{K}_{rf} + \underline{L}_{rl}\underline{K}_{lf})\underset{\sim}{u}_f + (\underline{K}_{rr} + \underline{K}_{rl}\underline{L}_{lr} + \underline{L}_{rl}\underline{K}_{lr} + \underline{L}_{rl}\underline{K}_{ll}\underline{L}_{lr})\underset{\sim}{u}_r = \underset{\sim}{f}_r + \underline{L}_{rl}\underset{\sim}{f}_l \end{array}\right\} \rightarrow$$

$$\begin{bmatrix} \underline{K}_{ff} & \underline{K}_{fr} + \underline{K}_{fl}\underline{L}_{rl} \\ \underline{K}_{rf} + \underline{L}_{rl}\underline{K}_{lf} & \underline{K}_{rr} + \underline{K}_{rl}\underline{L}_{lr} + \underline{L}_{rl}\underline{K}_{lr} + \underline{L}_{rl}\underline{K}_{ll}\underline{L}_{lr} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u}_f \\ \underset{\sim}{u}_r \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}_f \\ \underset{\sim}{f}_r + \underline{L}_{rl}\underset{\sim}{f}_l \end{bmatrix} \rightarrow$$

$$\underline{K}\underset{\sim}{u} = \underset{\sim}{f}$$

Program structure

A finite element program starts with reading data from an input file and initialization of variables and databases.

Subsquently, a loop over all elements is started to calculate $\underline{K}^e$ for each individual element and place it at the proper location in the structural matrix $\underline{K}$ (assembly). After taking into account the link relations and boundary conditions, the unknown nodal displacements are calculated.

Subsequently, another loop over all elements is entered to calculate the element strains, stresses and internal nodal forces $\underset{\sim}{f}_i^e$. The latter are assembled in the column $\underset{\sim}{f}_i$, which then contains the reaction forces of the system.

Finally some calculated values are stored for post-processing.

```
read input data from input file
calculate additional variables from input data
initialize values and arrays

for all elements
   calculate initial element stiffness matrix
   assemble global stiffness matrix
end element loop

determine external load from input

take tyings into account
take boundary conditions into account

calculate nodal displacements

for all elements
   calculate stresses from material behavior
   calculate element internal nodal forces
   assemble global internal load column
end element loop

store data for post-processing
```

2.3.1 FE program `tr2dL`

The Matlab program **`tr2dL`** is used to model and analyze two-dimensional truss structures. It is described in detail in appendix A. The input data, which must be provided by the user, are a.o. the coordinates of the nodes, the location of the trusses between nodes, element material data, link relations and prescribed nodal displacements and forces.

In this section, a few examples of two-dimensional truss structures are shown, which are modelled and analyzed with the program.

Simple two-dimensional truss structure

A simple truss structure is shown in the left figure below. The length of the horizontal truss [1] is 100 mm and the length of truss [2] is $200/\sqrt{3}$ mm. Cross-sectional areas are 10 and 20 mm^2, respectively. Young's modulus is 200 and 150 GPa and Poisson's ratio is $\nu = 0.3$. The prescribed force $F = -100$ N leads to the deformation $\{u_{2x}, u_{2y}\} = \{-0.0071, -0.0222\}$ mm, which is shown in the right part of the figure. The real deformation is very small, which is in accordance with the theory, so it is enlarged 1000 times.

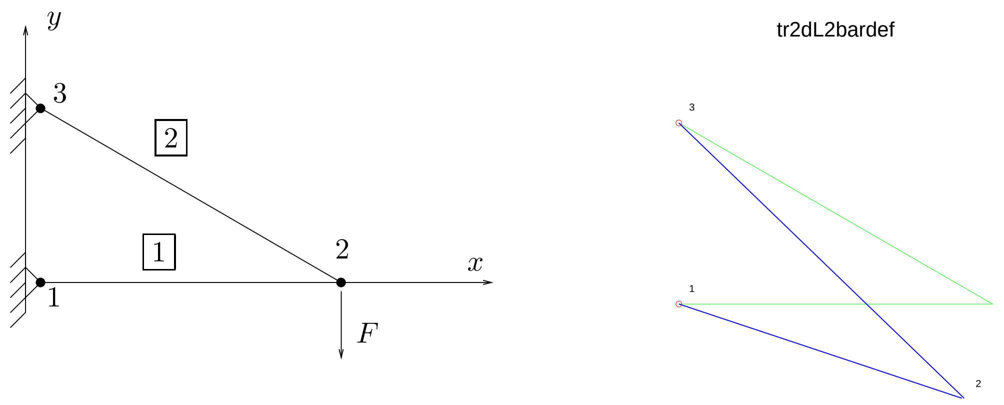


Fig. 2.9 : *Deformation of a truss structure (× 1000).*

Transformation of nodal coordinate system

It is possible to prescribe nodal displacements and/or forces in a local nodal coordinate system, which is rotated over an angle w.r.t. the global system. An example is shown below, where in node 2, the local coordinate axes are rotated over 45^o w.r.t. the xy-axes. The length of the horizontal truss is 100 mm and the length of the vertical truss is 50 mm. Cross-sectional areas are 1 mm^2. Young's moduli and Poisson's ratios are 100 GPa and 0.25. The external load is $F = 100$ N.

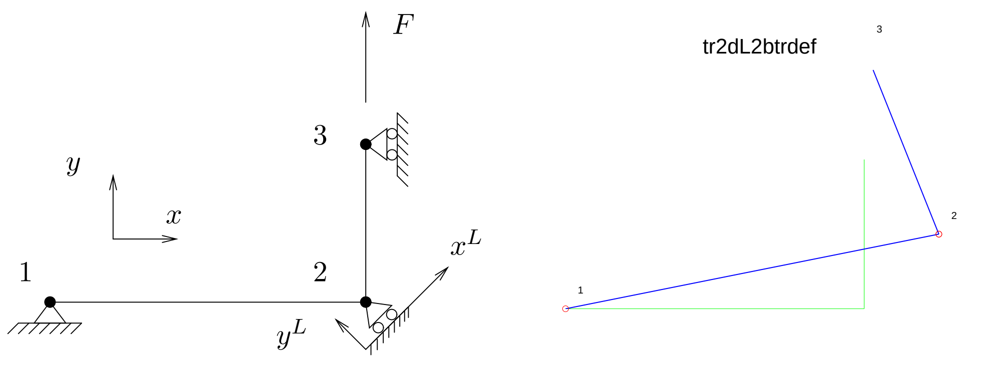


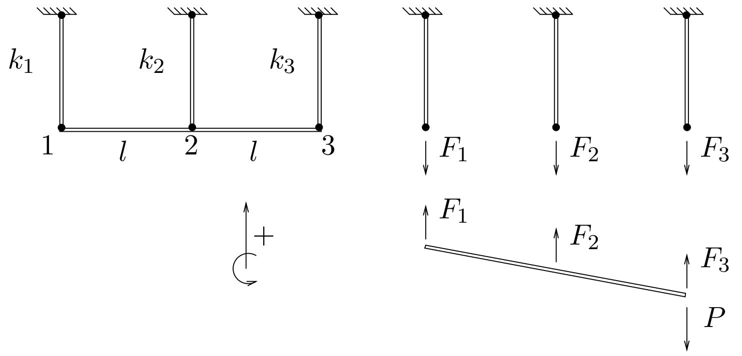
Fig. 2.10 : *Deformation of a truss structure (× 250), with a transformes nodal coordinate system.*

Tyings

The figure shows a rigid beam hanging from three trusses, which have equal stiffness k. The load P will cause an elongation of the trusses, which can be calculated, using link relations.

First the governing equations will be presented and solved analytically. Subsequently the solution of the finite element program will be presented.

The two equilibrium relations are not sufficient to solve the problem. Deformation and thus material behavior (= stiffness k) has to be taken into account. Still the final set of equations cannot be solved.



truss stiffness $\qquad k_1 = k_2 = k_3 = k$

equilibrium $\qquad F_1 + F_2 + F_3 - P = 0 \quad ; \quad -F_1 2l - F_2 l = 0$

deformation $\qquad v_1 = -\dfrac{F_1}{k} \quad ; \quad v_2 = -\dfrac{F_2}{k} \quad ; \quad v_3 = -\dfrac{F_3}{k}$

equilibrium equations in displacements

$$-kv_1 - kv_2 - kv_3 - P = 0 \qquad ; \qquad 2lkv_1 + lkv_2 = 0$$

Due to the rigidity of the beam, the displacements v_1, v_2 and v_3 are not independent. The dependency represents a link relation. Displacement v_2 is linked to the displacements v_1 and v_3. Displacement v_2 is eliminated from the equation system and v_1 and v_3 are retained.

link relation $\qquad v_2 = \frac{1}{2}(v_1 + v_3) \quad \rightarrow \quad v_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \end{bmatrix}$

elimination of v_2 $\quad \rightarrow \quad$ equation for retained displacements

$$\left. \begin{array}{l} -\frac{3}{2}kv_1 - \frac{3}{2}kv_3 - P = 0 \\ \frac{5}{2}lkv_1 + \frac{1}{2}lkv_3 = 0 \qquad \rightarrow \qquad v_1 = -\frac{1}{5}v_3 \end{array} \right\} \quad \rightarrow$$

$$\tfrac{3}{10}kv_3 - \tfrac{3}{2}kv_3 - P = 0 \quad \rightarrow \quad -\tfrac{6}{5}kv_3 - P = 0 \quad \rightarrow$$

$$v_3 = -\tfrac{5}{6}\frac{P}{k} \qquad \rightarrow \qquad v_1 = \tfrac{1}{6}\frac{P}{k}$$

$$\text{link} \quad \rightarrow \quad v_2 = -\tfrac{1}{3}\frac{P}{k}$$

The finite element solution is calculated. The undeformed and deformed structure is shown in the figure below. Both for the analytic and the numerical calculation, we find the next values for the nodal displacements, when setting $k = 100$ N/mm and $P = -10000$ N.

$$v_1 = \frac{100}{6} = 16.66mm \quad ; \quad v_2 = -\frac{100}{3} = -33.33mm \quad ; \quad v_3 = -\frac{500}{6} = -83.33mm$$

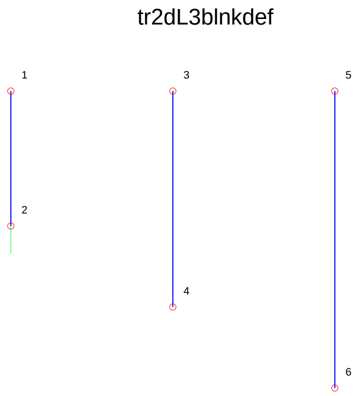


Fig. 2.11 : *Deformation of a truss structure with applied links (× 10).*

2.4 Nonlinear deformation

When deformation and/or rotation of the truss are large, various strains and stresses can be defined and related by material laws. The material behavior can be expected to be no longer linearly elastic.

2.4.1 Strains for large elongation

The deformation of the truss can be characterized uniquely by the two elongation factors λ and μ. However, it is common and useful to introduce deformation variables which are a function of the elongation factors : the strains. A wide variety of strain definitions is possible and used.

All strain definitions must obey some requirements, one of which is that they have to result in the same value for small elongations, being the value of the linear strain. When we plot the various strains as a function of the elongation factor, it is immediately clear that the strains, which are defined here, obey this requirement.

It is obvious that one and the same strain definition must be used throughout the same specimen and analysis. This implies that the contraction strain is defined analogously to the elongational strain. These strains are related by a material parameter, the Poisson's ratio ν. It is assumed, until stated otherwise, that this parameter is constant.

linear strain	$\varepsilon = \varepsilon_l = \lambda - 1$
logarithmic strain	$\varepsilon = \varepsilon_{ln} = \ln(\lambda)$
Green-Lagrange strain	$\varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$

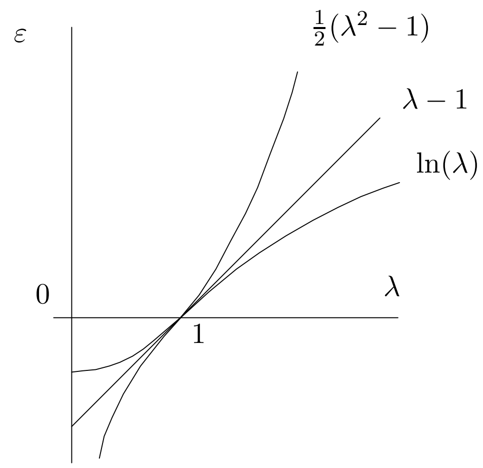


Fig. 2.12 : *Three strain definitions as function of the elongation factor*

Linear strain

The linear strain definition results in unrealistic contraction, when the elongation is too large. The cross-sectional area of the truss can become zero, which is of course not possible.

linear strain $\qquad \varepsilon = \varepsilon_l = \lambda - 1 = \dfrac{\Delta l}{l_0}$

contraction strain $\qquad \varepsilon_d = \mu - 1 = -\nu\varepsilon_l = -\nu(\lambda - 1)$

change of cross-sectional area

$$\mu = \sqrt{\frac{A}{A_0}} = 1 - \nu(\lambda - 1) \quad \rightarrow \quad A = A_0\{1 - \nu(\lambda - 1)\}^2$$

restriction of elongation

$$1 - \nu(\lambda - 1) > 0 \quad \rightarrow \quad \lambda - 1 < \frac{1}{\nu} \quad \rightarrow \quad \lambda < \frac{1 + \nu}{\nu}$$

Logarithmic strain

The logarithmic strain definition does not lead to unrealistic values for the contraction. Therefore it is very suitable to describe large deformations. [1]

logarithmic strain $\qquad \varepsilon = \varepsilon_{ln} = \ln(\lambda)$

contraction strain $\qquad \varepsilon_d = \ln(\mu) = -\nu\varepsilon_{ln} = -\nu \ln \lambda$

change of cross-sectional area

$$\mu = \sqrt{\frac{A}{A_0}} = e^{-\nu\varepsilon_{ln}} = e^{-\nu \ln(\lambda)} = \left[e^{\ln(\lambda)}\right]^{-\nu} = \lambda^{-\nu} \qquad \rightarrow \qquad A = A_0\lambda^{-2\nu}$$

[1] $\ln x = {}^e\log(x) = y \quad \rightarrow \quad x = e^y$

A deformation process may be executed in a number of steps, as is often done in forming processes. The start of a new step can be taken to be the reference state to calculate current strains. In that case the logarithmic strain is favorably used, because the subsequent strains can be added to determine the total strain w.r.t. the initial state.

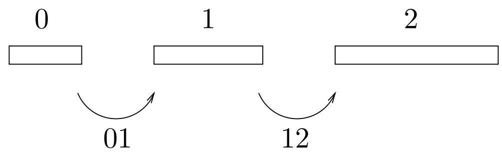


Fig. 2.13 : *Two-step deformation process*

$$
\begin{array}{lll}
l_0 \rightarrow l_1 & & \varepsilon_l(01) = \frac{l_1 - l_0}{l_0} \\
 & & \varepsilon_{ln}(01) = \ln(\frac{l_1}{l_0}) \\
 & & \\
l_1 \rightarrow l_2 & & \varepsilon_l(12) = \frac{l_2 - l_1}{l_1} \\
 & & \varepsilon_{ln}(12) = \ln(\frac{l_2}{l_1}) \\
 & & \\
l_0 \rightarrow l_2 & & \varepsilon_l(02) = \frac{l_2 - l_0}{l_0} \neq \varepsilon_l(01) + \varepsilon_l(12) \\
 & & \varepsilon_{ln}(02) = \ln(\frac{l_2}{l_0}) = \ln(\frac{l_2}{l_1}\frac{l_1}{l_0}) = \varepsilon_{ln}(01) + \varepsilon_{ln}(12)
\end{array}
$$

Green-Lagrange strain

Using the Green-Lagrange strain leads again to restrictions on the elongation to prevent the cross-sectional area to become zero.

Green-Lagrange strain $\qquad \varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$

contraction strain $\qquad \varepsilon_d = \frac{1}{2}(\mu^2 - 1) = -\nu\varepsilon_{ln} = -\nu\frac{1}{2}(\lambda^2 - 1)$

change of cross-sectional area

$$
1 - \nu(\lambda^2 - 1) > 0 \quad \rightarrow \quad \lambda < \sqrt{\frac{1+\nu}{\nu}}
$$

2.4.2 Mechanical power for an axially loaded truss

The figure shows a tensile bar which is elongated due to the action of an axial force F. Its undeformed cross-sectional area and length are A_0 and l_0, respectively. In the deformed configuration the cross-sectional area and length are A and l.

At constant force F an infinitesimal small increase in length is associated with a change in mechanical energy per unit of time (power) : $P = F\dot{l}$. The elongation rate $\dot{l}$ can be expressed in various strain rates.

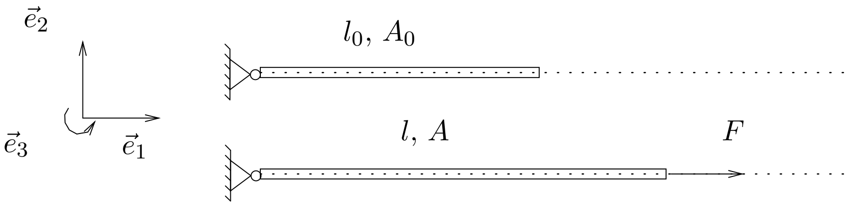


Fig. 2.14 : *Axial elongation of homogeneous truss*

linear strain	$\varepsilon_l = \lambda - 1$	$\rightarrow$	$\dot{\varepsilon}_l = \dot{\lambda} = \dfrac{\dot{l}}{l_0}$
logarithmic strain	$\varepsilon_{ln} = \ln(\lambda)$	$\rightarrow$	$\dot{\varepsilon}_{ln} = \dot{\lambda}\lambda^{-1} = \dfrac{\dot{l}}{l}$
Green-Lagrange strain	$\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$	$\rightarrow$	$\dot{\varepsilon}_{gl} = \dot{\lambda}\lambda = \lambda\,\dfrac{\dot{l}}{l_0} = \lambda^2\,\dfrac{\dot{l}}{l}$

$$\begin{array}{lllllllll}
P & = & F\dot{\ell} & = & F\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A_0}A_0\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A_0}V_0\dot{\varepsilon}_l \\
P & = & F\dot{\ell} & = & F\ell\dot{\varepsilon}_{ln} & = & \dfrac{F}{A}A\ell\dot{\varepsilon}_{ln} & = & \dfrac{F}{A}V\dot{\varepsilon}_{ln} \\
P & = & F\dot{\ell} & = & F\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A}A\ell\dfrac{\ell_0}{\ell}\dot{\varepsilon}_l & = & \dfrac{F}{A}V\lambda^{-1}\dot{\varepsilon}_l \\
P & = & F\dot{\ell} & = & F\ell\lambda^{-2}\dot{\varepsilon}_{gl} & = & \dfrac{F}{A}A\ell\lambda^{-2}\dot{\varepsilon}_{gl} & = & \dfrac{F}{A}V\lambda^{-2}\dot{\varepsilon}_{gl}
\end{array}$$

Various stress definitions automatically emerge when the mechanical power is considered in the undeformed volume $V_0 = A_0 l_0$ or the current volume $V = Al$ of the tensile bar. The stresses are :

σ	:	Cauchy or true stress
σ_n	:	engineering or nominal stress
σ_{p1}	:	1st Piola-Kirchhoff stress $= \sigma_n$
σ_κ	:	Kirchhoff stress
σ_{p2}	:	2nd Piola-Kirchhoff stress

$$\begin{array}{lllllll}
P & = & & = & & = & V_0\sigma_n\dot{\varepsilon}_l \\
P & = & V\sigma\dot{\varepsilon}_{ln} & = & V_0(J\sigma)\dot{\varepsilon}_{ln} & = & V_0\sigma_\kappa\dot{\varepsilon}_{ln} \\
P & = & V(\sigma\lambda^{-1})\dot{\varepsilon}_l & = & V_0(J\sigma\lambda^{-1})\dot{\varepsilon}_l & = & V_0\sigma_{p1}\dot{\varepsilon}_l \\
P & = & V(\sigma\lambda^{-2})\dot{\varepsilon}_{gl} & = & V_0(J\sigma\lambda^{-2})\dot{\varepsilon}_{gl} & = & V_0\sigma_{p2}\dot{\varepsilon}_{gl}
\end{array}$$

specific mechanical power : $\quad P = V_0\dot{W}_0 = V\dot{W}$

$$\begin{array}{lllllllll}
\dot{W}_0 & = & \sigma_n\dot{\varepsilon}_l & = & \sigma_\kappa\dot{\varepsilon}_{ln} & = & \sigma_{p1}\dot{\varepsilon}_l & = & \sigma_{p2}\dot{\varepsilon}_{gl} \\
\dot{W} & = & & = & \sigma\dot{\varepsilon}_{ln} & = & \sigma\lambda^{-1}\dot{\varepsilon}_l & = & \sigma\lambda^{-2}\dot{\varepsilon}_{gl}
\end{array}$$

2.4.3 Equilibrium

Deformations may be so large that the geometry changes considerably. This and/or nonlinear boundary conditions render the deformation problem nonlinear. Proportionality and superposition do not hold in that case. The internal force f_i is a nonlinear function of the elongation u.

Nonlinear material behavior may also result in a nonlinear function $f_i(u)$. This nonlinearity is almost always observed when deformation is large.

Solving the elongation from the equilibrium equation is only possible with an iterative solution procedure.

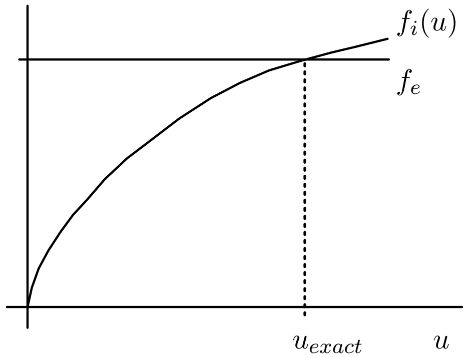


Fig. 2.15 : *Nonlinear internal load and constant external load*

external force	f_e
internal force	$f_i = \sigma A = f_i(u)$
equilibrium of point P	$f_i(u) = f_e$

2.4.4 Iterative solution procedure

It is assumed that an approximate solution u^* for the unknown exact solution u_{exact} exists. (Initially $u^* = 0$ is chosen.)

The *residual load* r^* is the difference between $f(u^*)$ and f_e. For the exact solution this residual is zero. What we want the iterative solution procedure to do, is generating better approximations for the exact solution so that the residual becomes very small (ideally zero).

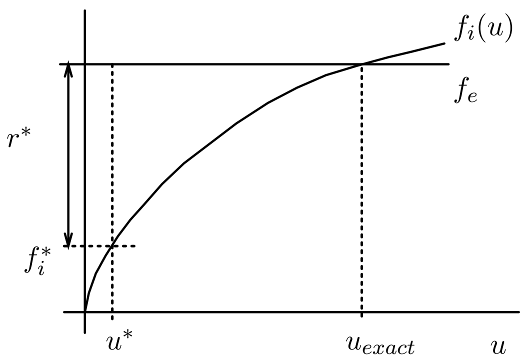


Fig. 2.16 : *Approximation of exact solution*

analytic solution	$f_i(u_{exact}) = f_e \quad \rightarrow \quad f_e - f_i(u_{exact}) = 0$
approximation u^*	$f_e - f_i(u^*) = r(u^*) \neq 0$
residual	$r^* = r(u^*)$

The unknown exact solution is written as the sum of the approximation and an unknown error δu. The internal force $f_i(u_{exact})$ is then written as a Taylor series expansion around u^* and linearized with respect to δu. The first derivative of f_i with respect to u is called the *tangential stiffness* K^*. Subsequently δu is solved from the linear iterative equation. The solution is called the *iterative displacement*.

$$\left.\begin{array}{l} f_i(u_{exact}) = f_e \\ u_{exact} = u^* + \delta u \end{array}\right\} \quad \rightarrow \quad f_i(u^* + \delta u) = f_e$$

$$f_i(u^*) + \left.\frac{df_i}{du}\right|_{u^*} \delta u = f_e \quad \rightarrow \quad f_i^* + K^* \delta u = f_e$$

$$K^* \, \delta u = f_e - f_i^* = r^* \quad \rightarrow \quad \delta u = \frac{1}{K^*} r^*$$

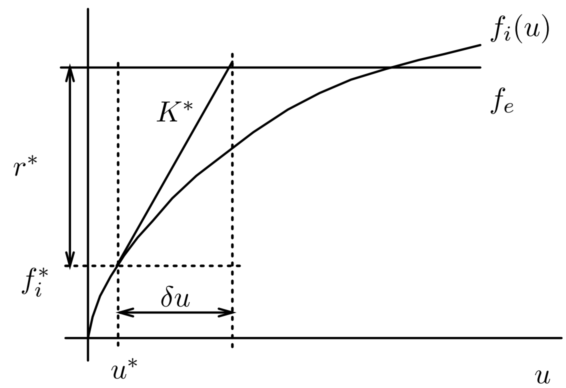


Fig. 2.17 : *Tangential stiffness and iterative solution*

With the iterative displacement δu a new approximate solution u^{**} can be determined by simply adding it to the known approximation.

When u^{**} is a better approximation than u^*, the iteration process is *converging*. As the exact solution is unknown, we cannot calculate the deviation of the approximation directly. There are several methods to quantify the *convergence*.

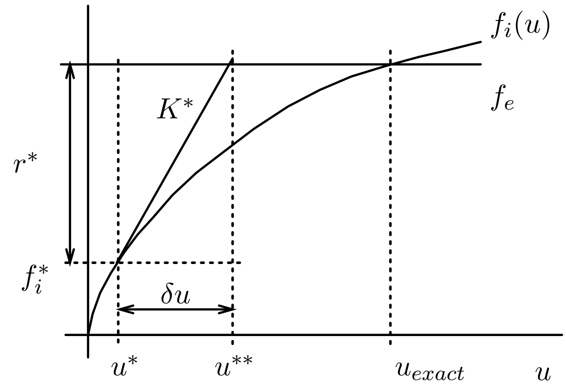


Fig. 2.18 : *New approximation of the exact solution*

new approximation	$u^{**} = u^* + \delta u$
error	$u_{exact} - u^{**}$
error smaller	$\rightarrow$ convergence

Convergence control

When the new approximation u^{**} is better than u^*, the residual r^{**} is smaller than r^*. If its value is not small enough, a new approximate solution is determined in a new iteration step. If its value is small enough, we are satisfied with the approximation u^{**} for the exact solution and the iteration process is terminated. To make this decision the residual is compared to a *convergence criterion* c_r. It is also possible to compare the iterative displacement δu with a convergence criterion c_u. If $\delta u < c_u$ it is assumed that the exact solution is determined close enough.

When the convergence criterion is satisfied, the displacement u will not satisfy the nodal equilibrium exactly, because the convergence limit is small but not zero. When incremental loading is applied, the difference between f_i and f_e is added to the load in the next increment, which is known as *residual load correction*.

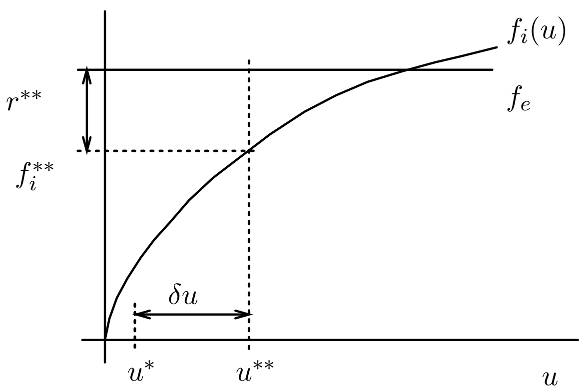


Fig. 2.19 : *New residual for approximate solution*

residual force	$\lvert r^{**}\rvert \leq c_r$	$\rightarrow$	stop iteration
iterative displacement	$\lvert \delta u\rvert \leq c_u$	$\rightarrow$	stop iteration

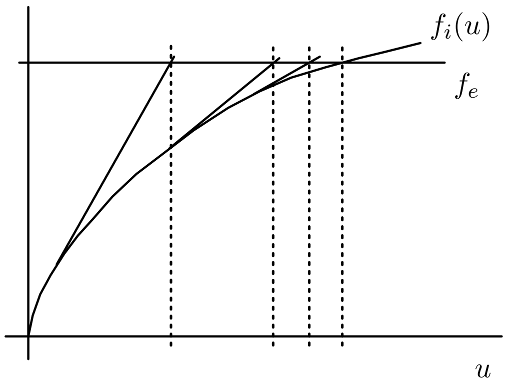


Fig. 2.20 : *Converging iteration process*

Residual and tangential stiffness

The residual and the tangential stiffness can be calculated from the material model, which describes the *material behavior*. It is assumed that this is a relation between the axial Cauchy stress σ and the elongation factor or stretch ratio $\lambda = \frac{l}{l_0}$: $\sigma = \sigma(\lambda)$. It is also necessary to now the relation between the cross-sectional area A and λ.

internal nodal force $\quad f_i^* = N(\lambda^*) = A^*\sigma^*$

tangential stiffness $\quad K^* = \left.\dfrac{\partial f_i}{\partial u}\right|_{u^*} = \left.\dfrac{\partial N(\lambda)}{\partial u}\right|_{u^*} = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{d\lambda}{du}$

geometry $\quad \lambda = 1 + \dfrac{\Delta l}{l_0} = 1 + \dfrac{1}{l_0}\,u \quad \rightarrow \quad \dfrac{d\lambda}{du} = \dfrac{1}{l_0}$

tangential stiffness $\quad K^* = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{\partial\lambda}{\partial u} = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{1}{l_0} = \left.\dfrac{dN}{d\lambda}\right|^* \dfrac{1}{l_0} = \dfrac{1}{l_0}\dfrac{d}{d\lambda}(\sigma A)\Big|^*$

$$K^* = \frac{1}{l_0}\left.\frac{d\sigma}{d\lambda}\right|^* A^* + \frac{1}{l_0}\,\sigma^* \left.\frac{dA}{d\lambda}\right|^*$$

Incremental loading

The external loading may be time-dependent. To determine the associated deformation, the time is discretized : the load is prescribed at subsequent, discrete moments in time and deformation is determined at these moments. A time interval between two discrete moments is called a *time increment* and the time dependent loading is referred to as *incremental loading.* This incremental loading is also applied for cases where the real time (seconds, hours) is not relevant, but when we want to prescribe the load gradually. One can than think of the "time" as a fictitious or virtual time.

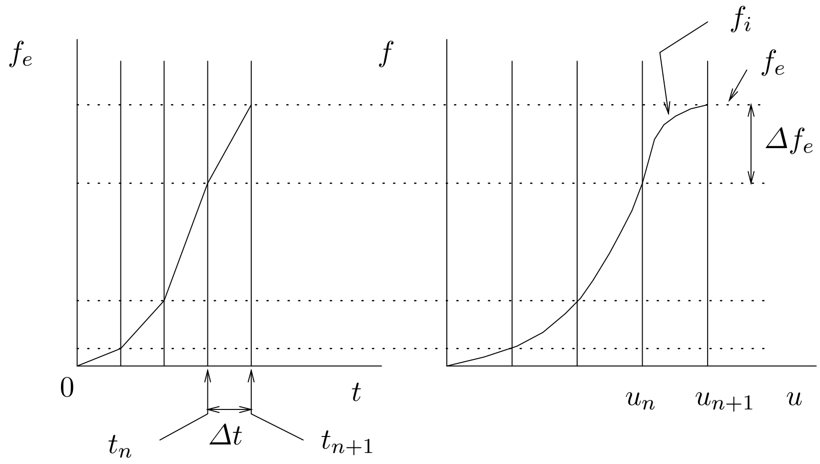


Fig. 2.21 : *Incremental loading*

Non-converging solution process

The iteration process is not always converging. Some illustrative examples are shown in the next figures.

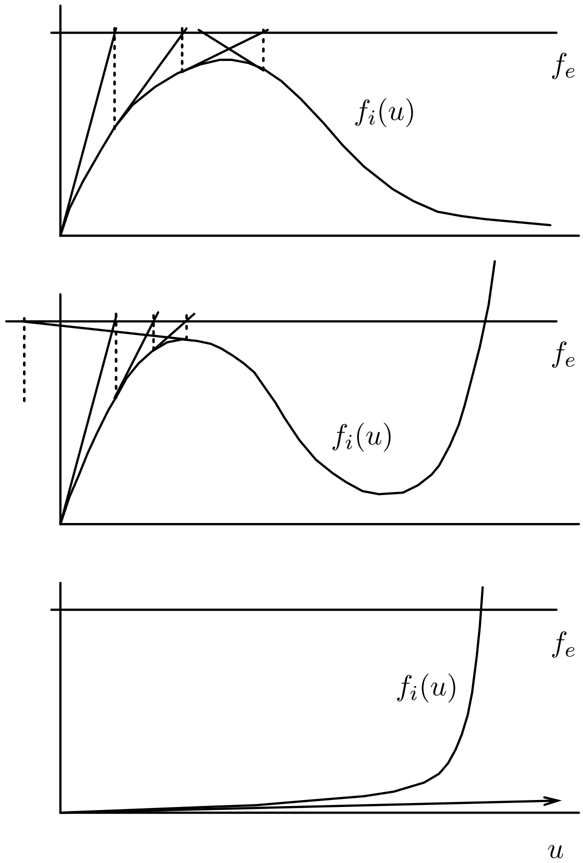


Fig. 2.22 : *Non-converging solution processes*

Modified Newton-Raphson procedure

Sometimes, it is possible to reach the exact solution by modifying the Newton-Raphson iteration process. The tangential stiffness is then not updated in every iteration step. Its initial value is used throughout the iterative procedure.

The figure shows a so-called "snap-through" problem, where no convergence can be reached due to a cycling *full* Newton-Raphson iteration process. With *modified* Newton-Raphson, iteration proceeds to the equilibrium $f_i = f_e$.

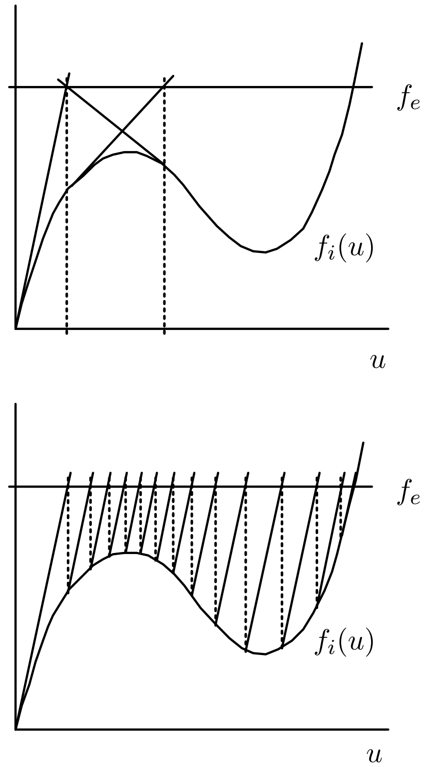


Fig. 2.23 : *Modified Newton-Raphson procedure*

Path-following solution algorithm

The iteration procedure can be combined with a "path-following" algorithm to control the load step.

Each increment starts with a load step $g_1 = \Delta f_e = \lambda_1 f_{ef}$ where f_{ef} is the final load and λ_1 is a known fraction. The initial iterative displacement δu_1 can be calculated and the initial incremental and total displacement approximations are known. The internal load f_{i1} and tangent stiffness K_1 can be calculated.

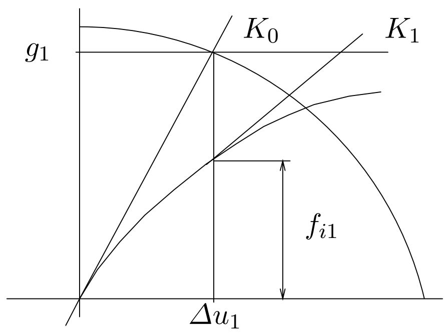


Fig. 2.24 : *Path-following solution procedure : first iterative step*

$$\begin{array}{l} K_0 \delta u_1 = f_{e0} + \lambda_1 f_{ef} = g_1 \quad \rightarrow \quad \delta u_1 = K_0^{-1} g_1 \quad \rightarrow \\ \Delta u_1 = \delta u_1 \quad ; \quad u_1 = u_0 + \Delta u_1 \quad \rightarrow \quad f_{i1} \ , \ K_1 \end{array}$$

In the second iteration step the incremental load is calculated again as a fraction of the final load, but now the fraction λ_2 is unknown. It can be calculated from the requirement that the length of $[\Delta u_2 \; g_2]$ has to be C, which is a known constant value for the current increment and is referred to as the arc-length. For the two resulting solutions the product of Δu_2 and Δu_1 is calculated. The solution resulting in the largest value is selected. With λ_2 known, the new approximate solution is determined. Obviously the incremental load step g_2 is changed compared to the initial value g_1.

The iteration process is continued until convergence is reached. The convergence norm can be calculated from the residual load, the iterative displacement or a combination of the two.

After convergence a new increment can be started with a modified initial fraction λ_1. The procedure is stopped when convergence is reached after applying the final load f_{ef}.

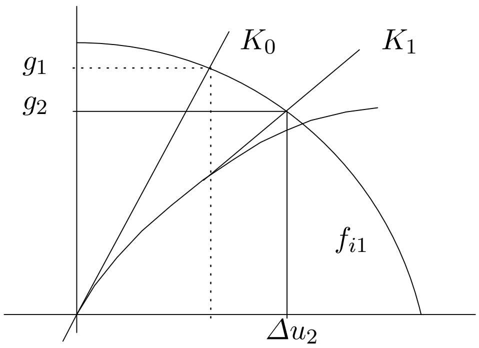


Fig. 2.25 : *Path-following solution procedure : reduction of external incremental load*

$$\begin{array}{l} K_1 \delta u_2 = g_2 - f_{i2} = f_{e0} + \lambda_2 f_{ef} - f_{i2} \quad \rightarrow \\ \delta u_2 = K_1^{-1} \left(f_{e0} + \lambda_2 f_{ef} - f_{i2} \right) \end{array}$$

$$\begin{array}{l} \left[\; \Delta u_2 \quad g_2 \; \right] \left[\begin{array}{c} \Delta u_2 \\ g_2 \end{array} \right] = (\Delta u_2)^2 + (g_2)^2 = C^2 \quad \rightarrow \\ \left(\Delta u_1 + K_1^{-1} f_{e0} + K_1^{-1} f_{ef} \lambda_2 - K_1^{-1} f_{i2} \right)^2 + \left(f_{e0} + \lambda_2 f_{ef} \right)^2 = C^2 \quad \rightarrow \\ \lambda_2 \quad \rightarrow \quad \delta u_2 \quad \rightarrow \quad \Delta u_2 \ , \ u_2 \quad \rightarrow \quad f_{i3} \ , \ K_2 \end{array}$$

2.5 Weighted residual formulation for nonlinear truss

In the initial configuration a truss has length ℓ_0. In the current configuration the truss is subjected to an axial load: concentrated forces N_0 and N_ℓ in begin and end point, and a volume load $q(s)$ per unit of length. It has length ℓ and is rotated with respect to the initial configuration. The coordinate along the truss axis is s and the direction of the axis is indicated by the unit vector $\vec{n}$.

In each point of the truss the equilibrium equation has to be satisfied. The equilibrium equation is derived under assumption of static loading conditions. It is a differential equation, for which analytical solutions do only exist for rather simple boundary conditions. For practical problems we have to be satisfied with an approximate solution.

The error represented by the approximation can be "smeared out" along the axis of the truss, by integrating the product of this error and a so-called *weighting function* over the length of the truss.

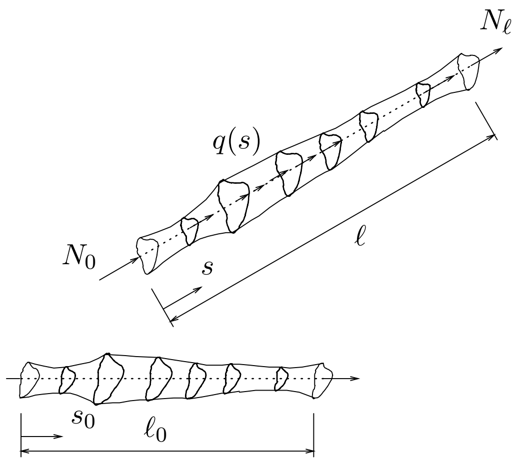


Fig. 2.26 : *Inhomogeneous truss*

equilibrium	$\dfrac{d\vec{N}}{ds} + \vec{q}(s) = \vec{0} \quad \rightarrow \quad \dfrac{d(\sigma A \vec{n})}{ds} + \vec{q}(s) = \vec{0} \quad \forall\ s \in [0, \ell]$
approximation	$\dfrac{d(\sigma^* A^* \vec{\bar{n}})}{ds} + \vec{q}(s) = \vec{\Delta}(s) \neq \vec{0} \quad \forall\ s \in [0, \ell]$
weighted error	$\vec{\Delta}(s)$ is "smeared out" over$[0, \ell]$ $\quad \rightarrow \quad \displaystyle\int_{s=0}^{s=\ell} \vec{w}(s) \cdot \vec{\Delta}(s)\, ds$

Weighted residual formulation

The product of the left-hand side of the equilibrium equation and a weighting function $\vec{w}(s)$ can be integrated over the element length, resulting in the weighted residual integral. The *principle of weighted residuals* now states that :

> the requirement that the equilibrium equation is satisfied in each point of the truss, is equivalent to the requirement that the weighted residual integral is zero for every possible weighting function.

The first term in the integral is integrated by parts to reduce the continuity requirements of the axial stress. This results in the so-called *weak form* of the weighted residual formulation. The right hand part of the resulting integral equation represents the contribution of the external loads.

$$\int_{s=0}^{s=\ell} \vec{w} \cdot \left\{ \frac{d(\sigma A \vec{n})}{ds} + \vec{q} \right\} ds = 0 \qquad \forall \quad \vec{w}(s) \quad \rightarrow$$

$$\int_{s=0}^{s=\ell} \frac{d\vec{w}}{ds} \cdot (\sigma A \vec{n})\, ds = \int_{s=0}^{s=\ell} \vec{w} \cdot \vec{q}\, ds + \left[\vec{w}(\ell) \cdot \vec{N}(\ell) - \vec{w}(0) \cdot \vec{N}(0) \right] = f_e(\vec{w}) \quad \forall \quad \vec{w}(s)$$

State transformation

Because the current length of the truss is not known, the integration can not be carried out. Also the derivatives with respect to the coordinate s can not be evaluated. These problems can be circumvented by a transformation. In this case we transform everything to the initial configuration on time t_0 where the truss is undeformed. This procedure is generally referred to as the *Total Lagrange* approach. When transformation is carried out to the last known configuration, we would have followed the *Updated Lagrange* approach, which will not be considered here.

$$\frac{d(\ )}{ds} = \frac{ds_0}{ds} \frac{d(\ )}{ds_0} = \frac{1}{\lambda} \frac{d(\ )}{ds_0} \qquad ; \qquad ds = \lambda ds_0 \qquad \Rightarrow$$

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot (\sigma A \vec{n})\, ds_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(s_0)$$

The current stress σ, cross-sectional area A and axis direction $\vec{n}$ have to be determined such that the integral is satisfied for every weighting function. Following a Newton-Raphson iteration procedure, the exact solutions are written as the sum of an approximation and a deviation. Subsequently linearisation is carried out.

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot (\sigma^* + \delta\sigma)(A^* + \delta A)(\vec{n}^* + \delta\vec{n})\, ds_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(s_0)$$

Linearisation with assumption $\delta A \approx 0$ leads to an iterative weighted residual integral.

$$\begin{aligned} \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \delta\sigma A^* \vec{n}^*\, ds_0 + \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \delta\vec{n}\, ds_0 \\ = f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \vec{n}^*\, ds_0 \qquad \forall \quad \vec{w}(s_0) \end{aligned}$$

Material model $\rightarrow$ iterative stress change

The material model relates the stress σ to the elongation λ. Using this relation the iterative change $\delta\sigma$ can be expressed in the iterative displacement $\delta\vec{u}$.

$$\sigma = \sigma(\lambda) \quad \rightarrow \quad \delta\sigma = \left.\frac{d\sigma}{d\lambda}\right|^{*} \delta\lambda = \left.\frac{d\sigma}{d\lambda}\right|^{*} \frac{d(\delta s)}{ds_0} = \left.\frac{d\sigma}{d\lambda}\right|^{*} \vec{n}^{*} \cdot \frac{d(\delta\vec{u})}{ds_0}$$

Rotation $\rightarrow$ iterative orientation change

Due to the rotation of the truss, the axis direction vector $\vec{n}$ is also a function of the iterative displacement. The vector $\vec{m}$ is a unit vector perpendicular to $\vec{n}$.

$$\begin{aligned}
\vec{n} &= \frac{d\vec{x}}{ds} = \frac{ds_0}{ds}\frac{d\vec{x}}{ds_0} = \frac{1}{\lambda}\frac{d\vec{x}}{ds_0} \\
\delta\vec{n} &= \left[-\frac{1}{\lambda^2}\frac{d\vec{x}}{ds_0}\right]^{*} \delta\lambda + \left[\frac{1}{\lambda}\right]^{*} \frac{d(\delta\vec{x})}{ds_0} = \left[-\frac{1}{\lambda}\vec{n}\right]^{*} \delta\lambda + \left[\frac{1}{\lambda}\right]^{*} \frac{d(\delta\vec{x})}{ds_0} \\
&= \left[-\frac{1}{\lambda}\vec{n}\vec{n}\right]^{*} \cdot \frac{d(\delta\vec{u})}{ds_0} + \left[\frac{1}{\lambda}\right]^{*} \frac{d(\delta\vec{u})}{ds_0} = \left[(\boldsymbol{I} - \vec{n}\vec{n})\frac{1}{\lambda}\right]^{*} \cdot \frac{d(\delta\vec{u})}{ds_0} \\
&= \left[\vec{m}\vec{m}\frac{1}{\lambda}\right]^{*} \cdot \frac{d(\delta\vec{u})}{ds_0}
\end{aligned}$$

Iterative weighted residual integral

The expressions for $\delta\sigma$ and $\delta\vec{n}$ are substituted into the iterative weighted residual integral.

$$\begin{aligned}
&\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \left(\left.\frac{d\sigma}{d\lambda}\right|^{*} \vec{n}^{*} \cdot \frac{d(\delta\vec{u})}{ds_0} \right) A^{*}\vec{n}^{*}\, ds_0 + \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^{*}A^{*} \left(\vec{m}^{*}\vec{m}^{*} \cdot \frac{1}{\lambda^{*}}\frac{d(\delta\vec{u})}{ds_0} \right) ds_0 \\
&\qquad = f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^{*}A^{*}\vec{n}^{*}\, ds_0 \qquad \forall \quad \vec{w}(s_0)
\end{aligned}$$

$$\begin{aligned}
&\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \vec{n}^{*} \left(\left.\frac{d\sigma}{d\lambda}\right|^{*} A^{*} \right) \vec{n}^{*} \cdot \frac{d(\delta\vec{u})}{ds_0}\, ds_0 + \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \vec{m}^{*} \left(\sigma^{*}A^{*}\frac{1}{\lambda^{*}} \right) \vec{m}^{*} \cdot \frac{d(\delta\vec{u})}{ds_0}\, ds_0 \\
&\qquad = f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^{*}A^{*}\vec{n}^{*}\, ds_0 \qquad \forall \quad \vec{w}(s_0)
\end{aligned}$$

2.6 Finite element method for nonlinear truss

The mechanical behavior of truss structures, which are build from nonlinear trusses, which may show large elongations and (thus) large rotations, can be analyzed with the finite element method. Individual truss elements are considered first, which means that the structure is discretized. Later the contributions of all trusses will be combined in an assembling procedure.

Element equation

We start with the weighted residual integral for one truss element, which length is ℓ_0^e in the initial state and ℓ^e in the current state. First, the global coordinate s_0 is replaced by a local element coordinate ξ.

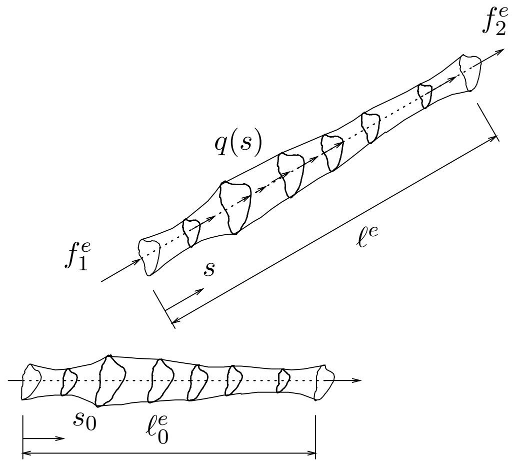


Fig. 2.27 : *Inhomogeneous truss element in undeformed state*

local coordinate : $\quad -1 \leq \xi \leq 1 \quad ; \quad ds_0 = \dfrac{l_0}{2}\, d\xi \quad ; \quad \dfrac{d(\;)}{ds_0} = \dfrac{2}{l_0}\dfrac{d(\;)}{d\xi}$

$$\int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \vec{n}^* \left(\left.\frac{d\sigma}{d\lambda}\right|^* A^* \frac{2}{l_0} \right) \vec{n}^* \cdot \frac{d(\delta\vec{u})}{d\xi}\, d\xi \; +$$

$$\int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \vec{m}^* \left(\sigma^* A^* \frac{1}{\lambda^*}\frac{2}{l_0} \right) \vec{m}^* \cdot \frac{d(\delta\vec{u})}{d\xi}\, d\xi = f_{e0}^e(\vec{w}) - \int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \sigma^* A^* \vec{n}^*\, d\xi$$

The vectors in the weighted residual integral are written in components with respect to a vector basis.

$$\int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \underset{\sim}{n}^* \left(\left.\frac{d\sigma}{d\lambda}\right|^* A^* \frac{2}{l_0} \right) \underset{\sim}{n}^{*T} \frac{d(\delta\underset{\sim}{u})}{d\xi}\, d\xi + \int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \underset{\sim}{m}^* \left(\sigma^* A^* \frac{1}{\lambda^*}\frac{2}{l_0} \right) \underset{\sim}{m}^{*T} \frac{d(\delta\underset{\sim}{u})}{d\xi}\, d\xi$$

$$= f_{e0}^e(\underset{\sim}{w}) - \int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \sigma^* A^* \underset{\sim}{n}^*\, d\xi$$

Interpolation

Both the iterative displacement and the weighting function components are interpolated between their values in the element nodes. Here we use a linear interpolation between two nodal values. The element nodes are located in the begin and end points of the element. Following the Galerkin procedure, the interpolation functions for $\delta \underset{\sim}{u}$ and $\underset{\sim}{w}$ are taken to be the same.

The derivatives of $\delta \underset{\sim}{u}$ and $\underset{\sim}{w}$ can also be interpolated directly.

$$
\begin{aligned}
\delta \underset{\sim}{u}^T &= \begin{bmatrix} \delta u_1 & \delta u_2 \end{bmatrix} = \begin{bmatrix} \delta u_{11}\psi^1 + \delta u_{21}\psi^2 & \delta u_{12}\psi^1 + \delta u_{22}\psi^2 \end{bmatrix} \\
\underset{\sim}{w}^T &= \begin{bmatrix} w_{11}\psi^1 + w_{21}\psi^2 & w_{12}\psi^1 + w_{22}\psi^2 \end{bmatrix}
\end{aligned}
$$

$$
\text{with} \qquad \psi^1(\xi) = \tfrac{1}{2}(1-\xi) \quad ; \quad \psi^2(\xi) = \tfrac{1}{2}(1+\xi)
$$

$$
\frac{d(\delta \underset{\sim}{u})}{d\xi} = \begin{bmatrix} \dfrac{d(\delta u_1)}{d\xi} \\ \dfrac{d(\delta u_2)}{d\xi} \end{bmatrix} = \begin{bmatrix} \frac{d\psi^1}{d\xi} & 0 & \frac{d\psi^2}{d\xi} & 0 \\ 0 & \frac{d\psi^1}{d\xi} & 0 & \frac{d\psi^2}{d\xi} \end{bmatrix} \begin{bmatrix} \delta u_{11} \\ \delta u_{12} \\ \delta u_{21} \\ \delta u_{22} \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \delta \underset{\sim}{u}^e
$$

$$
\frac{d\underset{\sim}{w}^T}{d\xi} = \begin{bmatrix} \dfrac{dw_1}{d\xi} & \dfrac{dw_2}{d\xi} \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \frac{d\psi^1}{d\xi} & 0 \\ 0 & \frac{d\psi^1}{d\xi} \\ \frac{d\psi^2}{d\xi} & 0 \\ 0 & \frac{d\psi^2}{d\xi} \end{bmatrix} = \underset{\sim}{w}^{eT} \tfrac{1}{2} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

Substitution of the interpolated variables leads to an element integral equation, where the internal nodal forces and the element tangential stiffness matrix can be recognized.

$$
\begin{aligned}
&\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix}^* \tfrac{1}{4} \left(\left. \frac{d\sigma}{d\lambda} \right|^* A^* \frac{2}{l_0} \right) \begin{bmatrix} c & s \end{bmatrix}^* \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} d\xi \, \delta \underset{\sim}{u}^e \; + \\
&\quad \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -s \\ c \end{bmatrix}^* \tfrac{1}{4} \left(\sigma^* A^* \frac{1}{\lambda^*} \frac{2}{l_0} \right) \begin{bmatrix} -s & c \end{bmatrix}^* \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} d\xi \, \delta \underset{\sim}{u}^e \\
&\qquad = \underset{\sim}{f}_{e0}^{e}(\underset{\sim}{w}^e) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tfrac{1}{2} \begin{bmatrix} c \\ s \end{bmatrix}^* (\sigma^* A^*) \, d\xi \\
&\qquad\quad \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \left. \frac{d\sigma}{d\lambda} \right|^* A^* \frac{1}{l_0} \right) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^* \begin{bmatrix} -c & -s & c & s \end{bmatrix}^* d\xi \, \delta \underset{\sim}{u}^e \; + \\
&\qquad\qquad \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \sigma^* A^* \frac{1}{\lambda^*} \frac{1}{l_0} \right) \begin{bmatrix} s \\ -c \\ -s \\ c \end{bmatrix}^* \begin{bmatrix} s & -c & -s & c \end{bmatrix}^* d\xi \, \delta \underset{\sim}{u}^e
\end{aligned}
$$

$$
= f^e_{e0}(\underset{\sim}{w}^e) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \tfrac{1}{2} \left(\sigma^* A^*\right) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^* d\xi
$$

$$
\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \left.\frac{d\sigma}{d\lambda}\right|^* A^* \frac{1}{l_0} \right) \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}^* d\xi \, \delta \underset{\sim}{u}^e +
$$

$$
\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \sigma^* A^* \frac{1}{\lambda^*} \frac{1}{l_0} \right) \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}^* d\xi \, \delta \underset{\sim}{u}^e
$$

$$
= f^e_{e0}(\underset{\sim}{w}^e) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \tfrac{1}{2} \left(\sigma^* A^*\right) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^* d\xi
$$

With the introduction of some proper matrices and columns, the element equation can be written in short form.

$$
\underset{\sim}{w}^{eT} \left[\int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \left.\frac{d\sigma}{d\lambda}\right|^* A^* \frac{1}{l_0} \right) d\xi \, \underline{M}^*_L \right] \delta \underset{\sim}{u}^e + \underset{\sim}{w}^{eT} \left[\int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \sigma^* A^* \frac{1}{\lambda^*} \frac{1}{l_0} \right) d\xi \, \underline{M}^*_N \right] \delta \underset{\sim}{u}^e
$$

$$
= f^e_{e0}(\underset{\sim}{w}^e) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \tfrac{1}{2} \left(\sigma^* A^*\right) \underset{\sim}{V}^* \, d\xi
$$

$$
\underset{\sim}{w}^{eT} \underline{K}^{e^*} \delta \underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}^e_{e0} - \underset{\sim}{w}^{eT} \underset{\sim}{f}^{e^*}_i = \underset{\sim}{w}^{eT} \underset{\sim}{r}^{e^*}
$$

Integration

Integation over the element length is needed to determine the element stiffness matrix $\underline{K}^{e^*}$ and the internal force column $\underset{\sim}{f}^{e^*}_i$.

For a homogeneous element, e.g. an element with uniform cross-sectional area and material properties, this leads to the following expressions.

$$
\underline{K}^{e^*} = \left(\left.\frac{d\sigma}{d\lambda}\right|^* A^* \frac{1}{l_0} \right) \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}^* + \left(\sigma^* A^* \frac{1}{l^*} \right) \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}^*
$$

$$
\underset{\sim}{f}^{e^*}_i = \sigma^* A^* \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^*
$$

Assembling

The contributions of the individual elements are added in the assembling procedure. The result is an integral equation for the total system, which, according to the principle of weighted residuals, has to be satisfied for every column with nodal weighting function values. This requirement leads to a system of algebraic equations from which the iterative nodal displacement components must be solved.

element contribution	$\underset{\sim}{w}^{eT} \underline{K}^{e^*} \delta \underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}^e_{e0} - \underset{\sim}{w}^{eT} \underset{\sim}{f}^{e^*}_i = \underset{\sim}{w}^{eT} \underset{\sim}{r}^{e^*}$
assembled equation	$\underset{\sim}{w}^T \underline{K}^* \delta \underset{\sim}{u} = \underset{\sim}{w}^T \underset{\sim}{f}_{e0} - \underset{\sim}{w}^T \underset{\sim}{f}^*_i = \underset{\sim}{w}^T \underset{\sim}{r}^* \qquad \forall \ \underset{\sim}{w}$
iterative equation system	$\underline{K}^* \delta \underset{\sim}{u} = \underset{\sim}{r}^*$

Boundary conditions

Boundary conditions are only applied at the beginning of an incremental step. Links – relations between degrees of freedom – can be incorporated as usual, but now of course for the iterative displacements.

Program structure

A finite element program starts with reading data from an input file and initialization of variables and databases.

The loading is prescribed as a function of the (fictitious) time in an incremental loop. In each increment the system of nonlinear equilibrium equations is solved iteratively.

In each iteration loop the system of equations is build. In a loop over all elements, the stresses are calculated and the material stiffness is updated. The element internal nodal force column and the element stiffness matrix are assembled into the global column and matrix.

After taking tyings and boundary conditions into account, the unknown nodal displacements and reaction forces are calculated.

When the convergence criterion is not reached, a new iteration step is performed. After convergence output data are stored and the next incremental step is carried out.

```
read input data from input file
calculate additional variables from input data
initialize values and arrays

while load increments to be done

   for all elements
      calculate initial element stiffness matrix
      assemble global stiffness matrix
   end element loop

   determine external incremental load from input
```

```
    while non-converged iteration step

       take tyings into account
       take boundary conditions into account

       calculate iterative nodal displacements
       calculate total deformation

       for all elements
          calculate stresses from material behavior
          calculate material stiffness from material behavior
          calculate element internal nodal forces
          calculate element stiffness matrix
          assemble global stiffness matrix
          assemble global internal load column
       end element loop

       calculate residual load column
       calculate convergence norm

    end iteration step

    store data for post-processing
 end load increment
```

2.6.1 FE program `tr2d`

The Matlab program **`tr2d`** is used to model and analyze two-dimensional truss structures, where large deformations and nonlinear material behavior may occur. The program is described in detail in appendix B.

In this section, examples of two-dimensional truss structures are shown. The material behavior is always elastic and described by a linear relation between the Cauchy stress and the linear strain. Other material models have also been implemented in the program, but this is the subject of the next chapter.

Large deformation of a truss structure

A structure is made of five trusses. The vertical truss is 0.5 m and the horizontal truss is 1 m in length. Cross-sectional areas are 100 mm^2. The modulus is 2.5 GPa. Contraction is not considered ($\nu = 0$). The vertical displacement of node 4 is prescribed to increase from 0 to -0.25 m. The reaction force, the horizontal displacement of node 4 and the vertical displacement of node 2 is plotted against the fictitious time t.

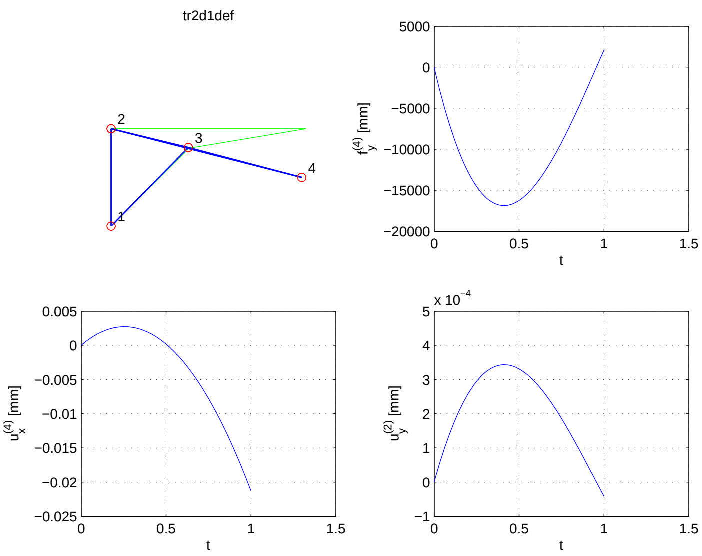


Fig. 2.28 : *Large deformation of a truss structure.*

Buckling

Large rotations occur when buckling leads to a sudden increase in deformation. The theoretical buckling load can be calculated analytically for a simple systems as shown here.

The numerical calculation starts with a very small imperfection being an initial vertical displacement of the inner node(s) of $\pm$ 0.0001 m. This allows us to reach not only the first and smallest buckled state, the symmetric shape, but also the second mode, the anti-symmetric shape. Also a larger imperfection is analized for both buckling modes.

The horizontal trusses have a high stiffness of $k_t = (EA)/l = (100e9)(100e-6)/1$ N/m, while the springs have a very low stiffness of $k = 1$ N/m. The displacement in node 4 is prescribed to increase from 0 to -0.02 m.

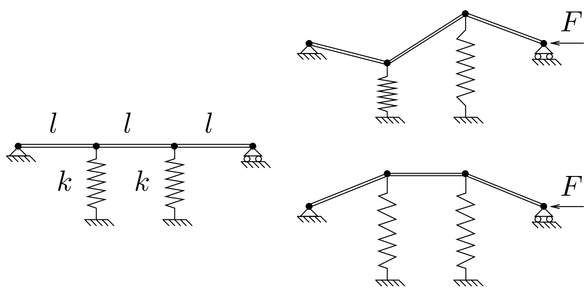


Fig. 2.29 : *Symmetric and anti-symmetric buckling.*

$$\text{symm} \; : \quad F_c = \frac{kl}{3} \qquad ; \qquad \text{anti-symm} \; : \quad F_c = kl$$

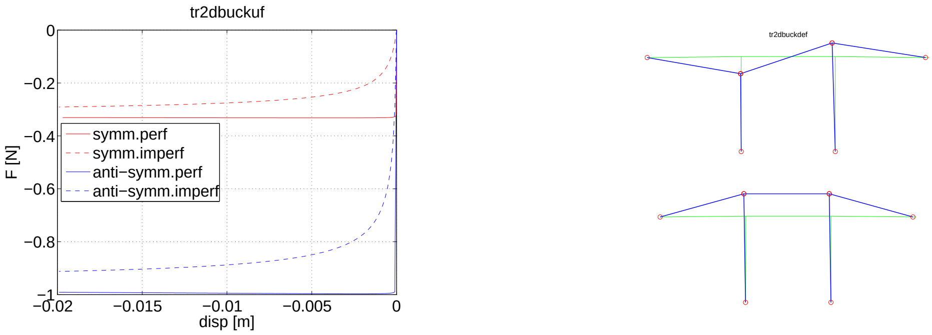


Fig. 2.30 : *Buckling forces versus displacement (left). Symmetric and anti-symmetric buckling shapes (right).*

Chapter 3

One-dimensional material models

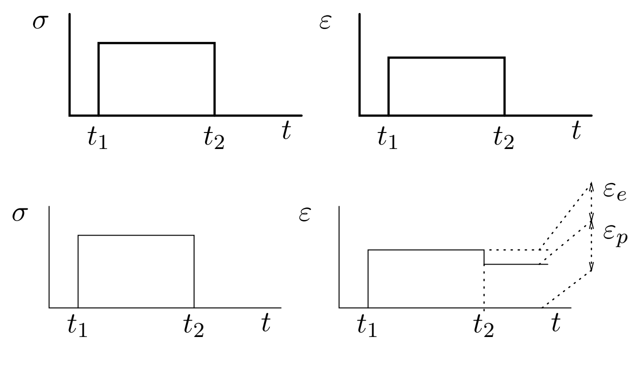
In the following sections material behavior is described in a one-dimensional context. and used to analyze the axial deformation of trusses. The material behavior is modeled, using a discrete mechanical model of springs, dashpots and friction sliders. The axial stress is related to the axial strain by one or more (differential) equation(s) from which the stress response must be calculated when the strain excitation is prescribed. This stress update procedure is implemented in Matlab files.

The various material models are incorporated in a finite element program, which is used to model and analyze the mechanical behavior of truss structures, subjected to prescribed displacements and/or forces. In the iterative solution procedure, the material stiffness plays an essential role and must be derived from the material law.

3.1 Material behavior

Characterization of the mechanical behavior of an unknown material almost always begins with performing a tensile experiment. A stepwise change in the axial stress σ may be prescribed and the strain ε of the tensile bar can be measured and plotted as a function of time. From these plots important conclusions can be drawn concerning the material behavior.

For *elastic* material behavior the strain follows the stress immediately and becomes zero after stress release. For *elastoplastic* material behavior the strain also follows the stress immediately, but there is permanent deformation after stress release. When the material is *viscoelastic* the strain shows time delayed response on a stress step, which indicates a time dependent behavior. When time dependent behavior is accompanied by permanent deformation, the behavior is referred to as *viscoplastic*.



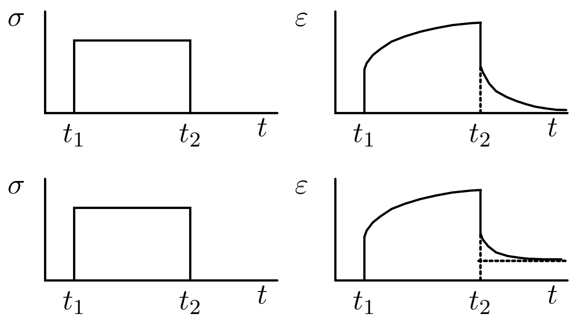


Fig. 3.1 : *Strain response for a stress-step for a) elastic b) elastoplastic, c) viscoelastic and d) viscoplastic material behavior*

Another way of representing the measurement data of the tensile experiment is by plotting the stress against the strain, resulting in the stress-strain curve. The relation between stress and strain may be *linear* or *nonlinear*. Also, the relation may be *history dependent*, due to changes in the material structure. Different behavior in tensile and compression may be observed.

Tensile curve : elastic behavior

When elastic behavior is well described by a linear relation between stress and strain, the elastic behavior is referred to as linear.

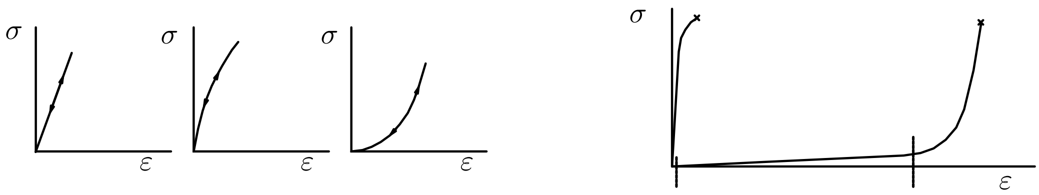


Fig. 3.2 : *Tensile curves for elastic material behavior*

Tensile curve : viscoelastic behavior

Viscoelastic behavior is time-dependent. The stress is a function of the strain rate. There is a phase difference between stress and strain, which results in a hysteresis loop when the loading is periodic.

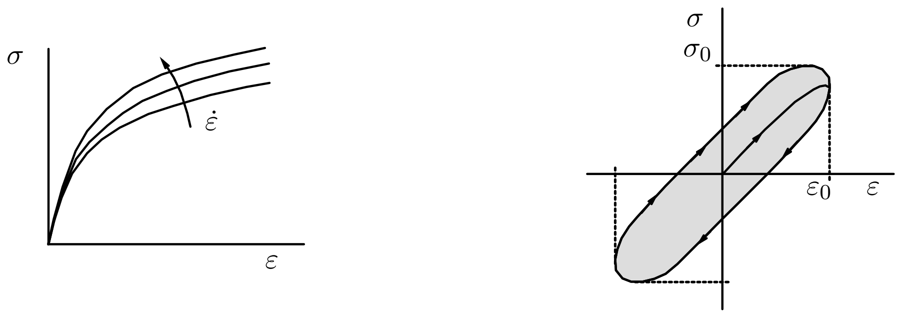


Fig. 3.3 : *Tensile curve and hysteresis loop for viscoelastic material behavior*

Tensile curve : elastoplastic behavior

When a material is loaded or deformed above a certain threshold, the resulting deformation will be permanent or plastic. When time (strain rate) is of no importance, the behavior is referred to as elastoplastic. Stress-strain curves may indicate different characteristics, especially when the loading is reversed from tensile to compressive.

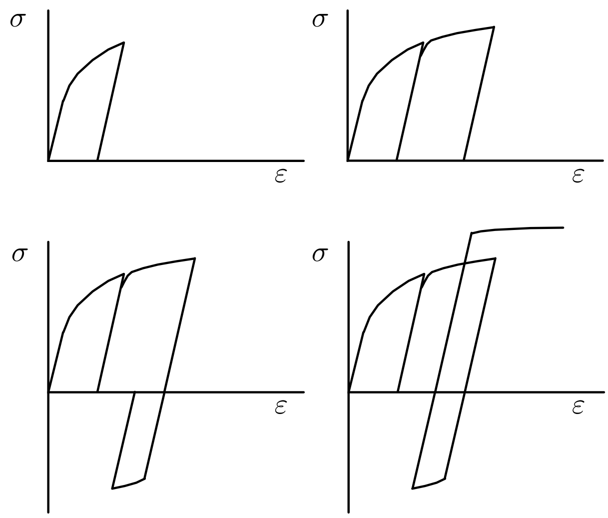


Fig. 3.4 : *Tensile curves for elastoplastic material behavior*

Tensile curve : viscoplastic behavior

A combination of plasticity and time-dependency is called viscoplastic behavior. This behavior is often observed for polymeric materials. For some polymers the stress reaches a maximum and subsequently drops with increasing strain. This phenomenon is referred to as *intrinsic softening.* In a tensile experiment it will provoke necking of the tensile bar.

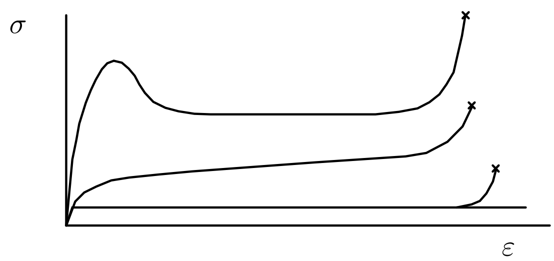


Fig. 3.5 : *Tensile curves for viscoplastic material behavior*

Tensile curve : damage

Structural damage influences the material properties. The onset and evolution of damage can be described with a damage model. For materials like concrete and ceramics, the onset and propagation of damage causes softening. Because damage is often associated with the initiation and growth of voids, the stress-strain curve is different for tensile and compressive loading.

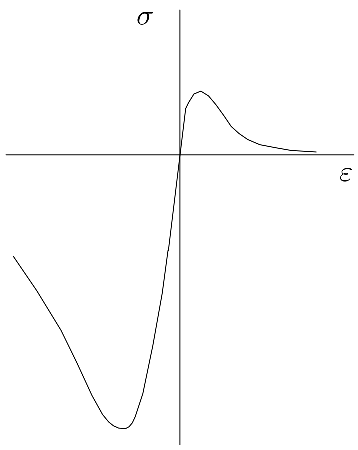
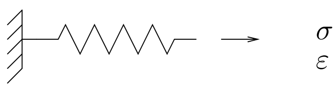


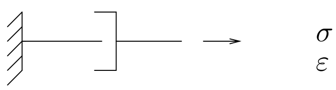
Fig. 3.6 : *Tensile curve for damaging material with different behavior in tension and compression*

Discrete material models

Material models relate stresses to deformation and possibly deformation rate. For three-dimensional continua the material model is often represented by a (large) number of coupled (differential) equations. As a simplified introduction, we will present material models first in a one-dimensional setting. The material behavior is represented by the behavior of a one-dimensional, discrete, mechanical system of springs, dashpots and friction sliders. For such a system the relation between the axial stress σ and the axial strain ε can be derived.

In the following sections models for elastic, elastoplastic, linear viscoelastic, creep, viscoplastic and nonlinear viscoelastic behavior will be presented.





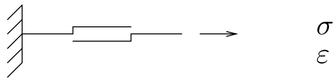


Fig. 3.7 : *Discrete elements : spring, dashpot and friction slider*

3.2 Elastic material behavior

When a material behaves elastically, the current stress can be calculated directly from the current strain, because there is no path and/or time dependency. When the stress is released, the strain will become zero, so there is no permanent deformation at zero stress. All stored strain energy is released and there is no energy dissipation. For the one-dimensional case of an axially loaded truss the elastic behavior is described by a relation between the stress σ and the elongation factor λ or the strain ε.

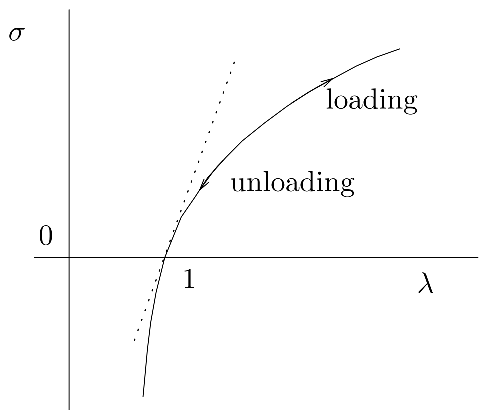


Fig. 3.8 : *Non-linear elastic material behavior*

Small strain elastic behavior

For small elongations, all strain definitions are the same, as are all stress definitions. The relation between stress and strain is linear and the constant material parameter is the Young's modulus.

strain $\qquad \varepsilon = \varepsilon_{gl} = \varepsilon_{ln} = \varepsilon_{l} = \lambda - 1$

stress $\qquad \sigma = \dfrac{F}{A} = \dfrac{F}{A_0} = \sigma_n$

linear elastic behavior $\qquad \sigma = E\varepsilon = E(\lambda - 1)$

modulus $\qquad E = \lim\limits_{\lambda \longrightarrow 1} \dfrac{d\sigma}{d\lambda} = \lim\limits_{\varepsilon \longrightarrow 0} \dfrac{d\sigma}{d\varepsilon}$

Large strain elastic behavior

For large deformations, nonlinear elastic behavior can be observed in polymers, elastomeric materials (rubbers) and, on a small scale, in atomic bonds, when a tensile/compression test is carried out and the axial force F is plotted as a function of λ. In a material model we want to describe such behavior with a mathematical relation between a stress and a strain. Consideration of the stored elastic energy per unit of volume learns that each stress definition is associated with a certain strain definition, so these should be combined in a material model. However, when the observed material behavior is described accurately by another stress/strain combination, it can be used as well.

For three-dimensional models more considerations have to be taken into account. Care has to be taken that the material model does not generate stresses for large rigid body rotations of the material, which is known as the requirement of *objectivity*.

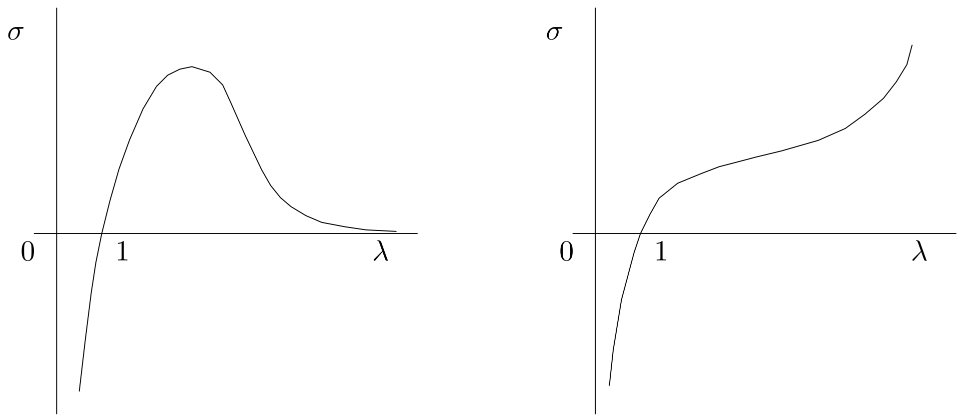


Fig. 3.9 : *Non-linear stress-strain relations for an atomic bond and for an elastomeric material*

Elasticity models

The discrete one-dimensional model for elastic material behavior is a spring. The behavior is modeled with a relation between the stress σ and the elongation factor λ or a strain ε. The material stiffness C_λ is the derivative of σ w.r.t. the stretch ratio λ. The derivative w.r.t. the strain ε results in the stiffness C_ε.

Consideration of the stored elastic energy per unit of material volume (see 2.4.2) learns that, in a material model, true stress σ should be combined with logarithmic strain ε_{ln}, engineering stress σ_n with linear strain ε_l or 2nd-Piola-Kirchhoff stress σ_{p2} with Green-Lagrange strain ε_{gl}. Experimentally observed tensile behavior can often be described with a linear relation between a certain stress and its associated strain.

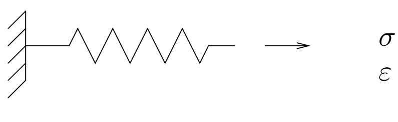


Fig. 3.10 : *Spring*

constitutive equation $\qquad \sigma = \sigma(\lambda)$

stiffness $\qquad C_\lambda = \dfrac{d\sigma}{d\lambda} = \dfrac{d\sigma}{d\varepsilon}\dfrac{d\varepsilon}{d\lambda} = C_\varepsilon \dfrac{d\varepsilon}{d\lambda}$

3.3 Hyper-elastic models

Elastomeric materials (rubbers) show very large elastic deformations (elongation up to $\lambda = 5$). The material models for these materials are therefore referred to as *hyper-elastic*. They are derived from an elastic energy function, which has to be determined experimentally. The three-dimensional versions of these so-called Rivlin or Mooney models are expressed in the principal elongation factors $\lambda_i, i = 1, 2, 3$. Experimental observations indicate that elastomeric materials are incompressible, so that we have $\lambda_1\lambda_2\lambda_3 = 1$.

$$W = \sum_i^n \sum_j^m C_{ij} \left(I_1 - 3\right)^i \left(I_2 - 3\right)^j \qquad \text{with} \quad C_{00} = 0$$
$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 = \frac{1}{\lambda_3^2} + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}$$

The incremental change of the elastically stored energy per unit of deformed volume, can be expressed in the principal stresses and the principal logarithmic strains.

$$dW = \sigma_1 d\varepsilon_{ln_1} + \sigma_2 d\varepsilon_{ln_2} + \sigma_3 d\varepsilon_{ln_3}$$

Mooney models

For incompressible materials like elastomer's (rubber) the stored elastic energy per unit of deformed volume is specified and fitted onto experimental data. Several specific energy functions are used.

Neo-Hookean $\quad W = C_{10}\left(I_1 - 3\right)$

Mooney-Rivlin $\quad W = C_{10}\left(I_1 - 3\right) + C_{01}\left(I_2 - 3\right)$

Signiorini $\quad W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2$

Yeoh $\quad W = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$

Klosner-Segal $\quad W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{03}(I_2 - 3)^3$

2-order invariant $\quad W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{20}(I_1 - 3)^2$

Third-order model of James, Green and Simpson

$$\begin{aligned} W = {} & C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + \\ & C_{20}(I_1 - 3)^2 + C_{02}(I_2 - 3)^2 + C_{21}(I_1 - 3)^2(I_2 - 3) + \\ & C_{30}(I_1 - 3)^3 + C_{03}(I_2 - 3)^3 + C_{12}(I_1 - 3)(I_2 - 3)^2 \end{aligned}$$

Ogden models

For 'slightly' compressible materials the Ogden specific energy functions are used. Because the volume change is not zero, these functions depend on the volume change factor $J = \lambda_1\lambda_2\lambda_3$. The second part of the energy function accounts for the volumetric deformation. Because the volumetric behavior is characterized by a constant bulk modulus K, the model is confined to slightly compressible deformation.

For 'highly' compressible materials like foams, specific energy functions also exist. The first part of the energy function also describes volume change.

$$\text{slightly compressible} \qquad W = \sum_{i=1}^{N} \frac{a_i}{b_i}\left[J^{-\frac{b_i}{3}}\left(\lambda_1^{b_i}+\lambda_2^{b_i}+\lambda_3^{b_i}\right)-3\right] + 4.5K\left(J^{\frac{1}{3}}-1\right)^2$$

$$\text{highly compressible} \qquad W = \sum_{i=1}^{N} \frac{a_i}{b_i}\left(\lambda_1^{b_i}+\lambda_2^{b_i}+\lambda_3^{b_i}-3\right) + \sum_{i=1}^{N} \frac{a_i}{c_i}\left(1-J^{c_i}\right)$$

One-dimensional models

For tensile (or compressive) loading of a homogeneous and isotropic truss, where the axial direction is taken to be the 1-direction, we have : $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$. In this case there is only an axial stress $\sigma_1 = \sigma$, so that we have

$$dW = \sigma\, d\varepsilon_{ln} \quad \longrightarrow \quad \sigma = \frac{dW}{d\varepsilon_{ln}} = \frac{dW}{d\lambda}\frac{d\lambda}{d\varepsilon_{ln}} = \frac{dW}{d\lambda}\lambda$$

The Neo-Hookean model is the simplest model as it contains only one material parameter. Axial stress σ and axial force F can be calculated easily. From statistical mechanics it is known that for an ideal rubber material the stress is :

$$\sigma = \frac{\rho R T}{M}\left(\lambda^2 - \frac{1}{\lambda}\right) \qquad \text{with} \qquad \begin{array}{lll} \rho & : & \text{density} \\ R & : & \text{gas constant} = 8.314\ \text{JK}^{-1}\text{mol}^{-1} \\ T & : & \text{absolute temperature} \\ M & : & \text{average molecular weight} \end{array}$$

Most rubber materials cannot be characterized well with the Neo-Hookean model. The more complex Mooney-Rivlin model yields better results. The stiffness C_λ is a function of the elongation factor λ. The initial stiffness E is often referred to as the modulus.

Neo – Hookean

$$\begin{aligned} W &= C_{10}\left(\lambda^2 + \tfrac{2}{\lambda} - 3\right) \\ \sigma &= C_{10}\left(2\lambda - \frac{2}{\lambda^2}\right)\lambda = 2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) \\ C_\lambda &= \frac{\partial\sigma}{\partial\lambda} = 2C_{10}\left(2\lambda + \frac{1}{\lambda^2}\right) \quad ; \quad E = \lim_{\lambda\to 1}\frac{\partial\sigma}{\partial\lambda} = 6C_{10} \\ F &= \sigma A = \sigma\frac{1}{\lambda}A_0 = 2C_{10}A_0\left(\lambda - \frac{1}{\lambda^2}\right) \end{aligned}$$

Mooney – Rivlin

$$
\begin{aligned}
W \;&= C_{10}\left(\lambda^2+\frac{2}{\lambda}-3\right)+C_{01}\left(\frac{1}{\lambda^2}+2\lambda-3\right)\\
\sigma \;&= 2C_{10}\left(\lambda^2-\frac{1}{\lambda}\right)+2C_{01}\left(\lambda^2-\frac{1}{\lambda}\right)\frac{1}{\lambda}\\
C_\lambda &= \frac{\partial\sigma}{\partial\lambda}=2C_{10}\left(2\lambda+\frac{1}{\lambda^2}\right)+2C_{01}\left(1+\frac{2}{\lambda^3}\right) \quad ; \quad E=\lim_{\lambda\to 1}\frac{\partial\sigma}{\partial\lambda}=6(C_{10}+C_{01})\\
F \;&= \sigma A=\sigma\frac{1}{\lambda}A_0=A_0\frac{1}{\lambda}\left[2C_{10}\left(\lambda^2-\frac{1}{\lambda}\right)+2C_{01}\left(\lambda^2-\frac{1}{\lambda}\right)\frac{1}{\lambda}\right]
\end{aligned}
$$

3.3.1 Stress update

The relation between σ and λ can be used to update the stress directly when the strain is known.

$$\sigma=\sigma(\lambda)$$

3.3.2 Stiffness

The material stiffness is determined by taking the derivative of the stress with respect to the elongation ratio or the strain.

$$C_\lambda=\frac{\partial\sigma}{\partial\lambda}$$

3.3.3 Implementation

See the files `tr2delas.m` and `tr2delam.m` for the implementation of the elastic and elastomeric material models.

3.3.4 Examples

A truss is loaded axially with a prescribed elongation/force. The initial length l_0 of the truss is 100 mm and the initial cross-sectional area A_0 is 10 mm^2. The axial force/elongation is calculated. The cross-sectional area will change as a function of the elongation.

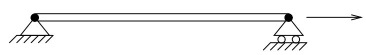


Fig. 3.11 : *Tensile loading of truss element*

For all elastic models the elastic constant is taken $C = 100000$ MPa and Poisson's ratio ν is 0.3. The stress-elongation results are shown in the next figures. The models with a linear relation between stress (σ or P) and Green-Lagrange strain, clearly lack a physically realistic description of the material behavior during compression.

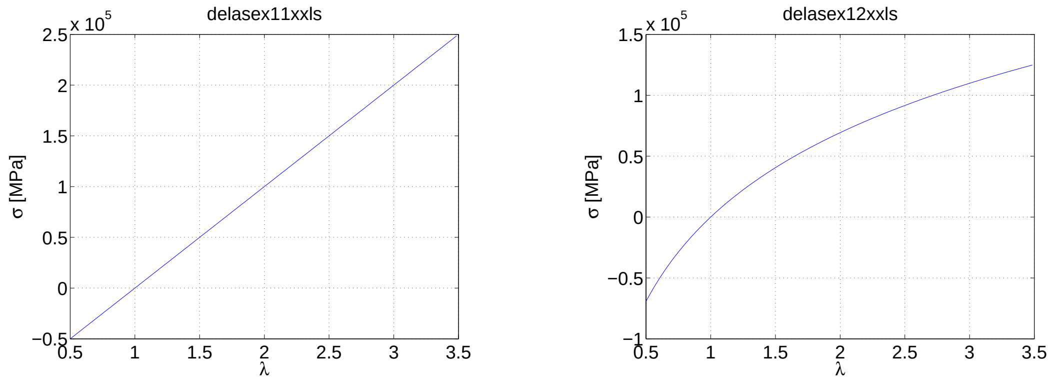


Fig. 3.12 : *Stress versus* λ *for* $\sigma \sim \varepsilon_l$ *and* $\sigma \sim \varepsilon_{ln}$ *models*

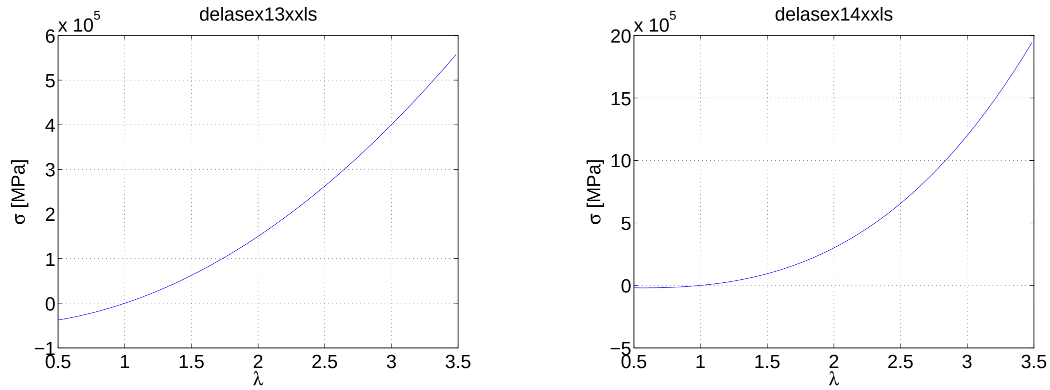


Fig. 3.13 : *Stress versus* λ *for* $\sigma \sim \varepsilon_{gl}$ *and* $P \sim \varepsilon_{gl}$ *models*

The axial force and the cross-sectional area are calculated and shown in the next figures as a function of the elongantion. The cross-sectional areas of some models become zero and even negative, which clearly shows the limited use of these models.

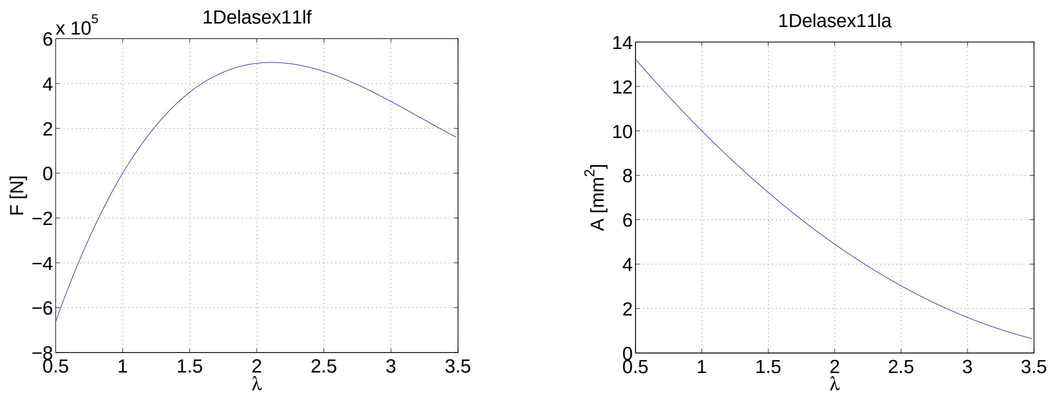


Fig. 3.14 : *Axial force and cross-sectional area versus the elongation for* $\sigma \sim \varepsilon_l$ *model*

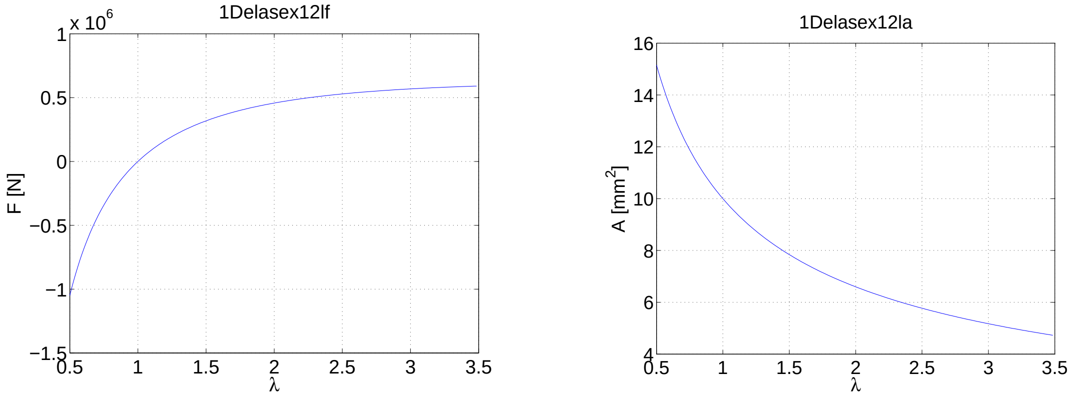


Fig. 3.15 : *Axial force and cross-sectional area versus the elongation for* $\sigma \sim \varepsilon_{ln}$ *model*

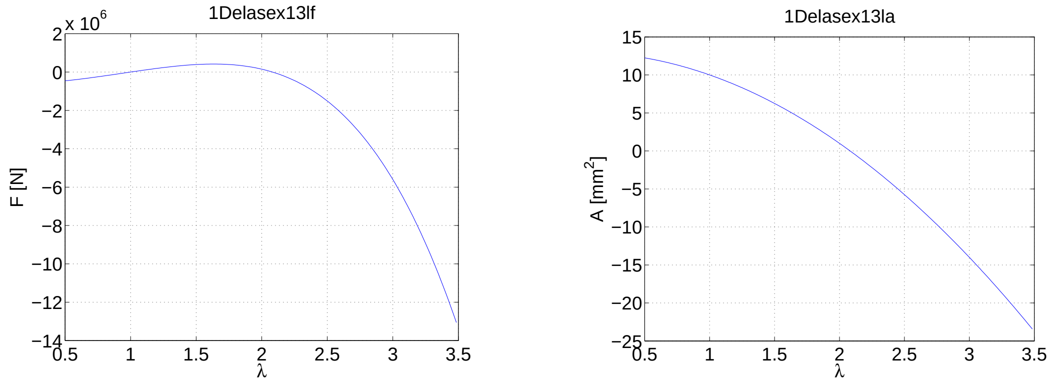


Fig. 3.16 : *Axial force and cross-sectional area versus the elongation for* $\sigma \sim \varepsilon_{gl}$ *model*

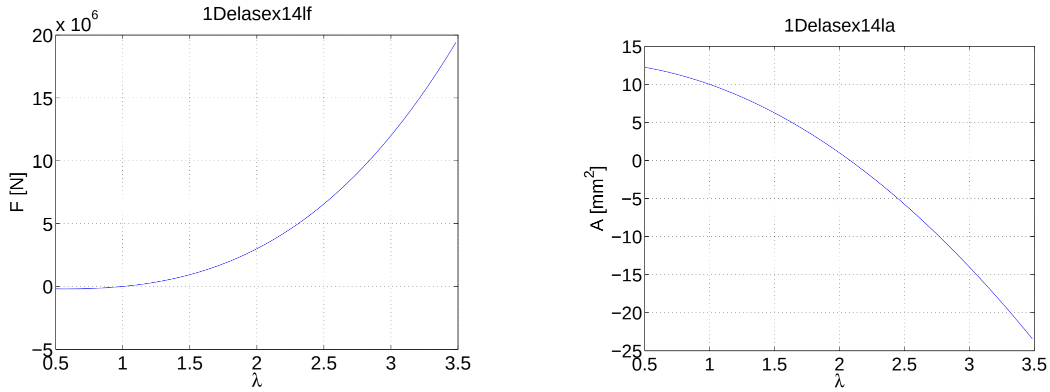


Fig. 3.17 : *Axial force and cross-sectional area versus the elongation for* $P \sim \varepsilon_{gl}$ *model*

For the elastomeric Neo-Hookean and Mooney-Rivlin models the material constants are : $C_{10} = 20000$ MPa and $C_{01} = 20000$ MPa. Stress versus elongation is shown.

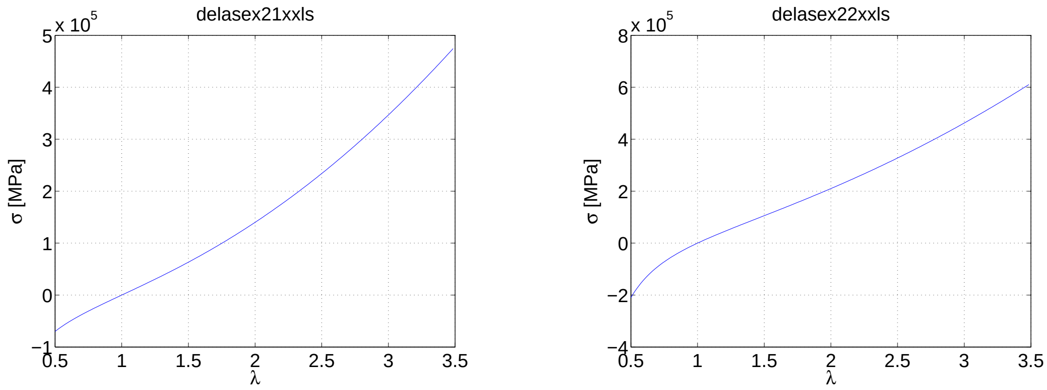


Fig. 3.18 : *Stress versus* λ *for Neo-Hookean and Mooney-Rivlin models*

The axial force is calculated for a prescribed axial elongation.

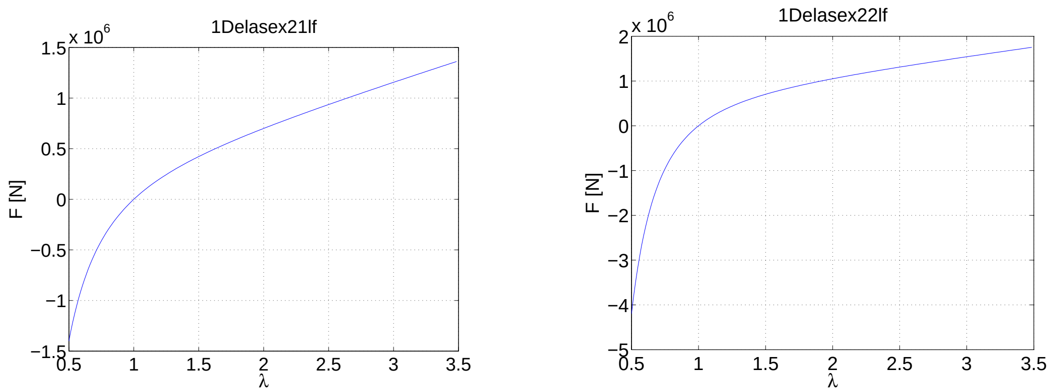


Fig. 3.19 : *Axial force and cross-sectional area versus the elongation for Neo-Hookean and Mooney-Rivlin model*

3.4 Elastoplastic material behavior

Below a certain load (stress) value, the deformation of all materials will be elastic. When the stress exceeds a limit value, plastic deformation occurs, which means that permanent elongation is observed after release of the load. At increased loading above the limit value, the stress generally increases with increasing elongation, a phenomenon referred to as *hardening*.

Reversed loading will first result in elastic deformation, but after reaching a limit value of the stress, plastic deformation will be observed again. Looking at the stress-strain curve after a few loading reversals, it can be seen that elastoplastic material behavior is history dependent: the stress is not uniquely related to the strain; its value depends on the deformation history. The total stress-strain history must be taken into account to determine the current stress.

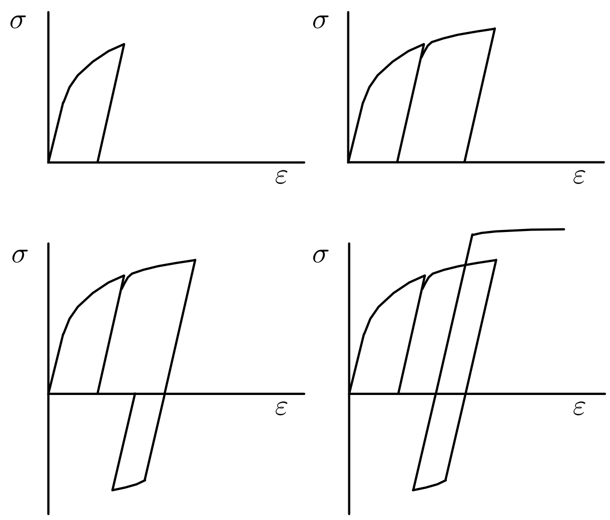


Fig. 3.20 : *Stress-strain curves for elastoplastic material behavior*

Tensile test

When a tensile bar, with undeformed length l_0 and cross-sectional area A_0, is subjected to a tensile test, the axial force F and the length l can be measured. The axial strain ε can be calculated from the elongation factor λ. To calculate the true stress $\sigma = \frac{F}{A}$, the actual cross-sectional area of the tensile bar must be measured during the experiment. The nominal stress $\sigma_n = \frac{F}{A_0}$ can be calculated straightforwardly. The nominal stress σ_n can be plotted against the linear strain $\varepsilon_l = \lambda - 1 = \frac{l-l_0}{l_0} = \frac{\Delta l}{l_0}$ resulting in the $\sigma_n - \varepsilon_l$ stress-strain curve.

Until the proportionality limit $\sigma_n = \sigma_P$ is reached, the material behavior is assumed to be linear elastic : $\sigma_n = E\varepsilon$, where E is Young's modulus. When the stress exceeds the initial yield stress $\sigma_{y0} > \sigma_P$, unloading will reveal permanent (= plastic) deformation of the bar. The exact value of σ_{y0} cannot be determined so in practice σ_{y0} is taken to be the stress where a plastic strain of 0.2 % remains. In the following however, we will assume that σ_{y0} is exactly known and that $\sigma_{y0} = \sigma_P$.

The axial force and therefore the nominal stress will reach a maximum value. At that point necking of the tensile bar will be observed. The maximum nominal stress is the tensile strength σ_T. In forming processes strains can be much higher than in a tensile test, because of the compression in certain directions.

After reaching the tensile strength the nominal stress will decrease while the strain is still increasing. Fracture occurs at the fracture stress $\sigma_n = \sigma_F$. The fracture strain ε_F is for metals and metal alloys about $10\% = 0.1$. This is a rather small elongation which means that for these materials we can assume $\sigma = \sigma_n$ and also that all strain definitions are approximately equivalent, so $\varepsilon = \varepsilon_l$.

Experiments have shown that during plastic deformation the volume of metals and metal alloys remains constant : plastic deformation is taken to be incompressible.

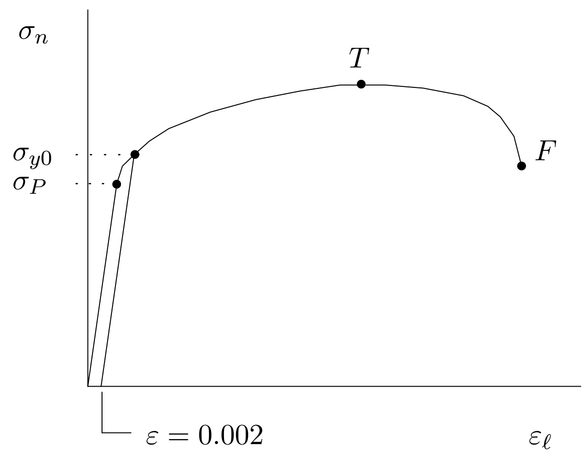


Fig. 3.21 : *Stress-strain curve during tensile test*

Compression test

For metal alloys a compression test instead of a tensile test will reveal that first yield will occur at $\sigma = \sigma_n = -\sigma_{y0}$. The initial material behavior is the same in tension and compression. In general terms the transition from purely elastic behavior to elastoplastic behavior is determined by a yield criterion. For the one-dimensional case this criterion says that first yielding will occur when :

$$f = \sigma^2 - \sigma_{y0}^2 = 0$$

The function f is the yield function.

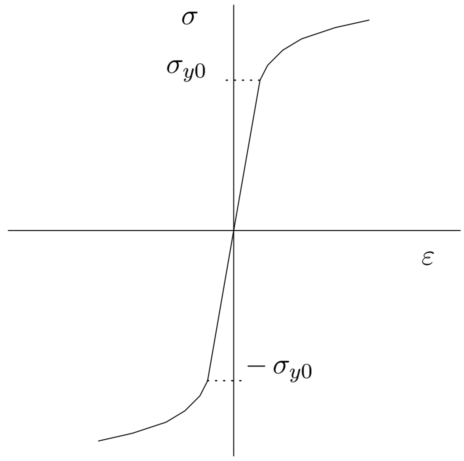


Fig. 3.22 : *Stress-strain curve during tensile or compression test*

Interrupted tensile test

When the axial load is released at σ_A (see figure below) with $\sigma_{y0} < \sigma_A < \sigma_T$, the unloading stress-strain path is elastic and characterized by the initial Young's modulus E. The permanent or plastic elongation is represented by the plastic strain ε_p. The difference between the total strain in point A and the plastic strain is the elastic strain $\varepsilon_e = \varepsilon_A - \varepsilon_p = \frac{\sigma_A}{E}$.

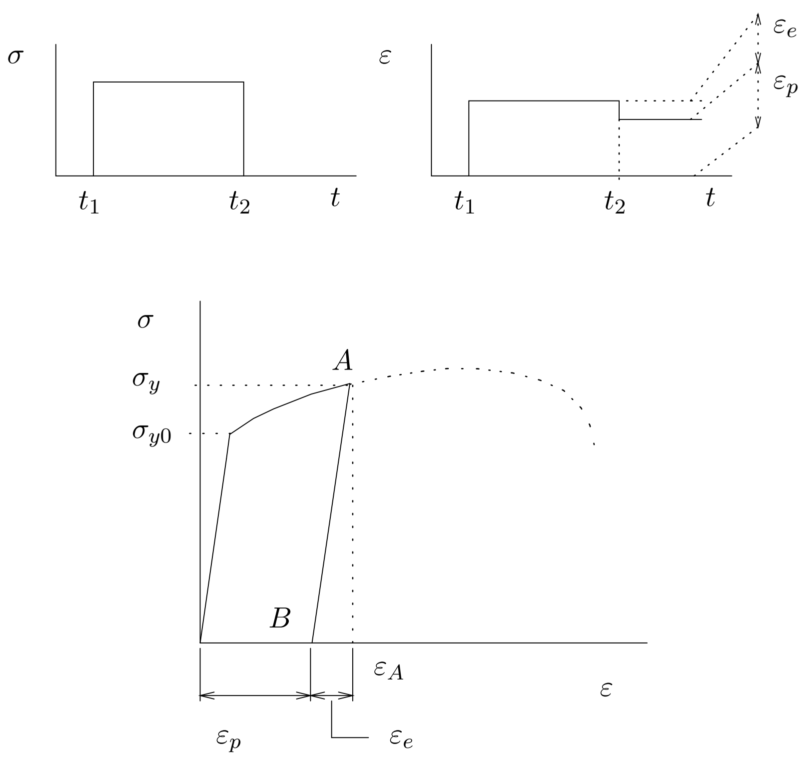


Fig. 3.23 : *Stress-strain curve after interrupted tensile test*

Resumed tensile test

When after unloading, the bar is again loaded with a tensile force, the elastic line BA will be followed where $\Delta\sigma = E\Delta\varepsilon = E\Delta\varepsilon_e$ holds. For $\sigma \geq \sigma_A(\varepsilon \geq \varepsilon_A)$ further elastoplastic deformation takes place and the stress-strain curve will be followed as if unloading were not occurred.

The stress σ_A is the *current yield stress* σ_y, which is generally larger then the initial yield stress σ_{y0}. The increase, referred to as *hardening*, is related to the plastic strain by a hardening law.

Hardening

To study the hardening phenomenon, the tensile bar is not reloaded in tension but in compression. Two extreme observations may be made, illustrated in the figure below.

In the first case the elastic trajectory increases in length due to plastic deformation : $AA' > Y_0Y_0'$. The elastic trajectory is symmetric about $\sigma = 0$ ($BA = BA'$). What we observe is *isotropic hardening.*

In the second case the elastic trajectory remains of constant length : $AA' = Y_0Y_0'$. It is symmetric about the line OC ($CA = CA'$). After unloading the yield stress under compression is different than the yield stress under tension. This is called *kinematic hardening.* The stress in point C, the center of the elastic trajectory, is the *shift stress* $\sigma = q$. This phenomenon is also referred to as the *Bauschinger effect.*

Real materials will show a combination of isotropic and kinematic hardening.

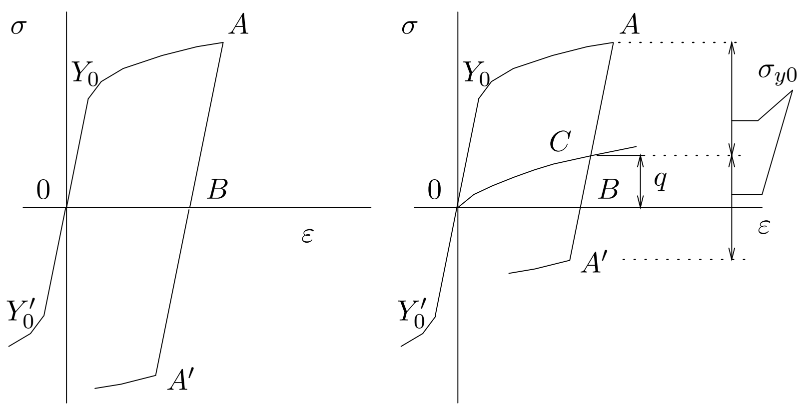


Fig. 3.24 : *Isotropic and kinematic hardening*

isotropic hardening : elastic area larger & symmetric w.r.t. $\sigma = 0$

$$\left.\begin{array}{lll} \text{tensile} & : & \sigma = \sigma_y \\ \text{compression} & : & \sigma = -\sigma_y \end{array}\right\} \quad \rightarrow \quad f = \sigma^2 - \sigma_y^2 = 0$$

kinematic hardening : elastic area constant & symmetric w.r.t. $\sigma = q$

$$\left.\begin{array}{lll} \text{tensile} & : & \sigma = q + \sigma_{y0} \\ \text{compression} & : & \sigma = q - \sigma_{y0} \end{array}\right\} \quad \rightarrow \quad f = (\sigma - q)^2 - \sigma_{y0}^2 = 0$$

combined isotropic/kinematic hardening

$$\left.\begin{array}{lll} \text{tensile} & : & \sigma = q + \sigma_y \\ \text{compression} & : & \sigma = q - \sigma_y \end{array}\right\} \quad \rightarrow \quad f = (\sigma - q)^2 - \sigma_y^2 = 0$$

Effective plastic strain

Isotropic hardening could be described by relating the yield stress σ_y to the plastic strain ε_p. However, as the figure below shows, this would lead to the unrealistic conclusion that the yield stress increases while the plastic strain decreases. To prevent this problem, the *effective plastic strain* $\bar{\varepsilon}_p$ is taken as the history parameter. It is a measure of the total plastic strain, be its change positive or negative, and as such cannot decrease.

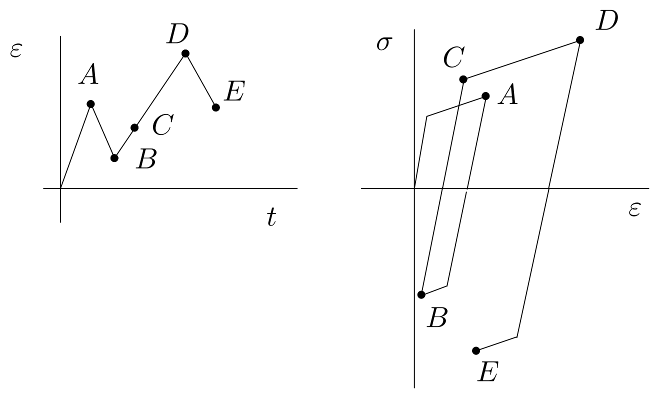


Fig. 3.25 : *Increasing yield stress at decreasing plastic strain*

$$\bar{\varepsilon}_p = \sum_{\varepsilon} |\Delta\varepsilon_p| = \sum_{\tau=0}^{\tau=t} \frac{|\Delta\varepsilon_p|}{\Delta t}\,\Delta t = \int_{\tau=0}^{t} |\dot{\varepsilon}_p|\,d\tau = \int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p\,d\tau$$

3.4.1 Hardening laws

For one-dimensional stress states encountered in the axial loading of a truss, several hardening laws are formulated, based on experimental observations. They can be generalized to three-dimensional stress-strain states. For isotropic hardening the current yield stress is related to the effective plastic strain and the initial yield stress. The isotropic hardening parameter is $H = \frac{d\sigma_y}{d\bar{\varepsilon}_p}$. For kinematic hardening the shift stress q is related to the plastic strain ε_p. The kinematic hardening parameter is $K = \frac{dq}{d\varepsilon_p}$.

Linear and power law hardening laws

Many hardening laws represent a linear or exponential relationship between stress and strain.

linear hardening $\qquad \sigma_y = \sigma_{y0} + H\bar{\varepsilon}_p$

Ludwik (1909) $\quad \sigma_y = \sigma_{y0} + \sigma_{y0}\left(\dfrac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^n \quad (0 \leq n \leq 1) \quad \rightarrow$

$$H = n\frac{\sigma_{y0}}{\varepsilon_{y0}}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^{n-1} = nE\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^{n-1}$$

mod. Ludwik $\quad \sigma_y = \sigma_{y0}\left(1 + m\bar{\varepsilon}_p^n\right) \quad \rightarrow \quad H = \sigma_{y0} m n \bar{\varepsilon}_p^{n-1}$

Swift (1952) $\quad \sigma_y = C(m + \bar{\varepsilon}_p)^n \quad \text{with} \quad C = \dfrac{\sigma_{y0}}{m^n}$

$$H = Cn\,(m + \bar{\varepsilon}_p)^{n-1}$$

Ramberg-Osgood (1943) $\quad \bar{\varepsilon}_p = \dfrac{\sigma_y}{E}\left[1 + \alpha\left(\dfrac{\sigma_y}{\sigma_{y0}}\right)^{m-1}\right] \quad (m \geq 0; \alpha \approx \frac{3}{7})$

Asymptotically perfect hardening laws

Some hardening laws are formulated in such a way as to result in no hardening (ideal plastic behavior) for large strain values.

ideal plastic $\quad \sigma_y = \sigma_{y0}$

Prager (1938) $\quad \sigma_y = \sigma_{y0} \tanh\left(\dfrac{E\bar{\varepsilon}_p}{\sigma_{y0}}\right)$

$$H = \frac{\sigma_{y0}}{\varepsilon_{y0}}\left[\operatorname{sech}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)\right]^2 = E\left[\operatorname{sech}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)\right]^2$$

Betten I (1975) $\quad \sigma_y = \sigma_{y0}\left[\tanh\left(\dfrac{E\bar{\varepsilon}_p}{\sigma_{y0}}\right)^m\right]^{1/m} \quad (m > 1)$

$$H = E\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^{m-1}\left[\tanh\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right]^{\frac{1}{m}-1}\left[\operatorname{sech}\left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right]^2$$

Voce (1949) $\quad \sigma_y = C\left(1 - ne^{-m\bar{\varepsilon}_p}\right) \quad \text{with} \quad C = \dfrac{\sigma_{y0}}{1-n} \quad (m > 1)$

$$H = Cnme^{-m\bar{\varepsilon}_p}$$

Betten II (1975) $\quad \sigma_y = \sigma_{y0} + (E\,\bar{\varepsilon}_p)\left[1 + \left(\dfrac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right]^{-1/m}$

$$H = E\left[1 + \left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right]^{-\frac{1}{m}}\left[1 - \left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\left\{1 + \left(\frac{\bar{\varepsilon}_p}{\varepsilon_{y0}}\right)^m\right\}^{-1}\right]$$

Cyclic load

A truss can be loaded with a prescribed strain $-\varepsilon_m \leq \varepsilon \leq \varepsilon_m$. It is assumed that the stress will reach values above the initial yield stress σ_{y0} and that linear hardening occurs.

For purely isotropic hardening the stress will increase after each load reversal and finally no further plastic deformation will take place.

For purely kinematic hardening the stress-strain path will be one single hysteresis loop, where the stress cycles, as does the strain, between two constant values $-\sigma_m \leq \sigma \leq \sigma_m$.

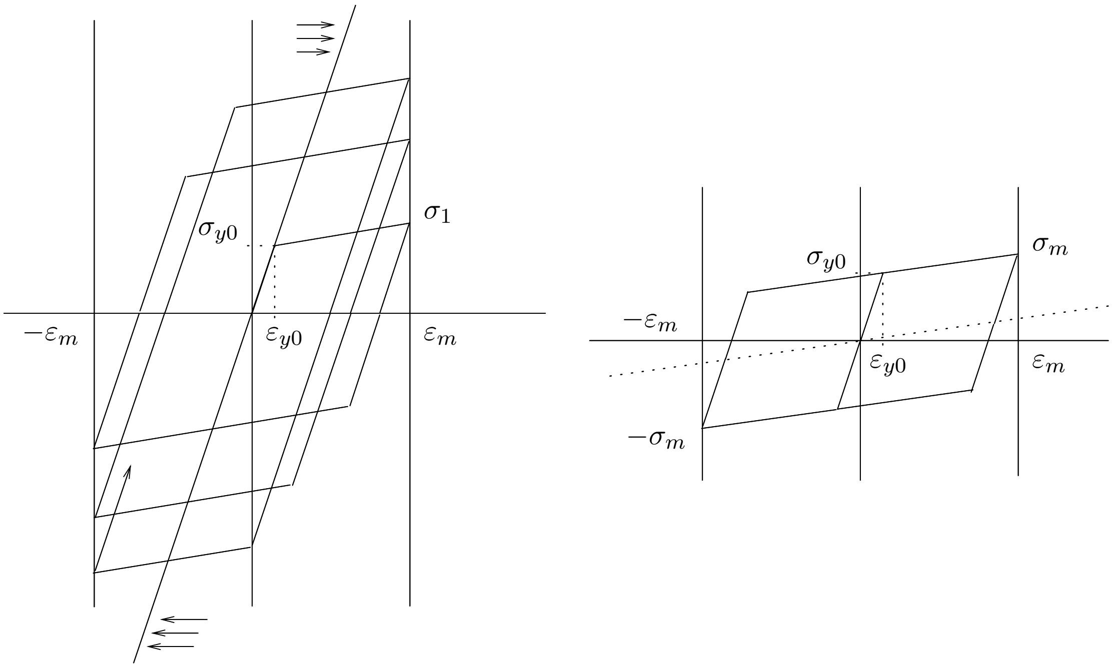


Fig. 3.26 : *Stress-strain curve during cyclic loading for isotropic and for kinematic hardening*

3.4.2 Elastoplastic model

The elastoplastic deformation characteristics can be represented by a discrete mechanical model. A friction element represents the yield limit and a hardening spring – stiffness H ($H > 0$) – provides the stiffness reduction after reaching the yield limit. The elastoplastic model describes rate-**in**dependent plasticity – there is no dashpot in the discrete model –, so the time is fictitious and "rate" is just referring to momentary change.

The yield criterion is used to decide at which stress level a purely elastic deformation will be followed by elastoplastic deformation. During elastoplastic deformation the total strain rate ($\dot{\varepsilon}$) is additively decomposed in an elastic ($\dot{\varepsilon}_e$) and a plastic ($\dot{\varepsilon}_p$) part. The plastic strain rate $\dot{\varepsilon}_p$ is related to $\frac{\partial f}{\partial \sigma}$ by the rate of the *plastic multiplier* λ, the so-called *consistency parameter* $\dot{\lambda}$. During ongoing plastic deformation the consistency equation $\dot{f} = 0$ must be satisfied, because f must remain zero.

The hardening law relates the current yield stress σ_y to the initial yield stress σ_{y0} and the effective plastic strain $\bar{\varepsilon}_p$. The shift stress q is related to the plastic strain ε_p.

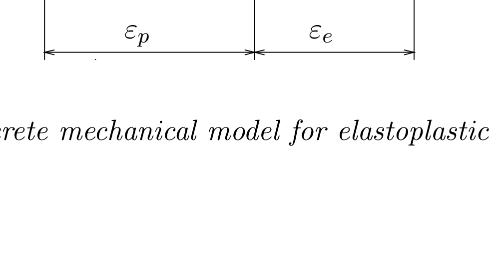


Fig. 3.27 : *Discrete mechanical model for elastoplastic material behavior*

- $f = (\sigma - q)^2 - \sigma_y^2$ with $f < 0 \quad | \quad f = 0 \wedge \dot{f} < 0 \rightarrow$ elastic; $f = 0 \wedge \dot{f} = 0 \rightarrow$ elastoplastic
- $\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p) \quad ; \quad q = q(\varepsilon_p)$
- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p$
- $\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\dot{\sigma}$
- $\dot{\varepsilon}_p = \dot{\lambda}\dfrac{\partial f}{\partial \sigma} = 2\dot{\lambda}(\sigma - q) \quad ; \quad \dot{\bar{\varepsilon}}_p = |\dot{\varepsilon}_p| = 2\dot{\lambda}\,|\sigma - q|$
- $\bar{\varepsilon}_p = \displaystyle\int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p \, d\tau = \sum_t |\Delta\varepsilon_p|$

Constitutive equations

From the constitutive relations a set of constitutive equations can be derived.

$$\left.\begin{array}{l} \dot{\sigma} = E\dot{\varepsilon}_e = E(\dot{\varepsilon} - \dot{\varepsilon}_p) = E\{\dot{\varepsilon} - 2\dot{\lambda}(\sigma - q)\} \\ f = 0 \end{array}\right\} \rightarrow$$

$$\left.\begin{array}{l} \dot{\sigma} + 2E(\sigma - q)\dot{\lambda} - E\dot{\varepsilon} = 0 \\ f = 0 \end{array}\right\}$$

The current stress has to be determined from these constitutive equations. The first one is a differential equation in pseudo-time. To solve it we use the incremental approach, where the total time is discretized and where we assume to have reached a solution for the begin-increment time t_n, i.e. values at the beginning if the current increment are known.

Although very general solution procedures can be used, we first consider a specisl case. It is assumed that the stress state at both the begin-increment time and the end-increment time is on the yield trajectory. Also linear hardening is considered, first isotropic then kinematic.

Linear isotropic hardening

For linear isotropic hardening with constant hardening parameter H, the stress increment can be derived straightforwardly. In a point of the elastic trajectory we know that $\Delta\sigma = E\Delta\varepsilon$ holds. In a point of the elastoplastic trajectory we can write $\Delta\sigma = S\Delta\varepsilon$, where the material stiffness $S = C_\varepsilon$ will depend on E and H.

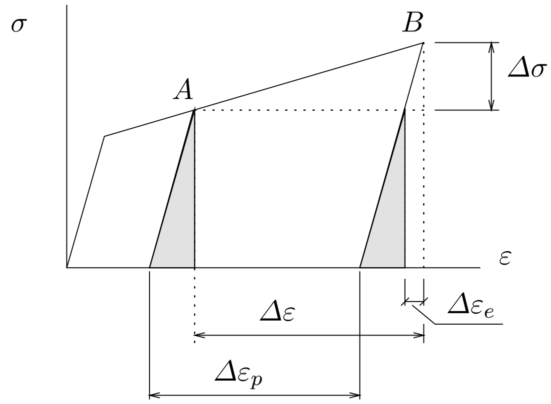


Fig. 3.28 : *Stress-strain curve for monotonic tensile loading*

$$\Delta\sigma = E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) = E\left(\Delta\varepsilon - \frac{\Delta\sigma_y}{H}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{H}\right) \quad \rightarrow$$

$$\Delta\sigma = \frac{EH}{E+H}\,\Delta\varepsilon = S\Delta\varepsilon \qquad ; \qquad \Delta\varepsilon_p = \frac{\Delta\sigma}{H} = \frac{E}{E+H}\,\Delta\varepsilon$$

Kinematic hardening

For linear kinematic hardening, the result is similarly derived.

$$\Delta\sigma = \frac{EK}{E+K}\,\Delta\varepsilon \quad ; \quad \Delta\varepsilon_p = \frac{1}{K}\,\Delta\sigma = \frac{E}{E+K}\,\Delta\varepsilon$$

Again these relations can be derived straightforwardly from the figure.

$$\Delta\sigma = E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) = E\left(\Delta\varepsilon - \frac{\Delta q}{K}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{K}\right) \quad \rightarrow$$

$$\Delta\sigma = \frac{EK}{E+K}\,\Delta\varepsilon = S\Delta\varepsilon \qquad ; \qquad \Delta\varepsilon_p = \frac{\Delta\sigma}{K} = \frac{E}{E+K}\,\Delta\varepsilon$$

Note that the stiffness equals Young's modulus when H (or K) approaches infinity.

3.4.3 Stress update

In a general case of elastoplastic deformation, the begin-increment state, indicated with index n, may reside on the elastic trajectory or on the elastoplastic trajectory. The end-increment state is indicated with an index $n+1$ but this is skipped furtheron Depending of $\Delta\varepsilon = \varepsilon - \varepsilon_n$ (further) elastoplastic deformation or elastic unloading can occur. Several possibilities are indicated in the figure below.

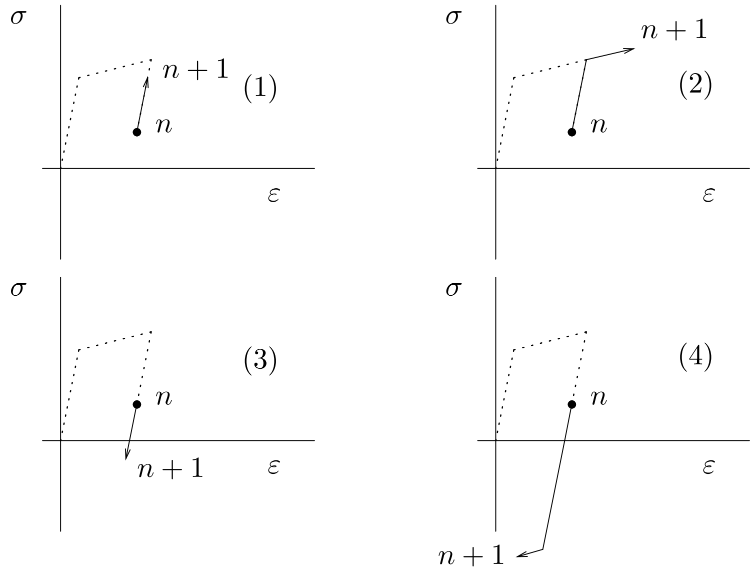


Fig. 3.29 : *Various incremental stress-strain changes*

Elastic stress predictor

Because it is not known a priori whether (ongoing) elastoplastic deformation or elastic unloading will have taken place in the current increment $t_n \rightarrow t_{n+1}$, the stress calculation starts from the assumption that the strain increment is completely elastic. The *elastic stress predictor* σ_e is calculated and subsequently the yield criterion is evaluated with the yield function f.

$$\sigma_e = \sigma_n + E(\varepsilon - \varepsilon_n)$$

- $f = (\sigma_e - q_n)^2 - \sigma_{y_n}^2 \leq 0 \quad \rightarrow \quad$ elastic increment
- $f = (\sigma_e - q_n)^2 - \sigma_{y_n}^2 > 0 \quad \rightarrow \quad$ elastoplastic increment

Elastic increment

When the increment is fully elastic, the end-increment stress equals the calculated elastic stress. As no plastic deformation has occurred during the increment, the effective plastic strain and the yield stress remain unchanged.

$$\sigma(t_{n+1}) = \sigma_e \quad ; \quad \bar{\varepsilon}_p(t_{n+1}) = \bar{\varepsilon}_p(t_n) = \bar{\varepsilon}_{p_n}$$
$$\sigma_y(t_{n+1}) = \sigma_y(t_n) = \sigma_{y_n} \quad ; \quad q(t_{n+1}) = q(t_n) = q_n$$

Elastoplastic increment

If the elastic stress predictor indicates that the yield criterion is violated, the increment is elastoplastic. The end-increment stress has to be determined by integration of the constitutive equations, such that at the end of the increment the stress satisfies the yield criterion as will be discussed.

There are many procedures which can be followed to solve the differential equation for the stress. They can be classified as *implicit* or *explicit*. The implicit methods are more accurate and more stable then the explicit methods.

We assume that the begin-increment state resides on the yield trajectory, so $f_n = 0$. In reality this is of course not always the case : the begin-increment case may be elastic ($f_n < 0$) and plastic deformation will develop during the increment. The implicit procedures can well cope with this phenomenon. Explicit procedures will need some correction.

Implicit solution procedure

In an implicit procedure we want to satisfy the constitutive equations at the current time, which is the end-increment time. Because various variables are unknown, the equations are non-linear and have to be solved iteratively. Remember that at the begin-increment time, the equations are satified – in the former increment – so all values are known and $f_n \leq 0$.

In an iterative approach, an unknown value is written as the sum of an approximate value and an iterative change, which is assumed to be very small, allowing linearisation. Concerning the yield function, it has to be recalled that it depends on the stress σ, the shift stress q and the yield stress σ_y and that both the yield stress and the shift stress depend on the plastic strain – through the hardening law – and thus on λ.

$$\left.\begin{array}{l} \sigma - \sigma_n + 2E(\sigma - q)(\lambda - \lambda_n) = E(\varepsilon - \varepsilon_n) \\ f - f_n = f = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} \sigma^* + \delta\sigma - \sigma_n + 2E(\sigma^* + \delta\sigma - q^* - \delta q)(\lambda^* + \delta\lambda - \lambda_n) = E(\varepsilon - \varepsilon_n) \\ f^* + \delta f = 0 \quad \rightarrow \quad f^* + \dfrac{\partial f}{\partial \sigma}\delta\sigma + \dfrac{\partial f}{\partial \lambda}\delta\lambda = 0 \end{array}\right\}$$

with

$$\begin{aligned} \frac{\partial f}{\partial \sigma} &= 2(\sigma - q) \\ \frac{\partial f}{\partial \lambda} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial \varepsilon_p}\frac{\partial \varepsilon_p}{\partial \lambda} + \frac{\partial f}{\partial \sigma_y}\frac{\partial \sigma_y}{\partial \bar{\varepsilon}_p}\frac{\partial \bar{\varepsilon}_p}{\partial \lambda} \\ &= [-2(\sigma - q)][K][2(\sigma - q)] + [-2\sigma_y][H][2|\sigma - q|] \\ &= -4K(\sigma - q)^2 - 4H\sigma_y|\sigma - q| \end{aligned}$$

this becomes

$$\left.\begin{array}{l} \sigma^* + \delta\sigma - \sigma_n + 2E(\sigma^* + \delta\sigma - q^* - \delta q)(\lambda^* + \delta\lambda - \lambda_n) = E(\varepsilon - \varepsilon_n) \\ f^* + 2(\sigma^* - q^*)\delta\sigma - [4K^*(\sigma^* - q^*)^2 + 4H^*\sigma_y^*|\sigma^* - q^*|]\delta\lambda = 0 \end{array}\right\}$$

After linearization and solving the unknown $\delta\sigma$ and $\delta\lambda$ until convergence is reached, the result is a new approximate value for σ, ε_p, q and σ_y at the end of the increment.

$$
\begin{aligned}
&\sigma^* = \sigma^* + \delta\sigma \\
&\lambda^* = \lambda^* + \delta\lambda \\
&\Delta\varepsilon_p = 2(\lambda^* - \lambda_n)(\sigma^* - q_n) \quad \rightarrow \quad \varepsilon_p \quad \rightarrow \quad q^*, K^* \\
&\Delta\bar{\varepsilon}_p = |\Delta\varepsilon_p| \quad \rightarrow \quad \bar{\varepsilon}_p \quad \rightarrow \quad \sigma_y^*, H^*
\end{aligned}
$$

Stiffness

From the solution procedure, also a relation for the material stiffness $C_\varepsilon = \frac{\partial\sigma}{\partial\varepsilon}$ can be derived.

$$
\begin{aligned}
&\left\{ \begin{array}{l} \sigma - \sigma_n + 2E(\sigma - q)(\lambda - \lambda_n) - E(\varepsilon - \varepsilon_n) = 0 \\ f = 0 \end{array} \right. \\
&\left\{ \begin{array}{l} \delta\sigma + 2E\delta\sigma(\lambda - \lambda_n) + 2E(\sigma - q)\delta\lambda - E\delta\varepsilon = 0 \\ (\sigma - q)\delta\sigma - 2K(\sigma - q)^2\delta\lambda - 2H\sigma_y|\sigma - q|\delta\lambda = 0 \end{array} \right. \\
&\left[1 + 2E(\lambda - \lambda_n) + \frac{2E(\sigma - q)^2}{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|} \right] \delta\sigma = E\delta\varepsilon \\
&C_\varepsilon = \frac{E\{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|\}}{\{1 + 2E(\lambda - \lambda_n)\}\{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|\} + 2E(\sigma - q)^2} \\
&\text{yield at } \tau = t = t_{n+1} \quad \rightarrow \quad (\sigma - q)^2 = \sigma_y^2 \text{ and } |\sigma - q| = \sigma_y \quad \rightarrow \\
&C_\varepsilon = \frac{E(K + H)}{E + K + H + 2E(K + H)(\lambda - \lambda_n)}
\end{aligned}
$$

Explicit solution procedure

An explicit procedure starts from the known state at the beginning of the increment and calculates incremental changes directly, assuming that values of some variables remain the same during the increment. Obviously, this is not through, so these procedures are not very accurate. The final solution may not satisfy the yield criterion $f = 0$ exactly, which calls for a correction, where the final state is projected onto the yield trajectory.

$$
\left. \begin{array}{l} \Delta\sigma + 2E(\sigma_n - q_n)\Delta\lambda = E\Delta\varepsilon \\ \Delta f = 0 \quad \rightarrow \quad \left.\dfrac{\partial f}{\partial\sigma}\right|_n \Delta\sigma + \left.\dfrac{\partial f}{\partial\lambda}\right|_n \Delta\lambda = 0 \end{array} \right\}
$$

$$
\left. \begin{array}{l} \Delta\sigma + 2E(\sigma_n - q_n)\Delta\lambda = E\Delta\varepsilon \\ \\ 2(\sigma_n - q_n)\Delta\sigma - 4K_n(\sigma_n - q_n)^2\Delta\lambda - 4H_n\sigma_{yn}|\sigma_n - q_n|\Delta\lambda = 0 \quad \rightarrow \\ \quad \Delta\lambda = \dfrac{(\sigma_n - q_n)}{2K_n(\sigma_n - q_n)^2 + 2H_n\sigma_{yn}|\sigma_n - q_n|}\Delta\sigma = \dfrac{1}{2K_n(\sigma_n - q_n) + 2H_n(\sigma_n - q_n)}\Delta\sigma \end{array} \right\} \rightarrow
$$

$$\Delta\sigma = \frac{E[K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n|]}{K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n| + E(\sigma_n - q_n)^2}\,\Delta\varepsilon$$

$$\Delta\varepsilon_p = 2(\sigma_n - q_n)\Delta\lambda = \frac{(\sigma_n - q_n)^2}{K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n|}$$

When we calculate the stress from the equations above, it is found that the result is not correct due to the bad transition from the elastic to the elastoplastic regime. To solve this problem, the elastoplastic increment is split in an elastic and an elastoplastic part.

A scaling factor β is calculated from the requirement that $\beta(\varepsilon - \varepsilon_n)$ brings us to the yield trajectory where $\varepsilon = \varepsilon^f$. This strain to yield ε^f is determined, and the current stress is calculated, using the current stiffness. Calculation of β is generalized for tension and pressure. Notice that $\operatorname{sign}(\alpha)$ is the *sign* of α.

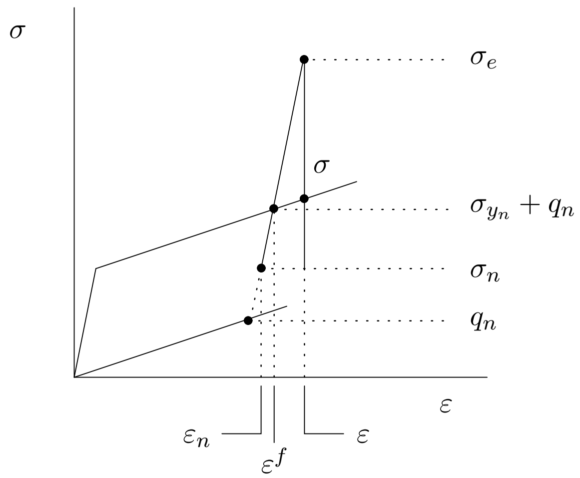


Fig. 3.30 : *Explicit stress update by increment splitting*

$$\sigma_e = \sigma_n + E(\varepsilon - \varepsilon_n) \qquad \rightarrow \qquad \Delta\sigma_e = \sigma_e - \sigma_n = E(\varepsilon - \varepsilon_n)$$

$$\beta = \frac{|\operatorname{sign}(\varepsilon - \varepsilon_n)\sigma_{y_n} - (\sigma_n - q_n)|}{|\sigma_e - \sigma_n|}$$

$$\varepsilon^f = \varepsilon_n + \beta(\varepsilon - \varepsilon_n) \qquad \rightarrow \qquad \Delta\varepsilon^f = \varepsilon - \varepsilon^f = (1-\beta)(\varepsilon - \varepsilon_n)$$

The elastoplastic stress increment $\Delta\sigma^f$ and the increment of λ is now calculated from the constitutive equations. The total stress increment and other state variables can then be calculated.

$$\left\{ \begin{array}{l} \Delta\sigma^f + 2E(\sigma_n - q)\Delta\lambda = E\Delta\varepsilon^f \\ 2(\sigma_n - q)\Delta\sigma^f - 4K_n(\sigma_n - q_n)^2\Delta\lambda - 4H_n\sigma_{yn}|\sigma_n - q_n|\Delta\lambda = 0 \end{array} \right.$$

The incremental changes are then :

$$
\begin{array}{l}
\Delta\sigma = \beta\Delta\sigma_e + \Delta\sigma^f \quad \rightarrow \quad \sigma = \sigma_n + \Delta\sigma \\
\lambda = \lambda_n + \Delta\lambda \\
\Delta\varepsilon_p = 2(\lambda - \lambda_n)(\sigma - q_n) \quad \rightarrow \quad \varepsilon_p \quad \rightarrow \quad q, K \\
\Delta\bar{\varepsilon}_p = |\Delta\varepsilon_p| \quad \rightarrow \quad \bar{\varepsilon}_p \quad \rightarrow \quad \sigma_y, H
\end{array}
$$

3.4.4 Implementation

See `tr2delpl.m` for the implementation.

3.4.5 Examples

Cyclic loading

The stress-strain behavior of a truss is calculated for a prescribed cyclic strain, for linear isotropic and linear kinematic hardening. The truss has initial length $l_0 = 100$ mm and cross-sectional area $A_0 = 10$ mm^2 and is loaded with a prescribed cyclic axial strain. The axial stress is calculated. Material parameters are :

Young's modulus	E	100000	MPa
Poisson's ratio	ν	0.3	-
initial yield stress	σ_{y0}	250	MPa
hardening coefficient	H	5000	MPa
hardening coefficient	K	5000	MPa

The isotropic hardening leads to an increasingly larger elastic trajectory. After many load reversals, the behavior will become purely elastic. The kinematic hardening results in a steady state hysteresis loop.

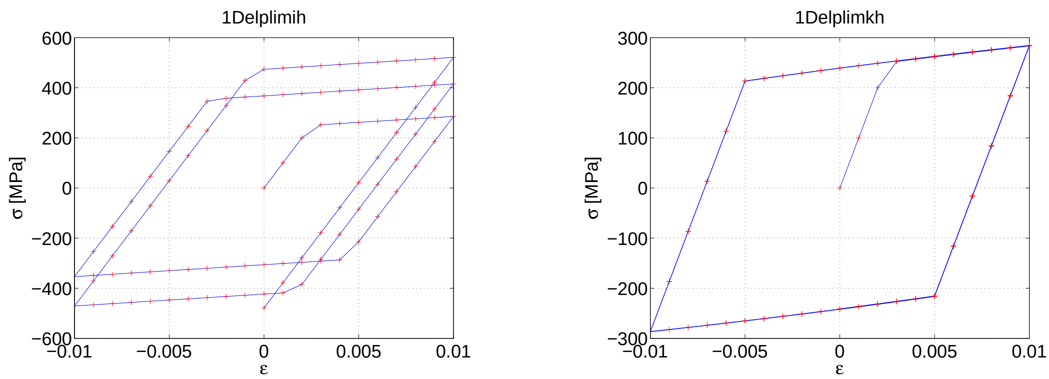
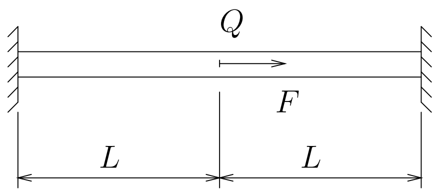


Fig. 3.31 : *Cyclic stress-strain behavior for linear isotropic and kinematic hardening*

Clamped truss

A prismatic truss is clamped between two rigid walls, as is shown in the figure. The length L of the truss is 2×1000 [mm], its cross-sectional area A is 100 mm^2. The material is elastic with Young's modulus $E = 200000$ N/mm^2 as long as the axial stress is below the initial yield stress of 200 N/mm^2. Above this value the material shows linear isotropic hardening with hardening coefficient $H = 1000$ M/mm^2.

In the middle of the truss, in point Q, a point load F is applied, which first increases and then is decreased to zero. The displacement of point Q is calculated with `tr2d`. The force F as function of the displacement is shown in the figure below.



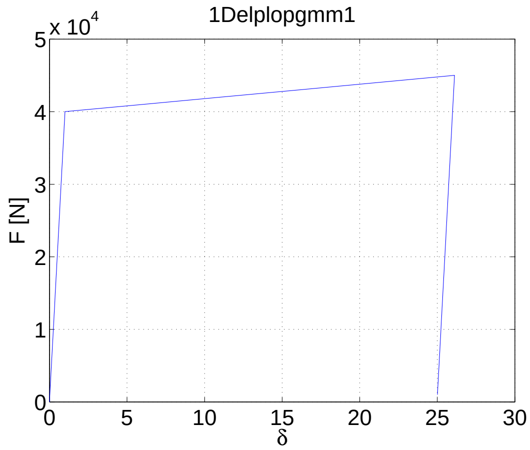
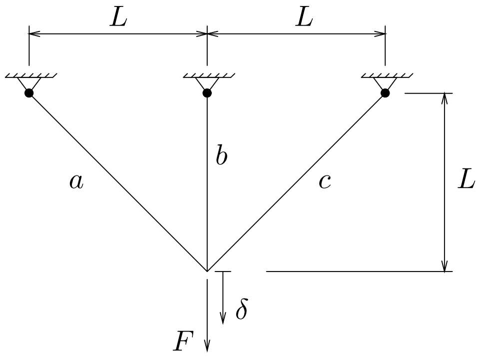


Fig. 3.32 : *Force F versus displacement of point Q.*

Truss structure

A structure of three trusses is loaded by a vertical displacement. When the axial stress exceeds a certain limit value, a trusses will deform plastically.



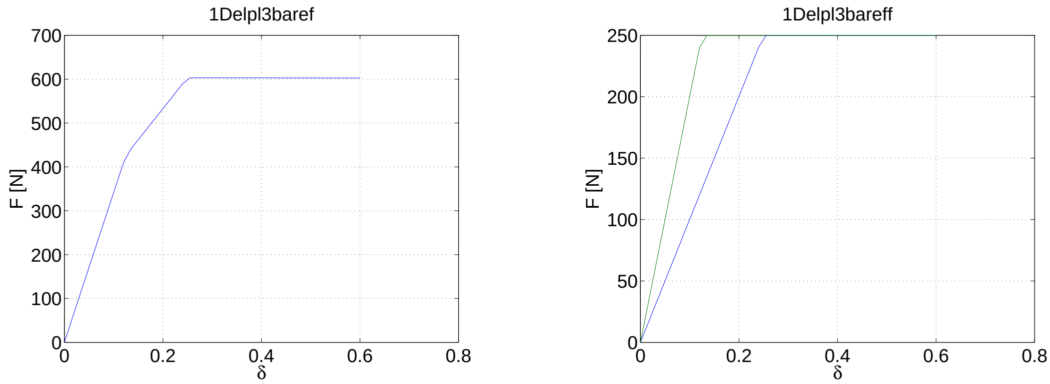


Fig. 3.33 : *Force F versus displacement δ.*

3.5 Linear viscoelastic material behavior

Viscoelastic materials show time dependent behavior. When during a tensile test the stress/-strain is prescribed stepwise, the strain/stress will not react immediately, but show a delayed response, which is called creep/relaxation. Viscoelastic material behavior is a combination of elastic and viscous behavior. Both cases will be illustrated first.

Here, we only consider linear viscoelastic behavior and also assume that strains are small.

Linear elastic material behavior

For a linear elastic material the stress is uniquely related to the strain by the Young's modulus E [Pa]. The linear elastic truss behaves like a spring with constant stiffness k [N/m].

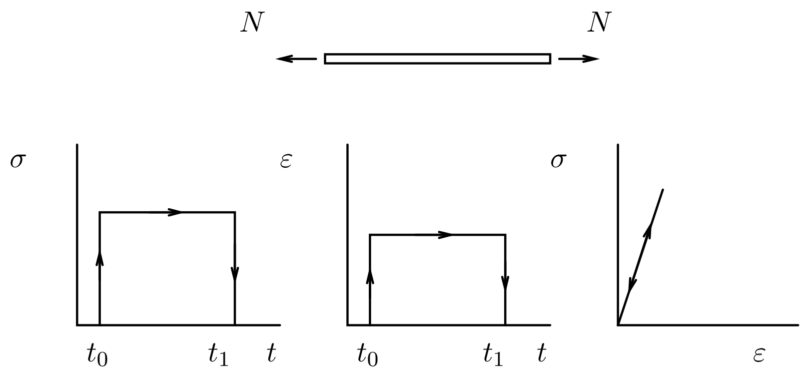


Fig. 3.34 : *Tensile experiment for linear elastic truss*

$$\varepsilon = \frac{1}{E}\,\sigma \qquad \rightarrow \qquad \sigma = E\varepsilon \qquad \rightarrow \qquad N = \sigma A = EA\varepsilon = \frac{EA}{l}\,\Delta l = k\,\Delta l$$

A linear elastic material can be subjected to a loading stress cycle. The work per unit of volume during the cycle appears to be zero indicating that there has been no dissipation. This is also obvious when looking at the stress-strain curve associated with the load cycle : the area below the curve is zero.

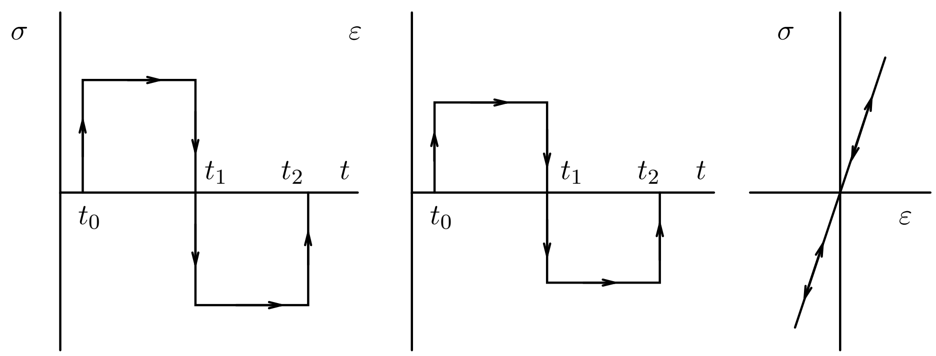


Fig. 3.35 : *Loading cycle applied to linear elastic truss*

$$
\begin{aligned}
U_d &= \int_{t_0}^{t_1} \sigma \, d\varepsilon + \int_{t_1}^{t_2} \sigma \, d\varepsilon = \int_{t_0}^{t_1} E\varepsilon \, d\varepsilon + \int_{t_1}^{t_2} E\varepsilon \, d\varepsilon \\
&= \tfrac{1}{2} E [\varepsilon_1^2 - \varepsilon_0^2 + \varepsilon_2^2 - \varepsilon_1^2] = 0
\end{aligned}
$$

Linear viscous material behavior

For a linear viscous material the stress is uniquely related to the strain rate by the viscosity η [Pa.s]. The linear viscous "truss" behaves like a dashpot with constant damping value b [Ns/m].

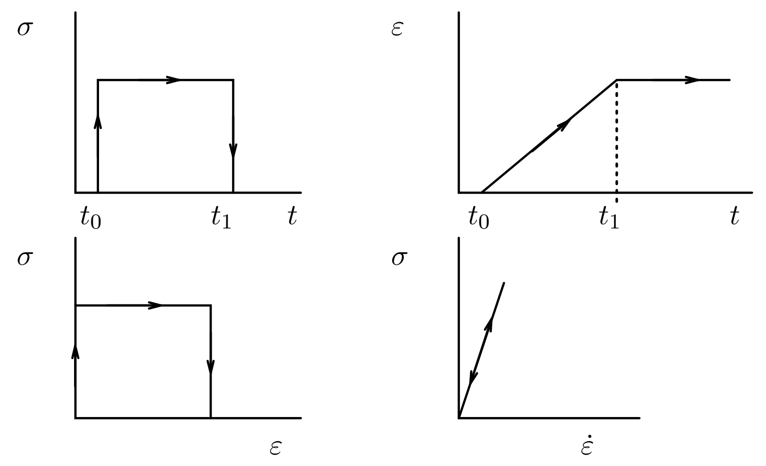


Fig. 3.36 : *Tensile experiment for linear viscous truss*

$$
\dot{\varepsilon} = \frac{1}{\eta}\,\sigma \qquad \rightarrow \qquad \sigma = \eta \dot{\varepsilon} \qquad \rightarrow \qquad N = \sigma A = \eta A \dot{\varepsilon} = \frac{\eta A}{l}\,\dot{\Delta l} = b\,\dot{\Delta l}
$$

The linear viscous material is subjected to a loading stress cycle. The work per unit of volume can be calculated and appears to be non-zero. All the work is dissipated as can be seen from the stress-strain curve : the area included by the stress-strain trajectory represents the specific dissipated energy.

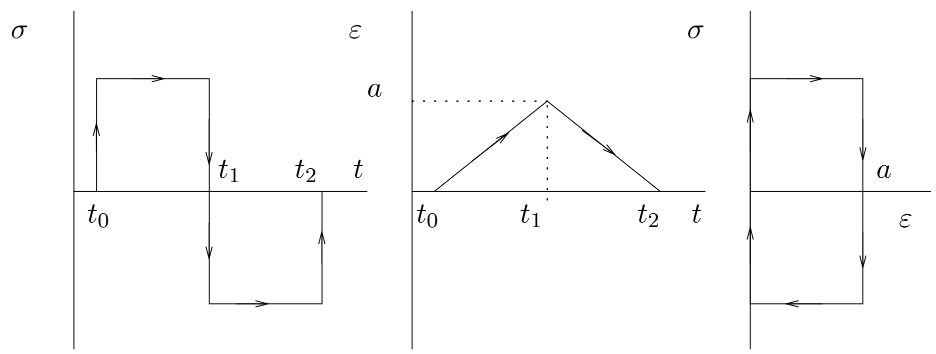


Fig. 3.37 : *Loading cycle applied to linear viscous truss*

$$U_d = \int_{t_0}^{t_1} \sigma \, d\varepsilon + \int_{t_1}^{t_2} \sigma \, d\varepsilon = \int_{t_0}^{t_1} \eta\dot{\varepsilon} \, d\varepsilon + \int_{t_1}^{t_2} \eta\dot{\varepsilon} \, d\varepsilon = \int_{t_0}^{t_1} \eta c \, d\varepsilon - \int_{t_1}^{t_2} \eta c \, d\varepsilon$$
$$= \eta c[\varepsilon_1 - \varepsilon_0 - \varepsilon_2 + \varepsilon_1] = 2\eta c a$$

Viscoelastic material behavior

Viscoelastic material behavior is a combination of elastic and viscous behavior. Part of the deformation energy will be dissipated, while the rest is stored as reversible elastic energy. Viscoelastic behavior can be characterized by mechanical models build from springs and dashpots.

We will assume the deformation to be small, so that the choice of stress and strain definitions is irrelevant. First the characteristics of the viscoelastic material behavior will be described, based on experimental observations. To predict the behavior, viscoelastic models are needed, which will be based on the behavior of springs and dashpots.

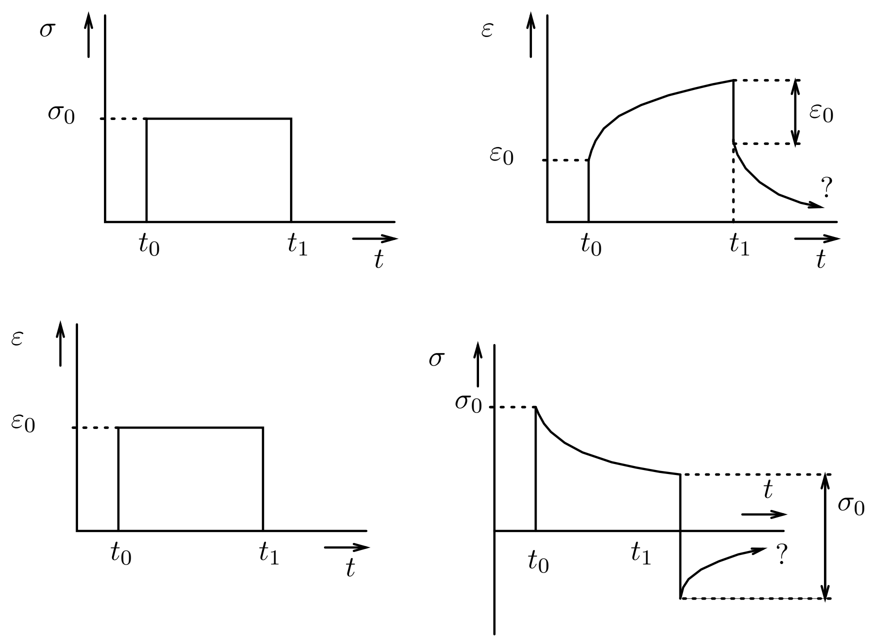


Fig. 3.38 : *Stress-strain curves for viscoelastic material behavior*

To investigate the characteristics of viscoelastic material behavior, a tensile test is carried out, where a tensile bar is loaded with stress excitations, which are prescribed as a step-function in time.

Proportionality

In a tensile test, a stress-step is applied and the strain is measured as a function of time. The test is repeated for increased stress amplitudes. From the measurement data, the strain values at the same time after loading are plotted against the stress amplitudes. The resulting plots are *isochrones*, because they represent the strain at the same time after loading.

For linear viscoelastic material behavior the isochrones are straight lines. This means that the strain as a function of time is *proportional* to the stress. The strain response can be written as the product of the stress amplitude $\Delta\sigma$ and a function of the time $D(t - t_0)$, whose value is zero for $t < t_0$.

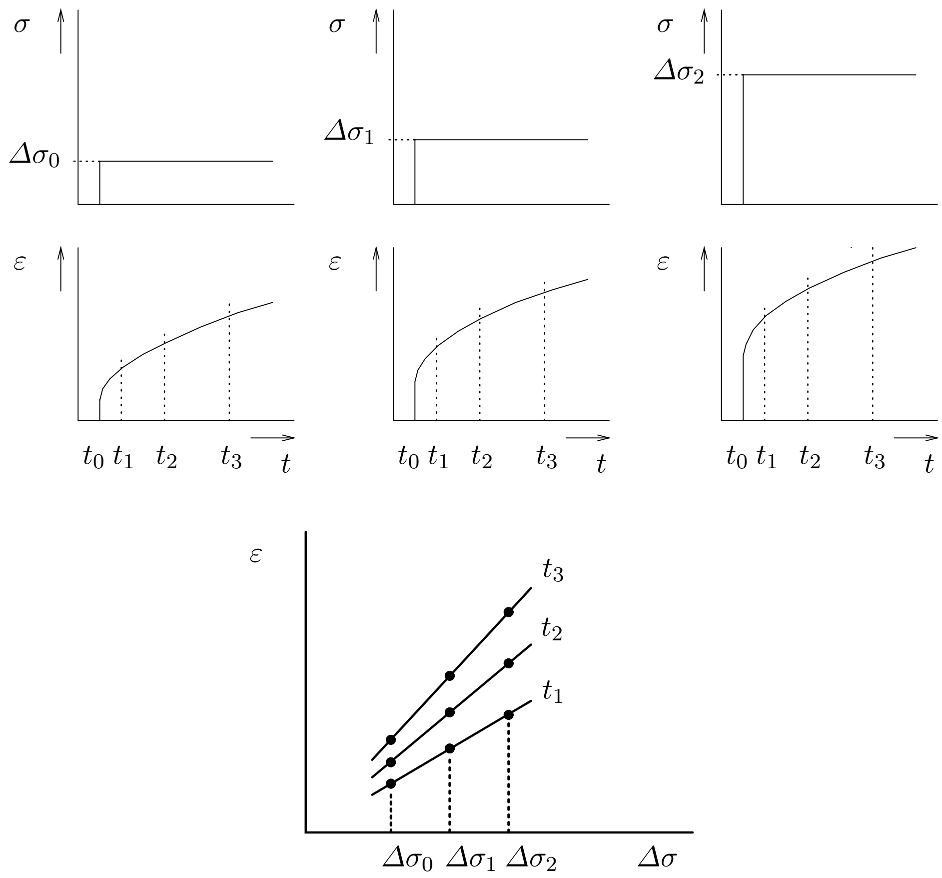


Fig. 3.39 : *Proportionality of strain response and stress excitation*

$$\varepsilon(t) = \Delta\sigma D(t - t_0) \qquad \text{for} \qquad \forall \quad t \geq t_0$$

Superposition

A tensile test is carried out three times. In the first two tests, a stress step with different amplitude is applied and the strain response is measured. Then, in the third experiment, the two stress steps are applied subsequently and again the strain response is measured.

For linear viscoelastic material behavior, the strain response in the third experiment is the sum of the separate responses in the first two experiments. This means that strain responses can be determined by *superposition*.

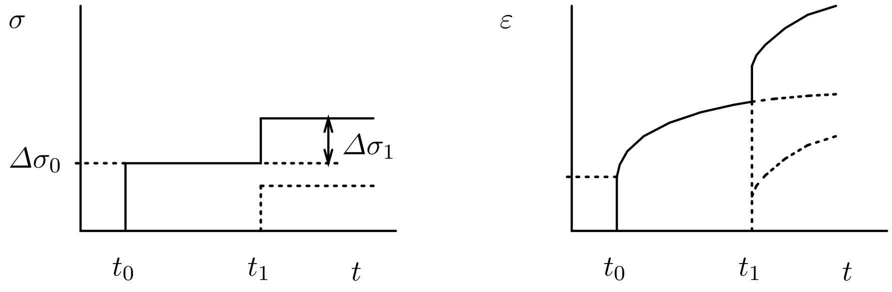


Fig. 3.40 : *Superposition of strain responses to two stress excitations*

separate excitations

$$
\begin{array}{llll}
\Delta\sigma = \Delta\sigma_0 & \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) & \text{for} & t > t_0 \\
\Delta\sigma = \Delta\sigma_1 & \rightarrow \quad \varepsilon(t) = \Delta\sigma_1 D(t - t_1) & \text{for} & t > t_1
\end{array}
$$

subsequent excitations

$$
\begin{array}{llll}
\Delta\sigma = \Delta\sigma_0 & \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) & \text{for} & t_0 < t < t_1 \\
\Delta\sigma = \Delta\sigma_0 + \Delta\sigma_1 & \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1) & \text{for} & t > t_1
\end{array}
$$

3.5.1 Boltzmann integral

Linear viscoelasticity is characterized by the two properties described in the previous sections.

1. proportionality : At every time the strain response is proportional to the amplitude of a constant stress step which is applied at $t = t_0$: $\varepsilon_i(t) = \Delta\sigma_i \, D(t - t_0)$ for $t > t_0$
2. superposition : The strain response to two subsequently (at time $t = t_0$ and $t = t_1$) applied constant amplitude ($\Delta\sigma_0$ and $\Delta\sigma_1$) stress steps equals the sum of the separate responses for $t > t_1$: $\varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1)$ for $t > t_1$

Every stress excitation can be seen as an infinite sequence of infinitesimal small stress steps. The superposition property then leads to the Boltzmann integral expressing the strain response. This integral is also called Duhamel or memory integral.

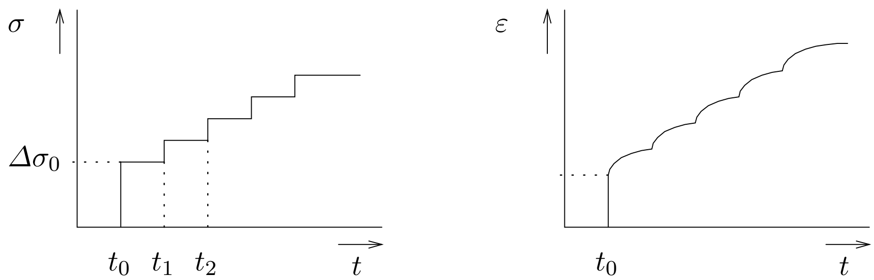


Fig. 3.41 : *Superposition of strain responses to subsequent stress excitations*

$$
\begin{aligned}
\varepsilon(t) &= \Delta\sigma_0 D(t-t_0) + \Delta\sigma_1 D(t-t_1) + \Delta\sigma_2 D(t-t_2) + .. \\
&= \sum_{i=1}^{n} \Delta\sigma_i D(t-t_i) \qquad \rightarrow \qquad \text{limit } n{\rightarrow}\infty \qquad\qquad (t{\rightarrow}\tau) \\
&= \int_{\tau=t_0^-}^{t} D(t-\tau)\, d\sigma(\tau) = \int_{\tau=t_0^-}^{t} D(t-\tau) \frac{d\sigma(\tau)}{d\tau}\, d\tau
\end{aligned}
$$

$$
\varepsilon(t) = \int_{\tau=t_0^-}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau
$$

For strain excitation and stress response the same observations can be made and we arrive at the Boltzmann integral for the stress response.

$$
\sigma(t) = \int_{\tau=t_0^-}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau
$$

3.5.2 Step excitations

Step excitations are important for the (experimental) characterization of viscoelastic materials. A unit step (amplitude = 1) can be described with the Heaviside function. The derivative of the unit step function is the Dirac function or unit pulse. It has the important property that integration of the product of a function $f(\tau)$ and $\delta(\tau, t^*)$ over an interval which contains $\tau = t^*$, the "location" of the Dirac pulse, results in the value $f(t^*)$.

Heaviside function $\qquad H(t,t^*) \quad \left\{ \begin{array}{ccc} t < t^* & : & H(t,t^*) = 0 \\ t > t^* & : & H(t,t^*) = 1 \end{array} \right\}$

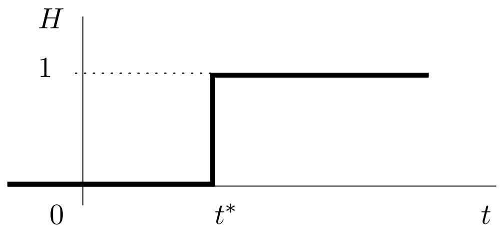


Fig. 3.42 : *Unit step or Heaviside function*

Dirac function $\qquad \delta(t,t^*) = \dfrac{d}{dt}\left\{H(t,t^*)\right\}$

$$
\int_{\tau=0}^{t>t^*} \delta(\tau,t^*)\, d\tau = 1 \quad ; \quad \int_{\tau=0}^{t>t^*} f(\tau)\delta(\tau,t^*)\, d\tau = f(t^*)
$$

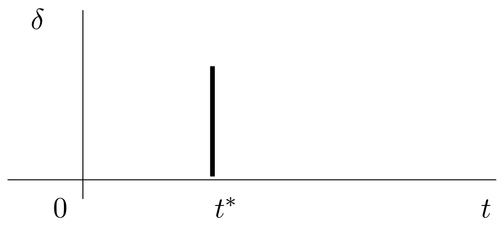


Fig. 3.43 : *Unit pulse or Dirac function*

Creep (retardation)

The strain response to a stress step excitation at $t = 0$ having an amplitude σ_0 equals $\sigma_0 D(t)$. The measured strain response can be used to fit a proposed model for $D(t)$. Experiments have revealed some characteristic properties.

$$\sigma(t) = \sigma_0 H(t, 0) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0 \delta(t, 0)$$

$$\varepsilon(t) = \int_{\tau=0^-}^{t} D(t - \tau)\dot{\sigma}(\tau)\, d\tau = \int_{\tau=0^-}^{t} D(t - \tau)\sigma_0 \delta(\tau, 0)\, d\tau = \sigma_0 D(t)$$

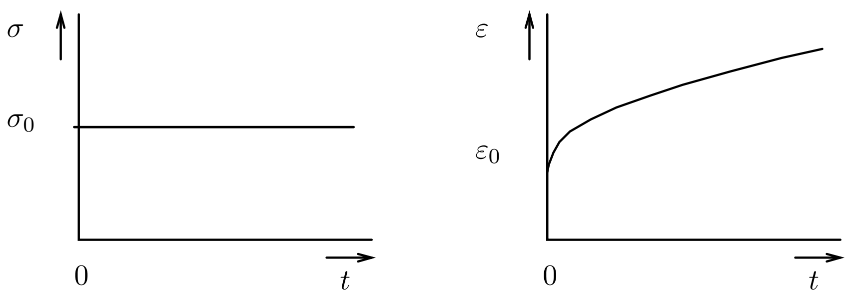


Fig. 3.44 : *Creep strain response to unit stress step*

- $\dot{D}(t) \geq 0 \qquad \forall \quad t \geq 0$
- $\ddot{D}(t) < 0 \qquad \forall \quad t \geq 0$

Relaxation

The stress response to a strain step excitation at $t = 0$ having an amplitude ε_0 equals $\varepsilon_0 E(t)$. The measured stress response can be used to fit a proposed model for $E(t)$. Experiments have revealed some characteristic properties.

$$\varepsilon(t) = \varepsilon_0 H(t, 0) \qquad \rightarrow \qquad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t, 0)$$

$$\sigma(t) = \int_{\tau=0^-}^{t} E(t - \tau)\dot{\varepsilon}(\tau)\, d\tau = \int_{\tau=0^-}^{t} E(t - \tau)\varepsilon_0 \delta(\tau, 0)\, d\tau = \varepsilon_0 E(t)$$

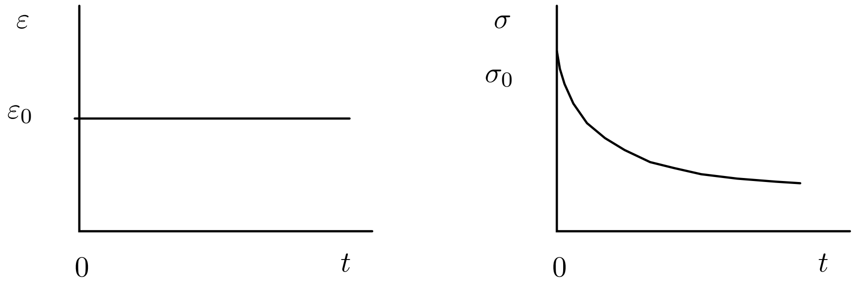


Fig. 3.45 : *Stress relaxation response to unit strain step*

- $\dot{E}(t) \leq 0 \qquad \forall \quad t \geq 0$
- $\ddot{E}(t) > 0 \qquad \forall \quad t \geq 0$
- $\displaystyle\int_{t=0}^{\infty} \dot{E}(t)\, dt \geq 0 \quad \rightarrow \quad \lim_{t\rightarrow\infty} \dot{E}(t) = 0$

3.5.3 Harmonic strain excitation

For the experimental characterization of viscoelastic materials, harmonic excitation is of great importance. We consider first a tensile test where the strain is prescribed harmonically with an angular frequency ω [rad s^{-1}] and amplitude ε_0.

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$

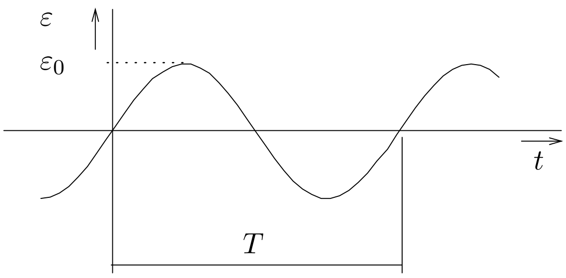


Fig. 3.46 : *Harmonic strain excitation*

Stress response

The stress response to the harmonic strain excitation can be calculated using the Boltzmann integral. Evaluating the integral involves transformation to another integration variable.

The result reveals two important viscoelastic material parameters : the *storage modulus* E' and the *loss modulus* E'', which both are a function of the angular frequency ω. They can be measured with a Dynamic Mechanical Analysis (DMA) experiment and transformed into a relaxation function $E(t)$. This experiment is more easy to perform and more accurate than the direct measurement of $E(t)$ in a relaxation experiment.

$$\begin{aligned}
\sigma(t) &= \int_{\tau=-\infty}^{t} E(t-\tau)\varepsilon_0\omega\cos(\omega\tau)\,d\tau = \varepsilon_0\omega\int_{\xi=-\infty}^{t} E(t-\tau)\cos(\omega\tau)\,d\tau \\
&\qquad t-\tau = s \quad \rightarrow \quad \tau = t-s \quad \rightarrow \quad d\tau = -ds \\
&= \varepsilon_0\omega\int_{s=0}^{\infty} E(s)\cos\{\omega(t-s)\}\,ds \\
&\qquad \cos(\omega t-\omega s) = \cos(\omega t)\cos(\omega s) + \sin(\omega t)\sin(\omega s) \\
&= \varepsilon_0\left[\omega\int_{s=0}^{\infty} E(s)\sin(\omega s)\,ds\right]\sin(\omega t) + \varepsilon_0\left[\omega\int_{s=0}^{\infty} E(s)\cos(\omega s)\,ds\right]\cos(\omega t) \\
&= \varepsilon_0 E'\sin(\omega t) + \varepsilon_0 E''\cos(\omega t)
\end{aligned}$$

$$\begin{aligned}
E'(\omega) &= \omega\int_{s=0}^{\infty} E(s)\sin(\omega s)\,ds &&: \text{ storage modulus} \\
E''(\omega) &= \omega\int_{s=0}^{\infty} E(s)\cos(\omega s)\,ds &&: \text{ loss modulus}
\end{aligned}$$

Energy dissipation

The dissipated energy per unit of volume during one period of the harmonic strain excitation can be calculated. This dissipated energy must always be positive. As shown below, it must be concluded that the loss modulus E'' is also positive.

Referring to the calculated stress response, we can conclude that the stress at time $t = 0$ where the strain was taken to be $\varepsilon = 0$, has a positive value. We thus have proved something which we already knew from experiments : there is a phase difference between strain and stress and the stress shows a gain w.r.t. the strain.

$$\begin{aligned}
U_d &= \int_{\varepsilon(0)}^{\varepsilon(T)} \sigma\,d\varepsilon = \int_{t=0}^{T} \sigma\dot{\varepsilon}\,dt \\
&= \int_{t=0}^{T} \left\{\varepsilon_0 E'\sin(\omega t) + \varepsilon_0 E''\cos(\omega t)\right\}\left\{\varepsilon_0\omega\cos(\omega t)\right\}\,dt \\
&= \int_{t=0}^{T} \varepsilon_0^2\omega\left\{E'\sin(\omega t)\cos(\omega t) + E''\cos^2(\omega t)\right\}\,dt \\
&= \int_{t=0}^{T} \varepsilon_0^2\omega\left\{\tfrac{1}{2}E'\sin(2\omega t) + \tfrac{1}{2}E'' + \tfrac{1}{2}E''\cos(2\omega t)\right\}\,dt \\
&= \tfrac{1}{2}\varepsilon_0^2\omega\left[-E'\frac{1}{2\omega}\cos(2\omega t) + E''t + E''\frac{1}{2\omega}\sin(2\omega t)\right]_0^{T=\frac{2\pi}{\omega}} \\
&= \tfrac{1}{2}\varepsilon_0^2\omega\left[-E'\frac{1}{2\omega} + E'\frac{1}{2\omega} + E''\frac{2\pi}{\omega}\right] \\
&= \pi\varepsilon_0^2 E'' > 0 \quad \Rightarrow \quad E'' > 0 \quad \rightarrow
\end{aligned}$$

$$\sigma(t=0) = \varepsilon_0 E'' > 0$$

The phase difference between stress and strain results in a so-called hysteresis loop, when a stress-strain diagram is drawn. The area enclosed by the hysteresis loop is a measure for the dissipated energy per unit of volume during one cycle.

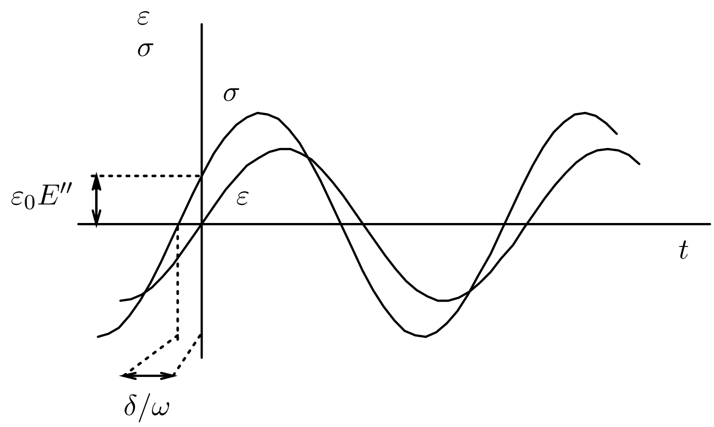


Fig. 3.47 : *Harmonic strain excitation and stress response*

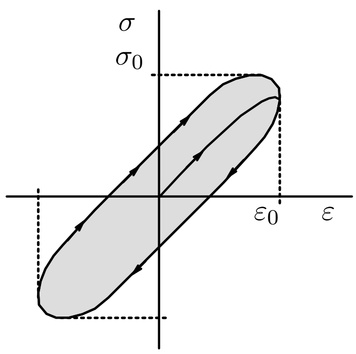


Fig. 3.48 : *Hysteresis of stress ans strain*

Relation between E', E'' and δ

Writing the stress response with two different relations, results in relations between E', E'' and δ. The amplitude σ_0 of the stress response can also be calculated.

$$\sigma(t) = \sigma_0 \sin(\omega t + \delta) = \sigma_0 \cos(\delta)\sin(\omega t) + \sigma_0 \sin(\delta)\cos(\omega t)$$
$$\sigma(t) = \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)$$

storage and loss modulus

$$\left.\begin{array}{l} E' = \dfrac{\sigma_0}{\varepsilon_0}\cos(\delta) \\[2ex] E'' = \dfrac{\sigma_0}{\varepsilon_0}\sin(\delta) \end{array}\right\} \quad \rightarrow \quad \left\{\begin{array}{l} \dfrac{E''}{E'} = \tan(\delta) \quad \rightarrow \\[2ex] \qquad\qquad \delta = \arctan\left(\dfrac{E''}{E'}\right) \end{array}\right.$$

amplitude $\sigma_0 = \varepsilon_0 \sqrt{(E')^2 + (E'')^2}$

Measured E', E'' and $\tan(\delta)$

Typical measured values for $E'(\omega)$, $E''(\omega)$ and $\tan(\delta)$ are shown in the plots below. For low and high frequencies, the loss modulus is zero, indicating that there is no dissipation and the material behaves elastically. For high frequencies, the "stiffness" E' is much higher than for low frequencies.

Storage and loss moduli can be measured accurately using DMA test equipment. From $E'(\omega)$ and $E''(\omega)$, the relaxation function $E(t)$ can be calculated using dedicated software.

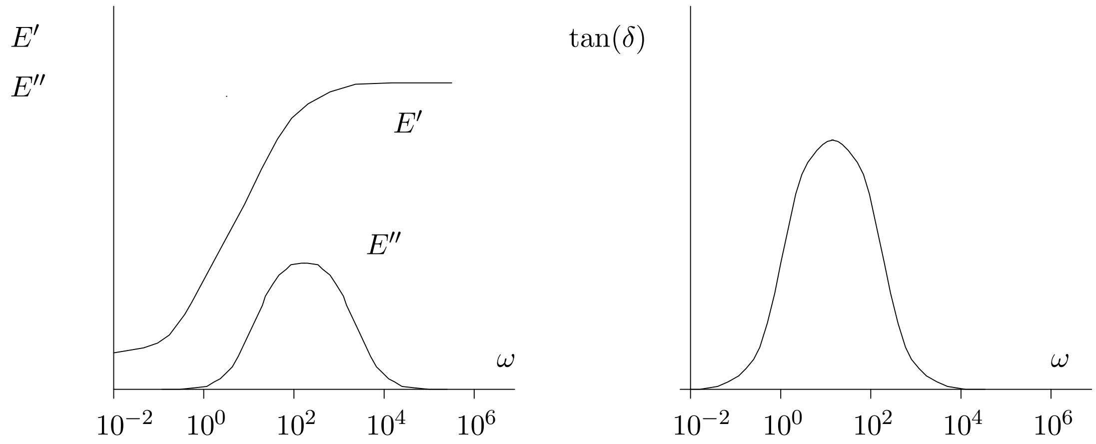


Fig. 3.49 : *Characteristic values of* E', E'' *and* $\tan(\delta)$

3.5.4 Harmonic stress excitation

The axial stress can be prescribed harmonically with an angular frequency ω [rad s^{-1}]. The strain response can be calculated with the Boltzmann integral and appears to be characterized by the *storage compliance* $D'(\omega)$ and the *loss compliance* $D''(\omega)$. Both compliances are positive for all ω. Because $\varepsilon(t=0) < 0$, the definition of D' includes a minus sign.

$$\sigma(t) = \sigma_0 \sin(\omega t) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

$$\begin{aligned}
\varepsilon(t) &= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\,d\tau = \int_{\tau=-\infty}^{t} D(t-\tau)\sigma_0\omega\cos(\omega\tau)\,d\tau \\
&= \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s)\sin(\omega s)\,ds\right] \sin(\omega t) + \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s)\cos(\omega s)\,ds\right] \cos(\omega t) \\
&= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}$$

$$\begin{aligned}
D'(\omega) &= \omega \int_{s=0}^{\infty} D(s)\sin(\omega s)\,ds \qquad &: \ \text{storage compliance} \\
D''(\omega) &= -\,\omega \int_{s=0}^{\infty} D(s)\cos(\omega s)\,ds \qquad &: \ \text{loss compliance}
\end{aligned}$$

Relation between D', D'' and δ

Writing the strain response with two different relations, results in relations between D', D'' and δ. The amplitude ε_0 of the strain response can also be calculated.

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t - \delta) = \varepsilon_0 \cos(\delta)\sin(\omega t) - \varepsilon_0 \sin(\delta)\cos(\omega t)$$
$$\varepsilon(t) = \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)$$

storage and loss compliance

$$\left.\begin{array}{l} D' = \dfrac{\varepsilon_0}{\sigma_0}\cos(\delta) \\[2ex] D'' = \dfrac{\varepsilon_0}{\sigma_0}\sin(\delta) \end{array}\right\} \quad \rightarrow \quad \left\{\begin{array}{l} \dfrac{D''}{D'} = \tan(\delta) \quad \rightarrow \\[2ex] \qquad\qquad \delta = \arctan\left(\dfrac{D''}{D'}\right) \end{array}\right.$$

amplitude $\qquad \varepsilon_0 = \sigma_0 \sqrt{(D')^2 + (D'')^2}$

Measured D' and D''

Typical measured values for $D'(\omega)$, $D''(\omega)$ are shown in the plots below. Again it is obvious that the loss compliance is zero for both very low and very high frequencies. The storage compliance is reversely proportional to the frequency.

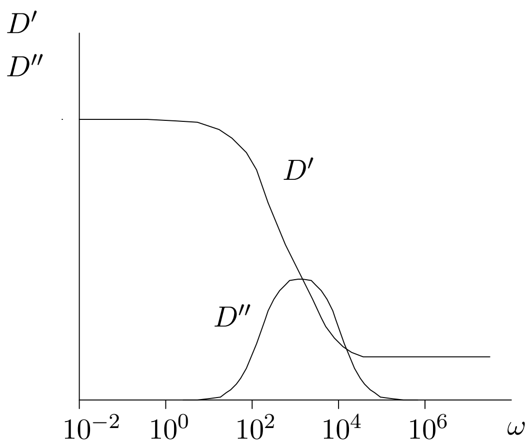


Fig. 3.50 : *Characteristic values of D' and D''*

Relation between (D', D'') and (E', E'')

There is a relation between storage and loss modulus on the one hand and storage and loss compliance on the other. Remember that there is **not** such a relation between the relaxation function and the creep function.

$$\left.\begin{array}{l} \sigma_0 = \varepsilon_0 \sqrt{(E')^2 + (E'')^2} \\ \\ \varepsilon_0 = \sigma_0 \sqrt{(D')^2 + (D'')^2} \end{array}\right\} \quad \rightarrow \quad [(E')^2 + (E'')^2][(D')^2 + (D'')^2] = 1 \tag{1}$$

$$\frac{D''}{D'} = \frac{E''}{E'} \quad \rightarrow \quad D'' = D' \frac{E''}{E'} \tag{2}$$

$$\begin{array}{lllll} (1) \ \& \ (2) & \rightarrow & D' = \dfrac{E'}{(E')^2 + (E'')^2} & ; & D'' = \dfrac{E''}{(E')^2 + (E'')^2} \\ \text{idem} & & E' = \dfrac{D'}{(D')^2 + (D'')^2} & ; & E'' = \dfrac{D''}{(D')^2 + (D'')^2} \end{array}$$

Complex variables

In literature on viscoelastic behavior and modeling, complex variables are often used. They can be derived easily by writing the strain excitation and the stress response as the real part of a complex number, where Euler's formula $e^{i\alpha x} = \cos(\alpha x) + i\sin(\alpha x)$ is used.

$$\begin{aligned} \varepsilon(t) &= \varepsilon_0 \sin(\omega t) = \varepsilon_0 \cos(\omega t - \tfrac{\pi}{2}) = Re\left[\varepsilon_0 e^{-i\frac{\pi}{2}} e^{i\omega t}\right] = Re\left[\varepsilon^* e^{i\omega t}\right] \\ \sigma(t) &= \sigma_0 \sin(\omega t + \delta) = \sigma_0 \cos(\omega t - \tfrac{\pi}{2} + \delta) = Re\left[\sigma_0 e^{i(\delta - \frac{\pi}{2})} e^{i\omega t}\right] = Re\left[\sigma^* e^{i\omega t}\right] \end{aligned}$$

complex modulus and compliance

$$\begin{aligned} E^* &= \frac{\sigma^*}{\varepsilon^*} = \frac{\sigma_0}{\varepsilon_0} e^{i\delta} = \frac{\sigma_0}{\varepsilon_0} \cos(\delta) + i \frac{\sigma_0}{\varepsilon_0} \sin(\delta) = E' + iE'' \\ D^* &= \frac{\varepsilon^*}{\sigma^*} = \frac{\varepsilon_0}{\sigma_0} e^{-i\delta} = \frac{\varepsilon_0}{\sigma_0} \cos(\delta) - i \frac{\varepsilon_0}{\sigma_0} \sin(\delta) = D' - iD'' \end{aligned}$$

dynamic modulus en compliance

$$\begin{aligned} E_d &= |E^*| = \sqrt{(E')^2 + (E'')^2} = \frac{\sigma_0}{\varepsilon_0} \\ D_d &= |D^*| = \sqrt{(D')^2 + (D'')^2} = \frac{\varepsilon_0}{\sigma_0} \end{aligned}$$

3.5.5 Viscoelastic models

The response of a viscoelastic material is given by the Boltzmann integral and to calculate it we need the creep and/or relaxation functions $D(t)$ and $E(t)$. Mathematical expressions can be chosen for these functions taking into account some general requirements. The chosen functions can than be fitted onto data from creep and relaxation tests. Instead of choosing rather arbitrary functions, they are generally derived from the behavior of one-dimensional mechanical spring-dashpot systems. Simple systems like the Maxwell, Kelvin-Voigt and Standard Solid element, are not always useful, because the lack of parameters prohibits a good fit of experimental data. In practice Generalized Maxwell or Generalized Kelvin-Voigt models are used.

Because creep and relaxation tests may need a long experimental time period and accuracy is not high, harmonic excitation tests are carried out to determine $D'(\omega)$, $D''(\omega)$, $E'(\omega)$ and $E''(\omega)$. These parameters can be converted to $D(t)$ and $E(t)$. These experiments are generally known as D(ynamic) M(echanical) A(nalysis) or D(ynamic) M(echanical) T(hermal) A(nalysis), because time-temperature superposition is mostly used.

In the following we will study some mechanical models. Their behavior is described by a differential equation. Solving this for stress or strain excitations results in the viscoelastic material functions.

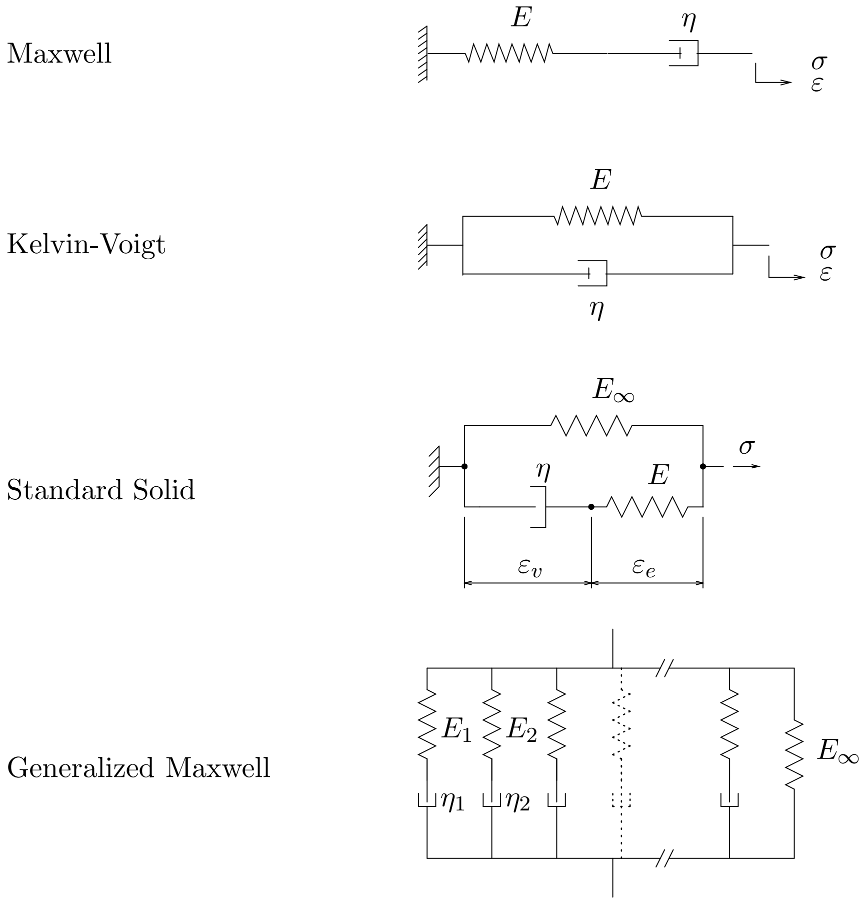


Fig. 3.51 : *Discrete mechanical models for viscoelastic material behavior*

Maxwell model

One of the simplest models to describe linear viscoelastic material behavior is the Maxwell model. It consists of a spring (modulus E) and a dashpot (viscosity η) in series.

The stress and strain in/of the Maxwell element is related by a first-order differential equation. For both stress and strain excitation, the differential equation can be solved, using appropriate initial conditions. General solutions – integrals for stress and strain – can be derived.

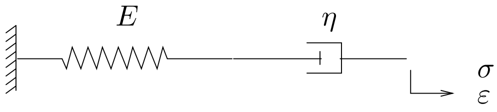


Fig. 3.52 : *Maxwell model*

$$\varepsilon = \varepsilon_E + \varepsilon_\eta \quad \rightarrow \quad \dot{\varepsilon} = \dot{\varepsilon}_E + \dot{\varepsilon}_\eta = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

Maxwell : step excitations

For step excitations of stress and strain the differential equation of the Maxwell model can be solved. The response represents the creep and relaxation functions, respectively.

$$\sigma(t) = \sigma_0 H(t,0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \delta(t,0)$$

$$\dot{\varepsilon}(t) = \frac{\sigma_0}{E}\,\delta(t,0) + \frac{\sigma_0}{\eta}$$

$$\varepsilon(t) = \frac{\sigma_0}{E} H(t,0) + \frac{\sigma_0}{\eta}\,t = \sigma_0 \left[\frac{1}{\eta}\left(t + \frac{\eta}{E} \right) \right] = \sigma_0 D(t)$$

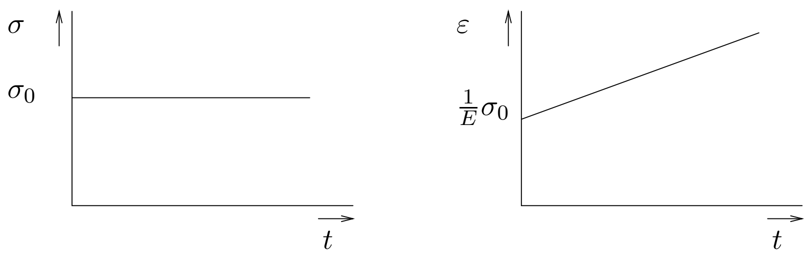


Fig. 3.53 : *Creep for a Maxwell model*

$$\varepsilon(t) = \varepsilon_0 H(t,0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0)$$

$$\sigma(t) = \varepsilon_0 E\, e^{-\frac{E}{\eta}t} = \varepsilon_0 E\, e^{-\frac{t}{\tau_m}} = \varepsilon_0 E(t)$$

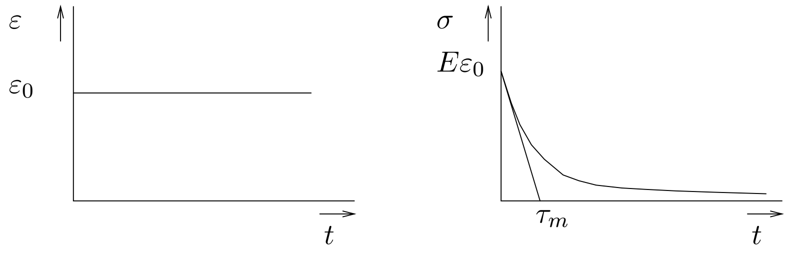


Fig. 3.54 : *Relaxation for a Maxwell model*

Maxwell : Boltzmann integrals

For a general stress and strain excitation the differential equation of the Maxwell model can also be solved. These general solutions are Boltzmann integrals, which can be used to calculate strain/stress responses to stress/strain excitations.

The creep and relaxation functions of the Maxwell model are readily recognized in the integrals. Response to step excitations reveals that the Maxwell model describes viscoelastic fluid behavior, characterized by a time constant $\tau = \frac{\eta}{E}$ [s].

$$\varepsilon(t) = \frac{1}{\eta}\int_{\tau=-\infty}^{t}\left\{(t-\tau)+\frac{\eta}{E}\right\}\dot{\sigma}(\tau)\,d\tau = \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\,d\tau$$

$$\sigma(t) = \int_{\tau=-\infty}^{t}\left\{Ee^{-\frac{E}{\eta}(t-\tau)}\right\}\dot{\varepsilon}(\tau)\,d\tau = \int_{\tau=-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\,d\tau$$

Maxwell : harmonic stress excitation

The strain response of the Maxwell model to an harmonic stress excitation is readily calculated from the differential equation. Storage and loss compliances are thus determined.

$$\sigma(t) = \sigma_0\sin(\omega t) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0\omega\cos(\omega t)$$

strain response

$$\begin{aligned}
\dot{\varepsilon}(t) &= = \frac{1}{E}\sigma_0\omega\cos(\omega t) + \frac{1}{\eta}\sigma_0\sin(\omega t) \\
\varepsilon(t) &= \sigma_0\left[\frac{1}{E}\right]\sin(\omega t) - \sigma_0\left[\frac{1}{\eta\omega}\right]\cos(\omega t) \\
&= \varepsilon_P(t) \qquad\qquad\qquad \varepsilon_H \text{ damps out} \\
&= \sigma_0 D'\sin(\omega t) - \sigma_0 D''\cos(\omega t)
\end{aligned}$$

dynamic quantities

$$D' = \frac{1}{E} \quad ; \quad D'' = \frac{1}{\eta\omega} \quad ; \quad \delta = \arctan\left(\frac{D''}{D'}\right) = \arctan\left(\frac{E}{\eta\omega}\right)$$

Maxwell : harmonic strain excitation

With the Boltzmann integral for the Maxwell model, the stress response to an harmonic strain excitation can be calculated. Storage and loss moduli are obtained as a function of ω. Comparing these functions with measured values reveals that the Maxwell model is generally not adequate to describe viscoelastic behavior of real materials.

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \qquad \rightarrow \qquad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$

stress response

$$\begin{aligned}
\sigma(t) &= \int_{\tau=-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau \\
&= E\varepsilon_0\omega e^{-\frac{E}{\eta}t} \int_{\tau=0}^{t} e^{\frac{E}{\eta}\tau} \cos(\omega\tau)\, d\tau \\
&= \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2 + \omega^2}\, \frac{E}{\eta} \right] e^{-\frac{E}{\eta}t} + \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2 + \omega^2}\, \omega \right] \sin(\omega t) + \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2 + \omega^2}\, \frac{E}{\eta} \right] \cos(\omega t) \\
&= \varepsilon_0 \left[\frac{E\omega}{(\frac{E}{\eta})^2 + \omega^2}\, \omega \right] \sin(\omega t) + \varepsilon_0 \left[\frac{E\omega}{(\frac{E}{\eta})^2 + \omega^2}\, \frac{E}{\eta} \right] \cos(\omega t) \qquad \text{for } t \geq 0 \\
&= \varepsilon_0 \left[\frac{E\omega^2\tau_m^2}{1+\omega^2\tau_m^2} \right] \sin(\omega t) + \varepsilon_0 \left[\frac{E\omega\tau_m}{1+\omega^2\tau_m^2} \right] \cos(\omega t) \\
&= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)
\end{aligned}$$

dynamic quantities

$$E' = \frac{E\omega^2}{(\frac{E}{\eta})^2 + \omega^2} \quad ; \quad E'' = \frac{E\omega(\frac{E}{\eta})}{(\frac{E}{\eta})^2 + \omega^2} \quad ; \quad \tan(\delta) = \frac{E''}{E'} = \frac{1}{\omega\tau_m}$$

Kelvin-Voigt model

The Kelvin-Voigt model is a simple model for the description of linear viscoelastic material behavior. It consists of a spring (modulus E) parallel to a dashpot (viscosity η).

The stress and strain in/of the Kelvin-Voigt element is related by a first-order differential equation. For strain excitation, this equation directly describes the stress response. For stress excitation, a general integral solution of the differential equation can be derived.

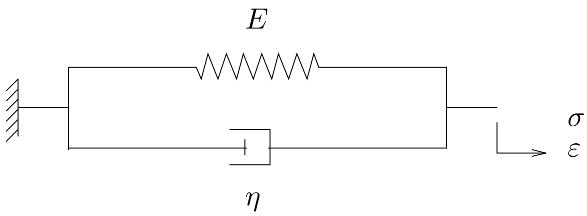


Fig. 3.55 : *Kelvin model*

$$\sigma = \sigma_E + \sigma_\eta = E\varepsilon + \eta\dot{\varepsilon}$$

Kelvin-Voigt : step excitations

Strain response to a step excitation of stress reveals that the Kelvin-Voigt model describes viscoelastic solid behavior, characterized by the time constant $\tau = \frac{\eta}{E}$ [s]. A stepwise strain excitation leads to infinite stress.

$$\sigma(t) = \sigma_0 H(t,0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \delta(t,0)$$

$$\left. \begin{array}{l} \eta\dot{\varepsilon}(t) + E\varepsilon(t) = \sigma(t) = \sigma_0 H(t,0) \\ \varepsilon(t) = \varepsilon_H(t) + \varepsilon_P = C\, e^{-\frac{E}{\eta}t} + \dfrac{\sigma_0}{E} \\ \varepsilon(t=0) = 0 \end{array} \right\} \quad \rightarrow \qquad C = -\frac{\sigma_0}{E}$$

$$\varepsilon(t) = \frac{\sigma_0}{E}\left[1 - e^{-\frac{E}{\eta}t}\right] = \sigma_0 D(t)$$

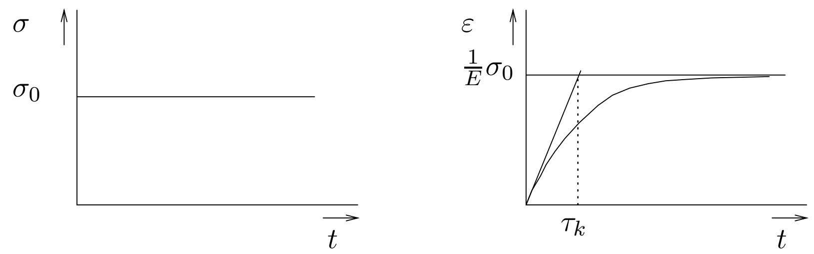


Fig. 3.56 : *Creep of a Kelvin model*

$$\varepsilon(t) = \varepsilon_0 H(t,0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0)$$

$$\begin{array}{l} \sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t) \\ \sigma(t) = E\varepsilon_0 + \eta\varepsilon_0\delta(t,0) = \varepsilon_0\left[E + \eta\delta(t,0)\right] = \infty \end{array}$$

Kelvin-Voigt : Boltzmann integral

The general solution for the strain response to a stress excitation is given by a Boltzmann integral, in which we recognize the creep function of the Kelvin-Voigt element. For a general strain excitation the stress response can be calculated directly from the Kelvin-Voigt element equation.

$$\varepsilon(t) = \frac{1}{E}\int_{\tau=-\infty}^{t} \left\{1 - e^{-\frac{E}{\eta}(t-\tau)}\right\} \dot{\sigma}(\tau)\, d\tau = \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau$$

Kelvin-Voigt : harmonic stress excitation

For the Kelvin-Voigt model, the storage and loss compliance can be calculated. The Boltzmann integral with the Kelvin-Voigt creep function is used to calculate the strain response for an harmonic stress excitation.

$$\sigma(t) = \sigma_0 \sin(\omega t) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

strain response

$$\begin{aligned} \varepsilon(t) &= \int_{\tau=0}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau \\ &= \sigma_0 \left[\frac{1}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta^2} \right] \sin(\omega t) - \sigma_0 \left[\frac{\omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{1}{\eta} \right] \cos(\omega t) \\ &= \sigma_0 \left[\frac{1}{E(1+\omega^2\tau_k^2)} \right] \sin(\omega t) - \sigma_0 \left[\frac{\omega\tau_k}{E(1+\omega^2\tau_k^2)} \right] \cos(\omega t) \\ &= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t) \end{aligned}$$

dynamic quantities

$$\begin{aligned} D'(\omega) &= \frac{1}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta^2} = \frac{1}{E(1+\omega^2\tau_k^2)} \\ D''(\omega) &= \frac{\omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{1}{\eta} = \frac{\omega\tau}{E(1+\omega^2\tau_k^2)} \\ \tan(\delta) &= \frac{D''}{D'} = \omega\tau_k \qquad \rightarrow \qquad \delta = arctan\left(\frac{\eta\omega}{E}\right) \end{aligned}$$

Standard Solid model

The Standard Solid model consists of a parallel arrangement of a Maxwell element (modulus E, viscosity η) and a linear spring (modulus E_∞). This model incorporates the Maxwell model ($E_\infty = 0$) and the Kelvin-Voigt model ($E = 0$). The stress-strain relation is described by a differential equation, which can be solved resulting in Boltzmann integrals for strain and stress.

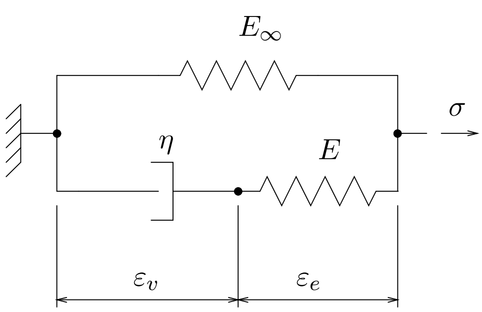


Fig. 3.57 : *Standard Solid model*

constitutive relations

- $\sigma = \sigma_\infty + \sigma_{ve}$
- $\dot{\varepsilon} = \dot{\varepsilon}_v + \dot{\varepsilon}_e$
- $\dot{\varepsilon}_v = \dfrac{1}{\eta}\,\sigma_{ve}$
- $\sigma_{ve} = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\,\dot{\sigma}_{ve}$
- $\varepsilon = \dfrac{1}{E_\infty}\,\sigma_\infty$

constitutive equation

$$
\begin{aligned}
\sigma &= \sigma_\infty + \sigma_{ve} = E_\infty\varepsilon + \eta\dot{\varepsilon}_v \\
&= E_\infty\varepsilon + \eta(\dot{\varepsilon} - \dot{\varepsilon}_e) = E_\infty\varepsilon + \eta\dot{\varepsilon} - \eta\,\frac{\dot{\sigma}_{ve}}{E} \\
&= E_\infty\varepsilon + \eta\dot{\varepsilon} - \frac{\eta}{E}\,(\dot{\sigma} - E_\infty\dot{\varepsilon}) \quad \rightarrow \\
\sigma + \frac{\eta}{E}\,\dot{\sigma} &= E_\infty\varepsilon + \frac{\eta(E+E_\infty)}{E}\,\dot{\varepsilon}
\end{aligned}
$$

Standard Solid : step excitations

Solutions for the differential equation when applying a step in the stress or a step in the strain can be derived. The time constant for creep is defined as $\tau_c = \frac{\eta}{E} + \frac{\eta}{E_\infty}$ and the time constant for relaxation as $\tau_r = \frac{\eta}{E}$. They represent the intersection point of the tangent to the creep/relaxation curve at $t = 0$ and the asymptote for strain ($\frac{\sigma_0}{E_\infty}$) and stress ($\varepsilon_0\,E_\infty$), respectively.

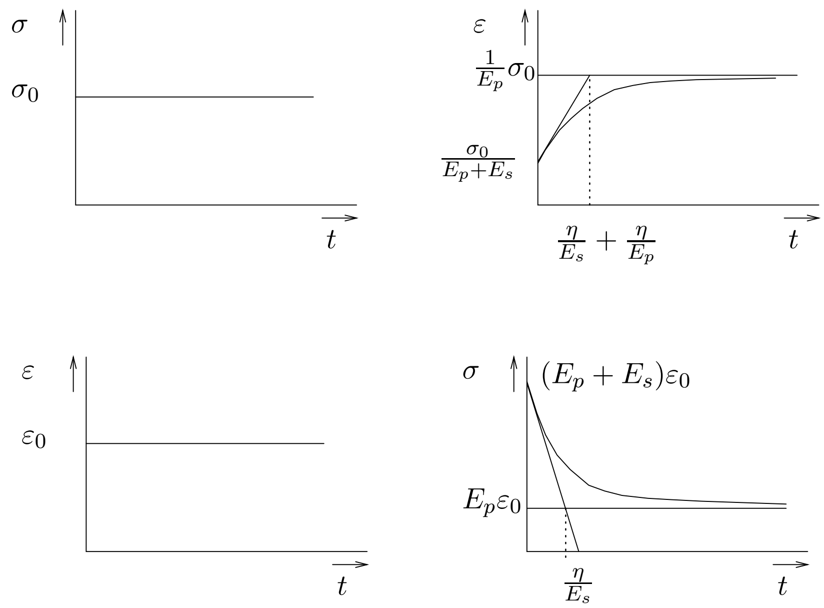


Fig. 3.58 : *Creep and relaxation of a Standard Solid model*

Standard Solid : Boltzmann integrals

In the Boltzmann integrals for strain and stress, the creep and relaxation functions of the Standard Solid element are readily recognized.

$$
\begin{aligned}
\varepsilon(t) &= \int_{\tau=-\infty}^{t} \left\{ \frac{1}{E_\infty} - \frac{E}{E_\infty(E_\infty + E)} e^{-\frac{E_\infty E}{\eta(E_\infty+E)}(t-\tau)} \right\} \dot{\sigma}(\tau)\, d\tau \\
&= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau \\
\sigma(t) &= \int_{\tau=-\infty}^{t} \left\{ E_\infty + E e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\varepsilon}(\tau)\, d\tau \\
&= \int_{\tau=-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau
\end{aligned}
$$

Generalized Maxwell model

Both the Maxwell and the Kelvin-Voigt models are too simple to describe the viscoelastic behavior of real materials. Combining a number of Maxwell elements in a parallel configuration, leads to the generalized Maxwell model, which mostly also has an extra parallel spring for the correct description of long-term behavior of viscoelastic solid materials. Such a model is generally used for experimental characterization of the behavior of linear viscoelastic materials in a Dynamic Mechanical (Thermal) Analysis (DM(T)A) test.

The creep function $E(t)$ is easily determined and has a number of time constants to characterize the viscoelastic material response. A model like the generalized Maxwell model is therefore also referred to as *multi mode*.

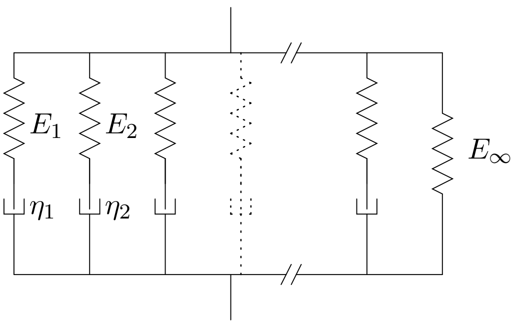


Fig. 3.59 : *Generalized Maxwell model*

$$
E(t) = E_\infty + \sum_i E_i e^{-\frac{t}{\tau_i}} \qquad ; \qquad \tau_i = \frac{\eta_i}{E_i}
$$

equilibrium modulus $\qquad E_\infty = \lim_{t \longrightarrow \infty} E(t)$

glass modulus $\qquad E_g = \lim_{t \longrightarrow 0} E(t) = E_\infty + \sum_i E_i$

Generalized Kelvin model

The generalized Kelvin model consists of a number of Kelvin-Voigt elements arranged in series. An extra spring – sometimes a dashpot – is also provided.

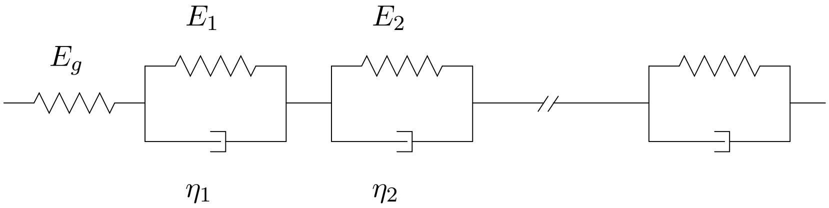


Fig. 3.60 : *Generalized Kelvin model*

$$
\begin{aligned}
D(t) &= \frac{1}{E_g} + \sum_i \frac{1}{E_i}(1 - e^{-\frac{t}{\tau_i}}) \qquad ; \qquad \tau_i = \frac{\eta_i}{E_i} \\
&= D_g + \sum_i D_i(1 - e^{-\frac{t}{\tau_i}})
\end{aligned}
$$

glass compliance $\quad D_g = \frac{1}{E_g} = \lim_{t \longrightarrow 0} D(t)$

equilibrium compliance $\quad D_\infty = \lim_{t \longrightarrow \infty} D(t) = D_g + \sum_i D_i$

3.5.6 Stress update

The current stress is given by a Boltzmann integral over the strain history.

Using a Generalized Maxwell model to specify the relaxation function $E(t)$, an expression for $\sigma(t)$ can be derived.

$$
\left.
\begin{aligned}
\sigma(t) &= \int\limits_{\tau=0}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau \\
E(t) &= E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{t}{\tau_i}}
\end{aligned}
\right\} \quad \rightarrow
$$

$$
\begin{aligned}
\sigma(t) &= \int\limits_{\tau=0}^{t} \left[E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{t-\tau}{\tau_i}} \right] \dot{\varepsilon}(\tau)\, d\tau = E_\infty \varepsilon(t) + \sum_{i=1}^{N} \int\limits_{\tau=0}^{t} E_i e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau)\, d\tau \\
&= E_\infty \varepsilon(t) + \sum_{i=1}^{N} \sigma_i(t)
\end{aligned}
$$

Time discretization

In the numerical analysis of the time dependent behavior, the total time interval $[0, t]$ is discretized :

$$[0, t] \quad \rightarrow \quad [t_1 = 0, t_2, t_3, .., t_n, t_{n+1} = t]$$

The timespan between two discrete moments in the time interval is a time increment. It is assumed that these increments are of equal length :

$$\Delta t = t_{j+1} - t_j \qquad ; \quad j = 1, ..., n$$

The Boltzmann integral is now split in an integral over $[0, t_n]$ and an integral over the last or current increment $[t_n, t_{n+1} = t]$.

$$\begin{aligned}
\sigma(t) &= E_\infty \varepsilon(t) + \sum_{i=1}^{N} \sigma_i(t) = E_\infty \varepsilon(t) + \sum_{i=1}^{N} \int_{\tau=0}^{t} E_i e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau)\, d\tau \\
&= E_\infty \varepsilon(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \int_{\tau=0}^{t_n} E_i e^{-\frac{t_n-\tau}{\tau_i}} \dot{\varepsilon}(\tau)\, d\tau + E_i \int_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau)\, d\tau \right]
\end{aligned}$$

Linear incremental strain

For further evaluation of $\sigma(t)$ it is assumed that the strain is a linear function of time in each time increment. For the current increment we have :

$$\varepsilon(\tau) = \varepsilon(t_n) + (\tau - t_n)\, \frac{\Delta\varepsilon}{\Delta t} \quad \rightarrow \quad \dot{\varepsilon}(\tau) = \frac{\Delta\varepsilon}{\Delta t}$$

The integral over the current increment can now be evaluated very easily.

$$\int_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau)\, d\tau = \frac{\Delta\varepsilon}{\Delta t} \int_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, d\tau = \frac{\Delta\varepsilon}{\Delta t}\, \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)$$

Stress

Calculating the current stress does not mean that the Boltzmann integral has to be evaluated over the total deformation history. When results are stored properly we can easily update the stress $\sigma(t)$.

$$\begin{aligned}
\sigma(t) &= E_\infty \varepsilon(t) + \sum_{i=1}^{N} \sigma_i(t) \\
&= E_\infty \varepsilon(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \int_{\tau=0}^{t_n} E_i e^{-\frac{t_n-\tau}{\tau_i}} \dot{\varepsilon}(\tau)\, d\tau + E_i \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta\varepsilon}{\Delta t} \right]
\end{aligned}$$

$$\sigma(t) = E_\infty \varepsilon(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \sigma_i(t_n) + E_i p_i \Delta\varepsilon \right]$$

$$\text{with} \qquad p_i = \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}} \right)$$

3.5.7 Stiffness

The current stiffness of the material is the derivative of the stress with respect to the stretch ratio. Because the linear strain is used here, the derivative w.r.t. strain has the same value.

$$\sigma(t) = E_\infty \varepsilon(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \sigma_i(t_n) + E_i p_i \Delta\varepsilon \right] \quad \rightarrow$$

$$\frac{\partial \sigma}{\partial \lambda} = C_\lambda = C_\varepsilon = E_\infty + \sum_{i=1}^{N} E_i p_i$$

3.5.8 Implementation

See `tr2dviel.m` for the implementation.

3.5.9 Examples

A truss with length 100 mm and cross-sectional area 10 mm^2 is loaded with a time dependent axial strain. The stress is calculated as a function of time.

Strain step

A strain step with an amplitude of 0.1 is applied and the stress response is calculated for the Maxwell and the Standard-Solid models. Rather fictitious values for the material parameters are chosen. The initial stress can be verified, using the strain amplitude and the initial stiffness. The final stress value can be verified, using the strain amplitude and the equilibrium modulus.

Maxwell	$E_\infty = 0$	$E_1 = 1$	$\tau_1 = 0.01$
Standard-Solid	$E_\infty = 1$	$E_1 = 1$	$\tau_1 = 0.01$

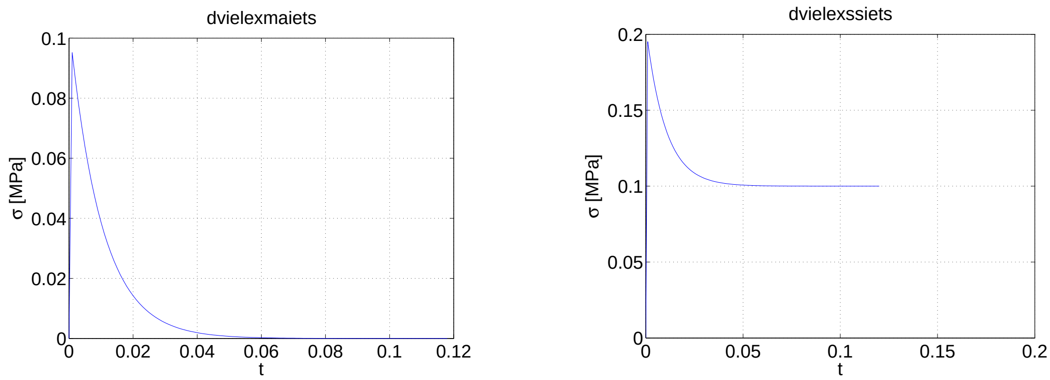


Fig. 3.61 : *Stress response for Maxwell and Standard-Solid model*

Linear viscoelastic models

When stress or strain is prescribed as a function of time, the strain or stress can be calculated. The examples show stress responses for a prescribed strain excitation, being a strain step ($\varepsilon_0 = 0.01$) followed by a constant strain rate ($\dot{\varepsilon} = 0.1$ [s^{-1}]).

The stress response is calculated, using a Maxwell, a Kelvin-Voigt, a Standard-Solid and a 2-mode model. Parameter values are listed in the table below.

	E_∞	E_1	τ_1	E_2	τ_2	ν
Maxwell	0	100	0.1	0	0	0
Kelvin-Voigt	100	10^{10}	10^{-12}	0	0	0
Standard-Solid	100	100	0.1	0	0	0
2-mode	100	100	0.1	100	0.1	0

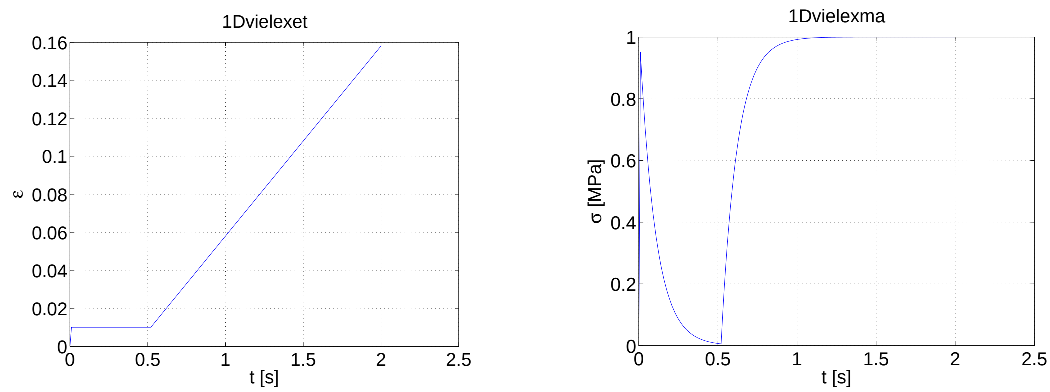


Fig. 3.62 : *Prescribed strain and stress response for Maxwell model*

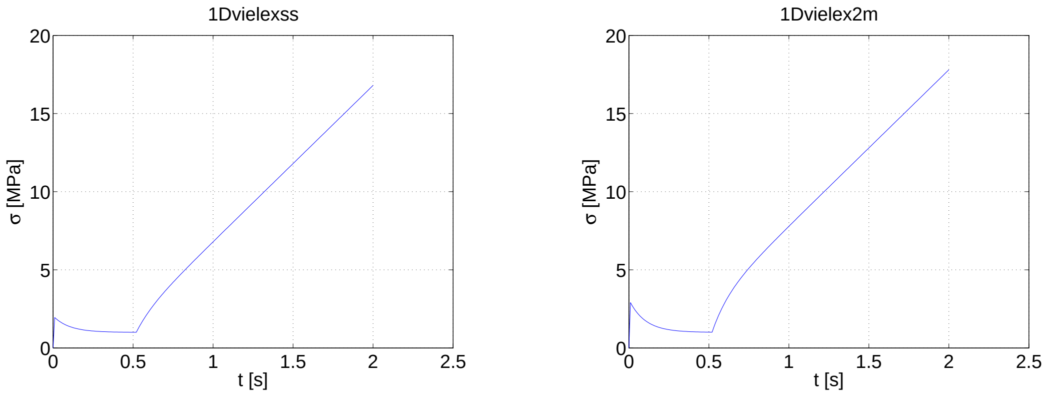


Fig. 3.63 : *Stress response for Standard-Solid and 2-mode model*

Multi-mode model response

An axial strain step with amplitude 0.01 is prescribed on an tensile bar with initial cross-sectional area $A_0 = 10$ mm^2. The stress response is calculated for a 12-mode generalized Maxwell model. The modal parameters are listed in the table below.

	E [MPa]	τ [s]		E [MPa]	τ [s]
1	3.0e6	3.1e-8	2	1.4e6	3.0e-7
3	3.9e6	3.0e-6	4	5.4e6	2.9e-5
5	1.3e6	2.8e-4	6	2.3e5	2.7e-3
7	7.6e4	2.6e-2	8	3.7e4	2.5e-1
9	3.3e4	2.5e+0	10	1.7e4	2.4e+1
11	8.0e3	2.3e+2	12	1.2e4	2.2e+3

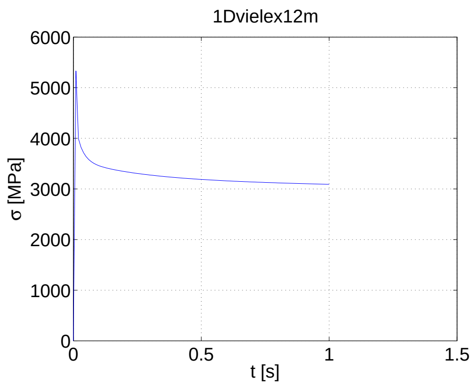


Fig. 3.64 : *Tensile stress versus time*

3.6 Creep behavior

The phenomenon called *creep* is the deformation under constant load. Viscoelastic material shows this behavior. The term *creep* however is especially reserved for deformation at temperatures, which are considered to be high with respect to the melting temperature T_m of the material, e.g. $T > 0.2T_m$. Such high temperatures are encountered in eg. (jet) engines and heat exchangers. Some materials with a low melting point, like lead (Pb), show creep at room temperature.

During the so-called stage I or primary creep, the strain rate decreases as a function of time. The strain rate is constant for stage II or secondary creep, also called steady state creep. Stage III or tertiary creep shows an increased strain rate and eventually leads to creep fracture or rupture.

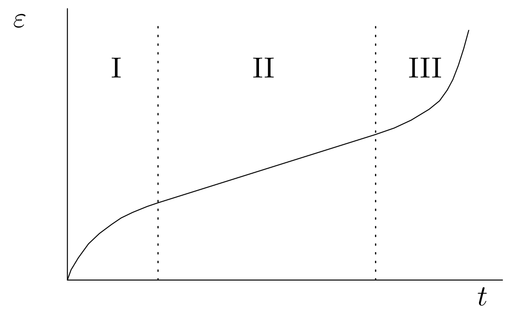


Fig. 3.65 : *Creep strain as a function of time at constant stress*

The creep behavior is influenced strongly by both stress and temperature, as is illustrated in the figure below. Obviously therefore, creep behavior is described by relating the creep strain rate $\dot{\varepsilon}_c$ to stress, temperature and time. The temperature dependency could be included by making material parameters a function of temperature. It is more convenient, however, to implement the temperature dependency explicitly in the creep model. Much used are so-called power law models.

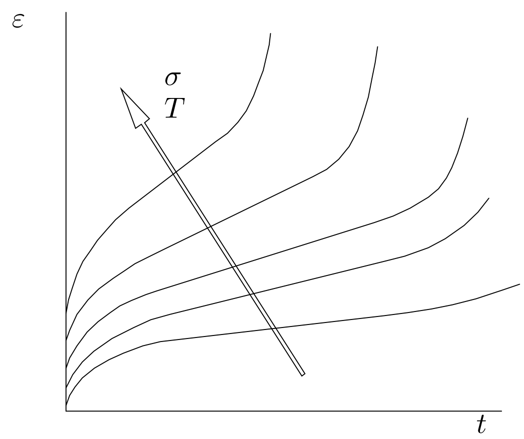


Fig. 3.66 : *The influence of stress and temperature on the creep strain rate*

general model $\qquad \dot{\varepsilon}_c = A\, f_\sigma(\sigma) f_\varepsilon(\varepsilon_c) f_T(T) f_t(t)$

power law model $\qquad \dot{\varepsilon}_c = A\sigma^m \varepsilon_c^n T^p \left(qt^{q-1}\right)$

Primary creep

Primary creep – also referred to as stage I, transient creep or delayed elastic effect – is observed at $T < 0.4T_m$. Mechanisms, which are associated with this behavior are dislocation coalescence and dislocation entanglement, leading to slip steps (jogs). Dislocations may pile-up at grain boundaries and impurities. All this leads to macroscopic hardening. When temperature is higher, $0.4T_m < T < 0.5T_m$, the thermal activity of dislocations is higher and a transition to secondary creep is seen.

Secondary creep

Secondary creep – also referred to as stage II, steady-state creep or viscous flow – is observed at $0.5T_m < T < 0.6T_m$. The hardening, which is apparent in primary creep is balanced by recovery, leading to thermal softening. The thermal energy leads to vacancy movement (self diffusion) and this causes dislocation movement (climb). The moving dislocations can annihilate, align and/or pass obstacles. More drastic recovery may be caused by recrystallization, which can occur when internal stresses exist.

The temperature dependency is generally included with an Arrhenius function $\exp(Q_c/RT)$, where Q_c is the creep actuation energy and R is Boltzmann's constant. The available evidence indicates that stage II creep is diffusion controlled, and so in the models the activation energy for creep, Q_c, can often be replaced by the activation energy for selfdiffusion Q_{sd}.

Most models for stage II creep are based on the *five-power-law* creep law. For temperatures below 0.5-0.6 T_m a transition toward primary creep is observed, which in reference to the modeling is called *power-law-breakdown*. Sometimes a threshold stress is introduced below which no creep can be measured.

Tertiary creep

Tertiary creep – also referred to as stage III or accelerating creep – is observed at $0.6T_m < T < 0.8T_m$ and is associated with geometric instabilities and damage.

One mechanism is grain boundary sliding and subsequent void initiation and coalescence, leading to inter granular cracks. Another mechanism is diffusional flow, which occurs mainly at higher temperatures and lower stresses. Two possibilities are : 1) diffusion through grains (*Nabarro-Herring creep*) with slow vacancy jump frequencies along many paths, and 2) diffusion along grain boundaries
(*Coble creep*) with high vacancy jump frequencies along a few paths.

Stage III creep is often modeled with continuum damage mechanics, where a damage variable is used to model internal damage, which influences the creep strain rate. An evolution equation is required to control the damage growth as a function of stress and/or strain.

Stress functions

Several authors have reported various functions f_σ to implement the influence of the stress.

Norton; Bailey (1929)	$\dot{\varepsilon}_c = K\sigma^n$
Hooke-Norton	$\dot{\varepsilon}_c = \dfrac{\dot{\sigma}}{E} + K\sigma^n$
Johnson et.al. (1963)	$\dot{\varepsilon}_c = D_1\sigma^{n_1} + D_2\sigma^{n_2}$
Dorn (1955)	$\dot{\varepsilon}_c = B\exp(\beta\sigma)$
Soderberg (1936)	$\dot{\varepsilon}_c = B\left[\exp\left(\dfrac{\sigma}{\sigma_0}\right) - 1\right]$
Prandtl (1928)	$\dot{\varepsilon}_c = A\sinh\left(\dfrac{\sigma}{\sigma_0}\right)$
Garofalo (1965)	$\dot{\varepsilon}_c = A\left[\sinh\left(\dfrac{\sigma}{\sigma_0}\right)\right]^n$
Lemaitre, Chaboche (1985)	$\dot{\varepsilon}_c = \left(\dfrac{\sigma}{\lambda_0}\right)^{N_0}\exp\left(\alpha\sigma^{N_0+1}\right)$

Temperature functions

Several authors have reported various functions f_T to implement the influence of the temperature. These creep models also take into account the dependency of stress and (sometimes) time.

Kauzmann (1941)	$\dot{\varepsilon}_c = A\ \exp\left(-\dfrac{\Delta H - \gamma\sigma}{RT}\right)$
Lifszic (1963)	$\dot{\varepsilon}_c = \dfrac{\sigma}{T}\ \exp\left(-\dfrac{\Delta H}{RT}\right)$
Dorn, Tietz (1949/55)	$\varepsilon_c = f\left(t\ \exp\left[-\dfrac{\Delta H}{RT}\right]\right) f_\sigma(\sigma)$
Penny, Marriott (1971)	$\varepsilon_c = \left(t\ \exp\left[-\dfrac{\Delta H}{RT}\right]\right)^n f_\sigma(\sigma)$
Boyle, Spence (1983)	$\varepsilon_c = C\ \exp\left(-\dfrac{\Delta H}{RT}\right) t^m\sigma^n$

Time functions

Several authors have reported various functions f_t to implement the influence of the time.

Andrade (1910)	$\varepsilon_c = \ln\left(1 + \beta t^{\frac{1}{3}}\right) + kt$
Andrade (small ε)	$\varepsilon_c = \beta t^{\frac{1}{3}} + kt \approx \beta t^{\frac{1}{3}}$

Bailey (1935)	$\varepsilon_c = F t^n$
Graham, Walles (1955)	$\varepsilon_c = \sum_{j=1}^{M} a_j t^{m_j}$
McVetty (1934)	$\varepsilon_c = G\,(1 - \exp(-qt)) + Ht$
Findley et.al. (1944)	$\varepsilon_c = \varepsilon_1 + \varepsilon_2 t^n \qquad (n < 1)$
Pugh (1975)	$\varepsilon_c = \dfrac{a_1 t}{1 + b_1 t} + \dfrac{a_2 t}{1 + b_2 t} + \dot{\varepsilon}_m t$
Garofalo	$\varepsilon_c = \varepsilon_t (1 - e^{-rt}) + \dot{\varepsilon}_s\, t$

3.6.1 Creep model

The discrete mechanical model for creep is a Maxwell element with a non-linear dashpot. The viscous or creep strain rate may be a function of stress σ, total creep strain ε_c, absolute temperature T and time t.

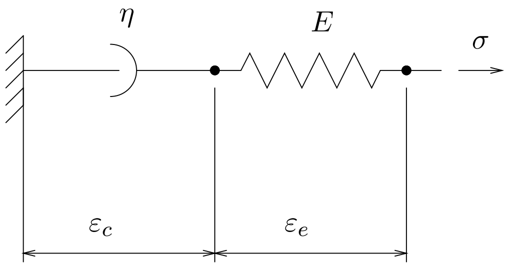


Fig. 3.67 : *Creep model*

constitutive relations

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_c$
- $\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\,\dot{\sigma}$
- $\dot{\varepsilon}_c = A\, f_\sigma(\sigma)\, f_{\varepsilon_c}(\varepsilon_c)\, f_T(T)\, f_t(t) = f(\sigma, \varepsilon_c, T, t)$

constitutive equation

$$\dot{\sigma} = E\dot{\varepsilon}_e = E\dot{\varepsilon} - E\dot{\varepsilon}_c = E\dot{\varepsilon} - Ef(\sigma, \varepsilon_c, T, t)$$

3.6.2 Stress update

The constitutive equation, can be solved explicitly or implicitly. For the latter case, a Newton iteration procedure must be implemented to calculate the stress.

$$
\begin{aligned}
&\dot{\sigma} = E\dot{\varepsilon} - Ef(\sigma, \varepsilon_c, T, t) \\
&\Delta\sigma = E\Delta\varepsilon - \Delta t Ef(\sigma, \varepsilon_c, T, t) \\
&\sigma - \sigma_n = E(\varepsilon - \varepsilon_n) - \Delta t Ef(\sigma, \varepsilon_c, T, t)
\end{aligned}
$$

Implicit stress update

In the implicit procedure the end-increment stress is determined iteratively.

$$
\begin{aligned}
\sigma - \sigma_n &= E(\varepsilon - \varepsilon_n) - \Delta t Ef(\sigma, \varepsilon_c, T, t) \\
\sigma^* + \delta\sigma - \sigma_n &= E(\varepsilon - \varepsilon_n) - \Delta t E(f^* + \delta f) = E(\varepsilon - \varepsilon_n) - \Delta t Ef^* - \Delta t E\delta f \\
&= E(\varepsilon - \varepsilon_n) - \Delta t Ef^* - \Delta t E\frac{\partial f}{\partial \sigma}\,\delta\sigma \quad \rightarrow \\
\left[1 + \Delta t E\,\frac{\partial f}{\partial \sigma}\right]\delta\sigma &= -\sigma^* + \sigma_n + E(\varepsilon - \varepsilon_n) - \Delta t Ef^*
\end{aligned}
$$

Explicit stress update

In the explicit procedure the end-increment stress is determined directly.

$$
\sigma = \sigma_n + E(\varepsilon - \varepsilon_n) - \Delta t\, Ef(\sigma_n, \varepsilon_{c_n}, T_n, t_n)
$$

3.6.3 Stiffness

The material stiffness C_ε is the ratio of the variation of stress and strain.

implicit

$$
\begin{aligned}
&\sigma - \sigma_n - E\varepsilon + E\varepsilon_n + \Delta t\, Ef(\sigma, \varepsilon_c, T, t) = 0 \\
&\delta\sigma + \Delta t\, E\,\left.\frac{\partial f}{\partial \sigma}\right|^* \delta\sigma - E\delta\varepsilon = 0 \\
&C_\varepsilon = \left(1 + \Delta t\, E\,\left.\frac{\partial f}{\partial \sigma}\right|^*\right)^{-1} E
\end{aligned}
$$

explicit

$$
\begin{aligned}
&\sigma - \sigma_n - E\varepsilon + E\varepsilon_n + \Delta t\, Ef(\sigma_n, \varepsilon_{c_n}, T_n, t_n) = 0 \\
&\delta\sigma = E\delta\varepsilon \quad \rightarrow \quad C_\varepsilon = E
\end{aligned}
$$

3.6.4 Implementation

See `tr2delvi.m` for the implementation.

3.6.5 Examples

In all examples a truss is subjected to an axial stress or strain.

Creep versus viscoelasticity

Linear viscoelastic behavior can be modeled with a multi-mode Maxwell model, represented by a mechanical system, which has a number of parallel Maxwell elements and one parallel spring. Springs and dashpots are linear.

Creep behavior is modeled with one Maxwell model with a nonlinear dashpot. The viscosity is a nonlinear function of stress, creep strain, temperature and time.

The Norton model for secondary creep can be made equivalent to the linear Maxwell model.

Maxwell model (E, η)

$$\varepsilon = \varepsilon_e + \varepsilon_c \quad ; \quad E(t) = Ee^{t/\tau} \quad ; \quad \tau = \frac{\eta}{E} \quad ; \quad \dot{\varepsilon}_c = \frac{\sigma}{\eta} \quad ; \quad \varepsilon_e = \frac{\sigma}{E}$$

Norton model (A, m)

$$\varepsilon = \varepsilon_e + \varepsilon_c \quad ; \quad \dot{\varepsilon}_c = f(\sigma, \varepsilon_c, T, t)\dot{\varepsilon}_c = A\sigma^m \quad ; \quad \varepsilon_e = \frac{\sigma}{E}$$

equivalence

Maxwell	$E = 10^9$	$\eta = 10^9$	$\tau = 1$
Norton	$E = 10^9$	$A = \frac{1}{\eta} = 10^{-9}$	$m = 1$

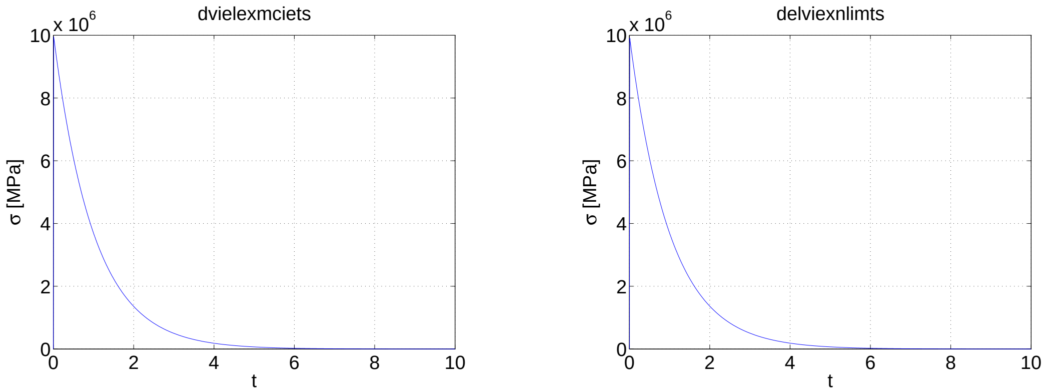


Fig. 3.68 : *Stress response for Maxwell viscoelastic and Norton creep model*

General creep model for SnAg-solder

Evans and Wilshire and later Maruyama and Oikawa proposed a general model, which describes the primary, secondary and tertiary creep of alloys. Parameters in the model must be fitted onto experimental data.

The creep strain at time t is described by two terms. The first one describes the hardening or primary creep stage and the second describes the weakening or tertiary creep stage. Combined, they characterize also the transition region, the secondary creep.

$$\begin{aligned}
&\varepsilon_c(t) = \varepsilon_0 + A(\sigma)\left[1 - e^{-\alpha(\sigma,T)t}\right] + B(\sigma,T)\left[e^{\alpha(\sigma,T)t} - 1\right] \\
&\alpha(\sigma,T) = c_1\left[\sinh(\beta\sigma)\right]^{n_1} e^{-\frac{Q_1}{RT}} \\
&A(\sigma) = c_2\sigma^{n_2} \qquad ; \qquad B(\sigma,T) = c_3\sigma^{n_3}e^{-\frac{Q_2}{RT}}
\end{aligned}$$

From the general model for the creep strain the creep strain rate $\dot{\varepsilon}_c$ can be calculated and subsequently the initial creep rate $\dot{\varepsilon}_{c,i}$, the time t_m for the minimum creep rate $\dot{\varepsilon}_{c,m}$ to occur and the strain $\varepsilon_{c,m}$ at that time.

With the universal gas constant $R = 8.314$ and stress in MPa and Q in kJ/mol, parameter values for SnAg-solder are fitted on experimental data and listed in the table below. The absolute temperature is assumed to be $T = 398$ [K].

$$\begin{aligned}
&\dot{\varepsilon}_c = A\alpha e^{-\alpha t} + B\alpha e^{\alpha t} \; ; \; \dot{\varepsilon}_{c,i} = \dot{\varepsilon}_c(t=0) = \alpha(A+B) \; ; \; t_m = \frac{1}{2\alpha}\ln\left(\frac{A}{B}\right) \\
&\dot{\varepsilon}_{c,m} = \dot{\varepsilon}_c(t=t_m) = 2\alpha\sqrt{AB} \quad ; \quad \varepsilon_{c,m} = \varepsilon_c(t=t_m) = \varepsilon_0 + A - B
\end{aligned}$$

ε_0	0		
c_1	1.73×10^5	n_1	4.66
β	0.095	Q_1	70
c_2	2.06×10^{-3}	n_2	1.1
c_3	9.65×10^{-4}	n_3	2.38
Q_2	17.8		

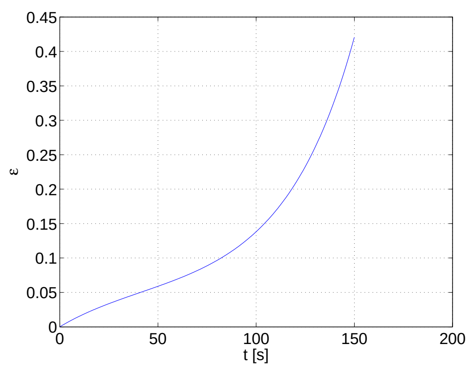


Fig. 3.69 : *Creep strain at constant stress* $\sigma = 20$ *MPa*

Special creep model for SnAg-solder

Several creep models for SnAg-solder have been published in literature. The 2-term model of Wiese (2005) is one of them and its parameter values have been fitted on experimental data for Sn4Ag0.5Cu solder material. Parameter values are listed in the table. Temperature (T) is in ^{o}K and equivalent stress (σ) is in MPa. The absolute temperature is assumed to be $T = 398$ [K].

$$\dot{\varepsilon}_c = A_1 \sigma^{m_1} e^{e_1/T} + A_2 \sigma^{m_2} e^{e_2/T}$$

$E = 59.533 - 66.667\,T$		
$A_1 = 4.10^{-7}$	$m_1 = 3$	$e_1 = -3223$
$A_1 = 1.10^{-12}$	$m_1 = 12$	$e_1 = -7348$

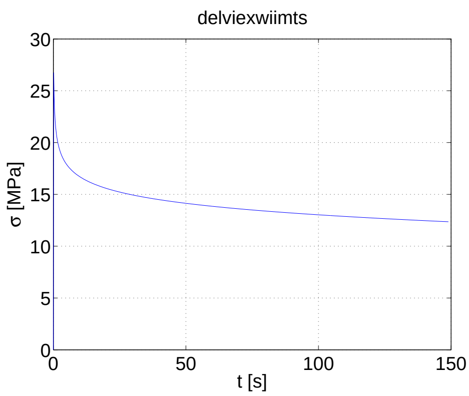


Fig. 3.70 : *Relaxation stress for constant strain* $\varepsilon = 0.001$

3.7 Viscoplastic material behavior

In many forming processes the deformation rates are small enough to consider the material behavior to be independent of strain rate and to use an elastoplastic material model. For high strain rates this assumption leads to faulty results. In a tensile test the yield stress is seen to increase with higher strain rates.

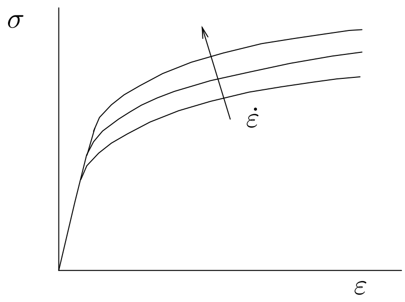


Fig. 3.71 : *Strain rate dependent plastic behavior*

Polymers and certain metallurgical alloys show softening behavior immediately after reaching the yield point. At larger strains the softening is followed by hardening. The complete stress-strain behavior is strain rate dependent, but the initial yield stress is constant.

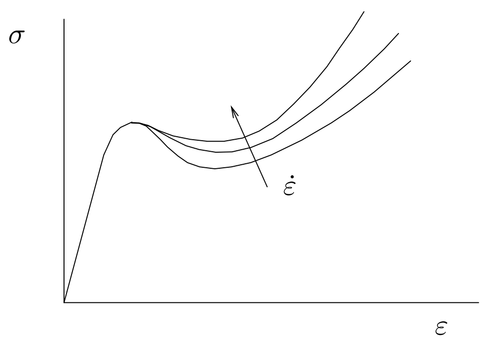


Fig. 3.72 : *Softening and resumed hardening*

3.7.1 Viscoplastic (Perzyna) model

The Perzyna model is a genuine viscoplastic model because it has a yield criterion, expressed with a yield function f. The model is called an "over-stress model" because $f > 0$ may occur. This is different compared to elastoplastic models, which always require $f \leq 0$. The rate of the *viscoplastic multiplier* λ, $\dot{\lambda}$, cannot be calculated from a consistency equation, but is given by a separate equation. We will only consider isotropic hardening.

The discrete mechanical model for viscoplastic material behavior consists of a spring E in series with a parallel arrangement of a hardening spring H, a linear dashpot η and a friction slider, opening at $\sigma = \sigma_y$.

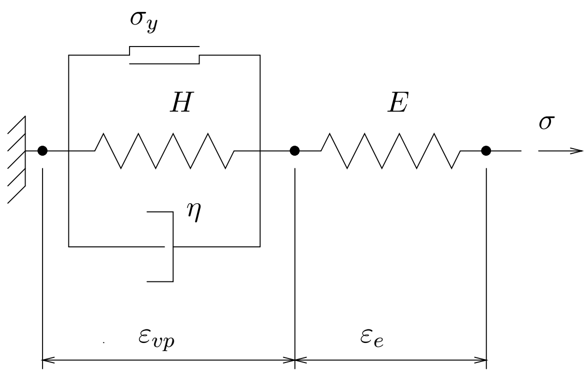


Fig. 3.73 : *Discrete model for viscoplastic material behavior*

- $f = \bar{\sigma} - \sigma_y$ with $f < 0 \rightarrow$ elastic; $f \geq 0 \rightarrow$ viscoplastic
- $\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_{vp})$
- $\bar{\sigma} = |\sigma|$
- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_{vp}$
- $\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\,\dot{\sigma}$
- $\dot{\varepsilon}_{vp} = \dot{\lambda}\,\dfrac{\partial f}{\partial \sigma} = \dot{\lambda}\,\dfrac{\sigma}{\bar{\sigma}} \quad ; \quad \dot{\bar{\varepsilon}}_{vp} = |\dot{\varepsilon}_{vp}|$
- $\bar{\varepsilon}_{vp} = \displaystyle\int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_{vp}\, d\tau$
- $\dot{\lambda} = \gamma\phi(f) = \gamma\,(f/\sigma_{y0})^N$

Hardening laws

Various hardening laws, which were earlier described in section **??**, could be used in the Perzyna model. The effective viscoplastic strain $\bar{\varepsilon}_{vp}$ is the history parameter used in the isotropic hardening models. Polymer materials may show softening, as is the case with Polycarbonate (PC). Parameters must be determined experimentally in a compression test, because the material softening poses problems of localization (necking) in a tensile test.

$$\sigma_y = \sigma_{y0} + H\bar{\varepsilon}_{vp} + a\bar{\varepsilon}_{vp}^{2} + b\bar{\varepsilon}_{vp}^{3} + c\bar{\varepsilon}_{vp}^{4} + d\bar{\varepsilon}_{vp}^{7}$$

Constitutive equations

From the constitutive relations a set of constitutive equations can be derived. The stress and the viscoplastic multiplier must be determined by integration of these equations.

$$\begin{cases} \dot{\sigma} = E\dot{\varepsilon}_e = E(\dot{\varepsilon} - \dot{\varepsilon}_{vp}) = E\{\dot{\varepsilon} - \dot{\lambda}\left(\dfrac{\sigma}{\bar{\sigma}}\right)\} \\ \dot{\lambda} = \gamma\phi \end{cases}$$

$$\begin{cases} \Delta\sigma = E\Delta\varepsilon - E\Delta\lambda\left(\dfrac{\sigma}{\bar{\sigma}}\right) \\ \Delta\lambda = \gamma\phi\Delta t \end{cases}$$

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\frac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \gamma\phi\Delta t \end{cases}$$

3.7.2 Stress update

In the viscoplastic Perzyna model the stress and viscoplastic multiplier have to be solved from a set of differential equations. This equations are nonlinear although the viscosity in the model is constant.

Numerical analysis of mechanical behavior must be done iteratively, e.g. with a Newton-Raphson scheme. Following an incremental procedure the total loading time is subdivided into a discrete number of increments, which we assume to be of equal length Δt. All relevant variables $\{\sigma, \varepsilon, \varepsilon_{vp}, \bar{\varepsilon}_{vp}, \sigma_y\}$ are assumed to be known at the beginning t_n of the current increment.

Elastic stress predictor

Because it is not known a priori whether (ongoing) elastoviscoplastic deformation or elastic unloading will occur in the current increment $t_n \rightarrow t_{n+1}$, the stress calculation starts from the assumption that the strain increment is completely elastic. The *elastic stress predictor* σ_e is calculated and subsequently the yield criterion is evaluated with the yield function f.

$$\sigma_e = \sigma_n + E(\varepsilon - \varepsilon_n)$$

- $f = \bar{\sigma}_e - \sigma_{y_n} \leq 0 \quad \rightarrow$ elastic increment
- $f = \bar{\sigma}_e - \sigma_{y_n} > 0 \quad \rightarrow$ elastoviscoplastic increment

Elastic increment

When the increment is fully elastic, the end-increment stress equals the calculated elastic stress. As no viscoplastic deformation has occurred during the increment, the effective viscoplastic strain and the yield stress remain unchanged.

$$\sigma(t_{n+1}) = \sigma_e$$
$$\bar{\varepsilon}_{vp}(t_{n+1}) = \bar{\varepsilon}_{vp}(t_n) = \bar{\varepsilon}_{vp_n}$$
$$\sigma_y(t_{n+1}) = \sigma_y(t_n) = \sigma_{y_n}$$

Elastoviscoplastic increment

If the elastic stress predictor indicates that the yield criterion is violated, the increment is elastoviscoplastic. The end-increment stress has to be determined by integration of the constitutive equations. Integration of the stress can be carried out following an *explicit* or an *implicit* method.

$$\begin{cases} \Delta\sigma = E\Delta\varepsilon - E\Delta\lambda \left(\frac{\sigma}{\bar{\sigma}}\right)\} \\ \Delta\lambda = \gamma\phi\Delta t \end{cases}$$

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\frac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \Delta t\gamma\phi \end{cases}$$

Implicit stress update

When the increment appears to be elastoviscoplastic, the end-increment stress must be updated from the elastic trial stress. The viscoplastic multiplier λ and the stress σ are determined such that the constitutive equations are satisfied. Because λ and σ are not independent, an iterative procedure has to be used.

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\frac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \Delta t\,\gamma\phi \end{cases}$$

$$\begin{cases} \sigma^* + \delta\sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda^* + \delta\lambda - \lambda_n)\left\{\left(\frac{\sigma}{\bar{\sigma}}\right)^* + \delta\left(\frac{\sigma}{\bar{\sigma}}\right)\right\} \\ \lambda^* + \delta\lambda - \lambda_n = \Delta t\,\gamma(\phi^* + \delta\phi) \end{cases}$$

linearization and reorganization

$$\begin{cases} \delta\sigma + \left[E\left(\frac{\sigma}{\bar{\sigma}}\right)^*\right]\delta\lambda \\ \qquad = -\sigma^* + \sigma_n + E\varepsilon - E\varepsilon_n - E(\lambda^* - \lambda_n)\left(\frac{\sigma}{\bar{\sigma}}\right)^* \\ \left[-\Delta t\,\gamma\frac{\partial\phi}{\partial\sigma}\right]\delta\sigma + \left[1 - \Delta t\,\gamma\frac{\partial\phi}{\partial\lambda}\right]\delta\lambda \\ \qquad = -\lambda^* + \lambda_n + \Delta t\gamma\phi^* \end{cases}$$

The variation of the function ϕ can be expressed in variations of σ and λ, using its definition and $\frac{\partial\bar{\varepsilon}_{vp}}{\partial\lambda} = \left(\frac{\sigma}{\bar{\sigma}}\right)$.

$$\begin{aligned} \frac{\partial\phi}{\partial\lambda} &= \frac{d\phi}{df}\frac{df}{d\sigma_y}\frac{d\sigma_y}{d\bar{\varepsilon}_{vp}}\frac{d\bar{\varepsilon}_{vp}}{d\lambda} = \frac{d\phi}{df}(-1)\,H\left(\frac{\sigma}{\bar{\sigma}}\right)^* = -\frac{d\phi}{df}\,H\left(\frac{\sigma}{\bar{\sigma}}\right)^* \\ \frac{\partial\phi}{\partial\sigma} &= \frac{d\phi}{df}\frac{df}{d\sigma} = \frac{d\phi}{df}\left(\frac{\sigma}{\bar{\sigma}}\right)^* \\ \frac{d\phi}{df} &= N\left(\frac{f}{\sigma_{y0}}\right)^{N-1}\frac{1}{\sigma_{y0}} \end{aligned}$$

In each iteration step, the stress, viscoplastic multiplier and other variables are updated.

Explicit stress update

In the explicit procedure, the end-increment values of λ and σ are determined directly.

$$\left\{\begin{array}{l} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\dfrac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \Delta t\, \gamma \phi \end{array}\right.$$

$$\left\{\begin{array}{l} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \\ \lambda - \lambda_n = \Delta t\, \gamma \phi_n \end{array}\right.$$

$$\left\{\begin{array}{l} \sigma + E\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right)\lambda = \sigma_n + E\varepsilon - E\varepsilon_n + E\lambda_n\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \\ \lambda = \lambda_n + \Delta t\, \gamma \phi_n \left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \end{array}\right.$$

The total stress increment and other state variables can now be calculated.

3.7.3 Stiffness

The material stiffness is calculated as the ratio of the stress variation and the strain variation: $C_\varepsilon = \frac{\delta\sigma}{\delta\varepsilon}$.

implicit

$$\left.\begin{array}{l} \sigma - \sigma_n = E(\varepsilon - \varepsilon_n) - E(\lambda - \lambda_n)\left(\dfrac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \Delta t\, \gamma\, \phi \end{array}\right\}$$

$$\left.\begin{array}{l} \delta\sigma = E\,\delta\varepsilon - E\,\delta\lambda\left(\dfrac{\sigma}{\bar{\sigma}}\right) - E(\lambda - \lambda_n)\left(\dfrac{1}{\bar{\sigma}}\right)\delta\sigma \\ \delta\lambda = \Delta t \gamma \delta\phi = \Delta t \gamma \dfrac{\partial \phi}{\partial \sigma}\delta\sigma + \Delta t \gamma \dfrac{\partial \phi}{\partial \lambda}\delta\lambda \end{array}\right\}$$

$$\delta\sigma = E\delta\varepsilon - E\left(\frac{\sigma}{\bar{\sigma}}\right)\frac{\gamma \Delta t \frac{\partial \phi}{\partial \sigma}}{1 - \gamma \Delta t \frac{\partial \phi}{\partial \lambda}}\delta\sigma - E(\lambda - \lambda_n)\left(\frac{1}{\bar{\sigma}}\right)\delta\sigma$$

$$C_\varepsilon = \frac{E\left\{1 - \gamma \Delta t \frac{\partial \phi}{\partial \lambda}\right\}}{\{1 - \gamma \Delta t \frac{\partial \phi}{\partial \lambda}\} + E\left(\frac{\sigma}{\bar{\sigma}}\right)\gamma \Delta t \frac{\partial \phi}{\partial \sigma} + E(\lambda - \lambda_n)\frac{1}{\bar{\sigma}}\left\{1 - \gamma \Delta t \frac{\partial \phi}{\partial \lambda}\right\}}$$

explicit

$$\left\{\begin{array}{l} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n)\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \\ \lambda - \lambda_n = \Delta t\, \gamma \phi_n \end{array}\right.$$

$$\left\{\begin{array}{l} \delta\sigma = E\delta\varepsilon - E\delta\lambda\left(\dfrac{\sigma_n}{\bar{\sigma}_n}\right) \\ \delta\lambda = 0 \end{array}\right.$$

$$C_\varepsilon = E$$

3.7.4 Implementation

See `tr2dperz.m` for the implementation.

3.7.5 Examples

Tensile test at various strain rates

A truss is loaded axially with a prescribed elongation. In the initial state the length of the truss is $l_0 = 100$ mm and its cross-sectional area is $A_0 = 10$ mm^2. The axial force/elongation is calculated for various material models. The cross-sectional area will change as a function of the elongation.

The Perzyna model is used to describe the viscoplastic material behavior. The hardening model and tabulated data for polycarbonate are used. The strain rate is varied.

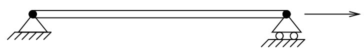


Fig. 3.74 : *Tensile loading of truss element*

$$\sigma_y = \sigma_{v0} + H\bar{\varepsilon}_{vp} + a\bar{\varepsilon}_{vp}^2 + b\bar{\varepsilon}_{vp}^3 + c\bar{\varepsilon}_{vp}^4 + d\bar{\varepsilon}_{vp}^7$$

E	1800	MPa	ν	0.37	-
σ_{y0}	37	MPa	H	-200	MPa
γ	0.001	1/s	N	3	-
a	500	MPa	b	700	MPa
c	800	MPa	d	30000	MPa

$$\dot{\varepsilon}_l = \{\, 0.01,\ 0.1,\ 1 \,\}$$

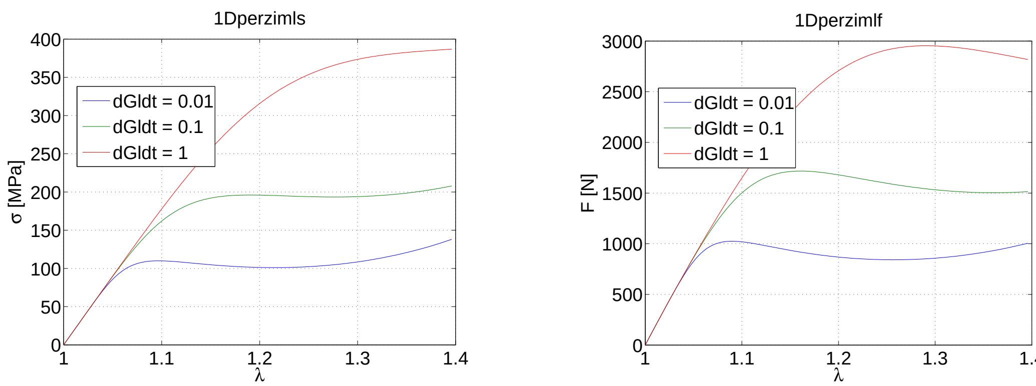


Fig. 3.75 : *Stress-stretch and force-elongation for PC*

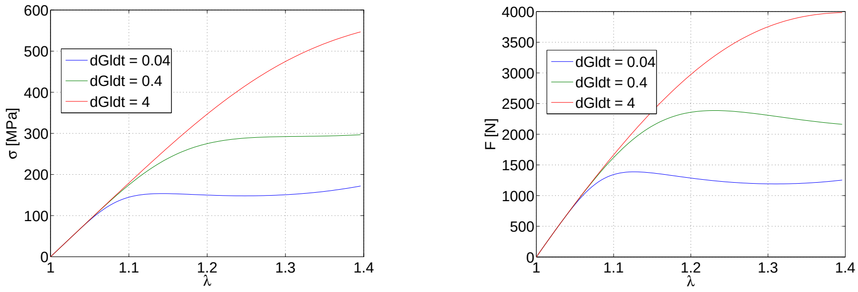


Fig. 3.76 : *Stress-stretch and force-elongation for PC; prescribed elongation*

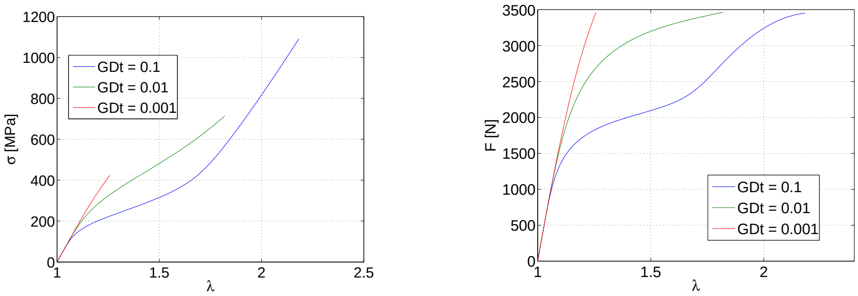


Fig. 3.77 : *Stress-stretch and force-elongation for PC; prescribed force*

3.8 Nonlinear viscoelastic material behavior

A polymeric material can be loaded in compression with a constant logarithmic strain rate. The true stress - absolute value - reaches a maximum value (B) after which softening occurs (BC) due to structural evolution. Subsequent hardening (CD) results in an increase of the stress, with increasing strain - absolute value - due to orientation of molecules.

Until the maximum stress level (B) is reached, the deformation is fully reversible. Initially the material behavior is linear viscoelastic (OA) but from a certain strain, nonlinear viscoelastic behavior (AB) is observed. After reaching the maximum stress (B), plastic flow occurs and therefore this stress is called the *yield stress* σ_y.

For a number of polymers, like polycarbonate (PC), polymethylmethacrylate (PMMA), polystyrene (PS) and polyethenetereftalate (PET), the above typical stress-strain behavior is observed.

Two time-dependent processes can be observed, one related to the deformation kinetics (strain rate dependency) and another related to the aging kinetics.

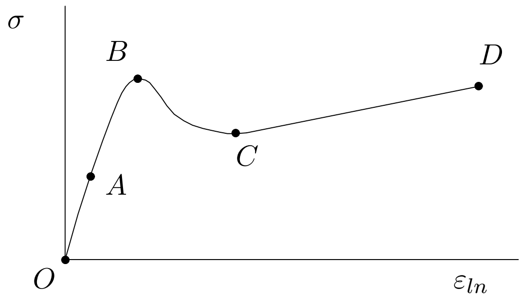


Fig. 3.78 : *Softening and resumed hardening*

When the uniaxial compression test for PC is carried out at a higher strain rate, the increase in stress is equal for each strain value.

This is shown in a graph, where the stress maximum σ_{yma}, the stress minimum after softening σ_{ymi} and the difference between those two values $\Delta\sigma_y$, are plotted against the logarithm of the true strain rate. The stress maximum is referred to as the *upper yield stress*, the stress minimum after softening is called the *lower yield stress* and the difference is the *yield drop*, which is constant for PC.

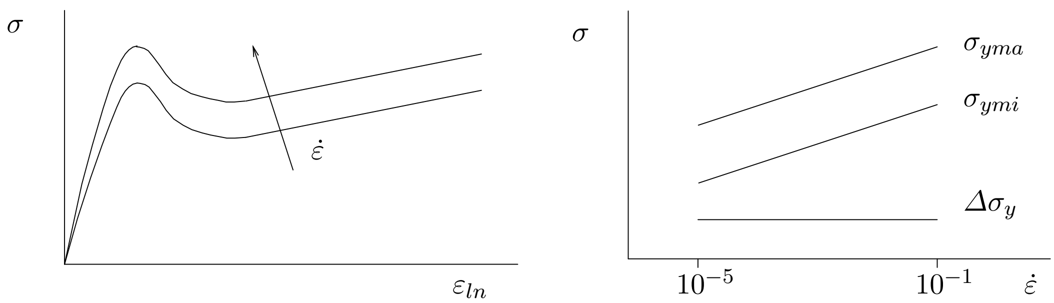


Fig. 3.79 : *Strain rate dependent stress-strain for PC*

The strain rate dependency of PMMA is different from that of PC. The increase of the stress with higher strain rates is not the same for each strain value. The upper yield stress increases more than the lower yield stress. The yield drop is a function of the strain rate.

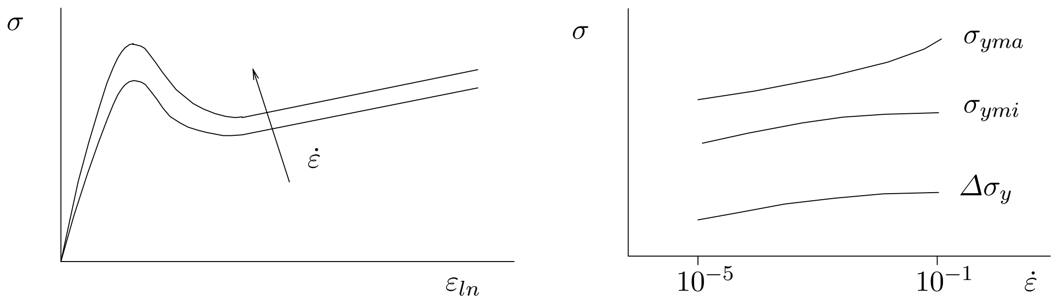


Fig. 3.80 : *Strain rate dependent stress-strain for PMMA*

The yield drop appears to be a function of the history of the material. When the specimen is quenched after processing, there is no yield drop. Softening is observed after a certain time, a phenomenon which is called *aging.* The time is characteristic for the polymer in question : 15 minutes for PS, 1 day for PMMA and about 3 weeks for PC. Aging and the resulting softening characteristic, can be neutralized by mechanical deformation, indicated as *rejuvenation.*

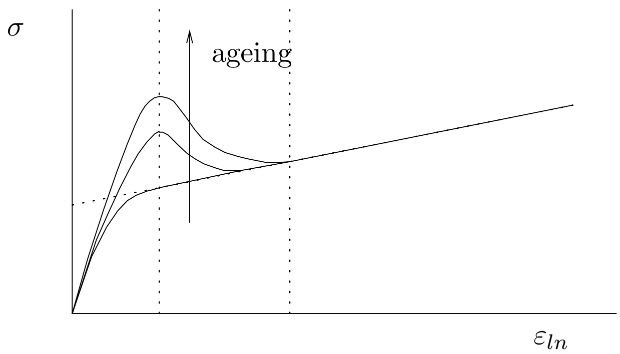


Fig. 3.81 : *Aging*

3.8.1 Nonlinear viscoelastic model

The complete model for nonlinear viscoelastic behavior is based on the models, which are used to describe the mechanical behavior at increasing stress level.

Linear viscoelastic behavior

For small strains the material behavior of polymers is linear viscoelastic and can be described by a Boltzmann integral with multi-mode Maxwell relaxation function. When more molecular processes are relevant, the relaxation functions for the separate processes can be added.

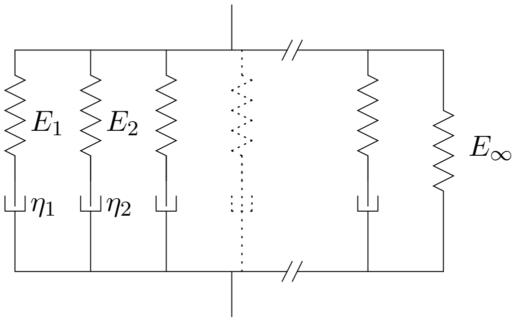


Fig. 3.82 : *Generalized Maxwell model for linear viscoelastic behavior*

$$\sigma(t) = \int\limits_{\xi=-\infty}^{t} E(t-\xi)\dot{\varepsilon}(\xi)\,d\xi \quad ; \quad E(x) = E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{x}{\tau_i}} \quad ; \quad \tau_i = \frac{\eta_i}{E_i}$$

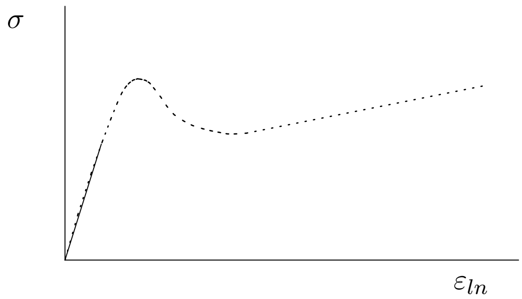


Fig. 3.83 : *Stress-strain relation in linear viscoelastic range*

Nonlinear viscoelastic behavior

For higher strains, but before yielding, the behavior is nonlinear viscoelastic and the relaxation function becomes a function of the stress. Fortunately this influence can generally be modeled by using time-stress superposition and adaptation of time variables using a time-stress shift factor.

$$\sigma(t) = \int\limits_{\xi=-\infty}^{t} E(\psi-\psi')\dot{\varepsilon}(\xi)\,d\xi$$

$$\psi = \int\limits_{\zeta=-\infty}^{t} \frac{d\zeta}{a_\sigma\{\sigma(\zeta)\}} \quad ; \quad \psi' = \int\limits_{\zeta=-\infty}^{\xi} \frac{d\zeta}{a_\sigma\{\sigma(\zeta)\}}$$

$$E(x) = E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{x}{\tau_i(\sigma)}} = E_\infty + \sum_{i=1}^{N} E_i e^{-\frac{x}{\tau_i a_\sigma(\sigma)}} \quad ; \quad a_\sigma = \frac{\sigma/\sigma_0}{\sinh(\sigma/\sigma_0)}$$

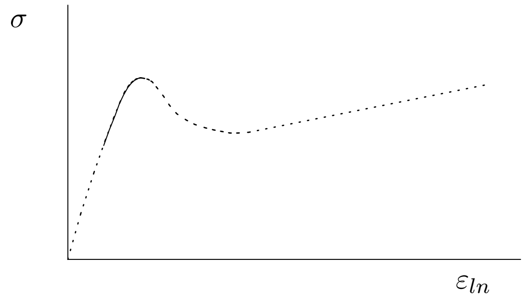


Fig. 3.84 : *Stress-strain relation in nonlinear viscoelastic range*

Creep

When the yield stress (= maximum stress) is reached, stress-activated plastic flow occurs, described by a semi-empirical relation for the viscous strain rate. The stress level depends on the strain rate and the temperature. When the stress is below the initial yield stress, the viscosity is very high and the material behavior is considered to be linear elastic with stiffness E. This behavior can be modeled with a Maxwell model with a linear spring (stiffness E) and a nonlinear dashpot (viscosity η). The total strain is additively decomposed in an elastic strain ε_e and a viscous strain ε_v. The viscous strain rate is given as a function of the equivalent viscoelastic stress $\bar{s}$ and temperature T.

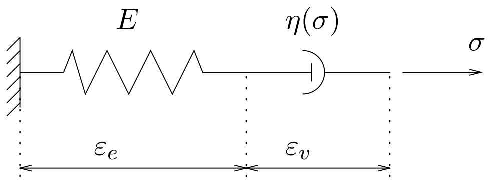


Fig. 3.85 : *Nonlinear creep model*

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\,\dot{\sigma}$
- $\dot{\varepsilon}_v = f(\bar{s}, T) = \dfrac{\sigma}{\eta(\bar{s}, T)} \qquad ; \qquad \bar{s} = |s|$

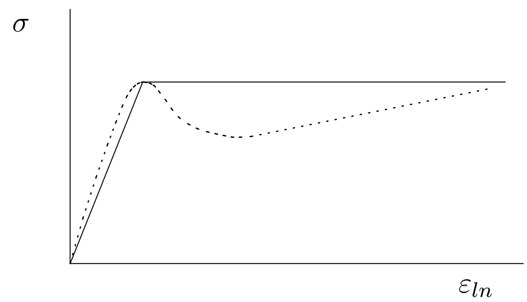


Fig. 3.86 : *Stress-strain relation for creep range*

Softening

After reaching the initial yield stress the stress decreases asymptotically toward a final value. Here, this softening is taken into account by decreasing the viscosity η with an internal (damage) variable D. Initially $D = D_0$ and finally D reaches a saturation value D_∞. The value of D is determined by an evolution equation, which relates $\dot{D}$ to the effective viscous strain rate $\dot{\bar{\varepsilon}}_v$, with $\dot{\bar{\varepsilon}}_v = |\dot{\varepsilon}_v|$ for the one-dimensional case.

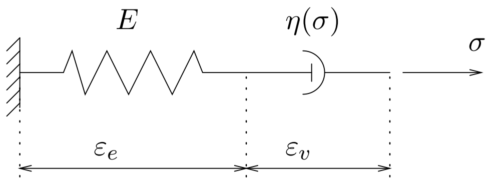


Fig. 3.87 : *Model for nonlinear creep*

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \dfrac{1}{E}\dot{\sigma}$
- $\dot{\varepsilon}_v = \dfrac{1}{\eta(\bar{s}, T, D)}\sigma \qquad ; \qquad \bar{s} = |s|$
- $\dot{D} = \left(1 - \dfrac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v \qquad ; \qquad \dot{\bar{\varepsilon}}_v = |\dot{\varepsilon}_v|$

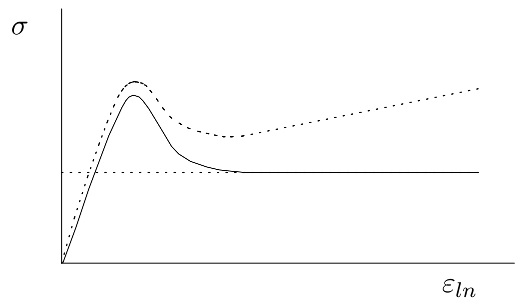


Fig. 3.88 : *Softening*

Hardening

In a compression test it is observed that the softening is followed by hardening. This can be modeled by decomposing the stress additively. In the discrete mechanical element a linear spring (stiffness H) is placed parallel to the Maxwell element with linear spring (stiffness E) and nonlinear dashpot (viscosity η).

The total axial stress σ is the sum of the viscoelastic stress s and the hardening stress w. The viscoelastic stress is related to the stiffness E, but also to the viscosity η.

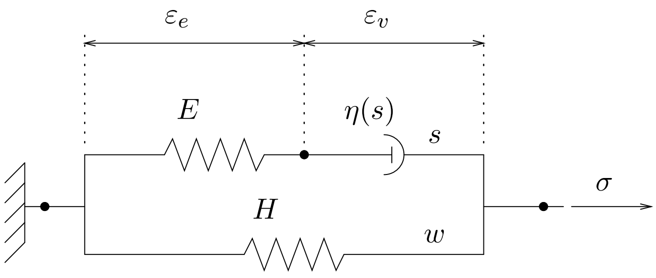


Fig. 3.89 : *Model for nonlinear viscoelastic behavior*

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = s + w = E\varepsilon_e + H\varepsilon$
- $\dot{\varepsilon}_v = \dfrac{1}{\eta(\bar{s}, T, D)} s \qquad ; \qquad \bar{s} = |s|$
- $\dot{D} = \left(1 - \dfrac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v \quad ; \quad \dot{\bar{\varepsilon}}_v = |\dot{\varepsilon}_v|$

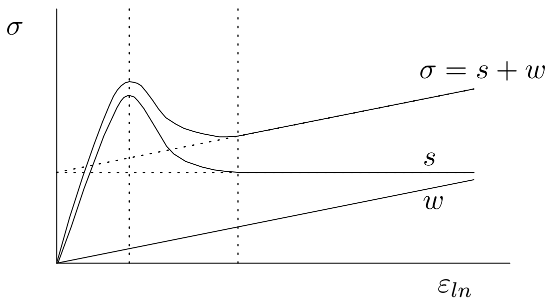


Fig. 3.90 : *Stress-strain curve for nonlinear viscoelastic behavior*

Aging and hardening

A different model to describe aging and softening is based on additive decomposition of the stress, where the total stress σ is the sum of the flow stress s, the hardening stress w and the aging stress $\Delta\sigma_y$, which is determined by an aging characteristic function $S(t, \bar{\varepsilon}_v)$. This function is taken to be the product of a time-dependent function $S_a(t)$, which describes the aging kinetics and a softening function $R_\gamma(\bar{\varepsilon}_v)$ which describes the softening kinetics. The viscosity is now a function of this function $S(t, \bar{\varepsilon}_v)$.

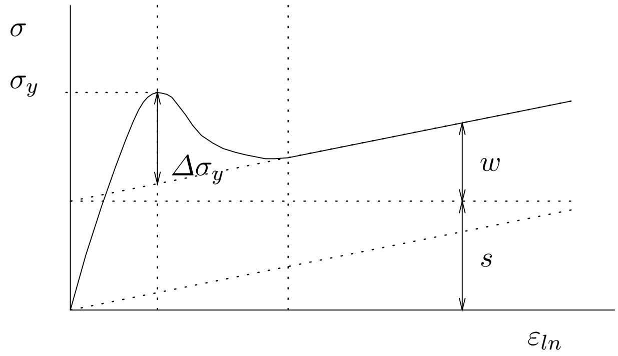


Fig. 3.91 : *Aging and hardening*

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = s + \Delta\sigma_y + w = E\varepsilon_e + \Delta\sigma_y + H\varepsilon$
- $\dot{\varepsilon}_v = \dfrac{1}{\eta(\bar{s}, T, S)}\, s \qquad ; \qquad \bar{s} = |s|$

- $S(t, \bar{\varepsilon}_v) = S_a(t) R_\gamma(\bar{\varepsilon}_v)$
- $R_\gamma(\bar{\varepsilon}_v) = \left[\left\{1 + \left(r_0 e^{\bar{\varepsilon}_v}\right)^{r_1}\right\} / \left\{1 + r_0^{r_1}\right\}\right]^{\frac{r_2 - 1}{r_1}} \quad ; \quad 0 < R < 1$
- $S_\alpha(t) = S_a(t_{eff}) = c_0 + c_1 \ln\left[\dfrac{t_{eff} + t_a}{t_0}\right]$
- $t_{eff}(T, \bar{s}) = \displaystyle\int_{\xi=0}^{t} \frac{d\xi}{\alpha_T(T(\xi))\alpha_\sigma(\bar{s}(\xi))}$
- $t_a = exp\left(\dfrac{S_\alpha(0) - c_0}{c_1}\right)$
- $\Delta\sigma_y = \sigma_y(t) - \sigma_{y0} = \dfrac{c}{c_1}\left\{S_\alpha(t) - c_0\right\}$

Viscosity

For each material the proper relation for the viscosity has to be chosen. For polymers the *Eyring viscosity* function is used.

$$\eta = A_0 \frac{\bar{s}}{\sqrt{3}\,\sinh\left(\bar{s}/(\sqrt{3}\tau_0)\right)}\, \exp\left[\frac{\Delta H}{RT} + \frac{\mu p}{\tau_0} - D\right]$$

$$\bar{s} = |s| \quad ; \quad p = -\tfrac{1}{3}s \quad ; \quad \tau_0 = \frac{RT}{V}$$

For metals the *Bodner-Partom viscosity* function is used.

$$\eta = \frac{\bar{s}}{\sqrt{12\Gamma_0}}\, \exp\left[\tfrac{1}{2}\left(\frac{Z}{\bar{s}}\right)^{2n}\right]$$

$$Z = Z_1 + (Z_0 - Z_1)\exp\left[-m\bar{\varepsilon}_p\right]$$

Nonlinear viscoelastic model

The nonlinear viscoelastic material behavior is described by some relations, which can be combined. The resulting constitutive equations must be solved simultaneously.

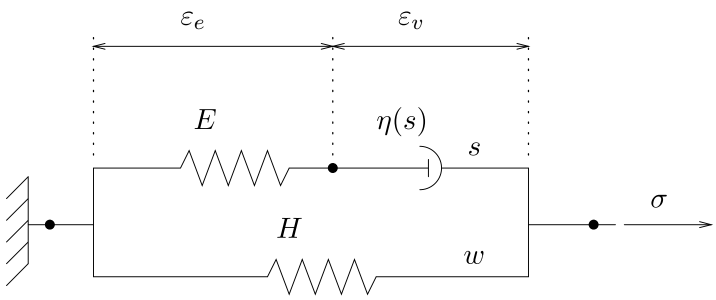


Fig. 3.92 : *Model for nonlinear viscoelastic behavior*

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = s + w = E\varepsilon_e + H\varepsilon$
- $\dot{\varepsilon}_v = \dfrac{1}{\eta(\bar{s}, T, D)} s \quad ; \quad \bar{s} = |s|$
- $\dot{D} = \left(1 - \dfrac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v \quad ; \quad \dot{\bar{\varepsilon}}_v = |\dot{\varepsilon}_v|$

constitutive equations

$$\left.\begin{array}{lcl}
\dot{\varepsilon}_e & = & \dot{\varepsilon} - \dot{\varepsilon}_v = \dot{\varepsilon} - \dfrac{1}{\eta(\bar{s}, T, D)} s = \dot{\varepsilon} - \dfrac{E}{\eta(\bar{s}, T, D)} \varepsilon_e \\
\sigma & = & s + w = E\varepsilon_e + H\varepsilon \\
\dot{D} & = & \left(1 - \dfrac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v
\end{array}\right\}$$

3.8.2 Stress update

The stress is related to the strain rate by a differential equation, which has to be solved together with the damage evolution equation. After updating ε_e the stress is calculated directly. In the following we use $\zeta = \frac{1}{\eta}$. An implicit or explicit procedure can be used to determine the elastic strain and the damage parameter.

$$\left\{\begin{array}{l}
\dot{\varepsilon}_e = \dot{\varepsilon} - E\zeta(\bar{s}, T, D)\varepsilon_e \\
\dot{D} = \left(1 - \dfrac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v
\end{array}\right.$$

$$\left\{\begin{array}{l}
\Delta\varepsilon_e = \Delta\varepsilon - \Delta t E\zeta(\bar{s}, T, D)\varepsilon_e \\
\Delta D = \left(1 - \dfrac{D}{D_\infty}\right) h\Delta\bar{\varepsilon}_v
\end{array}\right.$$

Implicit stress update

In the implicit update procedure the elastic strain ε_e and the damage parameter D are updated iteratively.

$$\left\{\begin{array}{l} \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}, T, D)\varepsilon_e \\ D - D_n = \left(1 - \dfrac{D}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{array}\right.$$

$$\left\{\begin{array}{l} \varepsilon_e^* + \delta\varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}, T, D^* + \delta D)(\varepsilon_e^* + \delta\varepsilon_e) \\ D^* + \delta D - D_n = \left(1 - \dfrac{D^* + \delta D}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{array}\right.$$

$$\left\{\begin{array}{l} \delta\varepsilon_e + \Delta t E \zeta(\bar{s}, T, D^*)\delta\varepsilon_e + \Delta t E \dfrac{\partial \zeta}{\partial D} \delta D \varepsilon_e^* \\ \qquad = -\varepsilon_e^* + \varepsilon_{en} + \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}, T, D^*)\varepsilon_e^* \\ \left[1 + \dfrac{h \Delta \bar{\varepsilon}_v}{D_\infty}\right] \delta D \\ \qquad = -D^* + D_n + \left(1 - \dfrac{D^*}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{array}\right.$$

Explicit stress update

In the explicit update procedure the elastic strain ε_e and the damage parameter D are determined directly.

$$\left\{\begin{array}{l} \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}_n, T, D_n)\varepsilon_{en} \\ D - D_n = \left(1 - \dfrac{D_n}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{array}\right.$$

$$\left\{\begin{array}{l} \varepsilon_e = \varepsilon - \varepsilon_n + \left\{1 - \Delta t E \zeta(\bar{s}_n, T, D_n)\right\} \varepsilon_{en} \\ D = D_n + \left(1 - \dfrac{D_n}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{array}\right.$$

3.8.3 Stiffness

The material stiffness is calculated as $C_\varepsilon = \frac{\delta\sigma}{\delta\varepsilon}$.

implicit

$$\left\{\begin{array}{l} \sigma = s + w = E\varepsilon_e + H\varepsilon \\ \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}, T, D)\varepsilon_e \\ D - D_n = \left(1 - \dfrac{D}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{array}\right.$$

$$\left\{\begin{array}{l} \delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \Delta t E \dfrac{\partial \zeta}{\partial D} \delta D \varepsilon_e - \Delta t E \zeta(\bar{s}, T, D)\delta\varepsilon_e \\ \delta D = -\dfrac{\delta D}{D_\infty} h \Delta \bar{\varepsilon}_v \qquad \rightarrow \quad \delta D = 0 \end{array}\right.$$

$$\left\{\begin{array}{l} \delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \Delta t \, E \zeta(\bar{s}, T, D)\delta\varepsilon_e \\ \delta D = 0 \end{array}\right.$$

$$C_\varepsilon = \frac{E + H\{1 + \varDelta t\, E\zeta(\bar{s}, T, D)\}}{1 + \varDelta t\, E\zeta(\bar{s}, T, D)}$$

explicit

$$\left\{\begin{array}{l} \sigma = s + w = E\varepsilon_e + H\varepsilon \\ \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \varDelta t E\zeta(\bar{s}_n, T, D_n)\varepsilon_e \\ D = D_n + \left(1 - \dfrac{D_n}{D_\infty}\right) h\varDelta\bar{\varepsilon}_v \end{array}\right.$$

$$\left\{\begin{array}{l} \delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \varDelta t E\zeta(\bar{s}_n, T, D_n)\delta\varepsilon_e \\ \delta D = 0 \end{array}\right.$$

$$\delta\sigma = \frac{E}{1 + \varDelta t E\zeta(\bar{s}_n, T, D_n)}\delta\varepsilon + H\delta\varepsilon$$

$$C_\varepsilon = \frac{E + H\{1 + \varDelta t E\zeta(\bar{s}_n, T, D_n)\}}{1 + \varDelta t E\zeta(\bar{s}_n, T, D_n)}$$

3.8.4 Implementation

See `tr2degp1.m` for the implementation.

3.8.5 Examples

Polymer materials are characterized by an Eyring viscosity. Parameters for various materials are experimentally determined and listed in the table. For the values in the table the temperature is chosen to be $T = 285$ K. The universal gas constant is $R = 8.314$ J/(mol.K).

	PET	PC	PS	PP	
E	2400	2305	3300	1092	MPa
ν	0.35	0.37	0.37	0.4	-
H	15	29	13	3	MPa
h	13	270	100	0	-
D_∞	11	19	14	-	-
A_0	3.8568E-27	9.7573E-27	4.2619E-34	2.0319E-29	s
$\varDelta H$	2.617E+05	2.9E+05	2.6E+5	2.2E+5	J/mol
μ	0.0625	0.06984	0.294	0.23	-
τ_0	0.927	0.72	2.1	1.0	MPa

Tensile test at various strain rates

A truss is loaded axially with a prescribed elongation. In the initial state the length of the truss is $l_0 = 100$ mm and its cross-sectional area is $A_0 = 10$ mm^2. The axial force/elongation is calculated for various material models. The cross-sectional area will change as a function of the elongation. The hardening model and tabulated data for polycarbonate are used. The strain rate is varied.

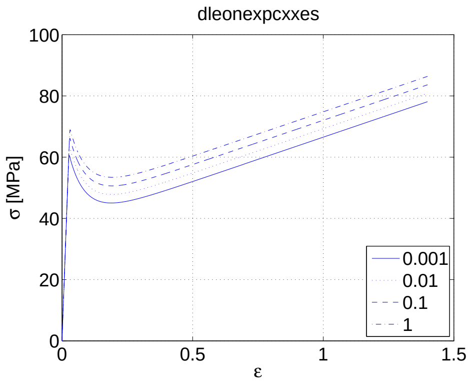


Fig. 3.93 : *Stress versus strain in polycarbonate for different strain rates*

Tensile test for various polymers

The truss is also loaded with a strain rate $\dot{\varepsilon} = 10^{-1}\ \mathrm{s}^{-1}$, using the tabulated parameter values for polycarbonate, polypropylene, polystyrene and PET.

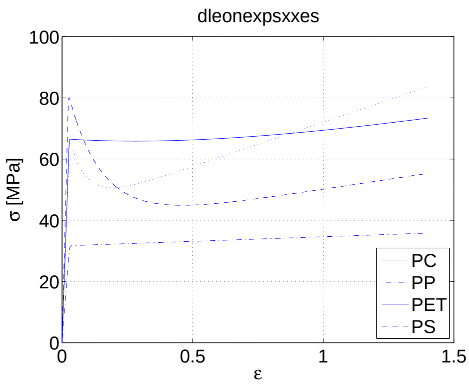


Fig. 3.94 : *Stress versus strain for different polymers at strain rate 0.1*

Chapter 4

Vectors, tensors, columns, matrices

In mechanics and other fields of physics, quantities are represented by vectors and tensors. Essential manipulations with these quantities will be summerized in this section. For quantitative calculations and programming, components of vectors and tensors are needed, which can be determined in a coordinate system with respect to a vector basis. The three components of a vector can be stored in a column. The nine components of a second-order tensor are generally stored in a three-by-three matrix.

A fourth-order tensor relates two second-order tensors. Matrix notation of such relations is only possible, when the 9 components of the second-order tensor are stored in columns. Doing so, the 81 components of a fourth-order tensor are stored in a 9×9 matrix. For some mathematical manipulations it is also advantageous to store the 9 components of a second-order tensor in a 9×9 matrix.

4.1 Summary of vector and tensor operations

In this section we give a summary of the most important manipulations and properties of vectors and tensors and their operations.

4.1.1 Vectors

Essential definitions and manipulations with vectors are summarized below. Three independent vectors in three-dimensional space constitute a vector base. In a Cartesian coordinate system, these base vectors are independent of the Cartesian coordinates $\{x, y, z\}$. In a cylindrical and a spherical coordinate system, some of the base vectors are a function of one of more coordinates.

vector: length and direction	$\vec{a} = \Vert a \Vert \vec{e} \quad ; \quad \Vert \vec{e} \Vert = 1$
scalar multiplication	$\alpha \vec{a} = \vec{b}$
summation	$\vec{a} + \vec{b} = \vec{c}$
scalar product[1]	$\vec{a} \cdot \vec{b} = \Vert \vec{a} \Vert \Vert \vec{b} \Vert \cos(\phi)$
vector product	$\vec{c} = \vec{a} * \vec{b} = \left\{ \Vert \vec{a} \Vert \Vert \vec{b} \Vert \right\} \sin(\phi)\, \vec{n} \quad ; \quad \Vert \vec{n} \Vert = 1$
triple product	$\vec{a} * \vec{b} \cdot \vec{c} = \left\{ \Vert \vec{a} \Vert \Vert \vec{b} \Vert \sin(\phi) \right\} \Vert \vec{c} \Vert \cos(\psi)$
tensor product[2]	$\vec{a}\vec{b} = \text{dyad} \quad ; \quad \vec{q} = \vec{a}\vec{b} \cdot \vec{p} = \vec{p} \cdot (\vec{a}\vec{b})^c$

orthonormal vector base	$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$; $\vec{e}_i \cdot \vec{e}_{j\neq i} = 0$; $\vec{e}_i \cdot \vec{e}_i = 1$
vector components $\rightarrow$ column[3]	$\vec{a} = \underset{\sim}{a}^T \underset{\sim}{\vec{e}} = \underset{\sim}{\vec{e}}^T \underset{\sim}{a}$

1) ϕ = smallest angle between $\vec{a}$ and $\vec{b}$

2) index c indicates conjugation

3) index T indicates transposition

4.1.2 Second-order tensors

Essential definitions and manipulations with second-order tensors are summarized below.

second-order tensor	$\boldsymbol{A} = \sum_i \alpha_i \vec{a}_i \vec{b}_i$; $\boldsymbol{A} \cdot \vec{p} = \vec{q}$
tensor components $\rightarrow$ matrix	$\boldsymbol{A} = \underset{\sim}{\vec{e}}^T \underline{A}\, \underset{\sim}{\vec{e}}$
unity tensor $\rightarrow$ unity matrix	$\boldsymbol{I} \cdot \vec{a} = \vec{a} \quad \forall \quad \vec{a} \quad \rightarrow \quad \boldsymbol{I} = \underset{\sim}{\vec{e}}^T \underline{I}\, \underset{\sim}{\vec{e}}$
conjugate tensor	$\boldsymbol{A}^c = \sum_i \alpha_i \vec{b}_i \vec{a}_i$; $\boldsymbol{A} \cdot \vec{p} = \vec{p} \cdot \boldsymbol{A}^c$
scalar product	$\alpha \boldsymbol{A} = \boldsymbol{B}$
summation	$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{C}$
inner product	$\boldsymbol{B} \cdot \boldsymbol{A} = \boldsymbol{C}$
double inner product	$\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{A}^c : \boldsymbol{B}^c = \text{scalar}$
1st invariant	$J_1(\boldsymbol{A}) = \text{tr}(\boldsymbol{A})$
2nd invariant	$J_2(\boldsymbol{A}) = \frac{1}{2}\left\{\text{tr}^2(\boldsymbol{A}) - \text{tr}(\boldsymbol{A} \cdot \boldsymbol{A})\right\}$
3rd invariant	$J_3(\boldsymbol{A}) = \det(\boldsymbol{A})$; $\det(\boldsymbol{A}) = 0 \rightarrow \boldsymbol{A} = \text{singular}$
inverse tensor	$\boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$; $\boldsymbol{A} = \text{regular}$
symmetric tensor	$\boldsymbol{A}^c = \boldsymbol{A}$
skew-symmetric tensor	$\boldsymbol{A}^c = -\boldsymbol{A}$
positive definite	$\vec{a} \cdot \boldsymbol{A} \cdot \vec{a} > 0 \quad \forall \quad \vec{a} \neq \vec{0}$
orthogonal tensor	$(\boldsymbol{A} \cdot \vec{a}) \cdot (\boldsymbol{A} \cdot \vec{b}) = \vec{a} \cdot \vec{b} \quad \forall \quad \vec{a}, \vec{b}$
adjugate tensor	$(\boldsymbol{A} \cdot \vec{a}) * (\boldsymbol{A} \cdot \vec{b}) = \boldsymbol{A}^a \cdot (\vec{a} * \vec{b}) \quad \forall \quad \vec{a}, \vec{b}$

4.1.3 Fourth-order tensors

Essential definitions and manipulations with fourth-order tensors are summarized below.

fourth-order tensor	${}^4\boldsymbol{A} = \sum_i \alpha_i \vec{a}_i \vec{b}_i \vec{c}_i \vec{d}_i$; ${}^4\boldsymbol{A} : \boldsymbol{B} = \boldsymbol{C}$
unity tensor	${}^4\boldsymbol{I} : \boldsymbol{A} = \boldsymbol{A} \quad \forall \quad \boldsymbol{A}$
inner product	${}^4\boldsymbol{A} \cdot \boldsymbol{B} = {}^4\boldsymbol{C}$

4.2 Column and matrix notation

Three-dimensional continuum mechanics is generally formulated initially without using a coordinate system, using vectors and tensors. For solving real problems or programming, we need to use components w.r.t. a vector basis. For a vector and a second-order tensor, the components can be stored in a column and a matrix. In this section a more extended column/matrix notation is introduced, which is especially useful, when things have to be programmed.

4.2.1 Matrix/column notation for second-order tensor

The components of a tensor $\boldsymbol{A}$ can be stored in a matrix $\underline{A}$. For later purposes it is very convenient to store these components in a column. To distinguish this new column from the normal column with components of a vector, we introduce a double "under-wave". In this new column $\underset{\approx}{A}$ the components of $\boldsymbol{A}$ are located on specific places.

As any other column, $\underset{\approx}{A}$ can be transposed. Transposition of the individual column elements is also possible. When this is the case we write : $\underset{\approx}{A}_t$.

3×3 matrix of a second-order tensor

$$\boldsymbol{A} = \vec{e}_i A_{ij} \vec{e}_j \quad \rightarrow \quad \underline{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

column notation

$$\begin{aligned} \underset{\approx}{A}^T &= \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{12} & A_{21} & A_{23} & A_{32} & A_{31} & A_{13} \end{bmatrix} \\ \underset{\approx}{A}_t^T &= \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{21} & A_{12} & A_{32} & A_{23} & A_{13} & A_{31} \end{bmatrix} \end{aligned}$$

conjugate tensor

$$\boldsymbol{A}^c \quad \rightarrow \quad A_{ji} \quad \rightarrow \quad \underline{A}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \quad \rightarrow \quad \underset{\approx}{A}_t$$

Column notation for $\boldsymbol{A} : \boldsymbol{B}$

With the column of components of a second-order tensor, it is now very straightforward to write the double product of two tensors as the product of their columns.

$$\begin{aligned} C &= \boldsymbol{A} : \boldsymbol{B} \\ &= \vec{e}_i A_{ij} \vec{e}_j : \vec{e}_k B_{kl} \vec{e}_l = A_{ij} \delta_{jk} \delta_{il} B_{kl} = A_{ij} B_{ji} \\ &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} + A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} + \\ &\qquad A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33} \\ &= \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{21} & A_{12} & A_{32} & A_{23} & A_{13} & A_{31} \end{bmatrix} \\ &\qquad \begin{bmatrix} B_{11} & B_{22} & B_{33} & B_{12} & B_{21} & B_{23} & B_{32} & B_{31} & B_{13} \end{bmatrix}^T \\ &= \underset{\approx}{A}_t^T \underset{\approx}{B} = \underset{\approx}{A}^T \underset{\approx}{B}_t \end{aligned}$$

idem

$$
\begin{array}{lcl}
C = \boldsymbol{A} : \boldsymbol{B}^c & \rightarrow & C = \underset{\sim}{A}{}_t^T \underset{\sim}{B}{}_t = \underset{\sim}{A}{}^T \underset{\sim}{B} \\
C = \boldsymbol{A}^c : \boldsymbol{B} & \rightarrow & C = \underset{\sim}{A}{}^T \underset{\sim}{B} = \underset{\sim}{A}{}_t^T \underset{\sim}{B}{}_t \\
C = \boldsymbol{A}^c : \boldsymbol{B}^c & \rightarrow & C = \underset{\sim}{A}{}_t^T \underset{\sim}{B} = \underset{\sim}{A}{}^T \underset{\sim}{B}{}_t
\end{array}
$$

Matrix/column notation $\quad \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$

The inner product of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is a new second-order tensor $\boldsymbol{C}$. The components of this new tensor can be stored in a 3×3 matrix $\underline{C}$, but of course also in a column $\underset{\sim}{C}$.

A matrix representation will result when the components of $\boldsymbol{A}$ and $\boldsymbol{B}$ can be isolated. We will store the components of $\boldsymbol{B}$ in a column $\underset{\sim}{B}$ and the components of $\boldsymbol{A}$ in a matrix.

$$
\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} = \vec{e}_i A_{ik} \vec{e}_k \cdot \vec{e}_l B_{lj} \vec{e}_j = \vec{e}_i A_{ik} \delta_{kl} B_{lj} \vec{e}_j = \vec{e}_i A_{ik} B_{kj} \vec{e}_j \quad \rightarrow
$$

$$
\underline{C} = \begin{bmatrix}
A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} & A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33} \\
A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} & A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} \\
A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} & A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} & A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33}
\end{bmatrix}
$$

$$
\underset{\sim}{C} = \begin{bmatrix} C_{11} \\ C_{22} \\ C_{33} \\ C_{12} \\ C_{21} \\ C_{23} \\ C_{32} \\ C_{31} \\ C_{13} \end{bmatrix} = \begin{bmatrix}
A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\
A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \\
A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33} \\
A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\
A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} \\
A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} \\
A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} \\
A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} \\
A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33}
\end{bmatrix}
$$

The column $\underset{\sim}{C}$ can be written as the product of a matrix $\underline{\underline{A}}$ and a column $\underset{\sim}{B}$ which contain the components of the tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. To distinguish the new matrix from the normal 3×3 matrix $\underline{A}$, which contains also the components of $\boldsymbol{A}$, we have introduced a double underline.

The matrix $\underline{\underline{A}}$ can of course be transposed, giving $\underline{\underline{A}}^T$. We have to introduce, however, three new manipulations concerning the matrix $\underline{\underline{A}}$. First it will be obvious that the individual matrix components can be transposed : $A_{ij} \rightarrow A_{ji}$. When we do this the result is written as : $\underline{\underline{A}}_t$, just as was done with a column $\underset{\sim}{C}$.

Two manipulations concern the interchange of columns or rows and are denoted as $(\ )_c$ and $(\ )_r$. It can be easily seen that not each row and/or column is interchanged, but only : $(4 \leftrightarrow 5)$, $(6 \leftrightarrow 7)$ and $(8 \leftrightarrow 9)$.

$$
\underset{\approx}{C} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & A_{12} & 0 & 0 & A_{13} & 0 \\ 0 & A_{22} & 0 & A_{21} & 0 & 0 & A_{23} & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & A_{32} & 0 & 0 & A_{31} \\ 0 & A_{12} & 0 & A_{11} & 0 & 0 & A_{13} & 0 & 0 \\ A_{21} & 0 & 0 & 0 & A_{22} & 0 & 0 & A_{23} & 0 \\ 0 & 0 & A_{23} & 0 & 0 & A_{22} & 0 & 0 & A_{21} \\ 0 & A_{32} & 0 & A_{31} & 0 & 0 & A_{33} & 0 & 0 \\ A_{31} & 0 & 0 & 0 & A_{32} & 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{13} & 0 & 0 & A_{12} & 0 & 0 & A_{11} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{22} \\ B_{33} \\ B_{12} \\ B_{21} \\ B_{23} \\ B_{32} \\ B_{31} \\ B_{13} \end{bmatrix} = \underline{\underline{A}}\,\underset{\approx}{B}
$$

idem

$$
\begin{array}{lcll}
\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} & \rightarrow & \underset{\approx}{C} = \underline{\underline{A}}\,\underset{\approx}{B} = \underline{\underline{A}}_c\,\underset{\approx}{B}_t & ; \quad \underset{\approx}{C}_t = \underline{\underline{A}}_r\,\underset{\approx}{B} = \underline{\underline{A}}_{rc}\,\underset{\approx}{B}_t \\
\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}^c & \rightarrow & \underset{\approx}{C} = \underline{\underline{A}}\,\underset{\approx}{B}_t = \underline{\underline{A}}_c\,\underset{\approx}{B} & \\
\boldsymbol{C} = \boldsymbol{A}^c \cdot \boldsymbol{B} & \rightarrow & \underset{\approx}{C} = \underline{\underline{A}}_t\,\underset{\approx}{B} = \underline{\underline{A}}_{tc}\,\underset{\approx}{B}_t & \\
\boldsymbol{C} = \boldsymbol{A}^c \cdot \boldsymbol{B}^c & \rightarrow & \underset{\approx}{C} = \underline{\underline{A}}_t\,\underset{\approx}{B}_t = \underline{\underline{A}}_{tc}\,\underset{\approx}{B} &
\end{array}
$$

4.2.2 Matrix notation of fourth-order tensor

The components of a fourth-order tensor can be stored in a 9×9 matrix. This matrix has to be defined and subsequently used in the proper way. We denote the matrix of ${}^4\boldsymbol{A}$ as $\underline{\underline{A}}$. When the matrix representation of ${}^4\boldsymbol{A}$ is $\underline{\underline{A}}$, it is easily seen that right- and left-conjugation results in matrices with swapped columns and rows, respectively.

$$
{}^4\boldsymbol{A} = \vec{e}_i\vec{e}_j A_{ijkl}\vec{e}_k\vec{e}_l \quad \rightarrow
$$

$$
\underline{\underline{A}} = \begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} & A_{1123} & A_{1132} & A_{1131} & A_{1113} \\ A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} & A_{2223} & A_{2232} & A_{2231} & A_{2213} \\ A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} & A_{3323} & A_{3332} & A_{3331} & A_{3313} \\ A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} & A_{1223} & A_{1232} & A_{1231} & A_{1213} \\ A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121} & A_{2123} & A_{2132} & A_{2131} & A_{2113} \\ A_{2311} & A_{2322} & A_{2333} & A_{2312} & A_{2321} & A_{2323} & A_{2332} & A_{2331} & A_{2313} \\ A_{3211} & A_{3222} & A_{3233} & A_{3212} & A_{3221} & A_{3223} & A_{3232} & A_{3231} & A_{3213} \\ A_{3111} & A_{3122} & A_{3133} & A_{3112} & A_{3121} & A_{3123} & A_{3132} & A_{3131} & A_{3113} \\ A_{1311} & A_{1322} & A_{1333} & A_{1312} & A_{1321} & A_{1323} & A_{1332} & A_{1331} & A_{1313} \end{bmatrix}
$$

$$
{}^4\boldsymbol{A}^c \;\rightarrow\; \underline{\underline{A}}^T \quad ; \quad {}^4\boldsymbol{A}^{rc} \;\rightarrow\; \underline{\underline{A}}_c \quad ; \quad {}^4\boldsymbol{A}^{lc} \;\rightarrow\; \underline{\underline{A}}_r
$$

Matrix/column notation $\boldsymbol{C} = {}^4\boldsymbol{A} : \boldsymbol{B}$

The double product of a fourth-order tensor ${}^4\boldsymbol{A}$ and a second-order tensor $\boldsymbol{B}$ is a second-order tensor, here denoted as $\boldsymbol{C}$. The components of $\boldsymbol{C}$ are stored in a column $\underset{\approx}{C}$, those of $\boldsymbol{B}$ in a column $\underset{\approx}{B}$. The components of ${}^4\boldsymbol{A}$ are stored in a 9×9 matrix. Using index-notation we can easily derive relations between the fore-mentioned columns.

$$\begin{aligned}
\boldsymbol{C} &= {}^4\boldsymbol{A} : \boldsymbol{B} \quad \rightarrow \\
\vec{e}_i C_{ij} \vec{e}_j &= \vec{e}_i \vec{e}_j A_{ijmn} \vec{e}_m \vec{e}_n : \vec{e}_p B_{pq} \vec{e}_q \\
&= \vec{e}_i \vec{e}_j A_{ijmn} \delta_{np} \delta_{mq} B_{pq} = \vec{e}_i \vec{e}_j A_{ijmn} B_{nm} \quad \rightarrow \\
\underset{\approx}{C} &= \underline{\underline{A}}_c \, \underset{\approx}{B} = \underline{\underline{A}} \, \underset{\approx}{B}_t
\end{aligned}$$

idem

$$\begin{aligned}
\boldsymbol{C} &= \boldsymbol{B} : {}^4\boldsymbol{A} \quad \rightarrow \\
\vec{e}_i C_{ij} \vec{e}_j &= \vec{e}_p B_{pq} \vec{e}_q : \vec{e}_m \vec{e}_n A_{mnij} \vec{e}_i \vec{e}_j \\
&= B_{pq} \delta_{qm} \delta_{pn} A_{mnij} \vec{e}_i \vec{e}_j = B_{nm} A_{mnij} \vec{e}_i \vec{e}_j \quad \rightarrow \\
\underset{\approx}{C}^T &= \underset{\approx}{B}^T \, \underline{\underline{A}}_r = \underset{\approx}{B}_t^T \, \underline{\underline{A}}
\end{aligned}$$

Matrix notation ${}^4\boldsymbol{C} = {}^4\boldsymbol{A} \cdot \boldsymbol{B}$

The inner product of a fourth-order tensor ${}^4\boldsymbol{A}$ and a second-order tensor $\boldsymbol{B}$ is a new fourth-order tensor, here denoted as ${}^4\boldsymbol{C}$. The components of all these tensors can be stored in matrices. For a three-dimensional physical problem, these would be of size 9×9. Here we only consider the 5×5 matrices, which would result in case of a two-dimensional problem.

$$\begin{aligned}
{}^4\boldsymbol{C} = {}^4\boldsymbol{A} \cdot \boldsymbol{B} &= \vec{e}_i \vec{e}_j A_{ijkl} \vec{e}_k \vec{e}_l \cdot \vec{e}_p B_{pq} \vec{e}_q \\
&= \vec{e}_i \vec{e}_j A_{ijkl} \vec{e}_k \delta_{lp} B_{pq} \vec{e}_q = \vec{e}_i \vec{e}_j A_{ijkl} B_{lq} \vec{e}_k \vec{e}_q \\
&= \vec{e}_i \vec{e}_j A_{ijkp} B_{pl} \vec{e}_k \vec{e}_l \quad \rightarrow
\end{aligned}$$

$$\begin{aligned}
\underline{\underline{C}} &= \begin{bmatrix}
A_{111p}B_{p1} & A_{112p}B_{p2} & A_{113p}B_{p3} & A_{111p}B_{p2} & A_{112p}B_{p1} \\
A_{221p}B_{p1} & A_{222p}B_{p2} & A_{223p}B_{p3} & A_{221p}B_{p2} & A_{222p}B_{p1} \\
A_{331p}B_{p1} & A_{332p}B_{p2} & A_{333p}B_{p3} & A_{331p}B_{p2} & A_{332p}B_{p1} \\
A_{121p}B_{p1} & A_{122p}B_{p2} & A_{123p}B_{p3} & A_{121p}B_{p2} & A_{122p}B_{p1} \\
A_{211p}B_{p1} & A_{212p}B_{p2} & A_{213p}B_{p3} & A_{211p}B_{p2} & A_{212p}B_{p1}
\end{bmatrix} \\
&= \begin{bmatrix}
A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} \\
A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} \\
A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} \\
A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} \\
A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121}
\end{bmatrix}
\begin{bmatrix}
B_{11} & 0 & 0 & B_{12} & 0 \\
0 & B_{22} & 0 & 0 & B_{21} \\
0 & 0 & B_{33} & 0 & 0 \\
B_{21} & 0 & 0 & B_{22} & 0 \\
0 & B_{12} & 0 & 0 & B_{11}
\end{bmatrix}
\end{aligned}$$

$$= \underline{\underline{A}} \, \underline{\underline{B}}_{cr} = \underline{\underline{A}}_c \, \underline{\underline{B}}_c \qquad \rightarrow \qquad \underline{\underline{C}}_r = \underline{\underline{A}}_r \, \underline{\underline{B}}_{cr} = \underline{\underline{A}}_{cr} \, \underline{\underline{B}}_c$$

Matrix notation $\,^4\boldsymbol{C} = \boldsymbol{B} \cdot \,^4\boldsymbol{A}$

The inner product of a second-order tensor and a fourth-order tensor can also be written as the product of the appropriate matrices.

$$
\begin{aligned}
^4\boldsymbol{C} = \boldsymbol{B} \cdot \,^4\boldsymbol{A} &= \vec{e}_i B_{ij} \vec{e}_j \cdot \vec{e}_p \vec{e}_q A_{pqrs} \vec{e}_r \vec{e}_s \\
&= \vec{e}_i B_{ij} \delta_{jp} \vec{e}_q A_{pqrs} \vec{e}_r \vec{e}_s = \vec{e}_i \vec{e}_q B_{ij} A_{jqrs} \vec{e}_r \vec{e}_s \\
&= \vec{e}_i \vec{e}_j B_{ip} A_{pjkl} \vec{e}_k \vec{e}_l \quad \rightarrow
\end{aligned}
$$

$$
\begin{aligned}
\underline{\underline{C}} &= \begin{bmatrix}
B_{1p}A_{p111} & B_{1p}A_{p122} & B_{1p}A_{p133} & B_{1p}A_{p112} & B_{1p}A_{p121} \\
B_{2p}A_{p211} & B_{2p}A_{p222} & B_{2p}A_{p233} & B_{2p}A_{p212} & B_{2p}A_{p221} \\
B_{3p}A_{p311} & B_{3p}A_{p322} & B_{3p}A_{p333} & B_{3p}A_{p312} & B_{3p}A_{p321} \\
B_{1p}A_{p211} & B_{1p}A_{p222} & B_{1p}A_{p233} & B_{1p}A_{p212} & B_{1p}A_{p221} \\
B_{2p}A_{p111} & B_{2p}A_{p122} & B_{2p}A_{p133} & B_{2p}A_{p112} & B_{2p}A_{p121}
\end{bmatrix} \\
&= \begin{bmatrix}
B_{11} & 0 & 0 & 0 & B_{12} \\
0 & B_{22} & 0 & B_{21} & 0 \\
0 & 0 & B_{33} & 0 & 0 \\
0 & B_{12} & 0 & B_{11} & 0 \\
B_{21} & 0 & 0 & 0 & B_{22}
\end{bmatrix}
\begin{bmatrix}
A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} \\
A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} \\
A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} \\
A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} \\
A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121}
\end{bmatrix} \\
&= \underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{B}}_c \underline{\underline{A}}_r \qquad \rightarrow \qquad \underline{\underline{C}}_r = \underline{\underline{B}}_r \underline{\underline{A}}_c = \underline{\underline{B}}_{cr} \underline{\underline{A}}_{cr}
\end{aligned}
$$

Matrix notation $\,^4\boldsymbol{C} = \,^4\boldsymbol{A} : \,^4\boldsymbol{B}$

The double inner product of two fourth-order tensors, $^4\boldsymbol{A}$ and $^4\boldsymbol{B}$, is again a fourth-order tensor $^4\boldsymbol{C}$. Its matrix, $\underline{\underline{C}}$, can be derived as the product of the matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$.

$$
\begin{aligned}
^4\boldsymbol{C} = \,^4\boldsymbol{A} : \,^4\boldsymbol{B} &= \vec{e}_i \vec{e}_j A_{ijkl} \vec{e}_k \vec{e}_l : \vec{e}_p \vec{e}_q B_{pqrs} \vec{e}_r \vec{e}_s \\
&= \vec{e}_i \vec{e}_j A_{ijkl} \delta_{lp} \delta_{kq} B_{pqrs} \vec{e}_r \vec{e}_s = \vec{e}_i \vec{e}_j A_{ijqp} B_{pqrs} \vec{e}_r \vec{e}_s \\
&= \vec{e}_i \vec{e}_j A_{ijqp} B_{pqkl} \vec{e}_k \vec{e}_l
\end{aligned}
$$

$$
\begin{aligned}
\underline{\underline{C}} &= \begin{bmatrix}
A_{11qp}B_{pq11} & A_{11qp}B_{pq22} & A_{11qp}B_{pq33} & A_{11qp}B_{pq12} & A_{11qp}B_{pq21} \\
A_{22qp}B_{pq11} & A_{22qp}B_{pq22} & A_{22qp}B_{pq33} & A_{22qp}B_{pq12} & A_{22qp}B_{pq21} \\
A_{33qp}B_{pq11} & A_{33qp}B_{pq22} & A_{33qp}B_{pq33} & A_{33qp}B_{pq12} & A_{33qp}B_{pq21} \\
A_{12qp}B_{pq11} & A_{12qp}B_{pq22} & A_{12qp}B_{pq33} & A_{12qp}B_{pq12} & A_{12qp}B_{pq21} \\
A_{21qp}B_{pq11} & A_{21qp}B_{pq22} & A_{21qp}B_{pq33} & A_{21qp}B_{pq12} & A_{21qp}B_{pq21}
\end{bmatrix} \\
&= \begin{bmatrix}
A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} \\
A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} \\
A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} \\
A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} \\
A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121}
\end{bmatrix}
\begin{bmatrix}
B_{1111} & B_{1122} & B_{1133} & B_{1112} & B_{1121} \\
B_{2211} & B_{2222} & B_{2233} & B_{2212} & B_{2221} \\
B_{3311} & B_{3322} & B_{3333} & B_{3312} & B_{3321} \\
B_{2111} & B_{2122} & B_{2133} & B_{2112} & B_{2121} \\
B_{1211} & B_{1222} & B_{1233} & B_{1212} & B_{1221}
\end{bmatrix} \\
&= \underline{\underline{A}}\,\underline{\underline{B}}_r = \underline{\underline{A}}_c \underline{\underline{B}}
\end{aligned}
$$

Matrix notation fourth-order unit tensor

The fourth-order unit tensor ${}^4\boldsymbol{I}$ can be written in matrix-notation. Following the definition of the matrix representation of a fourth-order tensor, the matrix $\underline{\underline{I}}$ may look a bit strange. The matrix representation of $\boldsymbol{A} = {}^4\boldsymbol{I} : \boldsymbol{A}$ is however consistently written as $\underset{\sim}{A} = \underline{\underline{I}}_c \, \underset{\sim}{A}$. In some situations the symmetric fourth-order unit tensor ${}^4\boldsymbol{I}^s$ is used.

$$
{}^4\boldsymbol{I} = \vec{e}_i\vec{e}_j\delta_{il}\delta_{jk}\vec{e}_k\vec{e}_l \quad \rightarrow
$$

$$
\underline{\underline{I}} =
$$

$$
\begin{bmatrix}
\delta_{11}\delta_{11} & \delta_{12}\delta_{12} & \delta_{13}\delta_{13} & \delta_{12}\delta_{11} & \delta_{11}\delta_{12} & \cdot \\
\delta_{21}\delta_{21} & \delta_{22}\delta_{22} & \delta_{23}\delta_{23} & \delta_{22}\delta_{21} & \delta_{21}\delta_{22} & \cdot \\
\delta_{31}\delta_{31} & \delta_{32}\delta_{32} & \delta_{33}\delta_{33} & \delta_{32}\delta_{31} & \delta_{31}\delta_{32} & \cdot \\
\delta_{11}\delta_{21} & \delta_{12}\delta_{22} & \delta_{13}\delta_{23} & \delta_{12}\delta_{21} & \delta_{11}\delta_{22} & \cdot \\
\delta_{21}\delta_{11} & \delta_{22}\delta_{12} & \delta_{23}\delta_{13} & \delta_{22}\delta_{11} & \delta_{21}\delta_{12} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

$$
{}^4\boldsymbol{I}^s = \tfrac{1}{2}\left({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}\right) \quad \rightarrow \quad \underline{\underline{I}}^s = \tfrac{1}{2}\left(\underline{\underline{I}} + \underline{\underline{I}}_c\right) = \tfrac{1}{2}
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & \cdot \\
0 & 2 & 0 & 0 & 0 & \cdot \\
0 & 0 & 2 & 0 & 0 & \cdot \\
0 & 0 & 0 & 1 & 1 & \cdot \\
0 & 0 & 0 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot &
\end{bmatrix}
$$

Matrix notation $\boldsymbol{II}$

In some relations the dyadic product $\boldsymbol{II}$ of the second-order unit tensor with itself appears. Its matrix representation can easily be written as the product of columns $\underset{\sim}{I}$ and its transposed.

$$
\boldsymbol{II} = \vec{e}_i\delta_{ij}\vec{e}_j\vec{e}_k\delta_{kl}\vec{e}_l = \vec{e}_i\vec{e}_j\delta_{ij}\delta_{kl}\vec{e}_k\vec{e}_l \quad \rightarrow
$$

$$
\underline{\underline{II}} =
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \underset{\sim}{I}\,\underset{\sim}{I}^T
$$

Matrix notation $\quad {}^4\boldsymbol{B} = {}^4\boldsymbol{I} \cdot \boldsymbol{A}$

The inner product of the fourth-order unit tensor ${}^4\boldsymbol{I}$ and a second-order tensor $\boldsymbol{A}$, can be elaborated using their matrices.

$$ {}^4\boldsymbol{B} = {}^4\boldsymbol{I} \cdot \boldsymbol{A} = \vec{e}_i\vec{e}_j\delta_{il}\delta_{jk}\vec{e}_k\vec{e}_l \cdot \vec{e}_p A_{pq}\vec{e}_q = \boldsymbol{A} \cdot {}^4\boldsymbol{I} \quad \rightarrow $$

$$ \underline{\underline{B}} = \begin{bmatrix} A_{11}\delta_{11} & A_{12}\delta_{12} & A_{13}\delta_{13} & A_{12}\delta_{11} & A_{11}\delta_{12} & .. \\ A_{21}\delta_{21} & A_{22}\delta_{22} & A_{23}\delta_{23} & A_{22}\delta_{21} & A_{21}\delta_{22} & .. \\ A_{31}\delta_{31} & A_{32}\delta_{32} & A_{33}\delta_{33} & A_{32}\delta_{31} & A_{31}\delta_{32} & .. \\ A_{11}\delta_{21} & A_{12}\delta_{22} & A_{13}\delta_{23} & A_{12}\delta_{21} & A_{11}\delta_{22} & .. \\ A_{21}\delta_{11} & A_{22}\delta_{12} & A_{23}\delta_{13} & A_{22}\delta_{11} & A_{21}\delta_{12} & .. \\ .. & .. & .. & .. & .. & .. \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & A_{12} & 0 & .. \\ 0 & A_{22} & 0 & 0 & A_{21} & .. \\ 0 & 0 & A_{33} & 0 & 0 & .. \\ 0 & A_{12} & 0 & 0 & A_{11} & .. \\ A_{21} & 0 & 0 & A_{22} & 0 & .. \\ .. & .. & .. & .. & .. & .. \end{bmatrix} $$

$$ = \underline{\underline{A}}_c $$

Summary and examples

Below the tensor/matrix transformation procedure is summarized and illustrated with a few examples. The storage of matrix components in columns or 'blown-up' matrices is easily done with the Matlab .m-files `m2cc.m` and `m2mm.m`. (See appendix C.)

$$ \begin{array}{ccl} \vec{x} & \rightarrow & \underset{\sim}{x} \\ \boldsymbol{A} & \rightarrow & \underline{A} \;\; ; \;\; \underset{\approx}{A} \;\; ; \;\; \underline{\underline{A}} \\ {}^4\boldsymbol{A} & \rightarrow & \underline{\underline{A}} \\ {}^4\boldsymbol{I} & \rightarrow & \underline{\underline{I}} \end{array} $$

$$ \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ; \quad \underline{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} $$

$$ \underset{\approx}{A} = \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{12} \\ A_{21} \\ .. \end{bmatrix} \quad ; \quad \underline{\underline{A}} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & A_{12} & .. \\ 0 & A_{22} & 0 & A_{21} & 0 & .. \\ 0 & 0 & A_{33} & 0 & 0 & .. \\ 0 & A_{12} & 0 & A_{11} & 0 & .. \\ A_{21} & 0 & 0 & 0 & A_{22} & .. \\ .. & .. & .. & .. & .. & .. \end{bmatrix} $$

$$ \underline{\underline{A}} = \begin{bmatrix} A_{1111} & A_{1122} & A_{1133} & A_{1112} & A_{1121} & .. \\ A_{2211} & A_{2222} & A_{2233} & A_{2212} & A_{2221} & .. \\ A_{3311} & A_{3322} & A_{3333} & A_{3312} & A_{3321} & .. \\ A_{1211} & A_{1222} & A_{1233} & A_{1212} & A_{1221} & .. \\ A_{2111} & A_{2122} & A_{2133} & A_{2112} & A_{2121} & .. \\ .. & .. & .. & .. & .. & .. \end{bmatrix} \; ; \; \underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & .. \\ 0 & 1 & 0 & 0 & 0 & .. \\ 0 & 0 & 1 & 0 & 0 & .. \\ 0 & 0 & 0 & 0 & 1 & .. \\ 0 & 0 & 0 & 1 & 0 & .. \\ .. & .. & .. & .. & .. & .. \end{bmatrix} $$

In de Matlab programs the name of a column or matrix indicates its structure as is indicated below. Some manipulations are introduced, which are easily done in Matlab.

$\boldsymbol{A} \rightarrow \underline{A}$	: 3×3 matrix	$\rightarrow$ `mA`
$\boldsymbol{A} \rightarrow \underset{\sim}{A}$	: column with components of $\underline{A}$	$\rightarrow$ `ccA = m2cc(mA,9)`
$\boldsymbol{A} \rightarrow \underline{\underline{A}}$	: 'blown-up' matrix	$\rightarrow$ `mmA = m2mm(mA,9)`
$\boldsymbol{A}^c \rightarrow \underline{A}^T$	: transpose	$\rightarrow$ `mAt = mA'`
$\boldsymbol{A}^c \rightarrow \underset{\sim}{A}_t$	: transpose all components	$\rightarrow$ `ccAt = m2cc(mAt,9)`
$\boldsymbol{A}^c \rightarrow \underline{\underline{A}}_t$	: transpose all components	$\rightarrow$ `mmAt = m2mm(mA')`
${}^4\boldsymbol{A}^{lc} \rightarrow \underline{\underline{A}}_r$	: interchange rows 4/5, 6/7, 8/9	$\rightarrow$ `mmAr = mmA([1 2 3 5 4 7 6 9 8],:)`
${}^4\boldsymbol{A}^{rc} \rightarrow \underline{\underline{A}}_c$	: interchange columns 4/5, 6/7, 8/9	$\rightarrow$ `mmAc = mmA(:,[1 2 3 5 4 7 6 9 8])`

4.2.3 Gradients

Gradient operators are used to differentiate w.r.t. coordinates and are as such associated with the coordinate system. The base vectors – unit tangent vectors to the coordinate axes – in the Cartesian system, $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$, are independent of the coordinates $\{x, y, z\}$. Two base vectors in the cylindrical coordinate system with coordinates $\{r, \theta, z\}$, are a function of the coordinate θ : $\{\vec{e}_r(\theta), \vec{e}_t(\theta), \vec{e}_z\}$. This dependency has ofcourse to be taken into account when writing gradients of vectors and tensors in components w.r.t. the coordinate system, using matrix/column notation.

The gradient of a vector is denoted as a conjugate tensor, whose components can be stored in a matrix or in a column : $\boldsymbol{L}_a = (\vec{\nabla}\vec{a})^c \quad \rightarrow \quad \underline{L}_a \quad \rightarrow \quad \underset{\sim}{L}_a$.

Cartesian
$$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = \begin{bmatrix} \vec{e}_x & \vec{e}_t & \vec{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \underset{\sim}{\vec{e}}^T \underset{\sim}{\nabla}$$

cylindrical
$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_t \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z} = \begin{bmatrix} \vec{e}_r & \vec{e}_t & \vec{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{bmatrix} = \underset{\sim}{\vec{e}}^T \underset{\sim}{\nabla}$$

Gradient of a vector in Cartesian coordinates

$$\begin{aligned}
\vec{\nabla}\vec{a} = \boldsymbol{L}_a^c &= \left(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \right) (a_x \vec{e}_x + a_y \vec{e}_y + a_z \vec{e}_z) \\
&= \vec{e}_x a_{x,x} \vec{e}_x + \vec{e}_x a_{y,x} \vec{e}_y + \vec{e}_x a_{z,x} \vec{e}_z + \vec{e}_y a_{x,y} \vec{e}_x + \\
&\qquad \vec{e}_y a_{y,y} \vec{e}_y + \vec{e}_y a_{z,y} \vec{e}_z + \vec{e}_z a_{x,z} \vec{e}_x + \vec{e}_z a_{y,z} \vec{e}_y + \vec{e}_z a_{z,z} \vec{e}_z
\end{aligned}$$

$$\underline{L}_a = \begin{bmatrix} a_{x,x} & a_{x,y} & a_{x,z} \\ a_{y,x} & a_{y,y} & a_{y,z} \\ a_{z,x} & a_{z,y} & a_{z,z} \end{bmatrix}$$

$$\underset{\approx}{L}_a^T = \begin{bmatrix} a_{x,x} & a_{y,y} & a_{z,z} & a_{x,y} & a_{y,x} & a_{y,z} & a_{z,y} & a_{z,x} & a_{x,z} \end{bmatrix}$$

Gradient of a vector in cylindrical coordinates

$$\begin{aligned}
\vec{\nabla}\vec{a} = \boldsymbol{L}_a^c &= \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_t \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z}\right)(a_r\vec{e}_r + a_t\vec{e}_t + a_z\vec{e}_z) \\
&= \vec{e}_r a_{r,r}\vec{e}_r + \vec{e}_r a_{t,r}\vec{e}_t + \vec{e}_r a_{z,r}\vec{e}_z + \vec{e}_t \frac{1}{r} a_{r,t}\vec{e}_r + \vec{e}_t \frac{1}{r} a_{t,t}\vec{e}_t + \vec{e}_t \frac{1}{r} a_{z,t}\vec{e}_z + \vec{e}_t \frac{1}{r} a_r\vec{e}_t - \vec{e}_t \frac{1}{r} a_t\vec{e}_r \\
&\quad \vec{e}_z a_{r,z}\vec{e}_r + \vec{e}_z a_{t,z}\vec{e}_t + \vec{e}_z a_{z,z}\vec{e}_z
\end{aligned}$$

$$\underline{L}_a = \begin{bmatrix} a_{r,r} & \frac{1}{r}a_{r,t} - \frac{1}{r}a_t & a_{r,z} \\ a_{t,r} & \frac{1}{r}a_{t,t} + \frac{1}{r}a_r & a_{t,z} \\ a_{z,r} & \frac{1}{r}a_{z,t} & a_{z,z} \end{bmatrix}$$

$$\underset{\approx}{L}_a^T = \begin{bmatrix} a_{r,r} & \frac{1}{r}a_{t,t} + \frac{1}{r}a_r & a_{z,z} & \frac{1}{r}a_{r,t} - \frac{1}{r}a_t & a_{t,r} & a_{t,z} & \frac{1}{r}a_{z,t} & a_{z,r} & a_{r,z} \end{bmatrix}$$

Divergence of a tensor in cylindrical coordinates

$$\begin{aligned}
\vec{\nabla} \cdot \boldsymbol{A} &= \vec{e}_i \cdot \nabla_i(\vec{e}_j A_{jk}\vec{e}_k) \\
&= \vec{e}_i \cdot (\nabla_i \vec{e}_j) A_{jk}\vec{e}_k + \vec{e}_i \cdot \vec{e}_j (\nabla_i A_{jk})\vec{e}_k + \vec{e}_i \cdot \vec{e}_j A_{jk}(\nabla_i \vec{e}_k) \\
&= \vec{e}_i \cdot (\nabla_i \vec{e}_j) A_{jk}\vec{e}_k + \delta_{ij}(\nabla_i A_{jk})\vec{e}_k + \delta_{ij} A_{jk}(\nabla_i \vec{e}_k)
\end{aligned}$$

$$\nabla_i \vec{e}_j = \delta_{i2}\delta_{1j}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2j}\,\frac{1}{r}\,\vec{e}_r$$

$$\begin{aligned}
&= \vec{e}_i \cdot (\delta_{i2}\delta_{1j}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2j}\,\frac{1}{r}\,\vec{e}_r) A_{jk}\vec{e}_k + \delta_{ij}(\nabla_i A_{jk})\vec{e}_k + \delta_{ij} A_{jk}(\delta_{i2}\delta_{1k}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2k}\,\frac{1}{r}\,\vec{e}_r) \\
&= \vec{e}_i \cdot (\delta_{i2}\delta_{1j}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2j}\,\frac{1}{r}\,\vec{e}_r) A_{jk}\vec{e}_k + \delta_{ij}(\nabla_i A_{jk})\vec{e}_k + (\delta_{i2}\delta_{1k}\,\frac{1}{r}\,\vec{e}_t - \delta_{i2}\delta_{2k}\,\frac{1}{r}\,\vec{e}_r) A_{jk}\delta_{ij} \\
&= \vec{e}_t \cdot (\delta_{1j}\,\frac{1}{r}\,\vec{e}_t - \delta_{2j}\,\frac{1}{r}\,\vec{e}_r) A_{jk}\vec{e}_k + (\nabla_j A_{jk})\vec{e}_k + (\delta_{j2}\delta_{1k}\,\frac{1}{r}\,\vec{e}_t - \delta_{j2}\delta_{2k}\,\frac{1}{r}\,\vec{e}_r) A_{jk} \\
&= \delta_{1j}\,\frac{1}{r}\,A_{jk}\vec{e}_k + (\nabla_j A_{jk})\vec{e}_k + (\delta_{j2}\delta_{1k}\,\frac{1}{r}\,\vec{e}_t - \delta_{j2}\delta_{2k}\,\frac{1}{r}\,\vec{e}_r) A_{jk} \\
&= \frac{1}{r} A_{1k}\vec{e}_k + (\nabla_j A_{jk})\vec{e}_k + \frac{1}{r}\,(A_{21}\vec{e}_t - A_{22}\vec{e}_r) \\
&= (\frac{1}{r}A_{11} - \frac{1}{r}A_{22})\vec{e}_1 + (\frac{1}{r}A_{12} + \frac{1}{r}A_{21})\vec{e}_2 + \frac{1}{r}A_{13}\vec{e}_3 + (\nabla_j A_{jk})\vec{e}_k \\
&= g_k\vec{e}_k + \nabla_j A_{jk}\vec{e}_k \\
&= \underset{\sim}{g}^T \underset{\sim}{\vec{e}} + (\underset{\sim}{\nabla}^T \underline{A})\underset{\sim}{\vec{e}} \\
&= (\underset{\sim}{\nabla}^T \underline{A})\underset{\sim}{\vec{e}} + \underset{\sim}{g}^T \underset{\sim}{\vec{e}} \qquad \text{with} \qquad \underset{\sim}{g}^T = \frac{1}{r}\begin{bmatrix} (A_{11} - A_{22}) & (A_{12} + A_{21}) & A_{33} \end{bmatrix}
\end{aligned}$$

Chapter 5

Kinematics

The motion and deformation of a three-dimensional continuum is studied in *continuum mechanics*. A continuum is an ideal material body, where the neighborhood of a material point is assumed to be dense and fully occupied with other material points. The real microstructure of the material (molecules, crystals, particles, ...) is not considered. The deformation is also continuous, which implies that the neighborhood of a material point always consists of the same collection of material points.

Kinematics describes the transformation of a material body from its undeformed to its deformed state without paying attention to the cause of deformation. In the mathematical formulation of kinematics a Lagrangian or an Eulerian approach can be chosen. (It is also possible to follow a so-called Arbitrary-Lagrange-Euler approach.)

The undeformed state is indicated as the state at time t_0 and the deformed state as the state at the current time t. When the deformation process is time- or rate-independent, the time variable must be considered to be a fictitious time, only used to indicate subsequent moments in the deformation process.

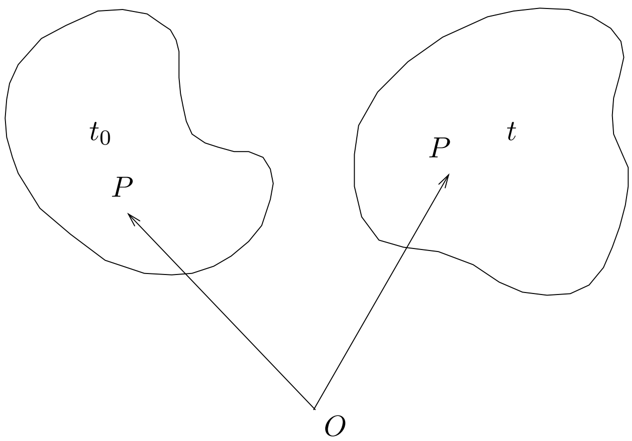


Fig. 5.1 : *Deformation of continuum*

5.1 Identification of points

Describing the deformation of a material body cannot be done without a proper identification of the individual material points.

5.1.1 Material coordinates

Each point of the material can be identified by or labeled with material coordinates. In a three-dimensional space three coordinates $\{\xi_1, \xi_2, \xi_3\}$ are needed and sufficient to identify a point uniquely. The material coordinates of a material point do never change. They can be stored in a column $\underset{\sim}{\xi}$: $\underset{\sim}{\xi}^T = \left[\begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \end{array} \right]$.

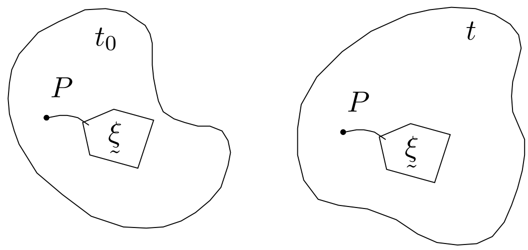


Fig. 5.2 : *Material coordinates*

5.1.2 Position vectors

A point of the material can also be identified with its position in space. Two position vectors can be chosen for this purpose : the position vector in the undeformed state, $\vec{x}_0$, or the position vector in the current, deformed state, $\vec{x}$. Both position vectors can be considered to be a function of the material coordinates $\underset{\sim}{\xi}$.

Each point is always identified with one position vector. One spatial position is always occupied by one material point. For a continuum the position vector is a continuous differentiable function.

Using a vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, components of the position vectors can be determined and stored in columns.

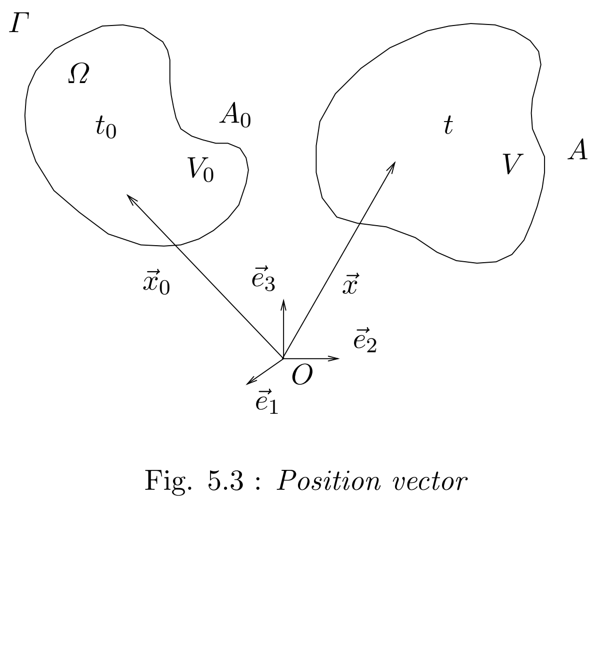


Fig. 5.3 : *Position vector*

undeformed configuration (t_0) $\vec{x}_0 = \vec{\chi}(\underset{\sim}{\xi}, t_0) = x_{01}\vec{e}_1 + x_{02}\vec{e}_2 + x_{03}\vec{e}_3$

deformed configuration (t) $\vec{x} = \vec{\chi}(\underset{\sim}{\xi}, t) = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$

5.1.3 Euler-Lagrange

When an *Eulerian formulation* is used, all variables are determined in material points which are identified in the deformed state with their *current* position vector $\vec{x}$. When a *Lagrangian formulation* is used to describe state transformation, all variables are determined in material points which are identified in the undeformed state with their *initial* position vector $\vec{x}_0$. For a scalar quantity a, this can be formally written with a function $\mathcal{A}_E$ or $\mathcal{A}_L$, respectively.

The difference da of a scalar quantity a in two adjacent points P and Q can be calculated in both the Eulerian and the Lagrangian framework. This leads to the definition of two gradient operators, $\vec{\nabla}$ and $\vec{\nabla}_0$, respectively.

For a vectorial quantity $\vec{a}$, the spatial difference $d\vec{a}$ in two adjacent points, can also be calculated, using either $\vec{\nabla}_0$ or $\vec{\nabla}$. For the position vectors, the gradients result in the unity tensor $\boldsymbol{I}$.

Euler : "observer" is fixed in space

$$a = \mathcal{A}_E(\vec{x}, t)$$

$$da = a_Q - a_P = \mathcal{A}_E(\vec{x} + d\vec{x}, t) - \mathcal{A}_E(\vec{x}, t) = d\vec{x} \cdot (\vec{\nabla} a)\Big|_t$$

$$\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3}$$

Lagrange : "observer" follows the material

$$a = \mathcal{A}_L(\vec{x}_0, t)$$

$$da = a_Q - a_P = \mathcal{A}_L(\vec{x}_0 + d\vec{x}_0, t) - \mathcal{A}_L(\vec{x}_0, t) = d\vec{x}_0 \cdot (\vec{\nabla}_0 a)\Big|_t$$

$$\vec{\nabla}_0 = \vec{e}_1 \frac{\partial}{\partial x_{01}} + \vec{e}_2 \frac{\partial}{\partial x_{02}} + \vec{e}_3 \frac{\partial}{\partial x_{03}}$$

position vectors

$$\vec{\nabla}\vec{x} = \boldsymbol{I} \quad ; \quad \vec{\nabla}_0\vec{x}_0 = \boldsymbol{I}$$

5.2 Time derivatives

A time derivative of a variable expresses the change of its value in time. This change can be measured in one and the same material point or in one and the same point in space. In the first case, the observer of the change follows the material, and, in the second case, he is located in a fixed spatial position.

This difference of observer position leads to two different time derivatives, the *material time derivative* and the *spatial time derivative*. Using a material time derivative is associated with the Lagrangian formulation, while in the Eulerian formulation the spatial time derivative is generally used. Below, we consider the time derivatives of a scalar vatiable a.

material time derivative $\quad \dfrac{Da}{Dt} = \dot{a} = \lim\limits_{\Delta t \longrightarrow 0} \dfrac{1}{\Delta t} \{A(\vec{x}_0, t + \Delta t) - A(\vec{x}_0, t)\}$

velocity of a material point $\quad \vec{v} = \vec{v}(\vec{x}_0) = \dot{\vec{x}}$

spatial time derivative $\quad \dfrac{\delta a}{\delta t} = \lim\limits_{\Delta t \longrightarrow 0} \dfrac{1}{\Delta t} \{\mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t)\}$

velocity *field* $\quad \vec{v} = \vec{v}(\vec{x}, t)$

A relation between the material and the spatial time derivative can be derived. The material velocity enters this relation and represents the velocity of the observer. The material time derivative can be written as the sum of the spatial time derivative and the convective time derivative.

$$
\begin{aligned}
\frac{Da}{Dt} &= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{A(\vec{x}_0, t + \Delta t) - A(\vec{x}_0, t)\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{\mathcal{A}(\vec{x} + d\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t)\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{\mathcal{A}(\vec{x} + d\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t + \Delta t) + \mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t)\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{d\vec{x} \cdot (\vec{\nabla} a)\Big|_{t+\Delta t} + \mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t)\} \\
&= \lim_{\Delta t \longrightarrow 0} \{\frac{d\vec{x}}{\Delta t} \cdot (\vec{\nabla} a)\Big|_{t+\Delta t}\} + \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \{\mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t)\} \\
&= \vec{v} \cdot (\vec{\nabla} a) + \frac{\delta a}{\delta t} \\
&= \text{(convective time derivative)} + \text{(spatial time derivative)} \\
&= \text{(material time derivative)}
\end{aligned}
$$

5.3 Deformation

Upon deformation, a material point changes position from $\vec{x}_0$ to $\vec{x}$. This is denoted with a displacement vector $\vec{u}$. In three-dimensional space this vector has three components : u_1, u_2 and u_3.

The deformation of the material can be described by the displacement vector of all the material points. This, however, is not a feasible procedure. Instead, we consider the deformation of an infinitesimal material volume in each point, which can be described with a *deformation tensor*.

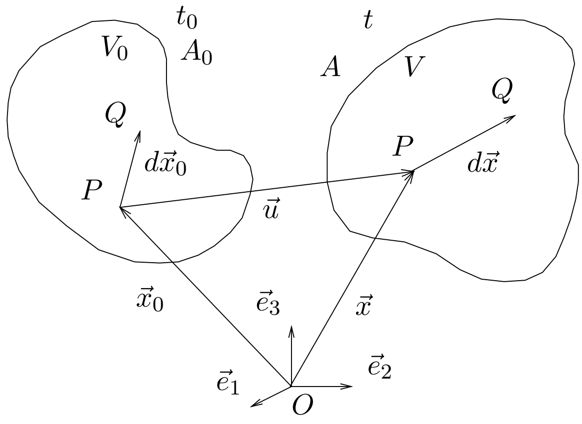


Fig. 5.4 : *Deformation of a continuum*

$$\vec{u} = \vec{x} - \vec{x}_0 = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$$

5.3.1 Deformation tensor

To introduce the deformation tensor, we first consider the deformation of an infinitesimal material line element, between two adjacent material points. The vector between these points in the undeformed state is $d\vec{x}_0$. Deformation results in a transformation of this vector to $d\vec{x}$, which can be denoted with a tensor, the deformation tensor $\boldsymbol{F}$. Using the gradient operator with respect to the undeformed state, the deformation tensor can be written as a gradient, which explains its much used name : *deformation gradient tensor.*

$$\begin{aligned} d\vec{x} &= \boldsymbol{F} \cdot d\vec{x}_0 \\ &= \vec{X}(\vec{x}_0 + d\vec{x}_0, \boldsymbol{t}) - \vec{X}(\vec{x}_0, \boldsymbol{t}) = d\vec{x}_0 \cdot \left(\vec{\nabla}_0\vec{x}\right) \\ &= \left(\vec{\nabla}_0\vec{x}\right)^c \cdot d\vec{x}_0 = \boldsymbol{F} \cdot d\vec{x}_0 \end{aligned}$$

$$\boldsymbol{F} = \left(\vec{\nabla}_0\vec{x}\right)^c = \left[\left(\vec{\nabla}_0\vec{x}_0\right)^c + \left(\vec{\nabla}_0\vec{u}\right)^c\right] = \boldsymbol{I} + \left(\vec{\nabla}_0\vec{u}\right)^c$$

In the undeformed configuration, an infinitesimal material volume is uniquely defined by three material line elements or material vectors $d\vec{x}_{01}$, $d\vec{x}_{02}$ and $d\vec{x}_{03}$. Using the deformation tensor $\boldsymbol{F}$, these vectors are transformed to the deformed state to become $d\vec{x}_1$, $d\vec{x}_2$ and $d\vec{x}_3$. These vectors span the deformed volume element, containing the same material points as in the initial volume element. It is thus obvious that $\boldsymbol{F}$ describes the transformation of the material.

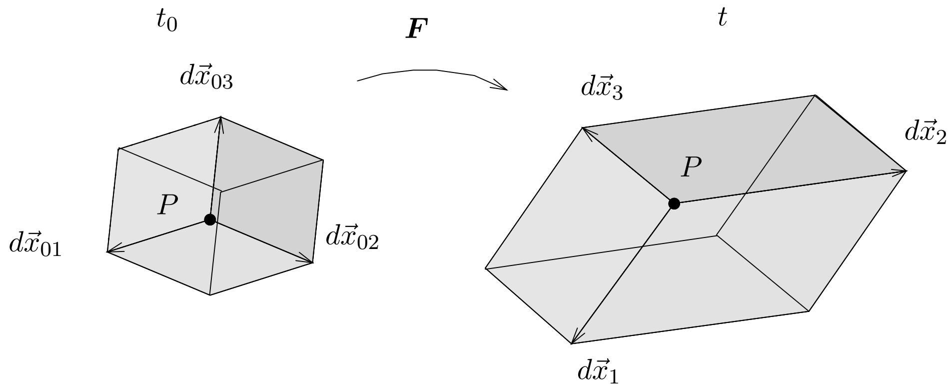


Fig. 5.5 : *Deformation tensor*

$$d\vec{x}_1 = \boldsymbol{F} \cdot d\vec{x}_{01} \qquad ; \qquad d\vec{x}_2 = \boldsymbol{F} \cdot d\vec{x}_{02} \qquad ; \qquad d\vec{x}_3 = \boldsymbol{F} \cdot d\vec{x}_{03}$$

Volume change

The three vectors which span the material element, can be combined in a triple product. The resulting scalar value is positive when the vectors are right-handed and represents the volume of the material element. In the undeformed state this volume is dV_0 and after deformation the volume is dV. Using the deformation tensor $\boldsymbol{F}$ and the definition of the determinant (third invariant) of a second-order tensor, the relation between dV and dV_0 can be derived.

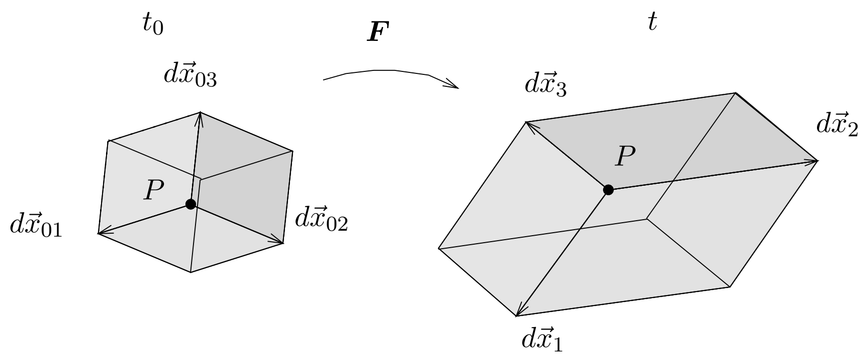


Fig. 5.6 : *Volume change*

$$\begin{aligned} dV &= d\vec{x}_1 * d\vec{x}_2 \cdot d\vec{x}_3 \\ &= (\boldsymbol{F} \cdot d\vec{x}_{01}) * (\boldsymbol{F} \cdot d\vec{x}_{02}) \cdot (\boldsymbol{F} \cdot d\vec{x}_{03}) \\ &= \det(\boldsymbol{F})\,(d\vec{x}_{01} * d\vec{x}_{02} \cdot d\vec{x}_{03}) \\ &= \det(\boldsymbol{F})\,dV_0 \\ &= J\,dV_0 \end{aligned}$$

Area change

The vector product of two vectors along two material line elements represents a vector, the length of which equals the area of the parallelogram spanned by the vectors. Using the deformation tensor $\boldsymbol{F}$, the change of area during deformation can be calculated.

$$
\begin{aligned}
dA\,\vec{n} &= d\vec{x}_1 * d\vec{x}_2 = (\boldsymbol{F}\cdot d\vec{x}_{01}) * (\boldsymbol{F}\cdot d\vec{x}_{02}) \\
dA\,\vec{n}\cdot(\boldsymbol{F}\cdot d\vec{x}_{03}) &= (\boldsymbol{F}\cdot d\vec{x}_{01}) * (\boldsymbol{F}\cdot d\vec{x}_{02})\cdot(\boldsymbol{F}\cdot d\vec{x}_{03}) \\
&= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot d\vec{x}_{03} \qquad \forall \quad d\vec{x}_{03} \quad \rightarrow \\
dA\,\vec{n}\cdot\boldsymbol{F} &= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02}) \\
dA\,\vec{n} &= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot\boldsymbol{F}^{-1} \\
&= \det(\boldsymbol{F})dA_0\,\vec{n}_0\cdot\boldsymbol{F}^{-1} \\
&= dA_0\,\vec{n}_0\cdot\left(\boldsymbol{F}^{-1}\det(\boldsymbol{F})\right)
\end{aligned}
$$

Inverse deformation

The determinant of the deformation tensor, being the quotient of two volumes, is always a positive number. This implies that the deformation tensor is regular and that the inverse $\boldsymbol{F}^{-1}$ exists. It represents the transformation of the deformed state to the undeformed state. The gradient operators $\vec{\nabla}$ and $\vec{\nabla}_0$ are related by the (inverse) deformation tensor.

$$
J = \frac{dV}{dV_0} = \det(\boldsymbol{F}) > 0 \quad \rightarrow \quad \boldsymbol{F} \text{ regular} \quad \rightarrow \quad d\vec{x}_0 = \boldsymbol{F}^{-1}\cdot d\vec{x}
$$

relation between gradient operators

$$
\boldsymbol{I} = \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{T} \rightarrow \left(\vec{\nabla}\vec{x}\right) = \boldsymbol{F}^{-T}\cdot\left(\vec{\nabla}_0\vec{x}\right) \quad \rightarrow \quad \vec{\nabla} = \boldsymbol{F}^{-T}\cdot\vec{\nabla}_0
$$

Homogeneous deformation

The deformation tensor describes the deformation of an infinitesimal material volume, initially located at position $\vec{x}_0$. The deformation tensor is generally a function of the position $\vec{x}_0$. When $\boldsymbol{F}$ is not a function of position $\vec{x}_0$, the deformation is referred to as being *homogeneous*. In that case, each infinitesimal material volume shows the same deformation. The current position vector $\vec{x}$ can be related to the initial position vector $\vec{x}_0$ and an unknown rigid body translation $\vec{t}$.

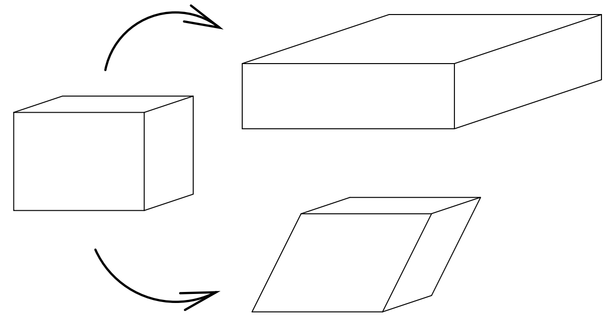


Fig. 5.7 : *Homogeneous deformation*

$$\vec{\nabla}_0 \vec{x} = \boldsymbol{F}^c = \text{uniform tensor} \quad \rightarrow \quad \vec{x} = (\vec{x}_0 \cdot \boldsymbol{F}^c) + \vec{t} = \boldsymbol{F} \cdot \vec{x}_0 + \vec{t}$$

5.3.2 Elongation and shear

During deformation a material line element $d\vec{x}_0$ is transformed to the line element $d\vec{x}$. The *elongation factor* or *stretch ratio* λ of the line element, is defined as the ratio of its length after and before deformation. The elongation factor can be expressed in $\boldsymbol{F}$ and $\vec{e}_0$, the unity direction vector of $d\vec{x}_0$. It follows that the elongation is calculated from the product $\boldsymbol{F}^c \cdot \boldsymbol{F}$, which is known as the right Cauchy-Green stretch tensor $\boldsymbol{C}$.

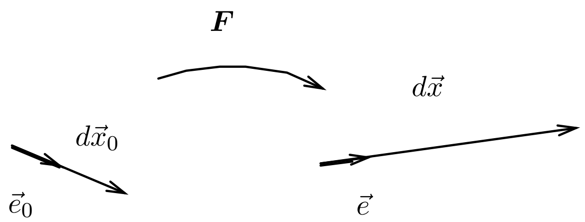


Fig. 5.8 : *Elongation of material line element*

$$\begin{aligned}
\lambda^2(\vec{e}_{01}) &= \frac{d\vec{x}_1 \cdot d\vec{x}_1}{d\vec{x}_{01} \cdot d\vec{x}_{01}} = \frac{d\vec{x}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot d\vec{x}_{01}}{d\vec{x}_{01} \cdot d\vec{x}_{01}} = \frac{||d\vec{x}_{01}||^2}{||d\vec{x}_{01}||^2} \left(\vec{e}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{e}_{01}\right) \\
&= \vec{e}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{e}_{01} = \vec{e}_{01} \cdot \boldsymbol{C} \cdot \vec{e}_{01}
\end{aligned}$$

We consider two material vectors in the undeformed state, $d\vec{x}_{01}$ and $d\vec{x}_{02}$, which are perpendicular. The shear deformation γ is defined as the cosine of θ, the angle between the two material vectors in the deformed state. The shear deformation can be expressed in $\boldsymbol{F}$ and $\vec{e}_{01}$ and $\vec{e}_{02}$, the unit direction vectors of $d\vec{x}_{01}$ and $d\vec{x}_{02}$. Again the shear is calculated from the right Cauchy-Green stretch tensor $\boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F}$.

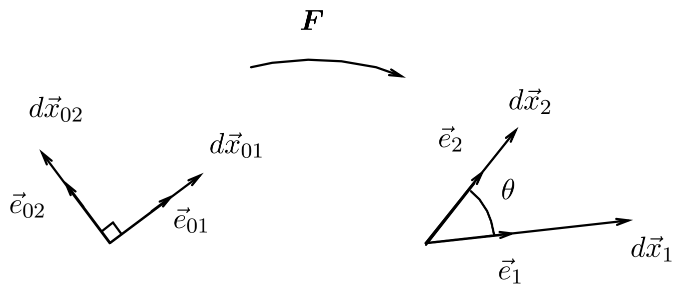


Fig. 5.9 : *Shear of two material line elements*

$$
\begin{aligned}
\gamma(\vec{e}_{01}, \vec{e}_{02}) &= \sin\left(\tfrac{\pi}{2} - \theta\right) = \cos(\theta) = \frac{d\vec{x}_1 \cdot d\vec{x}_2}{||d\vec{x}_1||\,||d\vec{x}_2||} = \frac{d\vec{x}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot d\vec{x}_{02}}{||d\vec{x}_1||\,||d\vec{x}_2||} \\
&= \frac{||d\vec{x}_{01}||\,||d\vec{x}_{02}||(\vec{e}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{e}_{02})}{\lambda(\vec{e}_{01})||d\vec{x}_{01}||\lambda(\vec{e}_{02})||d\vec{x}_{02}||} = \frac{\vec{e}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})} \\
&= \frac{\vec{e}_{01} \cdot \boldsymbol{C} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})}
\end{aligned}
$$

5.3.3 Principal directions of deformation

In each point P there is exactly one orthogonal material volume, which will not show any shear during deformation from t_0 to t. Rigid rotation may occur, although this is not shown in the figure.

The directions $\{1, 2, 3\}$ of the sides of the initial orthogonal volume are called *principal directions* of deformation and associated with them are the three *principal elongation factors* λ_1, λ_2 and λ_3. For this material volume the three principal elongation factors characterize the deformation uniquely. Be aware of the fact that the principal directions change when the deformation proceeds. They are a function of the time t.

The relative volume change J is the product of the three principal elongation factors. For incompressible material there is no volume change, so the above product will have value one.

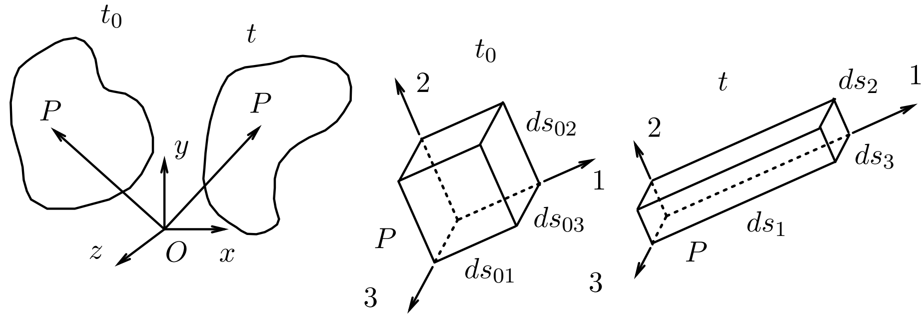


Fig. 5.10 : *Deformation of material cube with sides in principal directions*

$$\lambda_1 = \frac{ds_1}{ds_{01}} \quad ; \quad \lambda_2 = \frac{ds_2}{ds_{02}} \quad ; \quad \lambda_3 = \frac{ds_3}{ds_{03}} \quad ; \quad \gamma_{12} = \gamma_{23} = \gamma_{31} = 0$$

$$J = \frac{dV}{dV_0} = \frac{ds_1 ds_2 ds_3}{ds_{01} ds_{02} ds_{03}} = \lambda_1 \lambda_2 \lambda_3$$

5.3.4 Strains

The elongation of a material line element is completely described by the stretch ratio λ. When there is no deformation, we have $\lambda = 1$. It is often convenient to describe the elongation with a so-called elongational strain, which is zero when there is no deformation. A strain ε is defined as a function of λ, which has to satisfy certain requirements. Much used strain definitions are the linear, the logarithmic, the Green-Lagrange and the Euler-Almansi strain. One of the requirements of a strain definition is that it must linearize toward the linear strain, which is illustrated in the figure below.

linear	$\varepsilon_l = \lambda - 1$
logarithmic	$\varepsilon_{ln} = \ln(\lambda)$
Green-Lagrange	$\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$
Euler-Almansi	$\varepsilon_{ea} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$

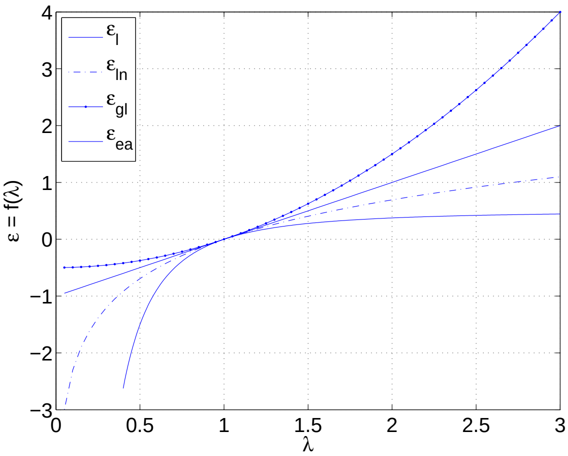


Fig. 5.11 : *Strain definitions*

5.3.5 Strain tensor

The Green-Lagrange strain of a line element with a known direction $\vec{e}_0$ in the undeformed state, can be calculated straightforwardly from the so-called Green-Lagrange strain tensor $\boldsymbol{E}$. Also the shear γ can be expressed in this tensor. For other strain definitions, different strain tensors are used, which are not descussed here.

$$\frac{1}{2}\left\{\lambda^2(\vec{e}_{01}) - 1\right\} = \vec{e}_{01} \cdot \left\{\frac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}\right)\right\} \cdot \vec{e}_{01} = \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{01}$$

$$\gamma(\vec{e}_{01}, \vec{e}_{02}) = \frac{\vec{e}_{01} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}) \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})} = \left[\frac{2}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})}\right] \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{02}$$

$$\left.\begin{array}{l} \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}\right) \\ \boldsymbol{F} = \left(\vec{\nabla}_0 \vec{x}\right)^T = \boldsymbol{I} + \left(\vec{\nabla}_0 \vec{u}\right)^T \end{array}\right\} \rightarrow \begin{array}{rl} \boldsymbol{E} & = \frac{1}{2}\left[\left\{\boldsymbol{I} + \left(\vec{\nabla}_0 \vec{u}\right)\right\} \cdot \left\{\boldsymbol{I} + \left(\vec{\nabla}_0 \vec{u}\right)^T\right\} - \boldsymbol{I}\right] \\ & = \frac{1}{2}\left[\left(\vec{\nabla}_0 \vec{u}\right)^T + \left(\vec{\nabla}_0 \vec{u}\right) + \left(\vec{\nabla}_0 \vec{u}\right) \cdot \left(\vec{\nabla}_0 \vec{u}\right)^T\right] \end{array}$$

5.3.6 Right Cauchy-Green deformation tensor

The general transformation of a material line element from the undeformed to the deformed state is uniquely described by the deformation (gradient) tensor $\boldsymbol{F}$. The true deformation consists of elongation of material line elements and mutual rotation of line elements, which is also referred to as shear.

The true deformation, represented by the expressions for λ and γ, is described by the product $\boldsymbol{F}^c \cdot \boldsymbol{F}$, which is called the *right Cauchy-Green deformation tensor* $\boldsymbol{C}$. This important tensor has two properties, which are easily recognized : 1) it is symmetric and 2) it is positive definite.

These properties imply that $\boldsymbol{C}$ has real-valued eigenvectors and eigenvalues, of which the latter must be positive. The eigenvectors are mutually perpendicular or can be chosen to be so. Taking them as a vector basis, the tensor $\boldsymbol{C}$ can be written in spectral form.

1. symmetric $\quad \boldsymbol{C}^c = \boldsymbol{C}$

2. positive definite

$$\begin{array}{l} \vec{a} \cdot \boldsymbol{C} \cdot \vec{a} = \vec{a} \cdot \boldsymbol{F}^c \cdot \boldsymbol{F} \cdot \vec{a} = (\boldsymbol{F} \cdot \vec{a}) \cdot (\boldsymbol{F} \cdot \vec{a}) \\ \boldsymbol{F} \text{ is regular} \quad \rightarrow \quad \boldsymbol{F} \cdot \vec{a} \neq \vec{0} \quad \text{if} \quad \vec{a} \neq \vec{0} \quad \rightarrow \\ \vec{a} \cdot \boldsymbol{C} \cdot \vec{a} > 0 \quad \forall \quad \vec{a} \neq \vec{0} \end{array}$$

3. $\left.\begin{array}{l} \text{eigenvalues and eigenvectors real} \\ \text{eigenvalues positive} \\ \text{eigenvectors } \perp \text{ (choice)} \end{array}\right\} \rightarrow \text{spectral representation}$

$$\boldsymbol{C} = \mu_1 \vec{m}_1 \vec{m}_1 + \mu_2 \vec{m}_2 \vec{m}_2 + \mu_3 \vec{m}_3 \vec{m}_3$$

Eigenvectors and eigenvalues

The physical meaning of the eigenvalues and eigenvectors of $\boldsymbol{C}$ becomes clear if we consider again the expressions for stretch and shear, but now using the spectral representation of $\boldsymbol{C}$. For these expressions to have a physical relevant meaning, the eigenvectors of $\boldsymbol{C}$ must characterize a material direction in the undeformed state. They are denoted as $\vec{n}_{0i}, i = 1, 2, 3$.

Two eigenvectors of $\boldsymbol{C}$ are mutually perpendicular and represent the direction of two material elements in the undeformed state. The shear deformation between these two material directions is zero. i.e. the material line elements remain perpendicular during deformation. They are called *principal directions of deformation* or *principal strain directions.*

The eigenvalues of $\boldsymbol{C}$ appear to be the squared stretch ratios of the material line elements oriented in the direction of the eigenvectors of $\boldsymbol{C}$. They are called the *principal elongation* *factors*. The right Cauchy-Green deformation tensor is fully defined in the undeformed state. It is therefore characterized as a Lagrangian tensor.

$$\boldsymbol{C} = \mu_1 \vec{m}_1 \vec{m}_1 + \mu_2 \vec{m}_2 \vec{m}_2 + \mu_3 \vec{m}_3 \vec{m}_3$$
$$\boldsymbol{C} = \mu_1 \vec{n}_{01} \vec{n}_{01} + \mu_2 \vec{n}_{02} \vec{n}_{02} + \mu_3 \vec{n}_{03} \vec{n}_{03}$$

$$\lambda(\vec{n}_{01}) = \sqrt{\vec{n}_{01} \cdot \boldsymbol{C} \cdot \vec{n}_{01}} = \sqrt{\mu_1} \quad ; \quad \gamma(\vec{n}_{01}, \vec{n}_{02}) = \frac{\vec{n}_{01} \cdot \boldsymbol{C} \cdot \vec{n}_{02}}{\sqrt{\vec{n}_{01} \cdot \boldsymbol{C} \cdot \vec{n}_{01}} \sqrt{\vec{n}_{02} \cdot \boldsymbol{C} \cdot \vec{n}_{02}}} = 0$$

$$\boldsymbol{C} = \lambda_1^2 \, \vec{n}_{01} \vec{n}_{01} + \lambda_2^2 \, \vec{n}_{02} \vec{n}_{02} + \lambda_3^2 \, \vec{n}_{03} \vec{n}_{03}$$

5.3.7 Right stretch tensor

Based on the right Cauchy-Green deformation tensor, a new tensor, the *right stretch tensor* $\boldsymbol{U}$, is simply defined as the square root of $\boldsymbol{C}$. It is obvious that $\boldsymbol{U}$, like $\boldsymbol{C}$, is symmetric, positive definite and regular.

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \lambda_1 \vec{n}_{01} \vec{n}_{01} + \lambda_2 \vec{n}_{02} \vec{n}_{02} + \lambda_3 \vec{n}_{03} \vec{n}_{03}$$

1. symmetric : $\boldsymbol{U}^c = \boldsymbol{U}$
2. positive definite : $\vec{a} \cdot \boldsymbol{U} \cdot \vec{a} > 0 \qquad \forall \quad \vec{a}$
3. regular : $\boldsymbol{U}^{-1} = \dfrac{1}{\lambda_1} \vec{n}_{01} \vec{n}_{01} + \dfrac{1}{\lambda_2} \vec{n}_{02} \vec{n}_{02} + \dfrac{1}{\lambda_3} \vec{n}_{03} \vec{n}_{03}$
4. $\det(\boldsymbol{C}) = \det(\boldsymbol{U} \cdot \boldsymbol{U}) = \det(\boldsymbol{F}^c \cdot \boldsymbol{F}) = \det{}^2(\boldsymbol{F}) \quad \rightarrow$
   $\det(\boldsymbol{U}) = \lambda_1 \lambda_2 \lambda_3 = \det(\boldsymbol{F}) = J$

The stretch tensor $\boldsymbol{U}$ can be used to transform perpendicular material line elements $d\vec{x}_{01}$, $d\vec{x}_{02}$ and $d\vec{x}_{03}$. The resulting material vectors $d\vec{x}^*_{01}$, $d\vec{x}^*_{02}$ and $d\vec{x}^*_{03}$, will have changed in length and will also be no longer perpendicular, when the original line elements do not coincide with the principal deformation directions. It can be concluded that $\boldsymbol{U}$ describes the real deformation, so elongation and shear.

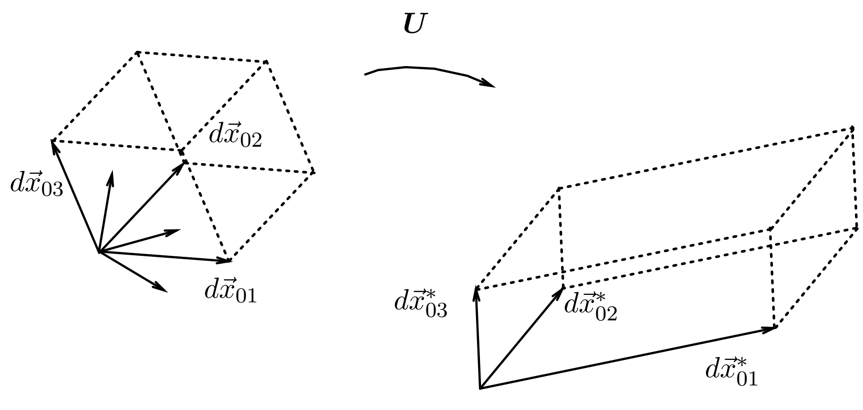


Fig. 5.12 : *Transformation by* $\boldsymbol{U}$

$$d\vec{x}^*_{01} = \boldsymbol{U} \cdot d\vec{x}_{01} \quad ; \quad d\vec{x}^*_{02} = \boldsymbol{U} \cdot d\vec{x}_{02} \quad ; \quad d\vec{x}^*_{03} = \boldsymbol{U} \cdot d\vec{x}_{03}$$

Total transformation

The total transformation from the undeformed to the deformed state, is not described by $\boldsymbol{U}$ but by $\boldsymbol{F}$. It seems that there must be another part of the total transformation, which is not described by $\boldsymbol{U}$. This missing link between $\boldsymbol{U}$ and $\boldsymbol{F}$ is a tensor $\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$.

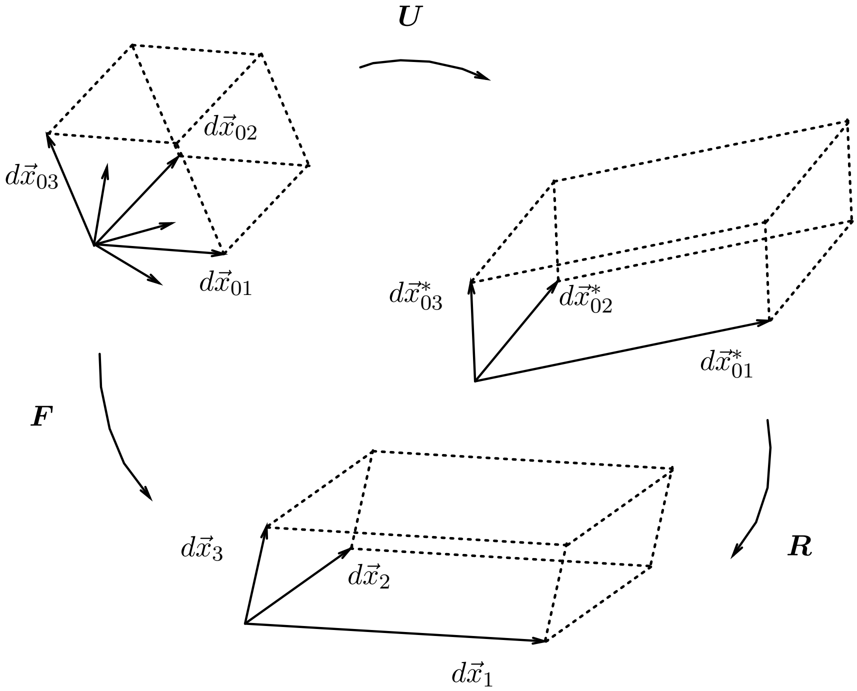


Fig. 5.13 : *Total transformation*

$$\left.\begin{array}{l} d\vec{x}^*_{01} = \boldsymbol{U} \cdot d\vec{x}_{01} \quad \rightarrow \quad d\vec{x}_{01} = \boldsymbol{U}^{-1} \cdot d\vec{x}^*_{01} \\ d\vec{x}_1 = \boldsymbol{F} \cdot d\vec{x}_{01} \end{array}\right\} \quad \rightarrow$$

$$d\vec{x}_1 = \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \cdot d\vec{x}^*_{01} = \boldsymbol{R} \cdot d\vec{x}^*_{01} \quad \rightarrow \quad \boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$$

5.3.8 Rotation tensor

The tensor $\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$ has some properties which renders it to have a physical meaning : it is a *rotation tensor* and describes the rigid body rotation of the material volume element during the transformation from the undeformed to the current, deformed state.

$$\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$$

1.

$$\begin{aligned} \boldsymbol{R}^c \cdot \boldsymbol{R} &= \boldsymbol{U}^{-c} \cdot \boldsymbol{F}^c \cdot \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \\ &= \boldsymbol{U}^{-c} \cdot \boldsymbol{U} \cdot \boldsymbol{U} \cdot \boldsymbol{U}^{-1} = \boldsymbol{U}^{-c} \cdot \boldsymbol{U}^c \cdot \boldsymbol{U} \cdot \boldsymbol{U}^{-1} \\ &= \boldsymbol{I} \qquad \rightarrow \qquad \boldsymbol{R} \text{ is orthogonal} \end{aligned}$$

2.

$$\begin{aligned}\det(\boldsymbol{R}) &= \det(\boldsymbol{F}\cdot\boldsymbol{U}^{-1}) \\ &= \det(\boldsymbol{U})\det(\boldsymbol{U}^{-1}) = \det(\boldsymbol{U}\cdot\boldsymbol{U}^{-1}) \\ &= \det(\boldsymbol{I}) = 1 \qquad \rightarrow \qquad \boldsymbol{R} \text{ is rotation tensor}\end{aligned}$$

5.3.9 Right polar decomposition

The total transformation described by $\boldsymbol{F}$ is decomposed into a true deformation, described by $\boldsymbol{U}$ and a rigid body rotation, described by $\boldsymbol{R}$. This decomposition is denoted as the *right polar decomposition* of the deformation tensor. This decomposition is unique and both $\boldsymbol{U}$ and $\boldsymbol{R}$ can be determined from $\boldsymbol{F}$.

$$\boldsymbol{F} = \boldsymbol{R}\cdot\boldsymbol{U}$$

5.3.10 Strain tensors

The stretch ratio of a material line element in the direction $\vec{e}_0$ could be determined using the right Cauchy-Green deformation tensor $\boldsymbol{C}$. For a strain definition $\varepsilon = f(\lambda)$ we would like to have a strain tensor $\boldsymbol{\varepsilon}$, such that the strain of a material line element in the direction $\vec{e}_0$ can be calculated according to : $\varepsilon(\vec{e}_0) = \vec{e}_0\cdot\boldsymbol{\varepsilon}\cdot\vec{e}_0$.

stretch ratio	$\lambda(\vec{e}_0) = \sqrt{\vec{e}_0\cdot\boldsymbol{C}\cdot\vec{e}_0}$
strain tensor	$\boldsymbol{\varepsilon}$
strain measure	$\varepsilon(\vec{e}_0) = \vec{e}_0\cdot\boldsymbol{\varepsilon}\cdot\vec{e}_0 = f(\lambda(\vec{e}_0))$
shear measure	$\gamma(\vec{e}_{01},\vec{e}_{02}) = \vec{e}_{01}\cdot\boldsymbol{\varepsilon}\cdot\vec{e}_{02}$

Linear strain tensor

The linear strain tensor $\boldsymbol{\mathcal{E}}$ is defined as $\boldsymbol{\mathcal{E}} = \boldsymbol{U} - \boldsymbol{I}$. The linear strain of a material line element in the direction $\vec{e}_0$ cannot be calculated with this tensor. This is only possible for a line element in a principal deformation direction $\vec{n}_{0i}$.

$$\boldsymbol{\mathcal{E}} = \boldsymbol{U} - \boldsymbol{I}$$

$$\vec{e}_0\cdot\boldsymbol{\mathcal{E}}\cdot\vec{e}_0 = \vec{e}_0\cdot\boldsymbol{U}\cdot\vec{e}_0 - \vec{e}_0\cdot\boldsymbol{I}\cdot\vec{e}_0 = \vec{e}_0\cdot\boldsymbol{U}\cdot\vec{e}_0 - 1 \neq \lambda(\vec{e}_0) - 1$$

$$\vec{n}_{0i}\cdot\boldsymbol{\mathcal{E}}\cdot\vec{n}_{0i} = \vec{n}_{0i}\cdot\boldsymbol{U}\cdot\vec{n}_{0i} - 1 = \lambda(\vec{n}_{0i}) - 1 = \lambda_i - 1$$

Logarithmic strain tensor

The logarithmic strain tensor Λ is defined as $\Lambda = \ln(\boldsymbol{U})$. The logarithmic strain of a material line element in the direction $\vec{e}_0$ cannot be calculated with this tensor. This is only possible for a line element in a principal deformation direction $\vec{n}_{0i}$.

$$\boldsymbol{\Lambda} = \ln(\boldsymbol{U})$$

$$\vec{e}_0 \cdot \boldsymbol{\Lambda} \cdot \vec{e}_0 = \vec{e}_0 \cdot \ln(\boldsymbol{U}) \cdot \vec{e}_0 \neq \ln(\lambda(\vec{e}_0))$$

$$\vec{n}_{0i} \cdot \boldsymbol{\Lambda} \cdot \vec{n}_{0i} = \vec{n}_{0i} \cdot \ln(\boldsymbol{U}) \cdot \vec{n}_{0i} = \ln(\lambda(\vec{n}_{0i})) = \ln(\lambda_i)$$

Green-Lagrange strain tensor

The Green-Lagrange strain tensor $\boldsymbol{E}$ is defined as $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$. For a material line element in the initial direction $\vec{e}_0$ the Green-Lagrange strain can be calculated using the Green-Lagrange strain tensor.

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$$

$$\vec{e}_0 \cdot \boldsymbol{E} \cdot \vec{e}_0 = \tfrac{1}{2}(\vec{e}_0 \cdot \boldsymbol{C} \cdot \vec{e}_0 - 1) = \tfrac{1}{2}\left(\lambda^2(\vec{e}_0) - 1\right)$$

Infinitesimal linear strain tensor

The infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is the linearized fraction of the Green-Lagrange strain tensor $\boldsymbol{E}$. For infinitesimal displacements, the first partial derivatives of the displacement components are so small that all involved squares and products are negligible with respect to the linear terms. The non-linear terms in $\boldsymbol{E}$ can than be neglected.

For infinitesimal displacements the change in position vector of a material point is not relevant. This means that the difference between gradient operators vanishes.

$$\begin{array}{c}
\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\left\{(\vec{\nabla}_0\vec{u}) + (\vec{\nabla}_0\vec{u})^c + (\vec{\nabla}_0\vec{u}) \cdot (\vec{\nabla}_0\vec{u})^c\right\} \\
\text{linearisation} \quad \rightarrow \quad \text{infinitesimal strain tensor} \\
\boldsymbol{\varepsilon} = \frac{1}{2}\left\{(\vec{\nabla}_0\vec{u}) + (\vec{\nabla}_0\vec{u})^c\right\} = \frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}^c) - \boldsymbol{I} = \frac{1}{2}\left\{(\vec{\nabla}\vec{u}) + (\vec{\nabla}\vec{u})^c\right\}
\end{array}$$

5.4 Deformation rate

The rate of deformation of a material line element is the material time derivative – we follow the same line element in time – of a material vector $d\vec{x}$ in the current state. This derivative can be related to $d\vec{x}$ with a tensor $\boldsymbol{L}$, the velocity gradient tensor. This tensor is decomposed into a symmetric and a skewsymmetric part, the *deformation rate tensor* $\boldsymbol{D}$ and the *spin tensor* $\boldsymbol{\Omega}$, respectively.

$$\begin{aligned} \dot{\overline{d\vec{x}}} &= \dot{\boldsymbol{F}} \cdot d\vec{x}_0 = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot d\vec{x} = \boldsymbol{L} \cdot d\vec{x} = (\vec{\nabla}\vec{v})^c \cdot d\vec{x} \\ &= \tfrac{1}{2}\{\boldsymbol{L} + \boldsymbol{L}^c\} \cdot d\vec{x} + \tfrac{1}{2}\{\boldsymbol{L} - \boldsymbol{L}^c\} \cdot d\vec{x} \\ &= \boldsymbol{D} \cdot d\vec{x} + \boldsymbol{\Omega} \cdot d\vec{x} \end{aligned}$$

5.4.1 Spin tensor

The spin tensor $\boldsymbol{\Omega}$ describes only rotation rate of the material line element. This follows directly from the fact that the spin tensor is skewsymmetric and has a unique associated axial vector $\vec{\omega}$.

$$\boldsymbol{\Omega} = \tfrac{1}{2}\left\{\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} - (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})^c\right\} = \tfrac{1}{2}\left\{\left(\vec{\nabla}\vec{v}\right)^c - \left(\vec{\nabla}\vec{v}\right)\right\}$$

$$\boldsymbol{\Omega} = \text{skewsymmetric} \quad \rightarrow \quad \boldsymbol{\Omega} \cdot d\vec{x} = \vec{\omega} * d\vec{x} = \text{velocity} \perp d\vec{x} = \text{rotation rate}$$

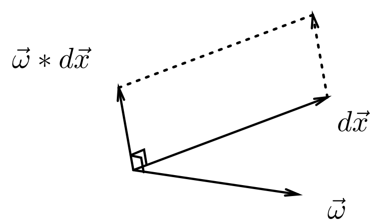


Fig. 5.14 : *Rotation rate of material line element*

The proof that a skewsymmetric tensor has an associated axial vector is repeated here.

$$\begin{aligned} &\vec{q} \cdot \boldsymbol{\Omega} \cdot \vec{q} = \vec{q} \cdot \boldsymbol{\Omega}^c \cdot \vec{q} = -\,\vec{q} \cdot \boldsymbol{\Omega} \cdot \vec{q} \quad \rightarrow \\ &\vec{q} \cdot \boldsymbol{\Omega} \cdot \vec{q} = 0 \quad \rightarrow \\ &\boldsymbol{\Omega} \cdot \vec{q} = \vec{p} \quad \rightarrow \\ &\vec{q} \cdot \vec{p} = 0 \quad \rightarrow \\ &\vec{q} \perp \vec{p} \quad \rightarrow \\ &\exists \quad \vec{\omega} \quad \text{zdd} \quad \vec{p} = \vec{\omega} * \vec{q} \quad \rightarrow \end{aligned}$$

$$\boldsymbol{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q}$$

The axial vector associated with a skewsymmetric tensor is unique. Its components can be determined and expressed in the components of the skewsymmetric tensor.

$$\boldsymbol{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q} \qquad\qquad \forall \quad \vec{q}$$

$$\boldsymbol{\Omega} \cdot \vec{q} = \underset{\sim}{\vec{e}}^T \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underset{\sim}{\vec{e}}^T \begin{bmatrix} \Omega_{11} q_1 + \Omega_{12} q_2 + \Omega_{13} q_3 \\ \Omega_{21} q_1 + \Omega_{22} q_2 + \Omega_{23} q_3 \\ \Omega_{31} q_1 + \Omega_{32} q_2 + \Omega_{33} q_3 \end{bmatrix}$$

$$\begin{aligned} \vec{\omega} * \vec{q} &= (\omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3) * (q_1 \vec{e}_1 + q_2 \vec{e}_2 + q_3 \vec{e}_3) \\ &= \omega_1 q_2 (\vec{e}_3) + \omega_1 q_3 (-\vec{e}_2) + \omega_2 q_1 (-\vec{e}_3) + \omega_2 q_3 (\vec{e}_1) + \omega_3 q_1 (\vec{e}_2) + \omega_3 q_2 (-\vec{e}_1) \\ &= [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3] \begin{bmatrix} \omega_2 q_3 - \omega_3 q_2 \\ \omega_3 q_1 - \omega_1 q_3 \\ \omega_1 q_2 - \omega_2 q_1 \end{bmatrix} \end{aligned}$$

$$\boldsymbol{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q} \qquad \forall \vec{q} \qquad \rightarrow \qquad \underline{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

5.4.2 Deformation rate tensor

The deformation rate tensor does not what its name suggests. For a random material vector $d\vec{x}$ the product $\boldsymbol{D} \cdot d\vec{x}$ is a vector which is not along $d\vec{x}$. The deformation rate tensor describes the rate of elongation but also partly the rate of rotation of $d\vec{x}$. Only for material line elements in the direction of one if its eigenvectors the tensor $\boldsymbol{D}$ describes purely elongation rate.

$$\boldsymbol{D} = \tfrac{1}{2} \left\{ \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} + (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})^c \right\} = \left\{ \left(\vec{\nabla} \vec{v} \right)^c + \left(\vec{\nabla} \vec{v} \right) \right\}$$

$$\begin{array}{ll} \boldsymbol{D} = \boldsymbol{D}^c \quad \rightarrow \quad \boldsymbol{D} = \nu_1 \vec{\eta}_1 \vec{\eta}_1 + \nu_2 \vec{\eta}_2 \vec{\eta}_2 + \nu_3 \vec{\eta}_3 \vec{\eta}_3 & \\ 1.: \quad \text{vector } d\vec{x} \text{ along } \vec{\eta}_1 \quad : & d\vec{x} = dx_1 \vec{\eta}_1 \\ \boldsymbol{D} \cdot d\vec{x} = dx_1 \boldsymbol{D} \cdot \vec{\eta}_1 = dx_1 \nu_1 \vec{\eta}_1 = \nu_1 d\vec{x} & \\ 2.: \quad \text{random vector} \quad : & d\vec{x} = dx_1 \vec{\eta}_1 + dx_2 \vec{\eta}_2 + dx_3 \vec{\eta}_3 \\ \boldsymbol{D} \cdot d\vec{x} = dx_1 \nu_1 \vec{\eta}_1 + dx_2 \nu_2 \vec{\eta}_2 + dx_3 \nu_3 \vec{\eta}_3 & \end{array}$$

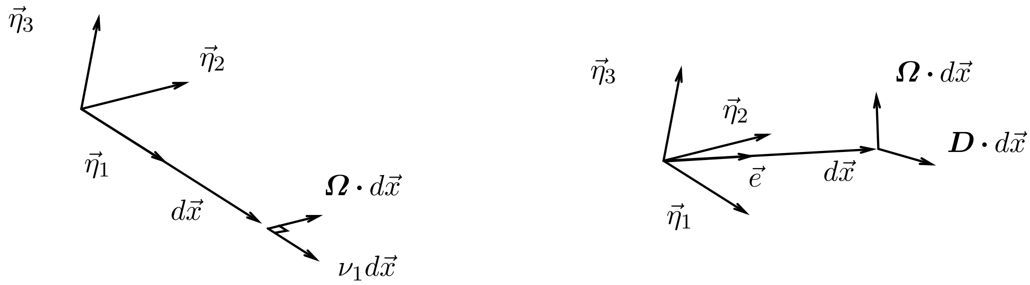


Fig. 5.15 : *Deformation rate of material line element*

5.4.3 Elongation rate

The elongation rate of a material line element can be expressed in the time derivative of the elongation factor λ.

$$\begin{aligned}
\lambda^2 = \vec{e}_0 \cdot \boldsymbol{C} \cdot \vec{e}_0 \qquad &\rightarrow \quad \frac{D}{Dt}(\lambda^2) = \frac{D}{Dt}(\vec{e}_0 \cdot \boldsymbol{C} \cdot \vec{e}_0) \quad \rightarrow \\
2\lambda\dot{\lambda} &= \vec{e}_0 \cdot \frac{D}{Dt}(\boldsymbol{C}) \cdot \vec{e}_0 = \vec{e}_0 \cdot \frac{D}{Dt}(\boldsymbol{F}^c \cdot \boldsymbol{F}) \cdot \vec{e}_0 \\
&= \vec{e}_0 \cdot \{\dot{\boldsymbol{F}}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \dot{\boldsymbol{F}}\} \cdot \vec{e}_0 \\
&= \vec{e}_0 \cdot \boldsymbol{F}^c \cdot \{\boldsymbol{F}^{-c} \cdot \dot{\boldsymbol{F}}^c + \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\} \cdot \boldsymbol{F} \cdot \vec{e}_0 \\
&= (\boldsymbol{F} \cdot \vec{e}_0) \cdot \{(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})^c + \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\} \cdot (\boldsymbol{F} \cdot \vec{e}_0) \\
&= (\lambda\vec{e}) \cdot (2\,\boldsymbol{D}) \cdot (\lambda\vec{e}) \quad \rightarrow
\end{aligned}$$

$$\frac{\dot{\lambda}}{\lambda} = \vec{e} \cdot \boldsymbol{D} \cdot \vec{e}$$

5.4.4 Volume change rate

The rate of change of the material volume, the material time derivative of the volume change factor J, is the product of J itself and the trace of the deformation rate tensor $\boldsymbol{D}$. To derive this relation, we consider a material volume element in the undeformed and the deformed state. In the undeformed state the sides of the element coincide with the pricipal deformation directions $\{\vec{n}_{01}, \vec{n}_{02}, \vec{n}_{03}\}$.

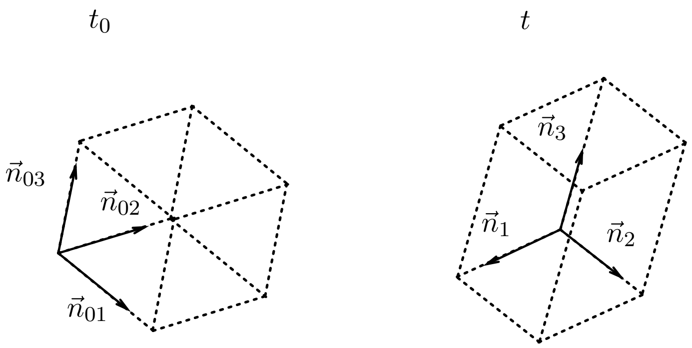


Fig. 5.16 : *Volume change rate of material cube*

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{D}) &= \vec{n}_1 \cdot \boldsymbol{D} \cdot \vec{n}_1 + \vec{n}_2 \cdot \boldsymbol{D} \cdot \vec{n}_2 + \vec{n}_3 \cdot \boldsymbol{D} \cdot \vec{n}_3 \\
&= \frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{\lambda}_3}{\lambda_3} = \frac{D}{Dt}\{\ln(\lambda_1) + \ln(\lambda_2) + \ln(\lambda_3)\} = \frac{D}{Dt}\{\ln(\lambda_1\lambda_2\lambda_3)\} \\
&= \frac{D}{Dt}[\ln\{\det(\boldsymbol{U})\}] = \frac{D}{Dt}[\ln\{\det(\boldsymbol{F})\}] = \frac{D}{Dt}\{\ln(J)\} = \frac{\dot{J}}{J} \quad \rightarrow
\end{aligned}$$

$$\dot{J} = J\mathrm{tr}(\boldsymbol{D}) = J\left(\vec{\nabla} \cdot \vec{v}\right)$$

5.4.5 Area change rate

The rate of change of a material area dA with unit normal vector $\vec{n}$ can also be expressed in the velocity gradient tensor $\boldsymbol{L}$.

$$
\begin{aligned}
\frac{D}{Dt}(dA\,\vec{n}) &= \frac{D}{Dt}\left\{\det(\boldsymbol{F})dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1}\right\} \\
&= \frac{D}{Dt}\left\{\det(\boldsymbol{F})\right\} dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} + \det(\boldsymbol{F})dA_0\vec{n}_0 \cdot \dot{\boldsymbol{F}}^{-1} \\
&= \dot{J}\,dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} - J\,dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{L} \\
&= \mathrm{tr}(\boldsymbol{L})J dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} - J\,dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{L} \\
&= J\,\mathrm{tr}(\boldsymbol{L})\boldsymbol{F}^{-c} \cdot dA_0\vec{n}_0 - J\,\boldsymbol{L}^c \cdot \boldsymbol{F}^{-c} \cdot dA_0\vec{n}_0 \\
&= J\,\left(\mathrm{tr}(\boldsymbol{L})\boldsymbol{I} - \boldsymbol{L}^c\right) \cdot \boldsymbol{F}^{-c} \cdot dA_0\vec{n}_0 \\
&= \left(\mathrm{tr}(\boldsymbol{L})\boldsymbol{I} - \boldsymbol{L}^c\right) dA\,\vec{n}
\end{aligned}
$$

5.5 Linear deformation

In linear elasticity theory deformations are very small. All kind of relations from general continuum mechanics theory may be linearized, resulting for instance in the linear strain tensor $\boldsymbol{\varepsilon}$, which is then fully expressed in the gradient of the displacement. The deformations are in fact so small that the geometry of the material body in the deformed state approximately equals that of the undeformed state.

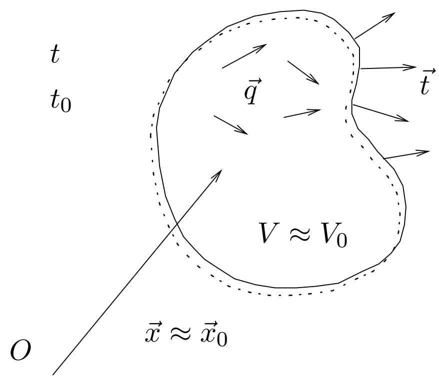


Fig. 5.17 : *Small deformation*

$$
\left.
\begin{array}{l}
\boldsymbol{E} = \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T + \left(\vec{\nabla}_0\vec{u}\right) + \left(\vec{\nabla}_0\vec{u}\right) \cdot \left(\vec{\nabla}_0\vec{u}\right)^T\right] \\
\text{small deformation} \quad \rightarrow \quad \left(\vec{\nabla}_0\vec{u}\right)^T = \boldsymbol{F} - \boldsymbol{I} \approx \boldsymbol{O}
\end{array}
\right\} \quad \rightarrow
$$

$$
\boldsymbol{E} \approx \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T + \left(\vec{\nabla}_0\vec{u}\right)\right] \approx \frac{1}{2}\left[\left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right)\right] = \boldsymbol{\varepsilon} \qquad \text{symm!}
$$

Not only straining and shearing must be small to allow the use of linear strains, also the rigid body rotation must be small. This is immediately clear, when we consider the rigid rotation of a material line element PQ around the fixed point P. The x- and y-displacement of point Q, u and v respectively, are expressed in the rotation angle ϕ and the length of the line element dx_0. The nonlinear Green-Lagrange strain is always zero. The linear strain, however, is only zero for very small rotations.

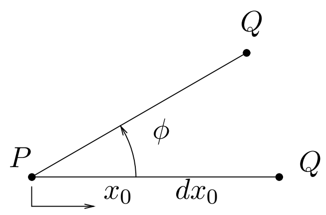


Fig. 5.18 : *Rigid rotation of a line element*

$$\left.\begin{array}{l} u = u_Q = -[dx_0 - dx_0 \cos(\phi)] = [\cos(\phi) - 1]dx_0 \\ v = v_Q = [\sin(\phi)]dx_0 \end{array}\right\} \quad \rightarrow$$

$$\frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \quad ; \quad \frac{\partial v}{\partial x_0} = \sin(\phi) \quad \rightarrow$$

$$\varepsilon_{gl} = \frac{\partial u}{\partial x_0} + \tfrac{1}{2}\left(\frac{\partial u}{\partial x_0}\right)^2 + \tfrac{1}{2}\left(\frac{\partial v}{\partial x_0}\right)^2 = 0$$

$$\varepsilon_l = \frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \neq 0 \quad !!$$

Elongational, shear and volume strain

For small deformations and rotations the elongational and shear strain can be linearized and expressed in the linear strain tensor $\boldsymbol{\varepsilon}$. The volume change ratio J can be expressed in linear strain components and also linearized.

$$
\begin{array}{llcl}
\text{elong. strain} & \frac{1}{2}\left(\lambda^2(\vec{e}_{01}) - 1\right) & = & \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{01} \\
& & \downarrow & \\
& \lambda(\vec{e}_{01}) - 1 & = & \vec{e}_{01} \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_{01} \\
\text{shear strain} & \gamma(\vec{e}_{01}, \vec{e}_{02}) = \sin\left(\frac{\pi}{2} - \theta\right) & = & \left(\dfrac{2}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})}\right) \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{02} \\
& & \downarrow & \\
& \frac{\pi}{2} - \theta & = & 2\,\vec{e}_{01} \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_{02} \\
\text{volume change} & J = \dfrac{dV}{dV_0} & = & \lambda_1\lambda_2\lambda_3 = (\varepsilon_1 + 1)(\varepsilon_2 + 1)(\varepsilon_2 + 1) \\
& & \downarrow & \\
& J & = & \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + 1 = \mathrm{tr}(\boldsymbol{\varepsilon}) + 1 \\
\text{volume strain} & J - 1 & = & \mathrm{tr}(\boldsymbol{\varepsilon})
\end{array}
$$

5.5.1 Linear strain matrix

With respect to an orthogonal basis, the linear strain tensor can be written in components, resulting in the linear strain matrix.

Because the linear strain tensor is symmetric, it has three real-valued eigenvalues $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and associated eigenvectors $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. The eigenvectors are normalized to have unit length and they are mutually perpendicular, so they constitute an orthonormal vector base. The strain matrix w.r.t. this vector base is diagonal.

The eigenvalues are referred to as the *principal strains* and the eigenvectors as the *principal strain directions*. They are equivalent to the *principal directions* of deformatieon. Line elements along these directions in the undeformed state t_0 do not show any shear during deformation towards the current state t.

$$
\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \quad \text{with} \quad \left\{ \begin{array}{l} \varepsilon_{21} = \varepsilon_{12} \\ \varepsilon_{32} = \varepsilon_{23} \\ \varepsilon_{31} = \varepsilon_{13} \end{array} \right.
$$

principal strain matrix $\qquad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$

spectral form $\qquad \boldsymbol{\varepsilon} = \varepsilon_1\vec{n}_1\vec{n}_1 + \varepsilon_2\vec{n}_2\vec{n}_2 + \varepsilon_3\vec{n}_3\vec{n}_3$

Cartesian components

The linear strain components w.r.t. a Cartesian coordinate system are easily derived using the component expressions for the gradient operator and the displacement vector. For derivatives a short notation is used : $(\ )_{i,j} = \dfrac{\partial (\ )_i}{\partial x_j}$.

gradient operator $\quad \vec{\nabla} = \vec{e}_x \dfrac{\partial}{\partial x} + \vec{e}_y \dfrac{\partial}{\partial y} + \vec{e}_z \dfrac{\partial}{\partial z}$

displacement vector $\quad \vec{u} = u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z$

linear strain tensor $\quad \boldsymbol{\varepsilon} = \frac{1}{2}\left\{(\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u})\right\} = \tilde{\vec{e}}^T \underline{\varepsilon}\, \tilde{\vec{e}}$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2u_{x,x} & u_{x,y} + u_{y,x} & u_{x,z} + u_{z,x} \\ u_{y,x} + u_{x,y} & 2u_{y,y} & u_{y,z} + u_{z,y} \\ u_{z,x} + u_{x,z} & u_{z,y} + u_{y,z} & 2u_{z,z} \end{bmatrix}$$

Cylindrical components

The linear strain components w.r.t. a cylindrical coordinate system are derived straightforwardly using the component expressions for the gradient operator and the displacement vector.

gradient operator $\quad \vec{\nabla} = \vec{e}_r \dfrac{\partial}{\partial r} + \vec{e}_t \dfrac{1}{r} \dfrac{\partial}{\partial \theta} + \vec{e}_z \dfrac{\partial}{\partial z}$

displacement vector $\quad \vec{u} = u_r \vec{e}_r(\theta) + u_t \vec{e}_t(\theta) + u_z \vec{e}_z$

linear strain tensor $\quad \boldsymbol{\varepsilon} = \frac{1}{2}\left\{(\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u})\right\} = \tilde{\vec{e}}^T \underline{\varepsilon}\, \tilde{\vec{e}}$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{rt} & \varepsilon_{rz} \\ \varepsilon_{tr} & \varepsilon_{tt} & \varepsilon_{tz} \\ \varepsilon_{zr} & \varepsilon_{zt} & \varepsilon_{zz} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ u_{z,r} + u_{r,z} & \frac{1}{r}u_{z,t} + u_{t,z} & 2u_{z,z} \end{bmatrix}$$

Compatibility conditions

The six independent strain components are related to only three displacement components. Therefore the strain components cannot be independent. Six relations can be derived, which are referred to as the compatibility conditions.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\,\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \qquad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial z}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\,\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \qquad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial y^2} = \frac{\partial^2 \varepsilon_{yx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial x}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\,\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \qquad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial z^2} = \frac{\partial^2 \varepsilon_{zy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial y}$$

$$\frac{1}{r^2}\,\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \varepsilon_{tt}}{\partial r^2} - \frac{2}{r}\,\frac{\partial^2 \varepsilon_{rt}}{\partial r \partial \theta} - \frac{1}{r}\,\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\,\frac{\partial \varepsilon_{tt}}{\partial r} - \frac{2}{r^2}\,\frac{\partial \varepsilon_{rt}}{\partial \theta} = 0$$

5.6 Special deformations

5.6.1 Planar deformation

It often happens that (part of) a structure is loaded in one plane. Moreover the load is often such that no bending out of that plane takes place. The resulting deformation is referred to as being planar.

Here it is assumed that the plane of deformation is the x_1x_2-plane. Note that in this planar deformation there still can be displacement perpendicular to the plane of deformation, which results in change of thickness.

The *in-plane* displacement components u_1 and u_2 are only a function of x_1 and x_2. The *out-of-plane* displacement u_3 may be a function of x_3 as well.

$$u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = u_3(x_1, x_2, x_3)$$

5.6.2 Plane strain

When the boundary conditions and the material behavior are such that displacement of material points are only in the x_1x_2-plane, the deformation is referred to as *plane strain* in the x_1x_2-plane. Only three relevant strain components remain.

$$u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = 0$$

$$\begin{array}{l} \varepsilon_{33} = 0 \quad ; \quad \gamma_{13} = \gamma_{23} = 0 \\ \text{compatibility} \quad : \quad \varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \end{array}$$

5.6.3 Axi-symmetric deformation

Many man-made and natural structures have an axi-symmetric geometry, which means that their shape and volume can be constructed by virtually rotating a cross section around the axis of revolution. Points are indicated with cylindrical coordinates $\{r, \theta, z\}$. When material properties and loading are also independent of the coordinate θ, the deformation and resulting stresses will be also independent of θ.

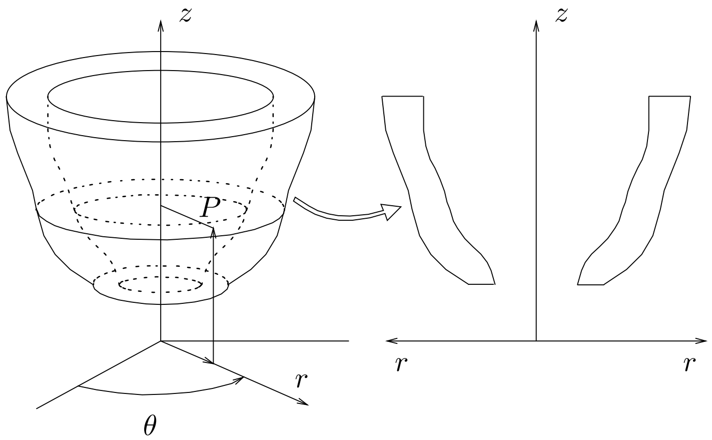


Fig. 5.19 : *Axi-symmetric deformation*

$$\frac{\partial}{\partial \theta}(\ ) = 0 \qquad \rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_t(r,z)\vec{e}_t(\theta) + u_z(r,z)\vec{e}_z$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & -\frac{1}{r}(u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ -\frac{1}{r}(u_t) + u_{t,r} & 2\frac{1}{r}(u_r) & u_{t,z} \\ u_{z,r} + u_{r,z} & u_{t,z} & 2u_{z,z} \end{bmatrix}$$

With the *additional* assumption that no rotation around the z-axis takes place ($u_t = 0$), all state variables can be studied in one half of the cross section through the z-axis.

$$\frac{\partial}{\partial \theta}(\ ) = 0 \text{ and } u_t = 0 \qquad \rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_z(r,z)\vec{e}_z$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & 0 & u_{r,z} + u_{z,r} \\ 0 & 2\frac{1}{r}(u_r) & 0 \\ u_{z,r} + u_{r,z} & 0 & 2u_{z,z} \end{bmatrix}$$

Axi-symmetric plane strain

When boundary conditions and material behavior are such that displacement of material points are only in the $r\theta$-plane, the deformation is referred to as *plane strain* in the $r\theta$-plane.

plane strain deformation

$$\left.\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta) \\ u_z = 0 \end{array}\right\} \quad \rightarrow \quad \varepsilon_{zz} = \gamma_{rz} = \gamma_{tz} = 0$$

linear strain matrix

$$\underline{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & u_{t,r} - \frac{1}{r}(u_t) & 0 \\ u_{t,r} - \frac{1}{r}(u_t) & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

plane strain deformation with $u_t = 0$

$$\left.\begin{array}{l} u_r = u_r(r) \\ u_z = 0 \end{array}\right\} \quad \rightarrow \quad \underline{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & 0 & 0 \\ 0 & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Chapter 6

Stresses

Kinematics describes the motion and deformation of a set of material points, considered here to be a continuous body. The cause of this deformation is not considered in kinematics. Motion and deformation may have various causes, which are collectively considered here to be external forces and moments.

Deformation of the material – not its motion alone – results in internal stresses. It is very important to calculate them accurately, because they may cause irreversible structural changes and even unallowable damage of the material.

6.1 Stress vector

Consider a material body in the deformed state, with edge and volume forces. The body is divided in two parts, where the cutting plane passes through the material point P. An edge load is introduced on both sides of the cutting plane to prevent separation of the two parts. In two associated points (= coinciding before the cut was made) in the cutting plane of both parts, these loads are of opposite sign, but have equal absolute value.

The resulting force on an area ΔA of the cutting plane in point P is $\Delta\vec{k}$. The resulting force per unit of area is the ratio of $\Delta\vec{k}$ and ΔA. The *stress vector* $\vec{p}$ in point P is defined as the limit value of this ratio for $\Delta A \rightarrow 0$. So, obviously, the stress vector is associated to both poin P and the cutting plane through this point.

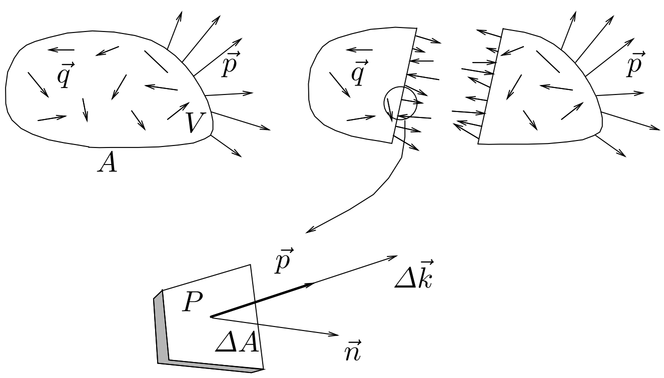


Fig. 6.1 : *Cross-sectional stresses and stress vector on a plane*

$$\vec{p} = \lim_{\Delta A \longrightarrow 0} \frac{\Delta \vec{k}}{\Delta A}$$

6.1.1 Normal stress and shear stress

The stress vector $\vec{p}$ can be written as the sum of two other vectors. The first is the *normal stress vector* $\vec{p}_n$ in the direction of the unity normal vector $\vec{n}$ on ΔA. The second vector is in the plane and is called the *shear stress vector* $\vec{p}_s$.

The length of the normal stress vector is the *normal stress* p_n and the length of the shear stress vector is the *shear stress* p_s.

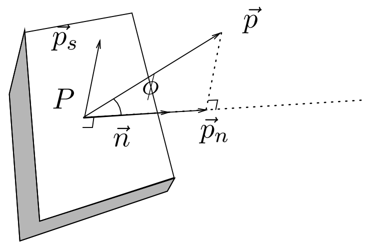


Fig. 6.2 : *Stress vector, normal strss and shear stress*

normal stress	:	$p_n = \vec{p} \cdot \vec{n}$
tensile stress	:	positive ($\phi < \frac{\pi}{2}$)
compression stress	:	negative ($\phi > \frac{\pi}{2}$)
normal stress vector	:	$\vec{p}_n = p_n \vec{n}$
shear stress vector	:	$\vec{p}_s = \vec{p} - \vec{p}_n$
shear stress	:	$p_s = \|\|\vec{p}_s\|\| = \sqrt{\|\|\vec{p}\|\|^2 - p_n^2}$

6.2 Cauchy stress tensor

The stress vector can be calculated, using the *stress tensor* $\boldsymbol{\sigma}$, which represents the stress state in point P. The plane is identified by its unity normal vector $\vec{n}$. The stress vector is calculated according to Cauchy's theorem, which states that in each material point such a stress tensor must uniquely exist. ($\exists!$: there exists only one.)

Theorem of Cauchy : $\quad \exists!$ tensor $\boldsymbol{\sigma}$ such that : $\vec{p} = \boldsymbol{\sigma} \cdot \vec{n}$

6.2.1 Cauchy stress matrix

With respect to an orthogonal basis, the Cauchy stress tensor $\boldsymbol{\sigma}$ can be written in components, resulting in the Cauchy stress matrix $\underline{\sigma}$, which stores the components of the Cauchy stress tensor w.r.t. an orthonormal vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. The components of the Cauchy stress matrix are components of stress vectors on the planes with unit normal vectors in the coordinate directions.

With our definition, the first index of a stress component indicates the direction of the stress vector and the second index indicates the normal of the plane where it is loaded. As an example, the stress vector on the plane with $\vec{n} = \vec{e}_1$ is considered.

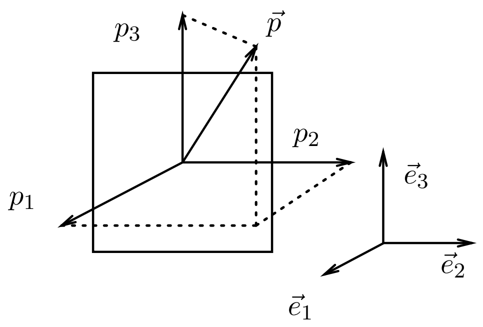
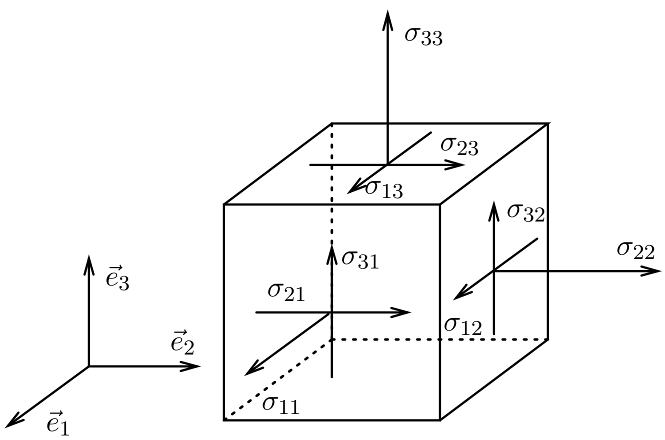


Fig. 6.3 : *Components of stress vector on a plane*

$$\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} \quad \rightarrow \quad \underset{\sim}{\vec{e}}^T \underset{\sim}{p} = \underset{\sim}{\vec{e}}^T \underline{\sigma}\ \underset{\sim}{\vec{e}} \cdot \underset{\sim}{\vec{e}}^T \underset{\sim}{n} = \underset{\sim}{\vec{e}}^T \underline{\sigma}\ \underset{\sim}{n}$$
$$\vec{n} = \vec{e}_1 \quad \rightarrow$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}$$

The components of the Cauchy stress matrix can be represented as normal and shear stresses on the side planes of a stress cube.

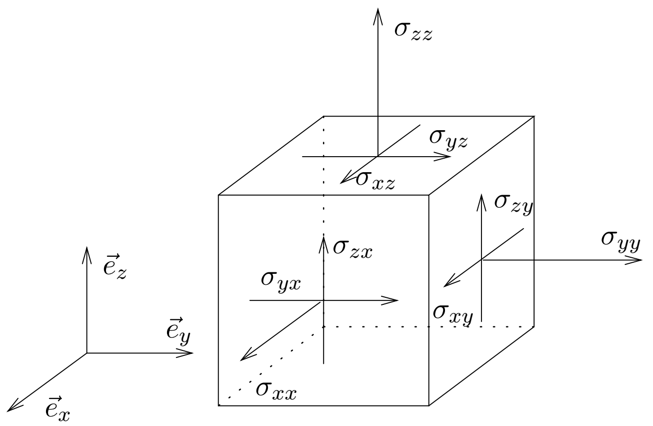


$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Fig. 6.4 : *Stress cube*

Cartesian components

In the Cartesian coordinate system the stress cube sides are parallel to the Cartesian coordinate axes. Stress components are indicated with the indices x, y and z.

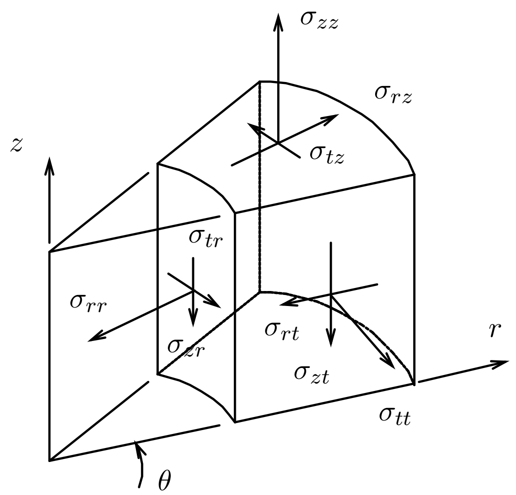


$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Fig. 6.5 : *Cartesian stress cube*

Cylindrical components

In the cylindrical coordinate system the stress 'cube' sides are parallel to the cylindrical coordinate axes. Stress components are indicated with the indices r, t and z.



$$\underline{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{rt} & \sigma_{rz} \\ \sigma_{tr} & \sigma_{tt} & \sigma_{tz} \\ \sigma_{zr} & \sigma_{zt} & \sigma_{zz} \end{bmatrix}$$

Fig. 6.6 : *Cylindrical stress "cube"*

6.2.2 Principal stresses and directions

It will be shown later that the stress tensor is symmetric. This means that it has three real-valued eigenvalues $\{\sigma_1, \sigma_2, \sigma_3\}$ and associated eigenvectors $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. The eigenvectors are normalized to have unit length and they are mutually perpendicular, so they constitute an orthonormal vector base. The stress matrix w.r.t. this vector base is diagonal.

The eigenvalues are referred to as the principal stresses and the eigenvectors as the principal stress directions. The stress cube with the normal principal stresses is referred to as the principal stress cube.

Using the spectral representation of $\boldsymbol{\sigma}$, it is easily shown that the stress tensor changes as a result of a rigid body rotation $\boldsymbol{Q}$.

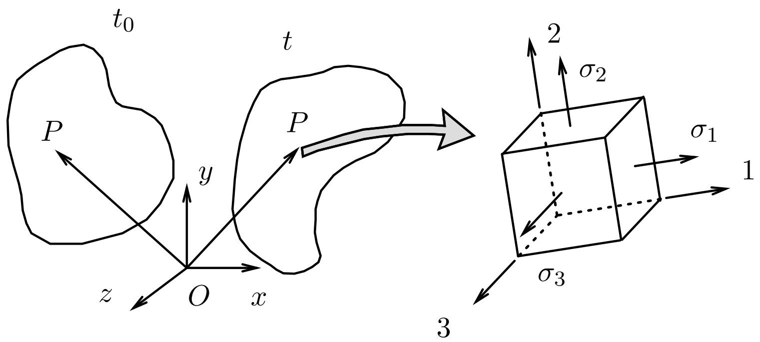


Fig. 6.7 : *Principal stress cube with principal stresses*

spectral form

$$\left.\begin{array}{l} \boldsymbol{\sigma} \cdot \vec{n}_1 = \sigma_1 \vec{n}_1 \\ \boldsymbol{\sigma} \cdot \vec{n}_2 = \sigma_2 \vec{n}_2 \\ \boldsymbol{\sigma} \cdot \vec{n}_3 = \sigma_3 \vec{n}_3 \end{array}\right\} \rightarrow \boldsymbol{\sigma} = \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3$$

principal stress matrix

$$\underline{\sigma}_P = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Stress transformation

We consider the two-dimensional plane with principal stress directions coinciding with the unity vectors $\vec{e}_1$ and $\vec{e}_2$. The principal stresses are σ_1 and σ_2. On a plane which is rotated anti-clockwise from $\vec{e}_1$ over an angle $\alpha < \frac{\pi}{2}$ the stress vector $\vec{p}$ and its normal and shear components can be calculated. They are indicated as σ_α and τ_α respectively.

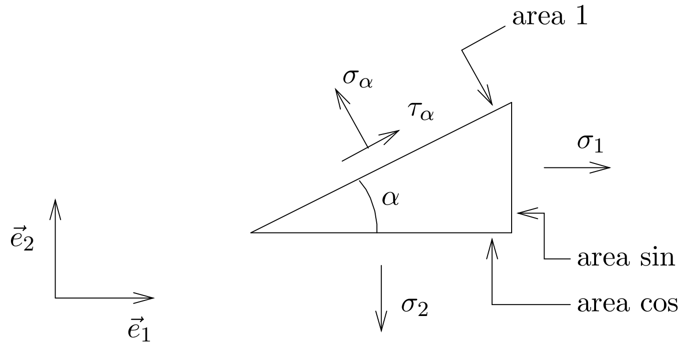


Fig. 6.8 : *Normal and shear stress on a plane*

$$
\begin{aligned}
&\boldsymbol{\sigma} = \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 \\
&\vec{n} = -\sin(\alpha)\vec{e}_1 + \cos(\alpha)\vec{e}_2 \\
&\vec{p} = \boldsymbol{\sigma}\cdot\vec{n} = -\sigma_1\sin(\alpha)\vec{e}_1 + \sigma_2\cos(\alpha)\vec{e}_2 \\
&\sigma_\alpha = \sigma_1\sin^2(\alpha) + \sigma_2\cos^2(\alpha) \\
&\tau_\alpha = (\sigma_2 - \sigma_1)\sin(\alpha)\cos(\alpha)
\end{aligned}
$$

Mohr's circles of stress

From the relations for the normal and shear stress on a plane in between two principal stress planes, a relation between these two stresses and the principal stresses can be derived. The resulting relation is the equation of a circle in the $\sigma_\alpha\tau_\alpha$-plane, referred to as Mohr's circle for stress. The radius of the circle is $\frac{1}{2}(\sigma_1 - \sigma_2)$. The coordinates of its center are $\{\frac{1}{2}(\sigma_1 + \sigma_2), 0\}$.

Stresses on a plane, which is rotated over α w.r.t. a principal stress plane, can be found in the circle by rotation over 2α.

Because there are three principal stresses and principal stress planes, there are also three stress circles. It can be proven that each stress state is located on one of the circles or in the shaded area.

$$
\begin{aligned}
\sigma_\alpha &= \sigma_1\sin^2(\alpha) + \sigma_2\cos^2(\alpha) \\
&= \sigma_1(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\alpha)) + \sigma_2(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\alpha)) \\
&= \tfrac{1}{2}(\sigma_1 + \sigma_2) - \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos(2\alpha) \quad \rightarrow
\end{aligned}
$$

$$
(1) \qquad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2\cos^2(2\alpha)
$$

$$
\tau_\alpha = -\cos(\alpha)\sin(\alpha)\sigma_1 + \cos(\alpha)\sin(\alpha)\sigma_2 = \tfrac{1}{2}(\sigma_2 - \sigma_1)\sin(2\alpha) \quad \rightarrow
$$

$$
(2) \qquad \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_2 - \sigma_1)\right\}^2\sin^2(2\alpha)
$$

$$
(1) + (2) \quad \rightarrow \quad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 + \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2
$$

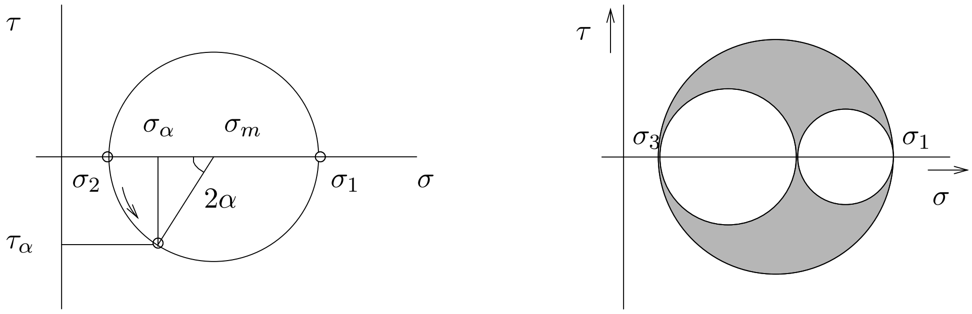


Fig. 6.9 : *Mohr's circles*

That there are three circles can be demonstrated by considering a random stress state $\{\sigma, \tau\}$ in the $\sigma\tau$-plane. The stress circles are subsequentially translated by superposition of a hydrostatic stress $-\frac{1}{2}(\sigma_1 + \sigma_3)$, $-\frac{1}{2}(\sigma_2 + \sigma_1)$ and $-\frac{1}{2}(\sigma_3 + \sigma_2)$. With the use of the stress vector $\vec{p}$ and the stress matrix $\underline{\sigma}^*$, resulting after superposition, it can be proven that the stress state is inside the largest stress circle and outside the other two.

inside σ_1, σ_3-circle

$$
\begin{aligned}
&\{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_3)\}^2 + \tau^2 = \sigma^2 + \tau^2 = ||\vec{p}||^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n} \\
&\qquad\qquad\qquad\qquad\qquad\qquad = n_1^2\alpha^2 + n_2^2\beta^2 + n_3^2\alpha^2 \\
&\text{with} \quad \beta^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right)^2 \leq \alpha^2 = \left(\sigma_1 - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \leq \alpha^2
\end{aligned}
$$

outside σ_2, σ_3-circle

$$
\begin{aligned}
&\{\sigma - \tfrac{1}{2}(\sigma_3 + \sigma_2)\}^2 + \tau^2 = \sigma^2 + \tau^2 = ||\vec{p}||^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n} \\
&\qquad\qquad\qquad\qquad\qquad\qquad = n_1^2\beta^2 + n_2^2\alpha^2 + n_3^2\alpha^2 \\
&\text{with} \quad \beta^2 = \left(\sigma_1 - \tfrac{1}{2}(\sigma_3 + \sigma_2)\right)^2 \geq \alpha^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_3 + \sigma_2)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \geq \alpha^2
\end{aligned}
$$

outside σ_1, σ_2-circle

$$
\begin{aligned}
&\{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_2)\}^2 + \tau^2 = \sigma^2 + \tau^2 = ||\vec{p}||^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n} \\
&\qquad\qquad\qquad\qquad\qquad\qquad = n_1^2\alpha^2 + n_2^2\alpha^2 + n_3^2\beta^2 \\
&\text{with} \quad \beta^2 = \left(\sigma_3 - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right)^2 \geq \alpha^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \geq \alpha^2
\end{aligned}
$$

6.3 Special stress states

Some special stress states are illustrated here. Stress components are considered in the Cartesian coordinate system.

6.3.1 Uni-axial stress

An unidirectional stress state is what we have in a tensile bar or truss. The axial load N in a cross-section (area A in the deformed state) is the integral of the axial stress σ over A. For homogeneous material the stress is uniform in the cross-section and is called the *true* or *Cauchy stress*. When it is assumed to be uniform in the cross-section, it is the ratio of N and A. The *engineering stress* is the ratio of N and the initial cross-sectional area A_0, which makes calculation easy, because A does not have to be known. For small deformations it is obvious that $A \approx A_0$ and thus that $\sigma \approx \sigma_n$.

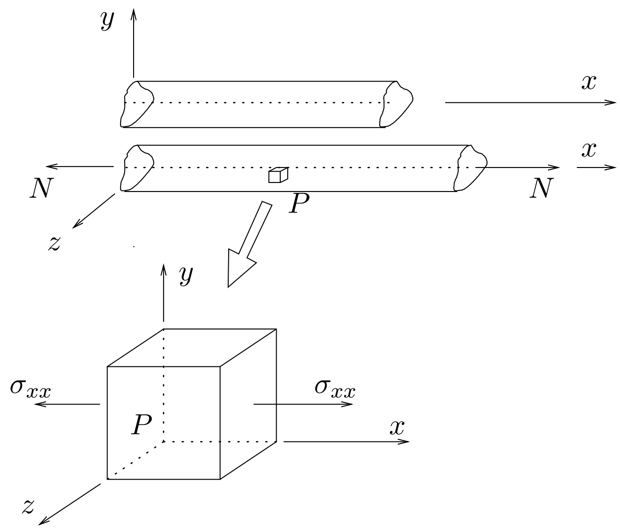


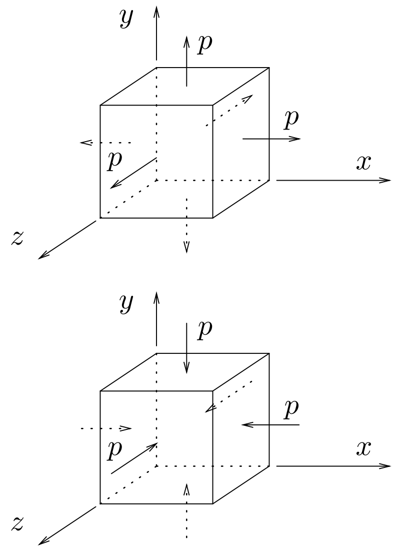
Fig. 6.10 : *Stresses on a small material volume in a tensile bar*

true or Cauchy stress $\qquad \sigma = \dfrac{N}{A} = \sigma_{xx} \quad \rightarrow \quad \boldsymbol{\sigma} = \sigma_{xx}\vec{e}_x\vec{e}_x$

engineering stress $\qquad \sigma_n = \dfrac{N}{A_0}$

6.3.2 Hydrostatic stress

A hydrostatic loading of the material body results in a hydrostatic stress state in each material point P. This can again be indicated by stresses (either tensile or compressive) on a stress cube. The three stress variables, with the same value, are normal to the faces of the stress cube.



$$\sigma_{xx} = p \qquad \sigma_{yy} = p \qquad \sigma_{zz} = p$$

$$\boldsymbol{\sigma} = p(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z)$$

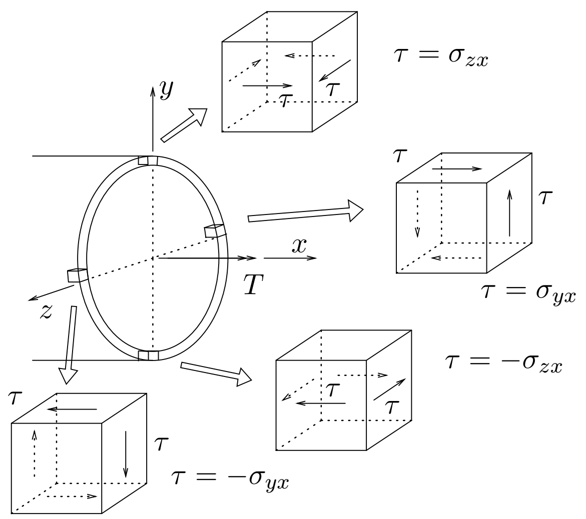
$$\boldsymbol{\sigma} = -p(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z)$$

$$\sigma_{xx} = -p \qquad \sigma_{yy} = -p \qquad \sigma_{zz} = -p$$

Fig. 6.11 : *Stresses on a material volume under hydrostatic loading*

6.3.3 Shear stress

The axial torsion of a thin-walled tube (radius R, wall thickness t) is the result of an axial torsional moment (torque) T. This load causes a shear stress τ in the cross-sectional wall. Although this shear stress has the same value in each point of the cross-section, the stress cube looks differently in each point because of the circumferential direction of τ.



$$\tau = \frac{T}{2\pi R^2 t}$$

Fig. 6.12 : *Stresses on a small material volume in the wall of a tube under shear loading*

$$\boldsymbol{\sigma} = \tau(\vec{e}_i\vec{e}_j + \vec{e}_j\vec{e}_i) \text{ with } i \neq j$$

6.3.4 Plane stress

When stresses on a plane perpendicular to the 3-direction are zero, the stress state is referred to as *plane stress* w.r.t. the 12-plane. Only three stress components are relevant in this case.

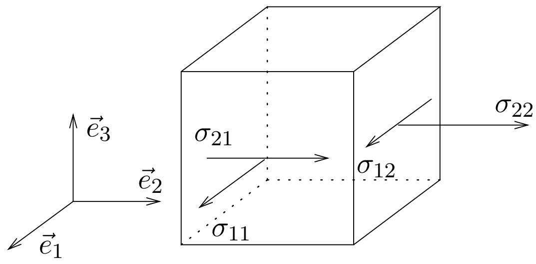


Fig. 6.13 : *Stress cube for plane stress in* 12*-plane*

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \quad \rightarrow \quad \boldsymbol{\sigma} \cdot \vec{e}_3 = \vec{0} \quad \rightarrow$$
$$\text{relevant stresses :} \quad \sigma_{11}, \sigma_{22}, \sigma_{12}$$

6.4 Resulting force on arbitrary material volume

A material body with volume V and surface area A is loaded with a volume load $\vec{q}$ per unit of mass and by a surface load $\vec{p}$ per unit of area. Taking a random part of the continuum with volume $\bar{V}$ and edge $\bar{A}$, the resulting force can be written as an integral over the volume, using Gauss' theorem. The load $\rho\vec{q}$ is a volume load per unit of volume, where ρ is the density of the material.

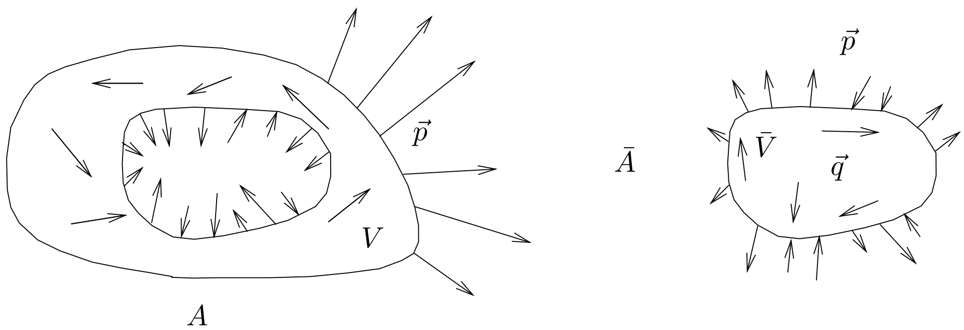


Fig. 6.14 : *Forces on a random section of a material body*

$$\vec{K} = \int_{\bar{V}} \rho\vec{q}\, dV + \int_{\bar{A}} \vec{p}\, dA = \int_{\bar{V}} \rho\vec{q}\, dV + \int_{\bar{A}} \vec{n} \cdot \boldsymbol{\sigma}^T \, dA$$

$$\text{Gauss theorem} \quad : \quad \int_{\bar{A}} \vec{n} \cdot (\,)\, dA = \int_{\bar{V}} \vec{\nabla} \cdot (\,)\, dV \quad \rightarrow$$

$$\vec{K} = \int_{\bar{V}} [\rho\vec{q} + \vec{\nabla} \cdot \boldsymbol{\sigma}^T]\, dV$$

6.5 Resulting moment on arbitrary material volume

The resulting moment about a fixed point of the forces working in volume and edge points of a random part of the continuum body can be calculated by integration.

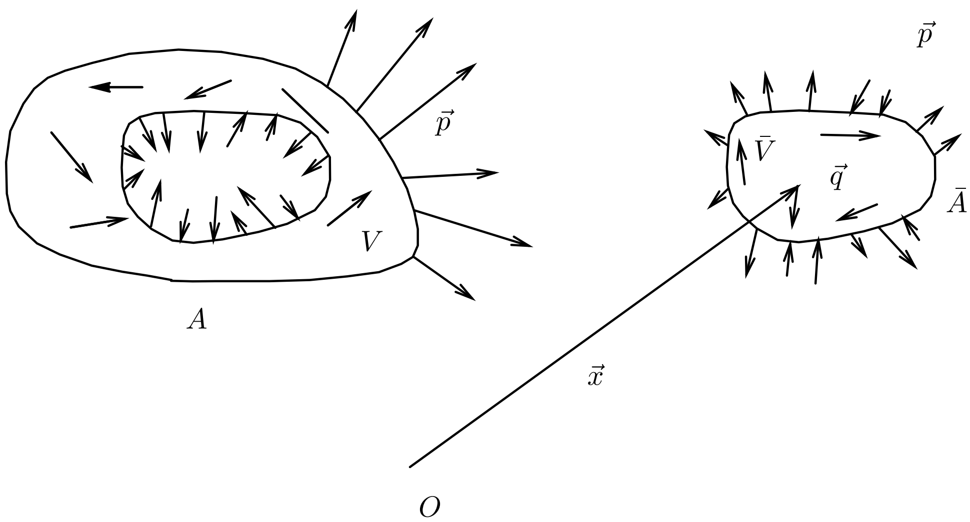


Fig. 6.15 : *Moments of forces on a random section of a material body*

$$\vec{M}_O = \int_{\bar{V}} \vec{x} * \rho\vec{q}\, dV + \int_{\bar{A}} \vec{x} * \vec{p}\, dA$$

Resulting moment on total body

Obviously we can also calculate the resulting moment for the whole material volume. By introducing a special point R other than the origin, the resulting moment can be expressed in the resulting moment about this point anf the moment of the resulting forces about this point. Often the resulting moment is considered with respect to the center of mass M with position $\vec{x}_M$.

$$\begin{aligned}
\vec{M}_O &= \int_{V} \vec{x} * \rho\vec{q}\, dV + \int_{A} \vec{x} * \vec{p}\, dA \\
&= \int_{V} (\vec{x}_R + \vec{r}) * \rho\vec{q}\, dV + \int_{A} (\vec{x}_R + \vec{r}) * \vec{p}\, dA
\end{aligned}$$

$$
\begin{aligned}
&= \vec{x}_R * \int\limits_V \rho \vec{q}\, dV + \vec{x}_R * \int\limits_A \vec{p}\, dA + \int\limits_V \vec{r} * \rho \vec{q}\, dV + \int\limits_A \vec{r} * \vec{p}\, dA \\
&= \vec{x}_R * \vec{K} + \vec{M}_R \\
&= \vec{x}_M * \vec{K} + \vec{M}_M \quad \rightarrow
\end{aligned}
$$

$$
\vec{M}_R = (\vec{x}_M - \vec{x}_R) * \vec{K} + \vec{M}_M = \vec{r}_M * \vec{K} + \vec{M}_M
$$

Chapter 7

Balance or conservation laws

In every physical process, so also during deformation of continuum bodies, some general accepted physical laws have to be obeyed : the conservation laws. During deformation the total *mass* has to be preserved and also the total *momentum* and *moment of momentum.* Because we do not consider dissipation and thermal effects, we will not discuss the conservation law for total energy.

7.1 Balance of mass

The mass of each finite, randomly chosen volume of material points in the continuum body must remain the same during the deformation process. Because we consider here a finite volume, this is the so-called *global* version of the mass conservation law.

From the requirement that this global law must hold for every randomly chosen volume, the *local version* of the conservation law can be derived. This derivation uses an integral transformation, where the integral over the volume $\bar{V}$ in the deformed state is transformed into an integral over the volume $\bar{V}_0$ in the undeformed state. From the requirement that the resulting integral equation has to be satisfied for each volume $\bar{V}_0$, the local version of the mass balance results.

The local version, which is also referred to as the *continuity equation*, can also be derived directly by considering the mass dM of the infinitesimal volume dV of material points.

The time derivative of the mass conservation law is also used frequently. Because we focus attention on the same material particles, a so-called *material time derivative* is used, which is indicated as $(\dot{\ })$.

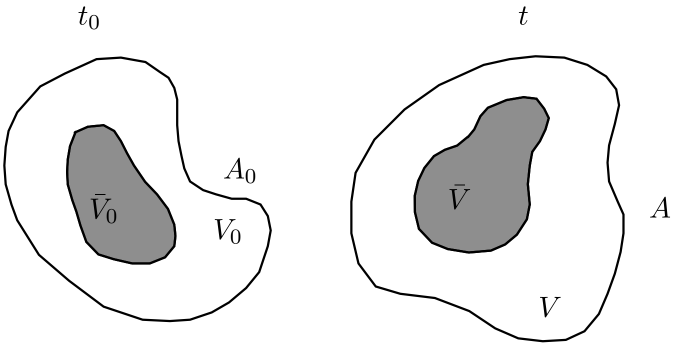


Fig. 7.1 : *Random volume in undeformed and deformed state*

$$\int_{\bar{V}} \rho \, dV = \int_{\bar{V}_0} \rho_0 \, dV_0 \quad \forall \, \bar{V} \quad \rightarrow \quad \int_{\bar{V}_0} (\rho J - \rho_0) \, dV_0 = 0 \quad \forall \, \bar{V}_0 \quad \rightarrow$$

$$\rho J = \rho_0 \qquad \forall \quad \vec{x} \in V(t)$$

$$dM = dM_0 \quad \rightarrow \quad \rho dV = \rho_0 dV_0 \quad \rightarrow \quad \rho J = \rho_0 \quad \forall \, \vec{x} \in V(t) \quad \rightarrow \quad \dot{\rho} J + \rho \dot{J} = 0$$

7.2 Balance of momentum

According to the balance of momentum law, a point mass m which has a velocity $\vec{v}$, will change its momentum $\vec{i} = m\vec{v}$ under the action of a force $\vec{K}$. Analogously, the total force working on a randomly chosen volume of material points equals the change of the total momentum of the material points inside the volume. In the balance law, again a material time derivative is used, because we consider the same material points. The total force can be written as a volume integral of volume forces and the divergence of the stress tensor.

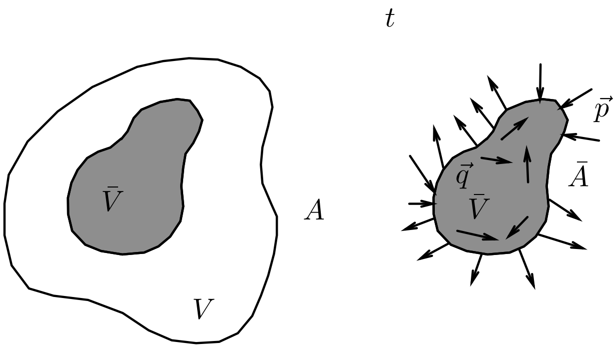


Fig. 7.2 : *Forces on random section of a material body*

$$
\begin{aligned}
\vec{K} &= \frac{D\vec{i}}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \rho\vec{v}\,dV \qquad \forall \quad \bar{V} \quad \rightarrow \\
&= \frac{D}{Dt}\int_{\bar{V}_0} \rho\vec{v}J\,dV_0 = \int_{\bar{V}_0} \frac{D}{Dt}\left(\rho\vec{v}J\right)\,dV_0 \qquad \forall \quad \bar{V}_0 \\
&= \int_{\bar{V}_0} \left(\dot{\rho}\vec{v}J + \rho\dot{\vec{v}}J + \rho\vec{v}\dot{J}\right)\,dV_0 \qquad \forall \quad \bar{V}_0 \\
&\qquad \text{mass balance} \quad : \quad \dot{\rho}J + \rho\dot{J} = 0 \quad \rightarrow \\
&= \int_{\bar{V}_0} \rho\dot{\vec{v}}J\,dV_0 = \int_{\bar{V}} \rho\dot{\vec{v}}\,dV \qquad \forall \quad \bar{V} \\
&\int_{\bar{V}} \left(\rho\vec{q} + \vec{\nabla}\cdot\boldsymbol{\sigma}^c\right)\,dV = \int_{\bar{V}} \rho\dot{\vec{v}}\,dV \qquad \forall \quad \bar{V}
\end{aligned}
$$

From the requirement that the global balance law must hold for every randomly chosen volume of material points, the local version of the balance of momentum can be derived, which must hold in every material point. In the derivation an integral transformation is used.

The local balance of momentum law is also called the *equation of motion*. For a stationary process, where the material velocity $\vec{v}$ in a fixed spatial point does not change, the equation is simplified. For a static process, where there is no acceleration of masses, the *equilibrium equation* results.

$$
\begin{array}{lll}
\text{local version : equation of motion} & \vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \rho\dot{\vec{v}} = \rho\dfrac{\delta\vec{v}}{\delta t} + \rho\vec{v}\cdot\left(\vec{\nabla}\vec{v}\right) & \forall\ \vec{x}\in V(t) \\
\text{stationary} \quad \left(\dfrac{\delta\vec{v}}{\delta t} = 0\right) & \vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \rho\vec{v}\cdot\left(\vec{\nabla}\vec{v}\right) & \\
\text{static : equilibrium equation} & \vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \vec{0} &
\end{array}
$$

7.2.1 Cartesian and cylindrical components

The equilibrium equation can be written in components w.r.t. a Cartesian vector basis. This results in three partial differential equations, one for each coordinate direction.

$$
\begin{aligned}
\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x &= 0 \\
\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y &= 0 \\
\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z &= 0
\end{aligned}
$$

Writing tensor and vectors in components w.r.t. a cylindrical vector basis is more elaborative because the cylindrical base vectors $\vec{e}_r$ and $\vec{e}_t$ are a function of the coordinate θ, so they have to be differentiated, when expanding the divergence term.

$$
\begin{aligned}
\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r &= 0 \\
\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t &= 0 \\
\sigma_{zr,r} + \frac{1}{r}\sigma_{zt,t} + \frac{1}{r}\sigma_{zr} + \sigma_{zz,z} + \rho q_z &= 0
\end{aligned}
$$

7.3 Balance of moment of momentum

The balance of moment of momentum states that the total moment about a fixed point of all forces working on a randomly chosen volume of material points ($\vec{M}_O$), equals the change of the total moment of momentum of the material points inside the volume, taken w.r.t. the same fixed point ($\vec{L}_O$).

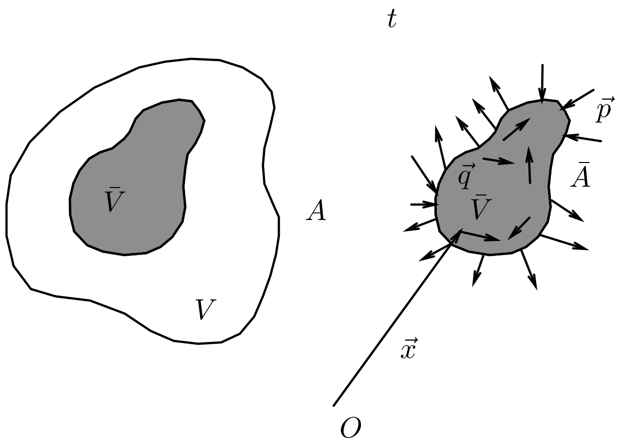


Fig. 7.3 : *Moment of forces on a random section of a material body*

$$
\begin{aligned}
\vec{M}_O &= \frac{D\vec{L}_O}{Dt} = \frac{D}{Dt}\int\limits_{\bar{V}} \vec{x} * \rho\vec{v}\, dV \qquad\qquad \forall \quad \bar{V} \\
&= \frac{D}{Dt}\int\limits_{\bar{V}_0} \vec{x} * \rho\vec{v}J\, dV_0 = \int\limits_{\bar{V}_0} \frac{D}{Dt}\left(\vec{x} * \rho\vec{v}J\right)\, dV_0 \qquad\qquad \forall \quad \bar{V}_0 \\
&= \int\limits_{\bar{V}_0} \left(\dot{\vec{x}} * \rho\vec{v}J + \vec{x} * \dot{\rho}\vec{v}J + \vec{x} * \rho\dot{\vec{v}}J + \vec{x} * \rho\vec{v}\dot{J}\right)\, dV_0 \qquad\qquad \forall \quad \bar{V}_0 \\
&\qquad\qquad \left.\begin{array}{l} \text{mass balance} \quad : \quad \dot{\rho}J + \rho\dot{J} = 0 \\ \dot{\vec{x}} * \vec{v} = \vec{v} * \vec{v} = \vec{0} \end{array}\right\} \quad \rightarrow \\
&= \int\limits_{\bar{V}_0} \vec{x} * \rho\dot{\vec{v}}J\, dV_0 = \int\limits_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \qquad\qquad \forall \quad \bar{V}
\end{aligned}
$$

$$
\int\limits_{\bar{V}} \vec{x} * \rho\vec{q}\, dV + \int\limits_{\bar{A}} \vec{x} * \vec{p}\, dA = \int\limits_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \qquad\qquad \forall \quad \bar{V}
$$

For the analysis of the dynamics of deformable and rigid bodies the balance law is often reformulated, such that its motion is the superposition a rotation about the center of rotation R and its translation. In appendix **??** this is elaborated.

To derive a local version, the integral over the area $\bar{A}$ has to be transformed to an integral over the enclosed volume $\bar{V}$. In this derivation, the operator ${}^3\boldsymbol{\epsilon}$ is used, which is defined such that

$$\vec{a} * \vec{b} = {}^3\boldsymbol{\epsilon} : \vec{a}\,\vec{b}$$

holds for all vectors $\vec{a}$ and $\vec{b}$.

$$\begin{aligned}
\int\limits_{\bar{A}} \vec{x} * \vec{p}\, dA &= \int\limits_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x}\,\vec{p})\, dA = \int\limits_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x}\,\boldsymbol{\sigma} \cdot \vec{n})\, dA = \int\limits_{\bar{A}} \vec{n} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x}\boldsymbol{\sigma})\}^c\, dA \\
&= \int\limits_{\bar{V}} \vec{\nabla} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x}\boldsymbol{\sigma})\}^c\, dV \\
&= \int\limits_{\bar{V}} \vec{\nabla} \cdot \{(\vec{x}\boldsymbol{\sigma})^c : {}^3\boldsymbol{\epsilon}^c\}\, dV = \int\limits_{\bar{V}} \vec{\nabla} \cdot \{(\boldsymbol{\sigma}^c\vec{x}) : {}^3\boldsymbol{\epsilon}^c\}\, dV \\
&= \int\limits_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} \cdot (\vec{\nabla} \cdot \vec{x}) : {}^3\boldsymbol{\epsilon}^c\right]\, dV \\
&= \int\limits_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} : {}^3\boldsymbol{\epsilon}^c\right]\, dV = \int\limits_{\bar{V}} \left[{}^3\boldsymbol{\epsilon} : \vec{x}(\vec{\nabla} \cdot \boldsymbol{\sigma}^c) + {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\right]\, dV \\
&= \int\limits_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c + \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\, dV
\end{aligned}$$

Substitution in the global version and using the local balance of momentum, leads to the local version of the balance of moment of momentum.

$$\begin{aligned}
&\int\limits_{\bar{V}} \vec{x} * \rho\vec{q}\, dV + \int\limits_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV + \int\limits_{\bar{V}} \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\, dV = \int\limits_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \quad \forall \quad \bar{V} \quad \rightarrow \\
&\int\limits_{\bar{V}} \vec{x} * \left[\rho\vec{q} + (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) - \rho\dot{\vec{v}}\right]\, dV + \int\limits_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow \\
&\int\limits_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow \quad {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c = \vec{0} \quad \forall \quad \vec{x} \in \bar{V}
\end{aligned}$$

$$\begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\sigma}^c = \boldsymbol{\sigma} \qquad\qquad \forall \quad \vec{x} \in V(t)$$

7.3.1 Cartesian and cylindrical components

With respect to a Cartesian or cylindrical basis the symmetry of the stress tensor results in three equations.

$$\underline{\sigma} = \underline{\sigma}^T \qquad \rightarrow$$

$$\begin{array}{llccccc} \text{Cartesian} & : & \sigma_{xy} = \sigma_{yx} & ; & \sigma_{yz} = \sigma_{zy} & ; & \sigma_{zx} = \sigma_{xz} \\ \text{cylindrical} & : & \sigma_{rt} = \sigma_{tr} & ; & \sigma_{tz} = \sigma_{zt} & ; & \sigma_{zr} = \sigma_{rz} \end{array}$$

7.4 Balance of energy

The first law of thermodynamics states that the total amount of energy supplied to a material body is converted to kinetic energy (U_k) and internal energy (U_i). The supplied energy is considered to be 1) work done by external mechanical loads (U_e), and 2) thermal energy supplied by internal sources or external fluxes (U_t). The internal energy can be of very different character, such as elastically stored energy and dissipated energy due to plastic deformation, viscous effects, crack growth, etcetera.

$$\frac{D}{Dt}(U_e + U_t) = \frac{D}{Dt}(U_k + U_i)$$

7.4.1 Mechanical energy

When a point load $\vec{k}$ is applied in a material point and the point moves with a velocity $\vec{v}$, the work of the load per unit of time is $\dot{U}_e = \vec{k} \cdot \vec{v}$. For a random volume $\bar{V}$ with edge $\bar{A}$ inside a material body the mechanical work of all loads per unit of time can be calculated. Using Gauss' theorem, this work can be written as an integral over the volume $\bar{V}$. Also the equation of motion is used to arrive at the final result.

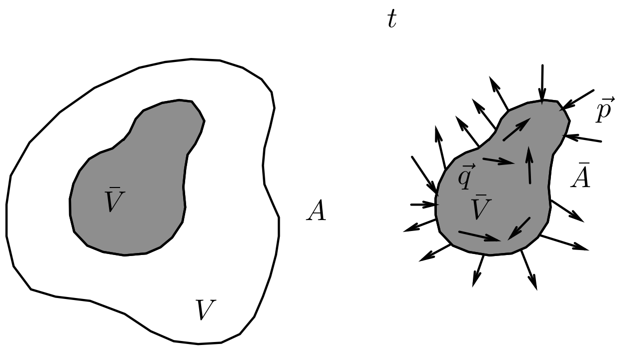


Fig. 7.4 : *Mechanical load on a material volume*

$$\dot{U}_e = \int_{\bar{V}} \rho \vec{q} \cdot \vec{v} \, dV + \int_{\bar{A}} \vec{p} \cdot \vec{v} \, dA = \int_{\bar{V}} \{ \rho \vec{q} \cdot \vec{v} + \vec{\nabla} \cdot (\boldsymbol{\sigma}^c \cdot \vec{v}) \} \, dV$$

$$\vec{\nabla} \cdot (\boldsymbol{\sigma}^c \cdot \vec{v}) = (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) \cdot \vec{v} + \boldsymbol{\sigma} : (\vec{\nabla} \vec{v})$$

$$= \rho\dot{\vec{v}} \cdot \vec{v} - \rho\vec{q} \cdot \vec{v} + \boldsymbol{\sigma} : \boldsymbol{D} + \boldsymbol{\sigma} : \boldsymbol{\Omega}$$

$$= \int_{\bar{V}} (\rho\dot{\vec{v}} \cdot \vec{v} + \boldsymbol{\sigma} : \boldsymbol{D})\, dV$$

7.4.2 Thermal energy

Thermal energy can be produced by internal sources. The heat production per unit of mass is r [J kg^{-1}].

Heat can flow in or out of a material body or in the body from one part to another. In a material point P the heat flux vector is $\vec{H}$ [J]. The heat flux density vector *in* P *through* a plane with area ΔA is

$$\vec{h} = \lim_{\Delta A \to 0} \frac{\vec{H}}{\Delta A} \qquad [\text{J m}^{-2}]$$

The resulting heat flux *in* P *through* the plane is $\vec{n} \cdot \vec{h}$ [J m^{-2}] , where $\vec{n}$ is the unit normal vector on the plane.

For a random volume $\bar{V}$ having edge $\bar{A}$ with unit normal outward vector $\vec{n}$, the increase in thermal energy at time t is $\dot{U}_t$.

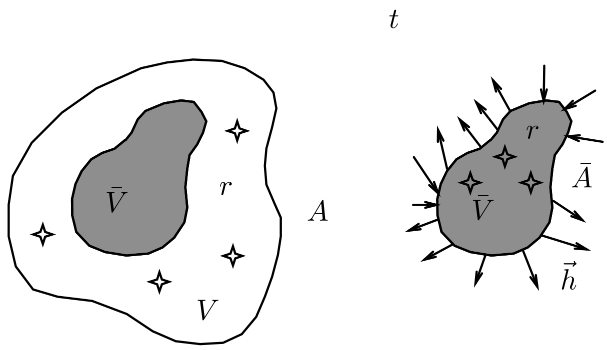


Fig. 7.5 : *Heat sources in and heat flux into a material volume*

$$\dot{U}_t = \int_{\bar{V}} \rho r\, dV - \int_{\bar{A}} \vec{n} \cdot \vec{h}\, dA = \int_{\bar{V}} (\rho r - \vec{\nabla} \cdot \vec{h})\, dV$$

7.4.3 Kinetic energy

The kinetic energy of a point mass m with velocity $\vec{v}$ is

$$U_k = \tfrac{1}{2}\, m\, ||\vec{v}||^2 = \tfrac{1}{2}\, m\, \vec{v} \cdot \vec{v}$$

For a random volume $\bar{V}$ of material points, having density ρ and velocity $\vec{v}$, the total kinetic energy U_k can be calculated by intergration.

$$U_k(t) = \int\limits_{\bar{V}} \tfrac{1}{2}\,\rho\,\vec{v}\cdot\vec{v}\,dV \quad \rightarrow \quad \dot{U}_k = \frac{D}{Dt}\int\limits_{\bar{V}} \tfrac{1}{2}\,\rho\,\vec{v}\cdot\vec{v}\,dV = \frac{D}{Dt}\int\limits_{\bar{V}_0} \tfrac{1}{2}\,\rho\,\vec{v}\cdot\vec{v}J\,dV_0$$

$$= \tfrac{1}{2}\int\limits_{\bar{V}_0} \left\{\dot{\rho}\,\vec{v}\cdot\vec{v}J + 2\rho\,\dot{\vec{v}}\cdot\vec{v}J + \rho\,\vec{v}\cdot\vec{v}\dot{J}\right\}\,dV_0$$

$$= \int\limits_{\bar{V}_0} \rho\,\dot{\vec{v}}\cdot\vec{v}J\,dV_0 = \int\limits_{\bar{V}} \rho\,\dot{\vec{v}}\cdot\vec{v}\,dV$$

7.4.4 Internal energy

The internal energy per unit of mass is ϕ. The total internal energy of all material points in a random volume $\bar{V}$ of a material body, U_i, can be calculated by integration.

$$U_i(t) = \int\limits_{\bar{V}} \rho\phi\,dV \quad \rightarrow \quad \dot{U}_i = \frac{D}{Dt}\int\limits_{\bar{V}} \rho\phi\,dV = \frac{D}{Dt}\int\limits_{\bar{V}_0} \rho\phi J\,dV_0$$

$$= \int\limits_{\bar{V}_0} \left\{\dot{\rho}\phi J + \rho\dot{\phi}J + \rho\phi\dot{J}\right\}\,dV_0$$

$$= \int\limits_{\bar{V}} \rho\dot{\phi}\,dV$$

7.4.5 Energy balance

The energy balance or first law of thermodynamics for a random volume of material points in a material body, can be written as an integral equation. It is the global form of the balance law, because a finite volume is considered.

$$\dot{U}_e + \dot{U}_t = \dot{U}_k + \dot{U}_i$$

$$\int\limits_{\bar{V}} (\rho\dot{\vec{v}}\cdot\vec{v} + \boldsymbol{\sigma} : \boldsymbol{D} + \rho r - \vec{\nabla}\cdot\vec{h})\,dV = \int\limits_{\bar{V}} (\rho\dot{\vec{v}}\cdot\vec{v} + \rho\dot{\phi})\,dV \qquad \forall \quad \bar{V}$$

$$\int\limits_{\bar{V}} \rho\dot{\phi}\,dV = \int\limits_{\bar{V}} (\boldsymbol{\sigma} : \boldsymbol{D} + \rho r - \vec{\nabla}\cdot\vec{h})\,dV \qquad \forall \quad \bar{V}$$

7.4.6 Energy equation

The local version of the energy balance, also called the *energy equation*, is easily derived by taking into account the fact that the global version must be valid for each volume $\bar{V}$.

The specific internal energy ϕ can be written as the product of the specific heat C_p (assumed to be constant here) and the absolute temperature T.

The heat flux density $\vec{h}$ is often related to the temperature gradient $\vec{\nabla}T$ accoding to

Fourier's law, where the thermal conductivity k (assumed to be constant here) is a material parameter.

$$\left.\begin{array}{ll} \rho\dot{\phi} = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r - \vec{\nabla} \cdot \vec{h} & \forall \quad \vec{x} \in V(t) \\ \dot{\phi} = C_p \dot{T} \quad (C_p : \text{specific heat}) & \end{array}\right\} \rightarrow$$

$$\left.\begin{array}{ll} \rho C_p \dot{T} = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r - \vec{\nabla} \cdot \vec{h} & \forall \quad \vec{x} \in V(t) \\ \vec{h} = -k\, \vec{\nabla} T \quad (k : \text{thermal conductivity}) & \end{array}\right\} \rightarrow$$

$$\rho C_p \dot{T} - k\, \nabla^2 T = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r \qquad \forall \quad \vec{x} \in V(t)$$

7.4.7 Mechanical power for three-dimensional deformation

Elastic deformation of a three-dimensional continuum leads to storage of elastic energy, which can be calculated per unit of undeformed (W_0) or deformed (W) volume. Different expressions for the strain rate can than be combined with different stress tensors, which are all a function of the Cauchy stress tensor $\boldsymbol{\sigma}$. The starting point is the change of stored energy per unit of deformed volume.

$$\dot{W} = \boldsymbol{\sigma} : \boldsymbol{D} \qquad \boldsymbol{\sigma} = \text{Cauchy stress tensor}$$

$$\begin{aligned} \dot{W}_0 &= [J\boldsymbol{\sigma}] : \boldsymbol{D} \\ &= \boldsymbol{\kappa} : \boldsymbol{D} \qquad \boldsymbol{\kappa} = \text{Kirchhoff stress tensor} \end{aligned}$$

$$\begin{aligned} \dot{W}_0 &= J\boldsymbol{\sigma} : \boldsymbol{D} = J\boldsymbol{\sigma} : \tfrac{1}{2}\left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} + (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})^c\right) = \\ &= J\boldsymbol{\sigma} : \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right) = J\left(\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}\right) : \dot{\boldsymbol{F}} = \boldsymbol{S} : \dot{\boldsymbol{F}} = \boldsymbol{S} : \dot{\boldsymbol{U}} \\ &= \boldsymbol{S} : \dot{\boldsymbol{\mathcal{E}}} \qquad \boldsymbol{S} = \text{1st-Piola-Kirchhoff stress tensor} \end{aligned}$$

$$\begin{aligned} \dot{W}_0 &= J\boldsymbol{\sigma} : \boldsymbol{D} = J\boldsymbol{\sigma} : \left(\boldsymbol{F}^{-c} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}\right) = J\left(\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-c}\right) : \dot{\boldsymbol{E}} \\ &= \boldsymbol{P} : \dot{\boldsymbol{E}} \qquad \boldsymbol{P} = \text{2nd-Piola-Kirchhoff stress tensor} \end{aligned}$$

7.5 Special equilibrium states

The three-dimensional equilibrium equations can be simplified for special deformation or stress states, such as plane strain, plane stress and axisymmetric cases.

Planar deformation

It is assumed here that the z-direction is the direction where either the strain or the stress is zero. Only stresses and strains in the plane perpendicular to the z-direction remain to be determined from equilibrium. The strain or stress in the z-direction can be calculated afterwards, either directly from the material law or iteratively during the solution procedure.

Cartesian components

$$\begin{aligned}
&\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = 0 \\
&\sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = 0 \\
&\sigma_{xy} = \sigma_{yx}
\end{aligned}$$

cylindrical components

$$\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \rho q_t = 0 \\
\\
&\sigma_{rt} = \sigma_{tr}
\end{aligned}$$

Axisymmetric deformation

In many cases the geometry, boundary conditions and material behavior is such that no state variable depends on the circumferential coordinate θ : $\frac{\partial}{\partial\theta} = 0$. For such axisymmetric deformations, the equilibrium equations can be simplified considerably.

In many axisymmetric deformations the boundary conditions are such that there is no displacement in the circumferential direction : $u_t = 0$. In these cases there are only four relevant strain and stress components and only three equilibrium equations.

When boundary conditions and material behavior are such that displacement of material points are only in the $r\theta$-plane, the deformation is referred to as *plane strain* in the $r\theta$-plane.

When stresses on a plane perpendicular to the z-direction are zero, the stress state is referred to as *plane stress* w.r.t. the $r\theta$-plane.

$$\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{2}{r}(\sigma_{tr}) + \sigma_{tz,z} + \rho q_t = 0 \qquad \text{(if } u_t \neq 0) \\
&\sigma_{zr,r} + \frac{1}{r}\sigma_{zr} + \sigma_{zz,z} + \rho q_z = 0 \\
&\sigma_{rt} = \sigma_{tr} \quad ; \quad \sigma_{tz} = \sigma_{zt} \qquad \text{(if } u_t \neq 0) \\
&\sigma_{zr} = \sigma_{rz}
\end{aligned}$$

planar

$$\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{2}{r}(\sigma_{tr}) + \rho q_t = 0 \qquad \text{(if } u_t \neq 0) \\
&\sigma_{rt} = \sigma_{tr} \qquad \text{(if } u_t \neq 0)
\end{aligned}$$

Chapter 8

Constitutive equations

Stresses must always satisfy the balance laws, which are considered to be laws of physics in the non-quantum world, where we live our lives together with our materials and structures. Balance laws must apply to each material, of which the deformation is studied. It is obvious, however, that various materials will behave very differently, when subjected to the same external loads. This behavior must be incorporated in the continuum mechanics theory and is therefore modelled mathematically. The resulting equations are referred to as *constitutive equations*. They can not fully be derived from physical principles, although the theory of *thermodynamics* tells us a lot of how they must look like. The real mathematical formulation of the material laws is however based on experimental observations of the deformation of the material.

In later sections, the behavior of a wide range of materials is modelled and used in a three-dimensional context. In this chapter, the more general aspects of constitutive equations are discussed.

8.1 Equations and unknowns

Although it is obvious that material laws must be incorporated to describe the behavior of different materials, they are also needed from a purely mathematical point of view. This has to do with the number of unknown variables and the number of equations, from which they must be solved. Obviously, the number of equations has to be the same as the number of unknowns.

The local balance laws for mass, momentum and moment of momentum have to be satisfied in every material point of the continuum body at every time during the deformation process.

The mass balance law is a scalar equation. The balance of momentum or equation of motion is a partial differential equation. It is a vector equation. The balance of moment of momentum is a tensor equation.

The unknown variables, which appear in the balance laws, are the density ρ of the material, the position vector $\vec{x}$ of the material point and the stress tensor $\boldsymbol{\sigma}$. The continuity equation can be used to express the density in the deformation tensor $\boldsymbol{F}$, which is known, when the position $\vec{x}$ of the material point is known. So we can skip the mass balance from our equation set and the density from the set of unknowns.

The moment of momentum equation can be used directly to state that there are only 6 unknown stress components instead of 9. So we loose three equations and three unknowns. The number of unknowns is now 9 and the number of equations is 3, so 6 constitutive equations are required. These equations are relations between the stress components and the components of the position vector.

mass	$\rho J = \rho_0$
momentum	$\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho\vec{q} = \rho\dot{\vec{v}}$
moment of momentum	$\boldsymbol{\sigma}^c = \boldsymbol{\sigma}$

density	ρ
position vector	$\vec{x}$
Cauchy stress tensor	$\boldsymbol{\sigma}$

The number of unknowns is now 9 and the number of equations is 3, so 6 constitutive equations are required. These equations are relations between the stress components and the components of the position vector.

$$\boldsymbol{\sigma} = \boldsymbol{N}(\vec{x})$$

8.2 General constitutive equation

The most general constitutive equation states that the stress tensor in point $\vec{x}$ at (the current) time t, is a function of the position of all material points at every previous time in the deformation process. This implies that the complete deformation history of all points is needed to calculate the current stress in each material point.

This constitutive equation is far to general to be useful. In the following it will be specified by incorporating assumptions about the material behavior. In practice these assumptions must of course be based on experimental observations.

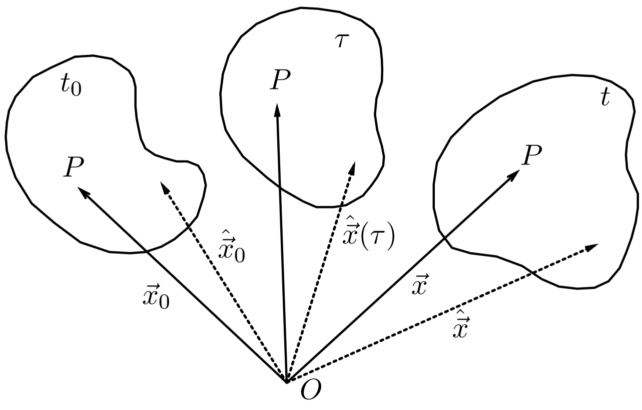


Fig. 8.1 : *Deformation history of a continuum*

$$\boldsymbol{\sigma}(\vec{x},t) = \boldsymbol{N}\{\hat{\vec{x}},\tau \mid \forall\ \hat{\vec{x}} \in V;\ \forall\ \tau \leq t\}$$

8.2.1 Locality

A wide range of materials and deformation processes allow the assumption of locality. In that case the stress in a point $\vec{x}$ is determined by the position of points in its direct neighborhood, so points with position vector $\vec{x} + d\vec{x}$. This can be written in terms of the deformation tensor $\boldsymbol{F}$.

$$\left.\begin{array}{l} \boldsymbol{\sigma}(\vec{x},t) = \boldsymbol{N}\{\hat{\vec{x}},\tau \mid \forall\ \hat{\vec{x}} \in V\ ;\ \forall\ \tau \leq t\} \\ \hat{\vec{x}} = \vec{x} + d\vec{x} = \vec{x} + \boldsymbol{F}(\vec{x}) \cdot d\vec{x}_0 \end{array}\right\} \quad \rightarrow \quad \boldsymbol{\sigma}(\vec{x},t) = \boldsymbol{N}(\vec{x},\boldsymbol{F}(\vec{x},\tau),\tau \mid \forall\ \tau \leq t)$$

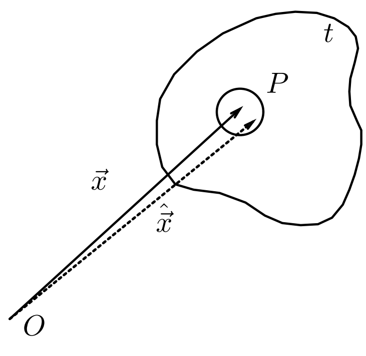


Fig. 8.2 : *Deformation with local influence*

8.2.2 Frame indifference

The stress state in a material point will not change when the material body is translated and/or rotated without (extra) deformation, i.e. when it moves as a rigid body. The variables in the constitutive equation may, however, change. The constitutive equation must be formulated such that these changes do not affect the stress state in a material point.

A rigid body translation is described with a displacement vector, which is equal for all material points. The stress is not allowed to change, so it is easily seen that the constitutive function $\boldsymbol{N}$ cannot depend on the position $\vec{x}$.

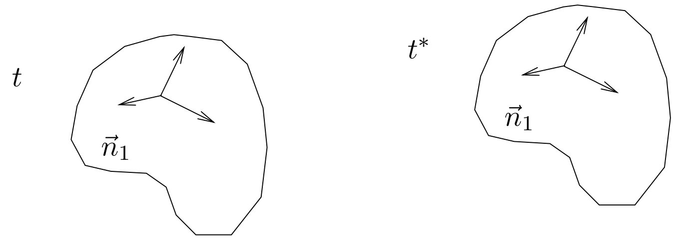


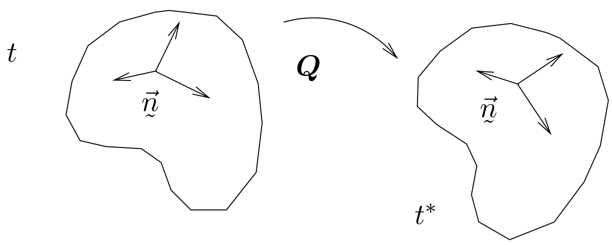
Fig. 8.3 : *Rigid body translation of a continuum*

$$\boldsymbol{\sigma}(\vec{x}, t) = \boldsymbol{N}(\boldsymbol{F}(\vec{x}, \tau), \tau \mid \forall\ \tau \leq t)$$

The symmetric Cauchy stress tensor can be written in spectral form. When the deformed body is subjected to a rigid rotation, described by the rotation tensor $\boldsymbol{Q}$, the principal stresses do not change, but the principal directions do. This means that the Cauchy stress tensor changes due to rigid rotation of the material.

The deformation tensor $\boldsymbol{F}$ will also change as a consequence of rigid rotation, which can be easily seen from the polar decomposition.

The relation between $\boldsymbol{\sigma}^*$ and $\boldsymbol{F}^*$ must be the same as that between $\boldsymbol{\sigma}$ and $\boldsymbol{F}$, which results in a requirement for the constitutive equation. (We skip the $\vec{x}$-dependency of $\boldsymbol{F}$.)



$$\vec{n}_1^* = \boldsymbol{Q} \cdot \vec{n}_1$$
$$\vec{n}_2^* = \boldsymbol{Q} \cdot \vec{n}_2$$
$$\vec{n}_3^* = \boldsymbol{Q} \cdot \vec{n}_3$$

Fig. 8.4 : *Rigid body rotation of a continuum*

$$\begin{aligned}
\boldsymbol{\sigma} &= \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3 \\
\boldsymbol{\sigma}^* &= \sigma_1 \vec{n}_1^* \vec{n}_1^* + \sigma_2 \vec{n}_2^* \vec{n}_2^* + \sigma_3 \vec{n}_3^* \vec{n}_3^* \\
&= \sigma_1 \boldsymbol{Q} \cdot \vec{n}_1 \vec{n}_1 \cdot \boldsymbol{Q}^c + \sigma_2 \boldsymbol{Q} \cdot \vec{n}_2 \vec{n}_2 \cdot \boldsymbol{Q}^c + \sigma_3 \boldsymbol{Q} \cdot \vec{n}_3 \vec{n}_3 \cdot \boldsymbol{Q}^c \\
&= \boldsymbol{Q} \cdot [\sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3] \cdot \boldsymbol{Q}^c = \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c
\end{aligned}$$

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} \quad \rightarrow \quad \boldsymbol{F}^* = \boldsymbol{R}^* \cdot \boldsymbol{U} = \boldsymbol{Q} \cdot \boldsymbol{R} \cdot \boldsymbol{U} \quad \rightarrow \quad \boldsymbol{F}^* = \boldsymbol{Q} \cdot \boldsymbol{F}$$

objectivity requirement

$$\boldsymbol{Q}(t) \cdot \boldsymbol{N}\left(\boldsymbol{F}(\tau) \mid \forall\ \tau \leq t\right) \cdot \boldsymbol{Q}^c(t) = \boldsymbol{N}\left(\boldsymbol{Q} \cdot \boldsymbol{F}(\tau) \mid \forall\ \tau \leq t\right) \qquad \forall \quad \boldsymbol{Q}$$

$$\sigma = C\boldsymbol{E} = C\tfrac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right) = C\tfrac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}\right)$$

$$\boldsymbol{\sigma}^* = \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^T$$

$$\begin{aligned} \boldsymbol{F}^* &= \boldsymbol{Q} \cdot \boldsymbol{F} \\ \boldsymbol{E}^* &= \tfrac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \boldsymbol{F} - \boldsymbol{I}\right) = \tfrac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}\right) = \boldsymbol{E} \end{aligned}$$

$$\boldsymbol{\sigma}^* = C\boldsymbol{E}$$

NOT OBJECTIVE

$$\sigma = C\boldsymbol{A} = C\tfrac{1}{2}\left(\boldsymbol{B} - \boldsymbol{I}\right) = C\tfrac{1}{2}\left(\boldsymbol{F} \cdot \boldsymbol{F}^T - \boldsymbol{I}\right)$$

$$\boldsymbol{\sigma}^* = \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^T$$

$$\begin{aligned} \boldsymbol{F}^* &= \boldsymbol{Q} \cdot \boldsymbol{F} \\ \boldsymbol{A}^* &= \tfrac{1}{2}\left(\boldsymbol{Q} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{Q}^T - \boldsymbol{I}\right) = \tfrac{1}{2}\boldsymbol{Q} \cdot \left(\boldsymbol{F} \cdot \boldsymbol{F}^T - \boldsymbol{I}\right) \cdot \boldsymbol{Q}^T = \boldsymbol{Q} \cdot \boldsymbol{A} \cdot \boldsymbol{Q}^c \end{aligned}$$

$$\boldsymbol{\sigma}^* = C\boldsymbol{A}^*$$

OBJECTIVE

$$\begin{aligned} \boldsymbol{\sigma} &= -p\boldsymbol{I} + 2\eta\boldsymbol{D} \\ \boldsymbol{D} &= \tfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^c) \qquad \text{with} \qquad \boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \end{aligned}$$

$$\boldsymbol{\sigma}^* = \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^T$$

$$\begin{aligned} \boldsymbol{F}^* &= \boldsymbol{Q} \cdot \boldsymbol{F} \quad ; \quad \boldsymbol{F}^{*^{-1}} = \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^c \quad ; \quad \dot{\boldsymbol{F}}^* = \dot{\boldsymbol{Q}} \cdot \boldsymbol{F} + \boldsymbol{Q} \cdot \dot{\boldsymbol{F}} \\ \boldsymbol{L}^* &= (\dot{\boldsymbol{Q}} \cdot \boldsymbol{F} + \boldsymbol{Q} \cdot \dot{\boldsymbol{F}}) \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^c = \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^c \\ \boldsymbol{D}^* &= \tfrac{1}{2}\left[\dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^c + \boldsymbol{Q} \cdot (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})^c \cdot \boldsymbol{Q}^c\right] \\ &\qquad \boldsymbol{Q} \cdot \boldsymbol{Q}^c = \boldsymbol{I} \quad \rightarrow \quad \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^c \\ &= \boldsymbol{Q} \cdot \boldsymbol{D} \cdot \boldsymbol{Q}^c \end{aligned}$$

$$\boldsymbol{\sigma}^* = -p\boldsymbol{I} + 2\eta\boldsymbol{D}^*$$

OBJECTIVE

8.3 Invariant stress tensor

For convenient constitutive modeling where stress (rate) is related to deformation (rate), we need stress tensors which are invariant with rigid rotation. Also their time derivative must answer this requirement.

A stress tensor $\boldsymbol{S} = \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c$ can be defined, where $\boldsymbol{A}$ is to be specified later, but always has to obey $\boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c$. It follows that the stress tensor $\boldsymbol{S}$ is invariant for rigid rotations.

$$\boldsymbol{S} = \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c$$

$$\left.\begin{array}{ll} \boldsymbol{S}^* = \boldsymbol{A}^* \cdot \boldsymbol{\sigma}^* \cdot \boldsymbol{A}^{*c} = \boldsymbol{A}^* \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{A}^{*c} & \\ \text{define} \qquad \boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c & \end{array}\right\} \quad \rightarrow$$

$$\boldsymbol{S}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c = \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c = \boldsymbol{S}$$

$$\boldsymbol{S} = \text{invariant for rigid rotation}$$

Also its time derivative $\dot{\boldsymbol{S}}$ is inveriant.

$$\begin{aligned}
\dot{\boldsymbol{S}} &= \dot{\boldsymbol{A}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{A}}^c \\
\dot{\boldsymbol{S}}^* &= \dot{\boldsymbol{A}}^* \cdot \boldsymbol{\sigma}^* \cdot \boldsymbol{A}^{*c} + \boldsymbol{A}^* \cdot \dot{\boldsymbol{\sigma}}^* \cdot \boldsymbol{A}^{*c} + \boldsymbol{A}^* \cdot \boldsymbol{\sigma}^* \cdot \dot{\boldsymbol{A}}^{*c} \\
&= (\dot{\boldsymbol{A}} \cdot \boldsymbol{Q}^c + \boldsymbol{A} \cdot \dot{\boldsymbol{Q}}^c) \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot (\dot{\boldsymbol{Q}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{Q}}^c) \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot (\boldsymbol{Q} \cdot \dot{\boldsymbol{A}}^c + \dot{\boldsymbol{Q}} \cdot \boldsymbol{A}^c) \\
&= \dot{\boldsymbol{A}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \dot{\boldsymbol{Q}}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{Q}}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{A}}^c + \\
&\qquad \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{A}^c \\
&= \dot{\boldsymbol{A}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \dot{\boldsymbol{Q}}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{Q}}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{A}^c \\
&= \dot{\boldsymbol{S}} \qquad \rightarrow \quad \dot{\boldsymbol{S}} = \text{invariant for rigid rotation}
\end{aligned}$$

The time derivative of $\boldsymbol{S}$ can also be expressed in the Cauchy stress tensor and its rate. As a short notation the Cauchy stress rate $\overset{\odot}{\boldsymbol{\sigma}}$ is introduced, which is a function of $\dot{\boldsymbol{\sigma}}$, $\boldsymbol{A}$ and $\dot{\boldsymbol{A}}$. This tensor has the same transformation upon rigid body rotation than the Cauchy stress tensor $\boldsymbol{\sigma}$.

$$\begin{aligned}
\boldsymbol{S} &= \boldsymbol{A}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{A}^c \\
\dot{\boldsymbol{S}} &= \dot{\boldsymbol{A}}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{A}^c + \boldsymbol{A}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{A}^c + \boldsymbol{A}\cdot\boldsymbol{\sigma}\cdot\dot{\boldsymbol{A}}^c \\
&= \boldsymbol{A}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma}\cdot\boldsymbol{A}^c + \boldsymbol{A}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{A}^c + \boldsymbol{A}\cdot\boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c\cdot\boldsymbol{A}^c \\
&= \boldsymbol{A}\cdot\left\{(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c + \dot{\boldsymbol{\sigma}}\right\}\cdot\boldsymbol{A}^c = \boldsymbol{A}\cdot\overset{\odot}{\boldsymbol{\sigma}}\cdot\boldsymbol{A}^c
\end{aligned}$$

$$\overset{\odot}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + (\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c$$

$$\begin{aligned}
\overset{\odot}{\boldsymbol{\sigma}}{}^* &= \dot{\boldsymbol{\sigma}}^* + (\boldsymbol{A}^{-1*}\cdot\dot{\boldsymbol{A}}^*)\cdot\boldsymbol{\sigma}^* + \boldsymbol{\sigma}^*\cdot(\boldsymbol{A}^{-1*}\cdot\dot{\boldsymbol{A}}^*)^c \\
&\qquad \boldsymbol{A}^* = \boldsymbol{A}\cdot\boldsymbol{Q}^c \quad\rightarrow\quad \boldsymbol{A}^{*-1} = \boldsymbol{A}^{-1*} = \boldsymbol{Q}\cdot\boldsymbol{A}^{-1} \\
&\qquad \dot{\boldsymbol{A}}^* = \dot{\boldsymbol{A}}\cdot\boldsymbol{Q}^c + \boldsymbol{A}\cdot\dot{\boldsymbol{Q}}^c \\
&\qquad \boldsymbol{A}^{-1*}\cdot\dot{\boldsymbol{A}}^* = \boldsymbol{Q}\cdot\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}}\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\dot{\boldsymbol{Q}}^c \\
&= \dot{\boldsymbol{\sigma}}^* + \boldsymbol{Q}\cdot\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}}\cdot\boldsymbol{Q}^c\cdot\boldsymbol{\sigma}^* + \boldsymbol{Q}\cdot\dot{\boldsymbol{Q}}^c\cdot\boldsymbol{\sigma}^* + \\
&\quad\ \boldsymbol{\sigma}^*\cdot\boldsymbol{Q}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c\cdot\boldsymbol{Q}^c + \boldsymbol{\sigma}^*\cdot\dot{\boldsymbol{Q}}\cdot\boldsymbol{Q}^c \\
&= \dot{\boldsymbol{Q}}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\dot{\boldsymbol{Q}}^c + \\
&\quad\ \boldsymbol{Q}\cdot\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\dot{\boldsymbol{Q}}^c\cdot\boldsymbol{Q}^c\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^c + \\
&\quad\ \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^c\cdot\dot{\boldsymbol{Q}}\cdot\boldsymbol{Q}^c \\
&= \boldsymbol{Q}\cdot[\dot{\boldsymbol{\sigma}} + (\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c]\cdot\boldsymbol{Q}^c \\
&= \boldsymbol{Q}\cdot\overset{\odot}{\boldsymbol{\sigma}}\cdot\boldsymbol{Q}^c \qquad\quad \rightarrow \quad \overset{\odot}{\boldsymbol{\sigma}} = \text{objective}
\end{aligned}$$

8.4 Objective rates and associated tensors

The tensor $\boldsymbol{A}$ is now specified, which results in some alternative invariant stress tensors. With each tensor a so-called objective rate of the Cauchy stress tensor is associated. choosing $\boldsymbol{A} \in \{\boldsymbol{F}^{-1}, \boldsymbol{Q}^{-1}, \boldsymbol{F}^c, \boldsymbol{R}^c\}$ results in the Truesdell, Jaumann, Cotter-Rivlin and Dienes tensor and rate.

general tensor $\quad \boldsymbol{S} = \boldsymbol{\sigma}_O = \boldsymbol{A}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{A}^c$

$\quad \dot{\boldsymbol{S}} = \dot{\boldsymbol{\sigma}}_O = \boldsymbol{A}\cdot\overset{\odot}{\boldsymbol{\sigma}}_O\cdot\boldsymbol{A}^c$

general rate $\quad \overset{\odot}{\boldsymbol{\sigma}}_O = \dot{\boldsymbol{\sigma}} + (\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c$

Truesdell tensor $\quad \boldsymbol{\sigma}_T = \boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-c}$

$\quad \dot{\boldsymbol{\sigma}}_T = \boldsymbol{F}^{-1}\cdot\overset{\odot}{\boldsymbol{\sigma}}_T\cdot\boldsymbol{F}^{-c}$

Truesdell rate $\quad \overset{\odot}{\boldsymbol{\sigma}}_T = \overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{L}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\boldsymbol{L}^c$

Jaumann tensor	$\boldsymbol{\sigma}_J = \boldsymbol{Q}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^{-c} \quad \text{with} \quad \dot{\boldsymbol{Q}} = \boldsymbol{\Omega} \cdot \boldsymbol{Q}$
	$\dot{\boldsymbol{\sigma}}_J = \boldsymbol{Q}^{-1} \cdot \overset{\odot}{\boldsymbol{\sigma}}_J \cdot \boldsymbol{Q}^{-c}$
Jaumann rate	$\overset{\odot}{\boldsymbol{\sigma}}_J = \overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\Omega}^c$
Cotter-Rivlin tensor	$\boldsymbol{\sigma}_C = \boldsymbol{F}^c \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}$
	$\dot{\boldsymbol{\sigma}}_C = \boldsymbol{F}^c \cdot \overset{\odot}{\boldsymbol{\sigma}}_C \cdot \boldsymbol{F}$
Cotter-Rivlin rate	$\overset{\odot}{\boldsymbol{\sigma}}_C = \overset{\triangle}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{L}^c \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \boldsymbol{L}$
Dienes tensor	$\boldsymbol{\sigma}_D = \boldsymbol{R}^c \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R} \quad \text{with} \quad \boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$
	$\dot{\boldsymbol{\sigma}}_D = \boldsymbol{R}^c \cdot \overset{\odot}{\boldsymbol{\sigma}}_D \cdot \boldsymbol{R}$
Dienes rate	$\overset{\odot}{\boldsymbol{\sigma}}_D = \overset{\diamond}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^c) \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot (\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^c)^c$

Chapter 9

Linear elastic material

For linear elastic material behavior the stress tensor $\boldsymbol{\sigma}$ is related to the linear strain tensor $\boldsymbol{\varepsilon}$ by the constant fourth-order stiffness tensor ${}^4\boldsymbol{C}$:

$$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}$$

The relevant components of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ w.r.t. an orthonormal vector basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are stored in columns $\underset{\approx}{\sigma}$ and $\underset{\approx}{\varepsilon}$. Note that we use double "waves" to indicate that the columns contain components of a second-order tensor.

$$\begin{aligned}
\underset{\approx}{\sigma}^T &= [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{21}\ \sigma_{23}\ \sigma_{32}\ \sigma_{31}\ \sigma_{13}] \\
\underset{\approx}{\varepsilon}^T &= [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \varepsilon_{12}\ \varepsilon_{21}\ \varepsilon_{23}\ \varepsilon_{32}\ \varepsilon_{31}\ \varepsilon_{13}]
\end{aligned}$$

The relation between these columns is given by the 9×9 matrix $\underline{\underline{C}}$, which stores the components of ${}^4\boldsymbol{C}$ and is referred to as the material stiffness matrix. Note again the use of double underscore to indicate that the matrix contains components of a fourth-order tensor.

$$\begin{array}{lll}
\text{tensor notation} & \boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} & \\
\text{index notation} & \sigma_{ij} = C_{ijkl}\varepsilon_{lk} \quad ; & i,j,k,l \in \{1,2,3\} \\
\text{matrix notation} & \underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon} &
\end{array}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\
C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\
C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\
C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\
C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\
C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\
C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\
C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\
C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331}
\end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

The stored energy per unit of volume is :

$$W = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} = \left[\tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}\right]^c = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C}^c : \boldsymbol{\varepsilon}$$

which implies that ${}^4\boldsymbol{C}$ is total-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^c$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^T$.

As the stress tensor is symmetric, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^c$, the tensor ${}^4\boldsymbol{C}$ must be left-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^{lc}$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^{LT}$. As also the strain tensor is symmetric, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^c$, the constitutive relation can be written with a 6×6 stiffness matrix.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

$$\text{specific energy} \qquad W = \tfrac{1}{2}\underset{\approx}{\varepsilon}^T \underline{\underline{C}}\, \underset{\approx}{\varepsilon} \quad \rightarrow \quad \text{symmetry} \qquad \boxed{\underline{\underline{C}} = \underline{\underline{C}}^T}$$

Symmetric stresses

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

Symmetric strains

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

$$\boxed{\varepsilon_{ij} = \varepsilon_{ji}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & [C_{1121} + C_{1112}] & [C_{1132} + C_{1123}] & [C_{1113} + C_{1131}] \\ C_{2211} & C_{2222} & C_{2233} & [C_{2221} + C_{2212}] & [C_{2232} + C_{2223}] & [C_{2213} + C_{2231}] \\ C_{3311} & C_{3322} & C_{3333} & [C_{3321} + C_{3312}] & [C_{3332} + C_{3323}] & [C_{3313} + C_{3331}] \\ C_{1211} & C_{1222} & C_{1233} & [C_{1221} + C_{1212}] & [C_{1232} + C_{1223}] & [C_{1213} + C_{1231}] \\ C_{2311} & C_{2322} & C_{2333} & [C_{2321} + C_{2312}] & [C_{2332} + C_{2323}] & [C_{2313} + C_{2331}] \\ C_{3111} & C_{3122} & C_{3133} & [C_{3121} + C_{3112}] & [C_{3132} + C_{3123}] & [C_{3113} + C_{3131}] \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Symmetric material parameters

The components of $\underline{\underline{C}}$ must be determined experimentally, by prescribing strains and measuring stresses and vice versa. It is clear that only the summation of the components in the 4th, 5th and 6th columns can be determined and for that reason, it is assumed that the stiffness tensor is right-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^{rc}$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^{RT}$.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & [C_{1121} + C_{1112}] & [C_{1132} + C_{1123}] & [C_{1113} + C_{1131}] \\ C_{2211} & C_{2222} & C_{2233} & [C_{2221} + C_{2212}] & [C_{2232} + C_{2223}] & [C_{2213} + C_{2231}] \\ C_{3311} & C_{3322} & C_{3333} & [C_{3321} + C_{3312}] & [C_{3332} + C_{3323}] & [C_{3313} + C_{3331}] \\ C_{1211} & C_{1222} & C_{1233} & [C_{1221} + C_{1212}] & [C_{1232} + C_{1223}] & [C_{1213} + C_{1231}] \\ C_{2311} & C_{2322} & C_{2333} & [C_{2321} + C_{2312}] & [C_{2332} + C_{2323}] & [C_{2313} + C_{2331}] \\ C_{3111} & C_{3122} & C_{3133} & [C_{3121} + C_{3112}] & [C_{3132} + C_{3123}] & [C_{3113} + C_{3131}] \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\boxed{C_{ijkl} = C_{ijlk}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Shear strain

To restore the symmetry of the stiffness matrix, the factor 2 in the last three columns is swapped to the column with the strain components. The shear components are replaced by the shear strains : $2\varepsilon_{ij} = \gamma_{ij}$. This leads to a symmetric stiffness matrix $\underline{\underline{C}}$ with 21 independent components.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\boxed{2\varepsilon_{ij} = \gamma_{ij}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

9.1 Material symmetry

Almost all materials have some material symmetry, originating from the micro structure, which implies that the number of independent material parameters is reduced. The following names refer to increasing material symmetry and thus to decreasing number of material parameters :

To reduce the number of elasticity parameters, we assume a coordinate system attached to the symmetry axes or planes in the material.

monoclinic → orthotropic → quadratic → transversal isotropic → cubic → isotropic

9.1.1 Triclinic

In a triclinic material there is no symmetry. Therefore there are 21 material parameters to be determined from independent experimental test setups. This is practically not feasible.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

$\boxed{21 \quad \text{material parameters}}$

9.1.2 Monoclinic

In each material point of a monoclinic material there is one symmetry plane, which we take here to be the 12-plane. Strain components w.r.t. two vector bases $\underset{\sim}{\vec{e}} = [\vec{e}_1 \; \vec{e}_2 \; \vec{e}_3]^T$ and $\underset{\sim}{\vec{e}}^* = [\vec{e}_1 \; \vec{e}_2 \; -\vec{e}_3]^T$ must result in the same stresses. It can be proved that all components of the stiffness matrix, with an odd total of the index 3, must be zero. This implies :

$$C_{2311} = C_{2322} = C_{2333} = C_{2321} = C_{3111} = C_{3122} = C_{3133} = C_{3121} = 0$$

A monoclinic material is characterized by 13 material parameters. In the figure the directions with equal properties are indicated with an equal number of lines.

Monoclinic symmetry is found in e.g. gypsum ($CaSO_4 2H_2O$).

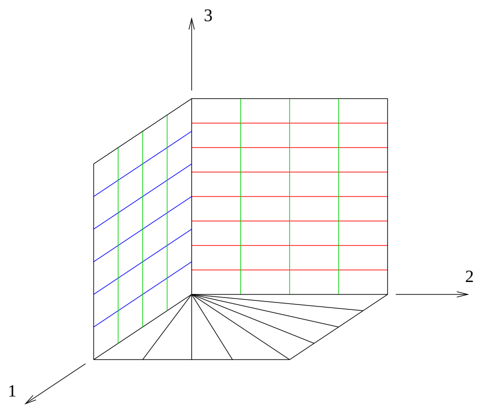


Fig. 9.1 : *One symmetry plane for monoclinic material symmetry*

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & 0 & 0 \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2332} & C_{2313} \\ 0 & 0 & 0 & 0 & C_{3132} & C_{3113} \end{bmatrix}$$

$\boxed{13 \quad \text{material parameters}}$

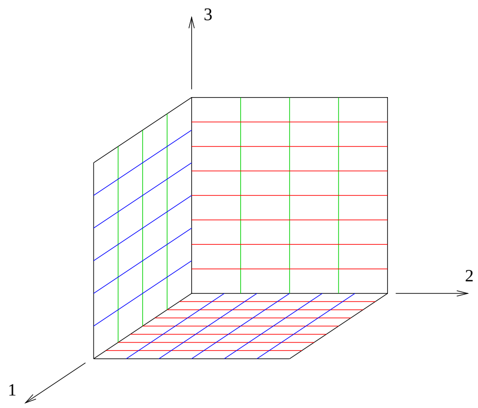
9.1.3 Orthotropic

In a point of an orthotropic material there are three symmetry planes which are perpendicular. We choose them here to coincide with the Cartesian coordinate planes. In addition to the implications for monoclinic symmetry, we can add the requirements

$$C_{1112} = C_{2212} = C_{3312} = C_{3123} = 0$$

An orthotropic material is characterized by 9 material parameters. In the stiffness matrix, they are now indicated as A, B, C, Q, R, S, K, L and M.

Orthotropic symmetry is found in orthorhombic crystals (e.g. cementite, Fe_3C) and in composites with fibers in three perpendicular directions.



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix}$$

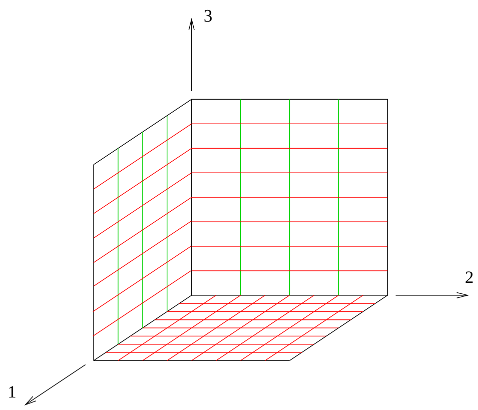
$\boxed{9 \quad \text{material parameters}}$

Fig. 9.2 : *Three symmetry planes for orthotropic material symmetry*

9.1.4 Quadratic

If in an orthotropic material the properties in two of the three symmetry planes are the same, the material is referred to as quadratic. Here we assume the behavior to be identical in the $\vec{e}_1$- and the $\vec{e}_2$-directions, however there is no isotropy in the 12-plane. This implies : $A = B$, $S = R$ and $M = L$. Only 6 material parameters are needed to describe the mechanical material behavior.

Quadratic symmetry is found in tetragonal crystals e.g. TiO_2 and white tin $Sn\beta$.



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$\boxed{6 \quad \text{material parameters}}$

Fig. 9.3 : *Quadratic material*

9.1.5 Transversal isotropic

When the material behavior in the 12-plane is isotropic, an additional relation between parameters can be deduced. To do this, we consider a pure shear deformation in the 12-plane, where a shear stress τ leads to a shear γ. The principal stress and strain directions coincide due to the isotropic behavior in the plane. In the principal directions the relation between principal stresses and strains follow from the material stiffness matrix.

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \quad \rightarrow \det(\underline{\sigma} - \sigma \underline{I}) = 0 \ \rightarrow \begin{cases} \sigma_1 = \tau \\ \sigma_2 = -\tau \end{cases}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} \rightarrow \det(\underline{\varepsilon} - \varepsilon \underline{I}) = 0 \ \rightarrow \begin{cases} \varepsilon_1 = \frac{1}{2}\gamma \\ \varepsilon_2 = -\frac{1}{2}\gamma \end{cases}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} A & Q \\ Q & A \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \rightarrow \quad \begin{array}{l} \sigma_1 = A\varepsilon_1 + Q\varepsilon_2 = \ \ \tau = \ \ K\gamma \\ \sigma_2 = Q\varepsilon_1 + A\varepsilon_2 = -\tau = -K\gamma \end{array} \quad \rightarrow$$

$$\left. \begin{array}{l} (A-Q)(\varepsilon_1 - \varepsilon_2) = 2K\gamma \\ \\ \varepsilon_1 = \frac{1}{2}\gamma \quad ; \quad \varepsilon_1 = -\frac{1}{2}\gamma \end{array} \right\} \quad \rightarrow \quad \boxed{K = \tfrac{1}{2}(A-Q)}$$

Examples of transversal isotropy are found in hexagonal crystals (CHP, Zn, Mg, Ti) and honeycomb composites. The material behavior of these materials can be described with 5 material parameters.

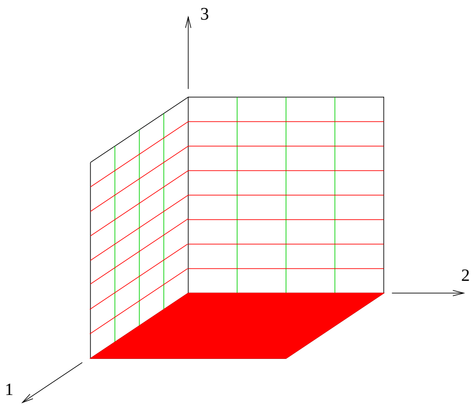


Fig. 9.4 : *Transversal material*

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$K = \tfrac{1}{2}(A-Q)$$

$$\boxed{5 \quad \text{material parameters}}$$

9.1.6 Cubic

In the three perpendicular material directions the material properties are the same. In the symmetry planes there is no isotropic behavior. Only 3 material parameters remain.

Examples of cubic symmetry are found in BCC and FCC crystals (e.g. in Ag, Cu, Au, Fe, NaCl).

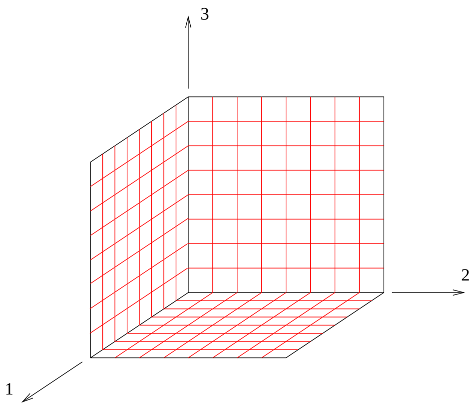


Fig. 9.5 : *Cubic material*

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$L \neq \tfrac{1}{2}(A - Q)$$

3	material parameters

9.1.7 Isotropic

In all three directions the properties are the same and in each plane the properties are isotropic. Only 2 material parameters remain.

Isotropic material behavior is found for materials having a microstructure, which is sufficiently randomly oriented and distributed on a very small scale. This applies to metals with a randomly oriented polycrystalline structure, ceramics with a random granular structure and composites with random fiber/particle orientation.

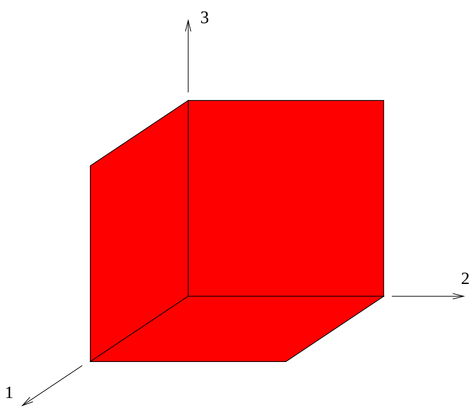


Fig. 9.6 : *Isotropic material*

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$L = \tfrac{1}{2}(A - Q)$$

2	material parameters

Engineering parameters

In engineering practice the linear elastic material behavior is characterized by Young's moduli, shear moduli and Poisson ratios. They have to be measured in tensile and shear experiments. In this section these parameters are introduced for an isotropic material.

For orthotropic and transversal isotropic material, the stiffness and compliance matrices, expressed in engineering parameters, can be found in appendix D.

To express the material constants A, Q and L in the parameters E, ν and G, two simple tests are considered : a tensile test along the 1-axis and a shear test in the 13-plane.

In a tensile test the contraction strain ε_d and the axial stress σ are related to the axial strain ε. The expressions for A, Q and L result after some simple mathematics.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \quad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon & \varepsilon_d & \varepsilon_d & 0 & 0 & 0 \end{bmatrix} \; ; \; \underset{\approx}{\sigma}^T = \begin{bmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \sigma = A\varepsilon + 2Q\varepsilon_d \\ 0 = Q\varepsilon + (A+Q)\varepsilon_d \rightarrow \varepsilon_d = -\dfrac{Q}{A+Q}\varepsilon \end{array}\right\} \rightarrow \sigma = \dfrac{A^2 + AQ - 2Q^2}{A+Q}\varepsilon \\ \varepsilon_d = -\nu\varepsilon \qquad ; \qquad \sigma = E\varepsilon \end{array}\right\} \rightarrow$$

$$A = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \qquad Q = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad L = \frac{E}{2(1+\nu)}$$

When we analyze a shear test, the relation between the shear strain γ and the shear stress τ is given by the shear modulus G. For isotropic material G is a function of E and ν. For non-isotropic materials, the shear moduli are independent parameters.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \quad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \; ; \; \underset{\approx}{\sigma}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \tau \end{bmatrix}$$

$$\tau = L\gamma = \frac{E}{2(1+\nu)}\gamma = G\gamma$$

For an isotropic material, a hydrostatic stress will only result in volume change. The relation between the volume strain and the hydrostatic stress is given by the bulk modulus K, which is a function of E and ν.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \quad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} J - 1 \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} &= \frac{1-2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &= \frac{3(1-2\nu)}{E}\,\tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{K}\,\tfrac{1}{3}\mathrm{tr}(\underline{\sigma}) \end{aligned}$$

The compliance and stiffness matrices for isotropic material can now be fully written in terms of the Young's modulus ad the Poisson's ratio.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

$$\text{with} \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}$$

Besides Young's modulus, shear modulus, bulk modulus and Poisson ratio in some formulations the so-called Lamé coefficients λ and μ are used, where $\mu = G$ and λ is a function of E and ν. The next tables list the relations between all these parameters.

	E,ν	λ,G	K,G	E,G	E,K
E	E	$\frac{(2G+3\lambda)G}{\lambda+G}$	$\frac{9KG}{3K+G}$	E	E
ν	ν	$\frac{\lambda}{2(\lambda+G)}$	$\frac{3K-2G}{2(3K+G)}$	$\frac{E-2G}{2G}$	$\frac{3K-E}{6K}$
G	$\frac{E}{2(1+\nu)}$	G	G	G	$\frac{3KE}{9K-E}$
K	$\frac{E}{3(1-2\nu)}$	$\frac{3\lambda+2G}{3}$	K	$\frac{EG}{3(3G-E)}$	K
λ	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	λ	$\frac{3K-2G}{3}$	$\frac{G(E-2G)}{3G-E}$	$\frac{3K(3K-E)}{9K-E}$

	E,λ	G,ν	λ,ν	λK	K,ν
E	E	$2G(1+\nu)$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$3K(1-2\nu)$
ν	$\frac{-E-\lambda+\sqrt{(E+\lambda)^2+8\lambda^2}}{4\lambda}$	ν	ν	$\frac{\lambda}{3K-\lambda}$	ν
G	$\frac{-3\lambda+E+\sqrt{(3\lambda-E)^2+8\lambda E}}{4}$	G	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{3(K-\lambda)}{2}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$
K	$\frac{E-3\lambda+\sqrt{(E-3\lambda)^2-12\lambda E}}{6}$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$\frac{\lambda(1+\nu)}{3\nu}$	K	K
λ	λ	$\frac{2G\nu}{1-2\nu}$	λ	λ	$\frac{3K\nu}{1+\nu}$

9.2 Isotropic material tensors

Isotropic linear elastic material behavior is characterized by only two independent material constants, for which we can choose Young's modulus E and Poisson's ratio ν. The isotropic material law can be written in tensorial form, where $\boldsymbol{\sigma}$ is related to $\boldsymbol{\varepsilon}$ with a fourth-order material stiffness tensor ${}^4\boldsymbol{C}$.
In column/matrix notation the strain components are related to the stress components by a 6×6 compliance matrix. Inversion leads to the 6×6 stiffness matrix, which relates strain components to stress components. It should be noted that shear strains are denoted as ε_{ij} and not as γ_{ij}, as was done before.

The stiffness matrix is written as the sum of two matrices, which can then be written in tensorial form.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\text{with} \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}$$

The stiffness matrix is rewritten as the sum of two matrices, the second of which is a unit matrix. Also the first one can be reduced to a matrix with ones and zeros only.

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{bmatrix} = \frac{E}{(1+\nu)}\begin{bmatrix}\frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0\\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0\\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix}$$

$$= \frac{E}{(1+\nu)}\left[\begin{bmatrix}\frac{(1-\nu)}{(1-2\nu)}-1 & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0\\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)}-1 & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0\\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)}-1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} + \right.$$

$$\left.\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\right]\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix}$$

$$= \left[\frac{E\nu}{(1+\nu)(1-2\nu)}\begin{bmatrix}1&1&1&0&0&0\\1&1&1&0&0&0\\1&1&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix} + \frac{E}{(1+\nu)}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\right]\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix}$$

Isotropic stiffness tensor

The first matrix is the matrix representation of the fourth-order tensor $\boldsymbol{II}$. The second matrix is the representation of the symmetric fourth-order tensor ${}^4\boldsymbol{I}^s$. The resulting fourth-order material stiffness tensor ${}^4\boldsymbol{C}$ contains two material constants c_0 and c_1. It is observed that $c_0 = \lambda$ and $c_1 = 2\mu$, where λ and μ are the Lamé coefficients introduced earlier.

$$\boldsymbol{\sigma} = \left[\frac{E\nu}{(1+\nu)(1-2\nu)}\right]\boldsymbol{I}\,\mathrm{tr}(\boldsymbol{\varepsilon}) + \left[\frac{E}{(1+\nu)}\right]\boldsymbol{\varepsilon}$$

$$
\begin{aligned}
&= Q\,\mathrm{tr}(\boldsymbol{\varepsilon})\,\boldsymbol{I} + 2L\boldsymbol{\varepsilon} \\
&= c_0\,\mathrm{tr}(\boldsymbol{\varepsilon})\,\boldsymbol{I} + c_1\,\boldsymbol{\varepsilon} \\
&= \left[c_0\boldsymbol{I}\boldsymbol{I} + c_1\,{}^4\boldsymbol{I}^s\right] : \boldsymbol{\varepsilon} \qquad \text{with} \quad {}^4\boldsymbol{I}^s = \tfrac{1}{2}({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}) \\
&= {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}
\end{aligned}
$$

Stiffness and compliance tensor

The strain and stress tensors can both be written as the sum of an hydrostatic - $(.)^h$ - and a deviatoric - $(.)^d$ - part. Doing so, the stress-strain relation can be easily inverted.

$$
\begin{aligned}
\boldsymbol{\sigma} &= {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \\
&= \left[c_0\boldsymbol{I}\boldsymbol{I} + c_1\,{}^4\boldsymbol{I}^s\right] : \boldsymbol{\varepsilon} \\
&\quad \text{with} \quad {}^4\boldsymbol{I}^s = \tfrac{1}{2}\left({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}\right) \\
&= c_0\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1\boldsymbol{\varepsilon} \\
&= c_0\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1\left\{\boldsymbol{\varepsilon}^d + \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}\right\} \\
&= (c_0 + \tfrac{1}{3}c_1)\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1\boldsymbol{\varepsilon}^d \\
&= (3c_0 + c_1)\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1\boldsymbol{\varepsilon}^d \\
&= (3c_0 + c_1)\boldsymbol{\varepsilon}^h + c_1\boldsymbol{\varepsilon}^d \\
&= \boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^d \\
&= \frac{1}{3c_0 + c_1}\,\boldsymbol{\sigma}^h + \frac{1}{c_1}\,\boldsymbol{\sigma}^d \\
&= \frac{1}{3c_0 + c_1}\,\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1}{c_1}\,\left\{\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right\} \\
&= -\,\frac{c_0}{(3c_0 + c_1)c_1}\,\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1}{c_1}\,\boldsymbol{\sigma} \\
&= \left[-\,\frac{c_0}{(3c_0 + c_1)c_1}\boldsymbol{I}\boldsymbol{I} + \frac{1}{c_1}\,{}^4\boldsymbol{I}^s\right] : \boldsymbol{\sigma} \\
&= \left[\gamma_0\boldsymbol{I}\boldsymbol{I} + \gamma_1\,{}^4\boldsymbol{I}^s\right] : \boldsymbol{\sigma} \\
&= {}^4\boldsymbol{S} : \boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= \frac{\nu E}{(1+\nu)(1-2\nu)} = Q \qquad ; \qquad c_1 = \frac{E}{1+\nu} = 2L \\
\gamma_0 &= -\frac{c_0}{(3c_0 + c_1)c_1} = -\frac{\nu}{E} = q \qquad ; \qquad \gamma_1 = \frac{1}{c_1} = \frac{1+\nu}{E} = \tfrac{1}{2}l
\end{aligned}
$$

The tensors can be written in components with respect to an orthonormal vector basis. This results in the relation between the stress and strain components, given below in index notation, where summation over equal indices is required (Einstein's convention).

$$
\begin{aligned}
\boldsymbol{\sigma} &= \left[c_0 \boldsymbol{I}\boldsymbol{I} + c_1\, {}^4\boldsymbol{I}^s\right] : \boldsymbol{\varepsilon} &
\boldsymbol{\varepsilon} &= \left[-\frac{c_0}{(3c_0+c_1)c_1}\boldsymbol{I}\boldsymbol{I} + \frac{1}{c_1}\, {}^4\boldsymbol{I}^s\right] : \boldsymbol{\sigma} \\
\sigma_{ij} &= \left[c_0\delta_{ij}\delta_{kl} + c_1\tfrac{1}{2}\left(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}\right)\right]\varepsilon_{lk} &
\varepsilon_{ij} &= \left[-\frac{c_0}{(3c_0+c_1)c_1}\,\delta_{ij}\delta_{kl} + \right. \\
&= c_0\delta_{ij}\varepsilon_{kk} + c_1\varepsilon_{ij} &
&\qquad \left. \frac{1}{c_1}\,\tfrac{1}{2}\left(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}\right)\right]\sigma_{lk} \\
&= c_1\left(\varepsilon_{ij} + \frac{c_0}{c_1}\,\delta_{ij}\varepsilon_{kk}\right) &
&= -\frac{c_0}{(3c_0+c_1)c_1}\,\delta_{ij}\sigma_{kk} + \frac{1}{c_1}\,\sigma_{ij} \\
&= \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\varepsilon_{kk}\right) &
&= \frac{1}{c_1}\left(\sigma_{ij} - \frac{c_0}{3c_0+c_1}\,\delta_{ij}\sigma_{kk}\right) \\
& & &= \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\,\delta_{ij}\sigma_{kk}\right)
\end{aligned}
$$

Specific elastic energy

The elastically stored energy per unit of volume (= the specific elastic energy) can be written as the sum of an hydrostatic and a deviatoric part. The hydrostatic part represents the specific energy associated with volume change. The deviatoric part indicates the specific energy needed for shape change.

$$
\begin{aligned}
W &= \tfrac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\sigma} : {}^4\boldsymbol{S} : \boldsymbol{\sigma} = \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : {}^4\boldsymbol{S} : (\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) \\
&= \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : \left(\gamma_0 \boldsymbol{I}\boldsymbol{I} + \gamma_1\, {}^4\boldsymbol{I}^s\right) : (\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) \\
&\qquad \gamma_0 \boldsymbol{I}\left[\boldsymbol{I} : \boldsymbol{\sigma}^h\right] = \gamma_0 \boldsymbol{I}\left[\boldsymbol{I} : \boldsymbol{I}\,\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\right] = \gamma_0 \boldsymbol{I}\left[\mathrm{tr}(\boldsymbol{\sigma})\right] = 3\gamma_0\boldsymbol{\sigma}^h \\
&\qquad \gamma_0 \boldsymbol{I}\left[\boldsymbol{I} : \boldsymbol{\sigma}^d\right] = \gamma_0\, \boldsymbol{I}\left[\mathrm{tr}(\boldsymbol{\sigma}^d)\right] = \gamma_0\, \boldsymbol{I}\left[0\right] = 0 \\
&= \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : (3\gamma_0\boldsymbol{\sigma}^h + \gamma_1\boldsymbol{\sigma}^h + \gamma_1\boldsymbol{\sigma}^d) \\
&\qquad \boldsymbol{\sigma}^h : \boldsymbol{\sigma}^h = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} : \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} = \tfrac{1}{9}\mathrm{tr}^2(\boldsymbol{\sigma})(3) = \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) \\
&\qquad \boldsymbol{\sigma}^h : \boldsymbol{\sigma}^d = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} : [\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}] = \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) - \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) = 0 \\
&= \left[\tfrac{1}{2}(\gamma_0 + \tfrac{1}{3}\,\gamma_1)\right]\mathrm{tr}^2(\boldsymbol{\sigma}) + \left[\tfrac{1}{2}\gamma_1\right]\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d \\
&= \left[\tfrac{1}{2}\frac{1-2\nu}{3E}\right]\mathrm{tr}^2(\boldsymbol{\sigma}) + \left[\tfrac{1}{2}\frac{1+\nu}{E}\right]\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{18K}\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d \\
&= W^h + W^d
\end{aligned}
$$

9.3 Thermo-elasticity

A temperature change ΔT of an unrestrained material invokes deformation. The total strain results from both mechanical and thermal effects and when deformations are small the total strain $\boldsymbol{\varepsilon}$ can be written as the sum of mechanical strains $\boldsymbol{\varepsilon}_m$ and thermal strains $\boldsymbol{\varepsilon}_T$. The thermal strains are related to the temperature change ΔT by the coefficient of thermal expansion tensor $\boldsymbol{A}$.

The stresses in terms of strains are derived by inversion of the compliance matrix $\underline{S}$. For thermally isotropic materials only the linear coefficient of thermal expansion α is relevant.

Anisotropic

$$\begin{aligned} \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_T = {}^4\boldsymbol{S} : \boldsymbol{\sigma} + \boldsymbol{A}\Delta T \quad &\rightarrow \quad \underset{\approx}{\varepsilon} = \underset{\approx}{\varepsilon}_m + \underset{\approx}{\varepsilon}_T = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \underline{\underline{A}}\,\Delta T \\ \boldsymbol{\sigma} = {}^4\boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{A}\Delta T) \quad &\rightarrow \quad \underset{\approx}{\sigma} = \underline{\underline{C}}\left(\underset{\approx}{\varepsilon} - \underline{\underline{A}}\,\Delta T\right) \end{aligned}$$

Isotropic

$$\begin{aligned} \boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \boldsymbol{\sigma} + \alpha\,\Delta T\boldsymbol{I} \quad &\rightarrow \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \alpha\,\Delta T\,\underset{\approx}{I} \\ \boldsymbol{\sigma} = {}^4\boldsymbol{C} : (\boldsymbol{\varepsilon} - \alpha\Delta T\boldsymbol{I}) \quad &\rightarrow \quad \underset{\approx}{\sigma} = \underline{\underline{C}}(\underset{\approx}{\varepsilon} - \alpha\,\Delta T\,\underset{\approx}{I}) \end{aligned}$$

For orthotropic material, this can be written in full matrix notation.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} A+Q+R \\ Q+B+S \\ R+S+C \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

9.4 Planar deformation

In many cases the state of strain or stress is planar. Both for plane strain and for plane stress, only strains and stresses in a plane are related by the material law. Here we assume that this plane is the 12-plane. For plane strain we than have $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, and for plane stress $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$. The material law for these planar situations can be derived from the general three-dimensional stress-strain relation, either from the stiffness matrix $\underline{\underline{C}}$

or from the compliance matrix $\underline{\underline{S}}$. Here the compliance and stiffness matrices are derived for the general orthotropic material. First the isothermal case is considered, subsequently planar relations are derived for thermo-elasticity. For cases with more material symmetry, the planar stress-strain relations can be simplified accordingly. The corresponding stiffness and compliance matrices can be found in appendix D, where they are specified in engineering constants.

9.4.1 Plane strain

For a plane strain state with $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, the stress σ_{33} can be expressed in the planar strains ε_{11} and ε_{22}. The material stiffness matrix $\underline{\underline{C}}_{\varepsilon}$ can be extracted directly from $\underline{\underline{C}}$. The material compliance matrix $\underline{\underline{S}}_{\varepsilon}$ has to be derived by inversion.

$$\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0 \quad \rightarrow \quad \sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22}$$

$$\tilde{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_{\varepsilon}\tilde{\varepsilon}$$

$$\tilde{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{AB - Q^2} \begin{bmatrix} B & -Q & 0 \\ -Q & A & 0 \\ 0 & 0 & \dfrac{AB - Q^2}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_{\varepsilon}\tilde{\sigma}$$

We can derive by substitution :

$$\sigma_{33} = \frac{1}{AB^2 - Q^2}\left[(BR - QS)\sigma_{11} + (AS - QR)\sigma_{22}\right]$$

Because the components of the three-dimensional compliance matrix $\underline{\underline{S}}$ are most conveniently expressed in Young's moduli, Poisson's ratios and shear moduli, this matrix is a good starting point to derive the planar matrices for specific cases. The plane strain stiffness matrix $\underline{\underline{C}}_{\varepsilon}$ must then be determined by inversion.

$$\varepsilon_{33} = 0 = r\sigma_{11} + s\sigma_{22} + c\sigma_{33} \quad \rightarrow \quad \sigma_{33} = -\frac{r}{c}\sigma_{11} - \frac{s}{c}\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} - \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \begin{bmatrix} \dfrac{r}{c} & \dfrac{s}{c} & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} ac - r^2 & qc - rs & 0 \\ qc - sr & bc - s^2 & 0 \\ 0 & 0 & kc \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underset{\approx}{S}_{\varepsilon} \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & 0 \\ -qc + rs & ac - r^2 & 0 \\ 0 & 0 & \dfrac{\Delta_s}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$$

$$= \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \underset{\approx}{C}_{\varepsilon} \underset{\sim}{\sigma}$$

We can now derive by substitution :

$$\sigma_{33} = -\frac{1}{\Delta_s} \left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

9.4.2 Plane stress

For the plane stress state, with $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$, the two-dimensional material law can be easily derived from the three-dimensional compliance matrix $\underset{\approx}{S}_{\varepsilon}$. The strain ε_{33} can be directly expressed in σ_{11} and σ_{22}. The material stiffness matrix has to be derived by inversion.

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \quad \rightarrow \quad \varepsilon_{33} = r\sigma_{11} + s\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \qquad = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underset{\approx}{S}_{\sigma} \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{ab - q^2} \begin{bmatrix} b & -q & 0 \\ -q & a & 0 \\ 0 & 0 & \dfrac{ab - q^2}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underset{\approx}{C}_{\sigma} \underset{\sim}{\varepsilon}$$

We can derive by substitution :

$$\varepsilon_{33} = \frac{1}{ab - q^2} \left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

The same relations can be derived from the three-dimensional stiffness matrix $\underset{\approx}{C}$.

$$\sigma_{33} = 0 = R\varepsilon_{11} + S\varepsilon_{22} + C\varepsilon_{33} \quad \rightarrow \quad \varepsilon_{33} = -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \begin{bmatrix} R \\ S \\ 0 \end{bmatrix} \begin{bmatrix} \frac{R}{C} & \frac{S}{C} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \frac{1}{C} \begin{bmatrix} AC - R^2 & QC - RS & 0 \\ QC - SR & BC - S^2 & 0 \\ 0 & 0 & KC \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{\Delta_c} \begin{bmatrix} BC - S^2 & -QC + RS & 0 \\ -QC + RS & AC - R^2 & 0 \\ 0 & 0 & \frac{\Delta_c}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_c = ABC - AS^2 - BR^2 - CQ^2 + 2QRS$$

$$= \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

9.4.3 Plane strain thermo-elastic

For thermo-elastic material behavior, the plane strain relations can be derived straightforwardly.

$$\begin{aligned} \sigma_{33} &= R\varepsilon_{11} + S\varepsilon_{22} - \alpha(R + S + C)\,\Delta T && (\text{from } \underline{\underline{C}}) \\ &= -\frac{r}{c}\,\sigma_{11} - \frac{s}{c}\,\sigma_{22} - \frac{\alpha}{c}\,\Delta T && (\text{from } \underline{\underline{S}}) \end{aligned}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} A + Q + R \\ B + Q + S \\ 0 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} &= \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} A + Q + R \\ B + Q + S \\ 0 \end{bmatrix} \right\} \\ &= \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} 1 + q_\varepsilon S + a_\varepsilon R \\ 1 + q_\varepsilon R + b_\varepsilon S \\ 0 \end{bmatrix} \end{aligned}$$

9.4.4 Plane stress thermo-elastic

For plane stress the thermo-elastic stress-strain relations can be derived again.

$$\begin{aligned}
\varepsilon_{33} &= r\sigma_{11} + s\sigma_{22} + \alpha\Delta T && \text{(from } \underline{\underline{S}}\text{)} \\
&= -\frac{R}{C}\,\varepsilon_{11} - \frac{S}{C}\,\varepsilon_{22} + \frac{1}{C}\,(R+S+C)\,\alpha\,\Delta T && \text{(from } \underline{\underline{C}}\text{)}
\end{aligned}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} &= \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \\
&= \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} A_\sigma + Q_\sigma \\ B_\sigma + Q_\sigma \\ 0 \end{bmatrix}
\end{aligned}$$

9.4.5 Plane strain/stress

In general we can write the stiffness and compliance matrix for planar deformation as a 3×3 matrix with components, which are specified for plane strain ($p = \varepsilon$) or plane stress ($p = \sigma$).

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} - \alpha\Delta T \begin{bmatrix} \Theta_{p1} \\ \Theta_{p2} \\ 0 \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix} + \alpha\Delta T \begin{bmatrix} \theta_{p1} \\ \theta_{p2} \\ 0 \end{bmatrix}$$

Chapter 10

Elastic limit criteria

Loading of a material body causes deformation of the structure and, consequently, strains and stresses in the material. When either strains or stresses (or both combined) become too large, the material will be damaged, which means that irreversible microstructural changes will result. The structural and/or functional requirements of the structure or product will be hampered, which is referred to as failure.

Their are several failure modes, listed in the table below, each of them associated with a failure mechanism. In the following we will only consider plastic yielding. When the stress state exceeds the yield limit, the material behavior will not be elastic any longer. Irreversible microstructural changes (crystallographic slip in metals) will cause permanent (= plastic) deformation.

failure mode	**mechanism**
plastic yielding	crystallographic slip (metals)
brittle fracture	(sudden) breakage of bonds
progressive damage	micro-cracks $\rightarrow$ growth $\rightarrow$ coalescence
fatigue	damage/fracture under cyclic loading
dynamic failure	vibration $\rightarrow$ resonance
thermal failure	creep / melting
elastic instabilities	buckling $\rightarrow$ plastic deformation

10.1 Yield function

In a one-dimensional stress state (tensile test), yielding will occur when the absolute value of the stress σ reaches the initial yield stress σ_{y0}. This can be tested with a yield criterion, where a yield function f is used. When $f < 0$ the material behaves elastically and when $f = 0$ yielding occurs. Values $f > 0$ cannot be reached.

$$f(\sigma) = \sigma^2 - \sigma_{y0}^2 = 0 \quad \rightarrow \quad g(\sigma) = \sigma^2 = \sigma_{y0}^2 = g_t = \text{limit in tensile test}$$

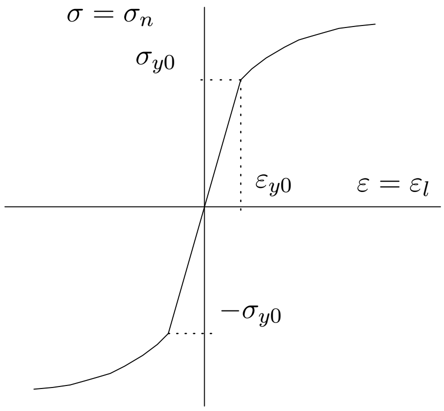


Fig. 10.1 : *Tensile curve with initial yield stress*

In a three-dimensional stress space, the yield criterion represents a yield surface. For elastic behavior ($f < 0$) the stress state is located inside the yield surface and for $f = 0$, the stress state is on the yield surface. Because $f > 0$ cannot be realized, stress states outside the yield surface can not exist. For isotropic material behavior, the yield function can be expressed in the principal stresses σ_1, σ_2 and σ_3. It can be visualized as a yield surface in the three-dimensional principal stress space.

$$\begin{array}{llll} f(\boldsymbol{\sigma}) = 0 & \rightarrow & g(\boldsymbol{\sigma}) = g_t & : \text{ yield surface in 6D stress space} \\ f(\sigma_1, \sigma_2, \sigma_3) = 0 & \rightarrow & g(\sigma_1, \sigma_2, \sigma_3) = g_t & : \text{ yield surface in 3D principal stress space} \end{array}$$

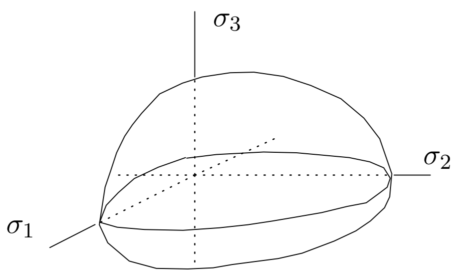


Fig. 10.2 : *Yield surface in three-dimensional principal stress space*

10.2 Principal stress space

The three-dimensional stress space is associated with a material point and has three axes, one for each principal stress value in that point. In the origin of the three-dimensional principal stress space, where $\sigma_1 = \sigma_2 = \sigma_3 = 0$, three orthonormal vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ constitute a vector base. The stress state in the material point is characterized by the principal stresses and thus by a point in stress space with "coordinates" σ_1, σ_2 and σ_3. This point can also be identified with a vector $\vec{\sigma}$, having components σ_1, σ_2 and σ_3 with respect to the vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

The hydrostatic axis, where $\sigma_1 = \sigma_2 = \sigma_3$ can be identified with a unit vector $\vec{e}_p$. Perpendicular to $\vec{e}_p$ in the $\vec{e}_1\vec{e}_p$-plane a unit vector $\vec{e}_q$ can be defined. Subsequently the unit vector $\vec{e}_r$ is defined perpendicular to the $\vec{e}_p\vec{e}_q$-plane.

The vectors $\vec{e}_q$ and $\vec{e}_r$ span the so-called Π-plane perpendicular to the hydrostatic axis. Vectors $\vec{e}_p$, $\vec{e}_q$ and $\vec{e}_r$ constitute a orthonormal vector base. A random unit vector $\vec{e}_t(\phi)$ in the Π-plane can be expressed in $\vec{e}_q$ and $\vec{e}_r$.

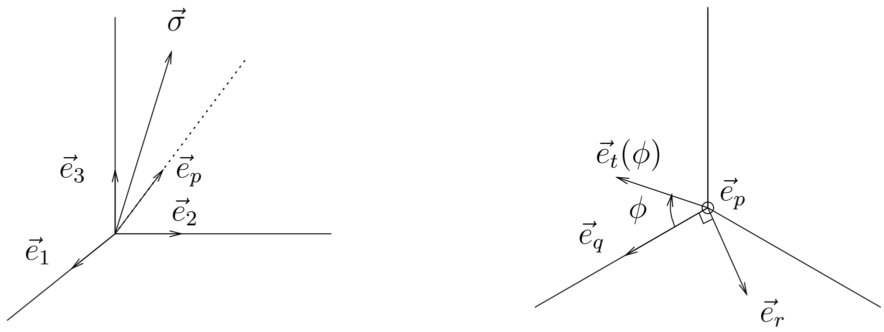


Fig. 10.3 : *Principal stress space*

hydrostatic axis $\quad \vec{e}_p = \frac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \quad \text{with} \quad ||\vec{e}_p|| = 1$

plane $\perp$ hydrostatic axis

$$\vec{e}_q^{\,*} = \vec{e}_1 - (\vec{e}_p \cdot \vec{e}_1)\vec{e}_p = \vec{e}_1 - \tfrac{1}{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) = \tfrac{1}{3}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3)$$
$$\vec{e}_q = \tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3)$$
$$\vec{e}_r = \vec{e}_p * \vec{e}_q = \tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) * \tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) = \tfrac{1}{2}\sqrt{2}(\vec{e}_2 - \vec{e}_3)$$

vector in Π-plane $\quad \vec{e}_t(\phi) = \cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r$

A stress state can be represented by a vector in the principal stress space. This vector can be written as the sum of a vector along the hydrostatic axis and a vector in the Π-plane. These vectors are referred to as the hydrostatic and the deviatoric part of the stress vector.

$$\vec{\sigma} = \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3 = \vec{\sigma}^h + \vec{\sigma}^d$$

$$
\begin{aligned}
\vec{\sigma}^h &= (\vec{\sigma}\cdot\vec{e}_p)\vec{e}_p = \sigma^h\vec{e}_p = \tfrac{1}{3}\sqrt{3}(\sigma_1+\sigma_2+\sigma_3)\vec{e}_p = \sqrt{3}\sigma_m\vec{e}_p \\
\sigma^h &= \tfrac{1}{3}\sqrt{3}(\sigma_1+\sigma_2+\sigma_3) \\
\vec{\sigma}^d &= \vec{\sigma} - (\vec{\sigma}\cdot\vec{e}_p)\vec{e}_p \\
&= \sigma_1\vec{e}_1+\sigma_2\vec{e}_2+\sigma_3\vec{e}_3 - \tfrac{1}{3}\sqrt{3}(\sigma_1+\sigma_2+\sigma_3)\tfrac{1}{3}\sqrt{3}(\vec{e}_1+\vec{e}_2+\vec{e}_3) \\
&= \sigma_1\vec{e}_1+\sigma_2\vec{e}_2+\sigma_3\vec{e}_3 - \tfrac{1}{3}(\sigma_1\vec{e}_1+\sigma_2\vec{e}_1+\sigma_3\vec{e}_1+\sigma_1\vec{e}_2+\sigma_2\vec{e}_2+\sigma_3\vec{e}_2+\sigma_1\vec{e}_3+\sigma_2\vec{e}_3+\sigma_3\vec{e}_3) \\
&= \tfrac{1}{3}\{(2\sigma_1-\sigma_2-\sigma_3)\vec{e}_1+(-\sigma_1+2\sigma_2-\sigma_3)\vec{e}_2+(-\sigma_1-\sigma_2+2\sigma_3)\vec{e}_3\} \\
\sigma^d &= ||\vec{\sigma}^d|| = \sqrt{\vec{\sigma}^d\cdot\vec{\sigma}^d} \\
&= \tfrac{1}{3}\sqrt{(2\sigma_1-\sigma_2-\sigma_3)^2+(-\sigma_1+2\sigma_2-\sigma_3)^2+(-\sigma_1-\sigma_2+2\sigma_3)^2} \\
&= \sqrt{\tfrac{2}{3}(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1)} \\
&= \sqrt{\boldsymbol{\sigma}^d:\boldsymbol{\sigma}^d}
\end{aligned}
$$

Because the stress vector in the principal stress space can also be written as the sum of three vectors along the base vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, the principal stresses can be expressed in σ^h and σ^d.

$$
\begin{aligned}
\vec{\sigma} &= \vec{\sigma}^h + \vec{\sigma}^d = \sigma^h\vec{e}_p + \sigma^d\vec{e}_t(\phi) \\
&= \sigma^h\vec{e}_p + \sigma^d\{\cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r\} \\
&= \sigma^h\tfrac{1}{3}\sqrt{3}(\vec{e}_1+\vec{e}_2+\vec{e}_3) + \sigma^d\{\cos(\phi)\tfrac{1}{6}\sqrt{6}(2\vec{e}_1-\vec{e}_2-\vec{e}_3) - \sin(\phi)\tfrac{1}{2}\sqrt{2}(\vec{e}_2-\vec{e}_3)\} \\
&= \{\tfrac{1}{3}\sqrt{3}\,\sigma^h + \tfrac{1}{3}\sqrt{6}\,\sigma^d\cos(\phi)\}\vec{e}_1 + \\
&\quad \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d\cos(\phi) - \tfrac{1}{2}\sqrt{2}\,\sigma^d\sin(\phi)\}\vec{e}_2 + \\
&\quad \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d\cos(\phi) + \tfrac{1}{2}\sqrt{2}\,\sigma^d\sin(\phi)\}\vec{e}_3 \\
&= \sigma_1\vec{e}_1+\sigma_2\vec{e}_2+\sigma_3\vec{e}_3
\end{aligned}
$$

10.3 Yield criteria

In the following sections, various yield criteria are presented. Each of them starts from a hypothesis, stating when the material will yield. Such a hypothesis is based on experimental observation and is valid for a specific (class of) material(s). The yield criteria can be visualized in several stress spaces:

- the two-dimensional (σ_1, σ_2)-space for plane stress states with $\sigma_3 = 0$,
- the three-dimensional $(\sigma_1, \sigma_2, \sigma_3)$-space,
- the Π-plane and
- the $\sigma\tau$-plane, where Mohr's circles are used.

10.3.1 Maximum stress/strain

The maximum stress/strain criterion states that

$$\boxed{\text{yielding occurs when one of the stress/strain components exceeds a limit value.}}$$

This criterion is used for orthotropic materials.

$$\sigma_{ij} = \sigma_{max} \quad | \quad \varepsilon_{ij} = \varepsilon_{max} \quad ; \quad \{i,j\} = \{1,2,3\} \qquad \text{(orthotropic materials)}$$

10.3.2 Rankine

The maximum principal stress (or Rankine) criterion states that

$$\boxed{\text{yielding occurs when the maximum principal stress reaches a limit value.}}$$

The Rankine criterion is used for brittle materials like cast iron. At failure these materials show *cleavage fracture.*

$$|\sigma_{max}| = max(|\sigma_i| \ ; \ i = 1,2,3) = \sigma_{max,t} = \sigma_{y0} \text{(brittle materials; cast iron)}$$

The figure shows the yield surface in the principal stress space for a plane stress state with $\sigma_3 = 0$.

In the three-dimensional stress space the yield surface is a cube with side-length $2\sigma_{y0}$.

In the (σ, τ)-space the Rankine criterion is visualized by to limits, which can not be exceeded by the absolute maximum of the principal stress.

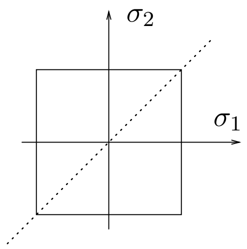


Fig. 10.4 : *Rankine yield surface in two-dimensional principal stress space*

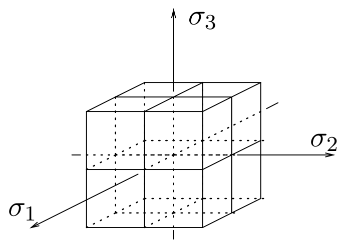


Fig. 10.5 : *Rankine yield surface in three-dimensional principal stress space*

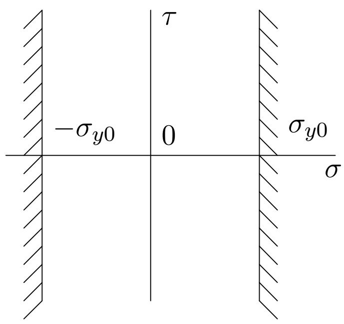


Fig. 10.6 : *Rankin yield limits in (σ, τ)-space*

10.3.3 De Saint Venant

The maximum principal strain (or De Saint Venant) criterion states that

yielding occurs when the maximum principal strain reaches a limit value.

From a tensile experiment this limit value appears to be the ratio of uni-axial yield stress and Young's modulus.

For $\sigma_1 > \sigma_2 > \sigma_3$, the maximum principal strain can be calculated from Hooke's law and its limit value can be expressed in the initial yield value σ_{yo} and Young's modulus E.

$$\varepsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 = \frac{\sigma_{y0}}{E} \quad \rightarrow \quad \sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma_{y0}$$

For other sequences of the principal stresses, relations are similar and can be used to construct the yield curve/surface in 2D/3D principal stress space.

$$\varepsilon_{max} = max(|\varepsilon_i| \ ; \ i = 1,2,3) = \varepsilon_{max_t} = \varepsilon_{y0} = \frac{\sigma_{y0}}{E}$$

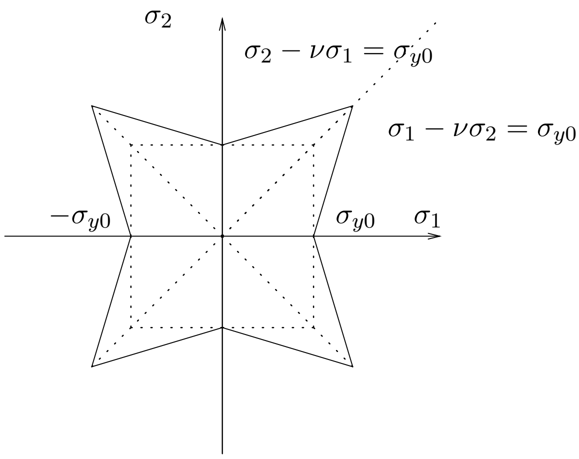


Fig. 10.7 : *Saint-Venant's yield curve in two-dimensional principal stress space*

10.3.4 Tresca

The Tresca criterion (Tresca, Coulomb, Mohr, Guest (1864)) states that

yielding occurs when the maximum shear stress reaches a limit value.

In a tensile test the limit value for the shear stress appears to be half the uni-axial yield stress.

$$\tau_{max} = \tfrac{1}{2}(\sigma_{max} - \sigma_{min}) = \tau_{max,t} = \tfrac{1}{2}\sigma_{y0} \rightarrow \bar{\sigma}_{TR} = \sigma_{max} - \sigma_{min} = \sigma_{y0}$$

Using Mohr's circles, it is easily seen how the maximum shear stress can be expressed in the maximum and minimum principal stresses.

For the plane stress case ($\sigma_3 = 0$) the yield curve in the $\sigma_1\sigma_2$-plane can be constructed using Mohr's circles. When both principal stresses are positive numbers, the yielding occurs when the largest reaches the one-dimensional yield stress σ_{y0}. When σ_1 is positive (= tensile stress), compression in the perpendicular direction, so a negative σ_2, implies that σ_1 must decrease to remain at the yield limit. Using Mohr's circles, this can easily be observed.

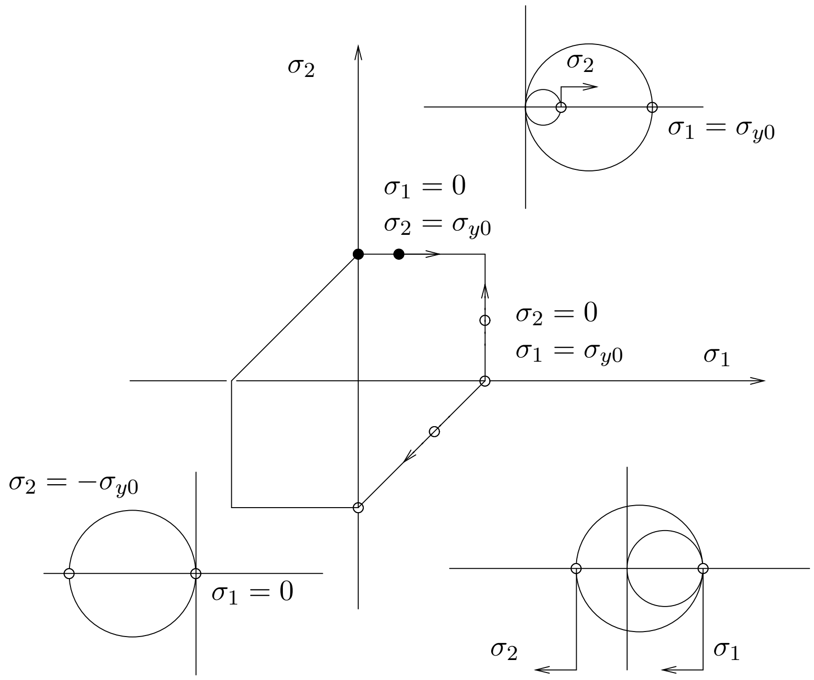


Fig. 10.8 : *Tresca yield curve in two-dimensional principal stress space*

Adding an extra hydrostatic stress state implies a translation in the three-dimensional principal stress space

$$\{\sigma_1, \sigma_2, \sigma_3\} \quad \rightarrow \quad \{\sigma_1 + c, \sigma_2 + c, \sigma_2 + c\}$$

i.e. a translation parallel to the hydrostatic axis where $\sigma_1 = \sigma_2 = \sigma_3$. This will never result in yielding or more plastic deformation, so the yield surface is a cylinder with its axis coinciding with (or parallel to) the hydrostatic axis.

In the *Π*-plane, the Tresca criterion is a regular 6-sided polygonal.

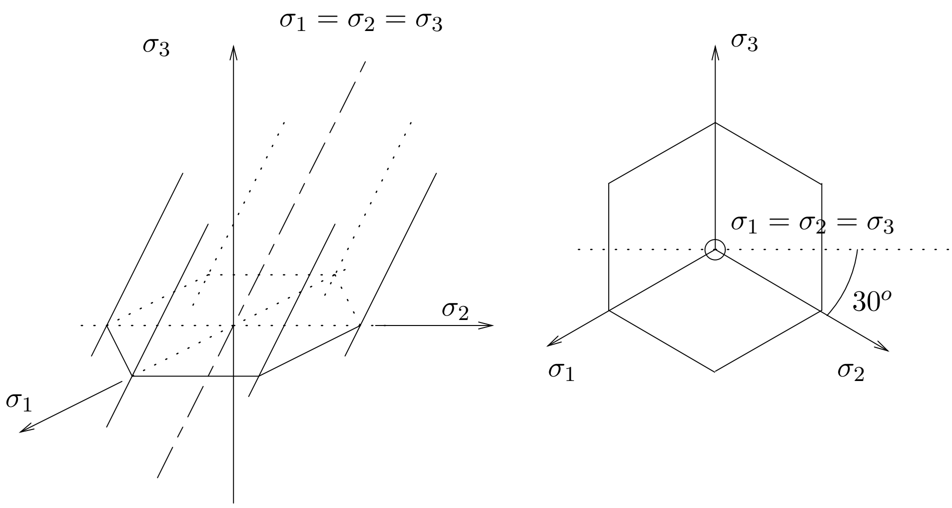


Fig. 10.9 : *Tresca yield surface in three-dimensional principal stress space and the Π-plane*

In the $\sigma\tau$-plane the Tresca yield criterion can be visualized with Mohr's circles.

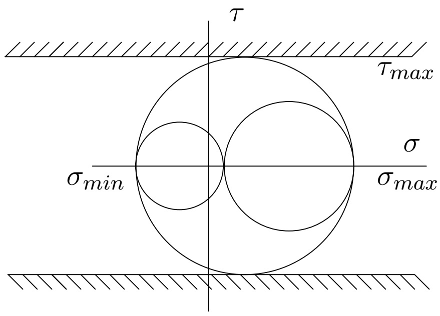


Fig. 10.10 : *Mohr's circles and Tresca yield limits in* (σ, τ)*-space*

10.3.5 Von Mises

According to the Von Mises elastic limit criterion (Von Mises, Hubert, Hencky (1918)),

> yielding occurs when the specific shape deformation elastic energy reaches a critical value.

The specific *shape deformation energy* is also referred to as *distortional energy* or *deviatoric energy* or *shear strain energy*. It can be derived by splitting up the total specific elastic energy W into a hydrostatic part W^h and a deviatoric part W^d. The deviatoric W^d can be expressed in $\boldsymbol{\sigma}^d$ and the hydrostatic W^h can be expressed in the mean stress $\sigma_m = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$. The deviatoric part can be expressed in the second invariant J_2 of the deviatoric stress tensor and in the principal stresses.

For the tensile test the shape deformation energy W_t^d can be expressed in the yield stress σ_{y0}. The Von Mises yield criterion $W^d = W_t^d$ can than be written as $\bar{\sigma}_{VM} = \sigma_{y0}$, where $\bar{\sigma}_{VM}$ is the equivalent or effective Von Mises stress, a function of all principal stresses. It is sometimes replaced by the octahedral shear stress $\tau_{oct} = \frac{1}{3}\sqrt{2}\bar{\sigma}_{VM}$.

$$W^d = W_t^d$$

$$\begin{aligned}
W^d &= \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{4G}\left\{\boldsymbol{\sigma} : \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma})\right\} \quad \left(= \; -\frac{1}{2G}\,J_2(\boldsymbol{\sigma}^d)\right) \\
&= \frac{1}{4G}\,(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{12G}\,(\sigma_1 + \sigma_2 + \sigma_3)^2 \\
&= \frac{1}{4G}\,\tfrac{1}{3}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}
\end{aligned}$$

$$W_t^d = \frac{1}{4G}\,\tfrac{1}{3}\,\left\{(\sigma - 0)^2 + (0 - 0)^2 + (0 - \sigma)^2\right\} = \frac{1}{4G}\,\tfrac{1}{3}2\sigma^2 = \frac{1}{4G}\,\tfrac{1}{3}\,2\sigma_{y0}^2$$

$$\bar{\sigma}_{VM} = \sqrt{\tfrac{1}{2}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}} = \sigma_{y0}$$

The Von Mises yield criterion can be expressed in Cartesian stress components.

$$\begin{aligned}
\bar{\sigma}_{VM}^2 &= \tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = 3J_2 \\
&= \tfrac{3}{2}\mathrm{tr}(\underline{\sigma}^d\underline{\sigma}^d) \quad \text{with } \underline{\sigma}^d = \underline{\sigma} - \tfrac{1}{3}\mathrm{tr}(\underline{\sigma})\underline{I} \\
&= \tfrac{3}{2}\Big\{\left(\tfrac{2}{3}\sigma_{xx}-\tfrac{1}{3}\sigma_{yy}-\tfrac{1}{3}\sigma_{zz}\right)^2+\sigma_{xy}^2+\sigma_{xz}^2+ \\
&\qquad\quad \left(\tfrac{2}{3}\sigma_{yy}-\tfrac{1}{3}\sigma_{zz}-\tfrac{1}{3}\sigma_{xx}\right)^2+\sigma_{yz}^2+\sigma_{yx}^2+ \\
&\qquad\quad \left(\tfrac{2}{3}\sigma_{zz}-\tfrac{1}{3}\sigma_{xx}-\tfrac{1}{3}\sigma_{yy}\right)^2+\sigma_{zx}^2+\sigma_{zy}^2\Big\} \\
&= \left(\sigma_{xx}^2+\sigma_{yy}^2+\sigma_{zz}^2\right)-\left(\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}\right)+2\left(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2\right) \\
&= \sigma_{y0}^2
\end{aligned}$$

For plane stress ($\sigma_3 = 0$), the yield curve is an ellipse in the $\sigma_1\sigma_2$-plane. The length of the principal axes of the ellipse is $\sqrt{2}\sigma_{y0}$ and $\sqrt{\frac{1}{3}}\sigma_{y0}$.

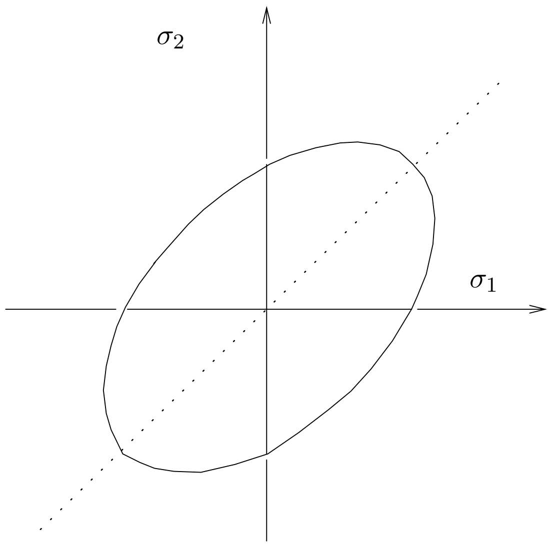


Fig. 10.11 : *Von Mises yield curve in two-dimensional principal stress space*

The three-dimensional Von Mises yield criterion is the equation of a cylindrical surface in three-dimensional principal stress space. Because hydrostatic stress does not influence yielding, the axis of the cylinder coincides with the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$.

In the Π-plane, the Von Mises criterion is a circle with radius $\sqrt{\frac{2}{3}}\sigma_{y0}$.

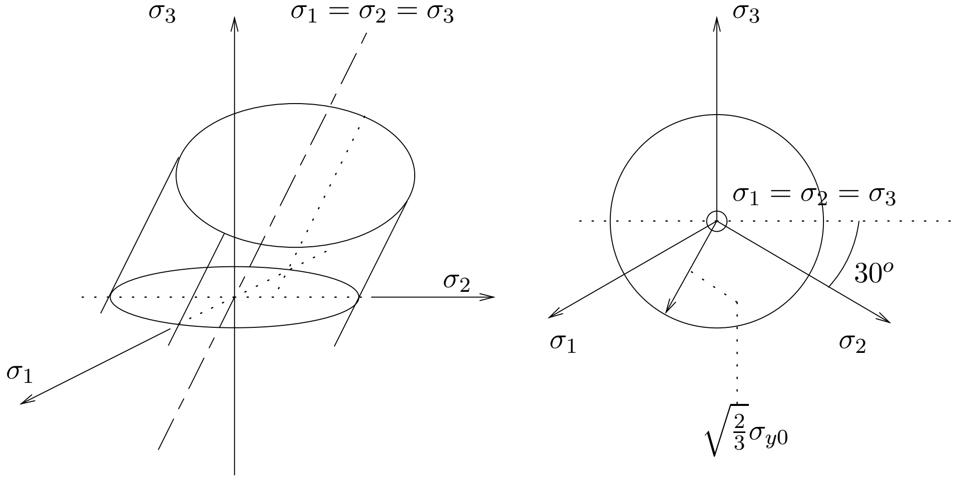


Fig. 10.12 : *Von Mises yield surface in three-dimensional principal stress space and the Π-plane*

10.3.6 Beltrami-Haigh

According to the elastic limit criterion of Beltrami-Haigh,

yielding occurs when the total specific elastic energy W reaches a critical value.

$$W = W_t$$

$$\begin{aligned} W = W^h + W^d &= \frac{1}{18K}\,\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d \\ &= \left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 + \frac{1}{4G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \\ W_t = &\left(\frac{1}{18K} - \frac{1}{12G}\right)\sigma^2 + \frac{1}{4G}\,\sigma^2 = \frac{1}{2E}\,\sigma^2 = \frac{1}{2E}\,{\sigma_{y0}}^2 \end{aligned}$$

$$2E\left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 + \frac{2E}{4G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) = \sigma_{y0}^2$$

The yield criterion contains elastic material parameters and thus depends on the elastic properties of the material. In three-dimensional principal stress space the yield surface is an ellipsoid. The longer axis coincides with (or is parallel to) the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$.

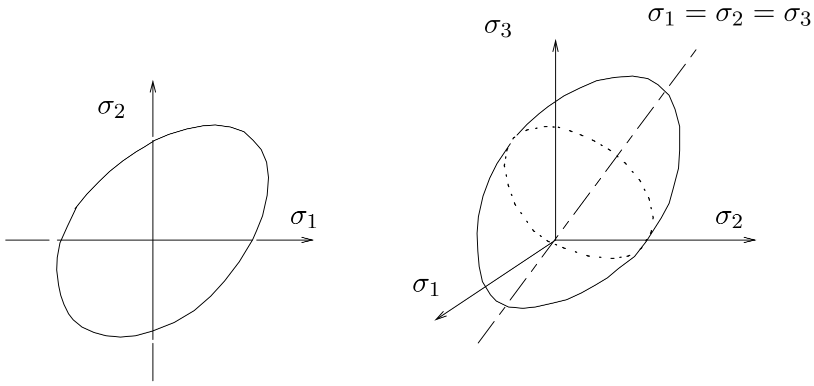


Fig. 10.13 : *Beltrami-Haigh yield curve and surface in principal stress space*

10.3.7 Mohr-Coulomb

A prominent difference in behavior under tensile and compression loading is seen in much materials, e.g. concrete, sand, soil and ceramics. In a tensile test such a material may have a yield stress σ_{ut} and in compression a yield stress σ_{uc} with $\sigma_{uc} > \sigma_{ut}$. The Mohr-Coulomb yield criterion states that

yielding occurs when the shear stress reaches a limit value.

For a plane stress state with $\sigma_3 = 0$ the yield contour in the $\sigma_1\sigma_2$-plane can be constructed in the same way as has been done for the Tresca criterion.

The yield surface in the three-dimensional principal stress space is a cone with axis along the hydrostatic axis.

The intersection with the plane $\sigma_3 = 0$ gives the yield contour for plane stress.

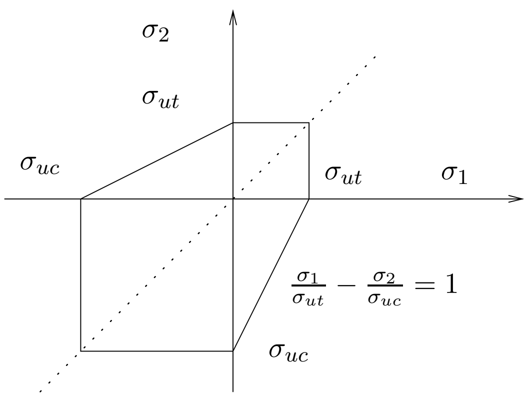


Fig. 10.14 : *Mohr-Coulomb yield curve in two-dimensional principal stress space*

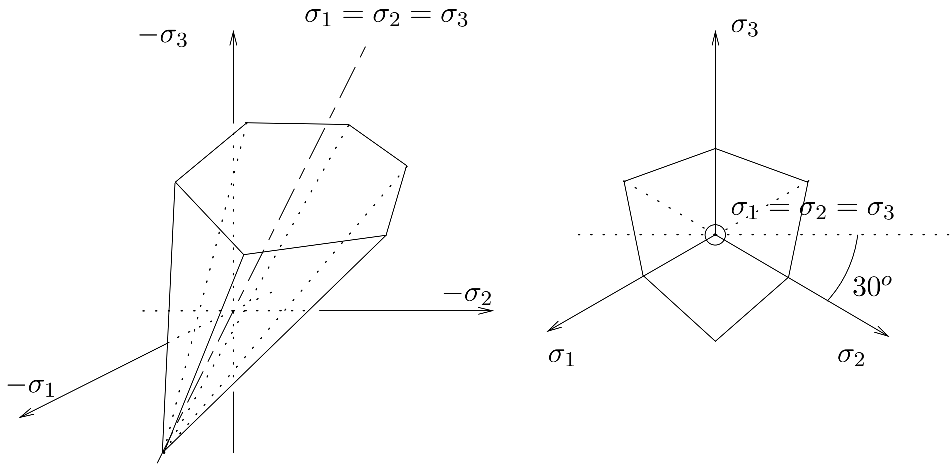


Fig. 10.15 : *Mohr-Coulomb yield surface in three-dimensional principal stress space and the Π-plane*

10.3.8 Drucker-Prager

For materials with internal friction and maximum adhesion, yielding can be described by the Drucker-Prager yield criterion. It relates to the Mohr-Coulomb criterion in the same way as the Von Mises criterion relates to the Tresca criterion.

For a plane stress state with $\sigma_3 = 0$ the Drucker-Prager yield contour in the $\sigma_1\sigma_2$-plane is a shifted ellipse.

In three-dimensional principal stress space the Drucker-Prager yield surface is a cone with circular cross-section.

$$\sqrt{\tfrac{2}{3}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} + \frac{6\sin(\phi)}{3-\sin(\phi)}\,p = \frac{6\cos(\phi)}{3-\sin(\phi)}\,C$$

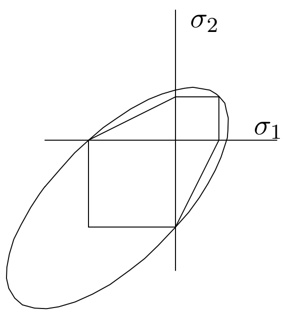


Fig. 10.16 : *Drucker-Prager yield curve in two-dimensional principal stress space*

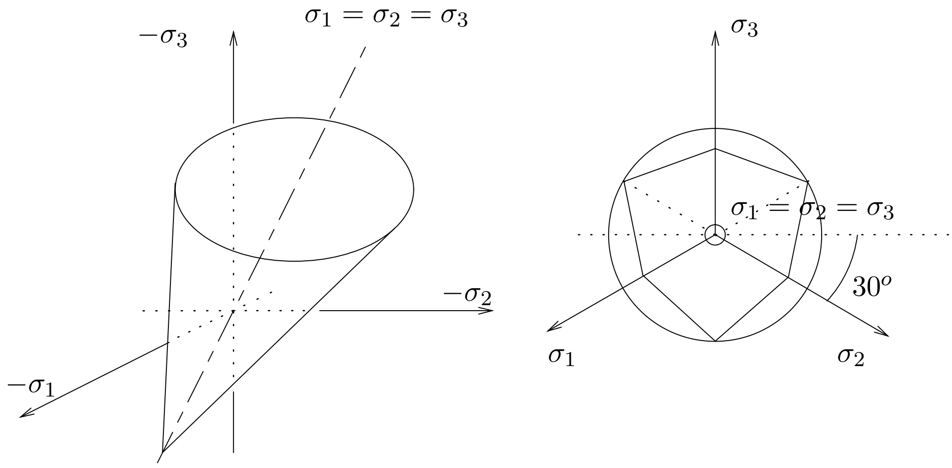


Fig. 10.17 : *Drucker-Prager yield surface in three-dimensional principal stress space and the Π-plane*

10.3.9 Other yield criteria

There are many more yield criteria, which are used for specific materials and loading conditions. The criteria of Hill, Hoffman and Tsai-Wu are used for orthotropic materials. In these criteria, there is a distinction between tensile and compressive stresses and their respective limit values.

parabolic Drucker-Prager $\left(3J_2 + \sqrt{3}\,\beta\sigma_{y0}J_1\right)^{\frac{1}{2}} = \sigma_{y0}$

Buyokozturk $\left(3J_2 + \sqrt{3}\,\beta\sigma_{y0}J_1 - 0.2J_1^2\right)^{\frac{1}{2}} = \sigma_{y0}$

Hill $\dfrac{\sigma_{11}^2}{X^2} - \dfrac{\sigma_{11}\sigma_{22}}{XY} + \dfrac{\sigma_{22}^2}{Y^2} + \dfrac{\sigma_{12}^2}{S^2}$

Hoffman

$$\left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_{11} + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_{22} + \left(\frac{1}{X_tX_c}\right)\sigma_{11}^2 + \left(\frac{1}{Y_tY_c}\right)\sigma_{22}^2 + \left(\frac{1}{S^2}\right)\sigma_{12}^2 - \left(\frac{1}{X_tX_c}\right)\sigma_{11}\sigma_{22} = 0$$

Tsai-Wu

$$\left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_{11} + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_{22} + \left(\frac{1}{X_tX_c}\right)\sigma_{11}^2 + \left(\frac{1}{Y_tY_c}\right)\sigma_{22}^2 + \left(\frac{1}{S^2}\right)\sigma_{12}^2 + 2F_{12}\,\sigma_{11}\sigma_{22} = 0$$

with $\quad F_{12}^2 > \dfrac{1}{X_tX_c}\,\dfrac{1}{Y_tY_c}$

Chapter 11

Governing equations

In this chapter we will recall the equations, which have to be solved to determine the deformation of a three-dimensional linear elastic material body under the influence of an external load. The equations will be written in component notation w.r.t. a Cartesian and a cylindrical vector base and simplified for plane strain, plane stress and axi-symmetry. The material behavior is assumed to be isotropic.

11.1 Vector/tensor equations

The deformed (current) state is determined by 12 state variables : 3 displacement components and 9 stress components. These unknown quantities must be solved from 12 equations : 6 equilibrium equations and 6 constitutive equations.

With proper boundary (and initial) conditions the equations can be solved, which, for practical problems, must generally be done numerically. The compatibility equations are generally satisfied for the chosen strain-displacement relation. In some solution approaches they are used instead of the equilibrium equations.

gradient operator	:	$\vec{\nabla} = \underset{\sim}{\nabla}^T \underset{\sim}{\vec{e}}$
position	:	$\vec{x} = \underset{\sim}{x}^T \underset{\sim}{\vec{e}}$
displacement	:	$\vec{u} = \underset{\sim}{u}^T \underset{\sim}{\vec{e}}$
strain	:	$\boldsymbol{\varepsilon} = \frac{1}{2}\left\{ \left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right) \right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$
compatibility	:	$\nabla^2 \{\mathrm{tr}(\boldsymbol{\varepsilon})\} - \vec{\nabla} \cdot (\vec{\nabla} \cdot \boldsymbol{\varepsilon})^T = 0$
stress	:	$\boldsymbol{\sigma} = \underset{\sim}{\vec{e}}^T \underline{\sigma}\, \underset{\sim}{\vec{e}}$
balance laws	:	$\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
material law	:	$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \quad ; \quad \boldsymbol{\varepsilon} = {}^4\boldsymbol{C}^{-1} : \boldsymbol{\sigma} = {}^4\boldsymbol{S} : \boldsymbol{\sigma}$
th.mech. mat. law	:	$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : (\boldsymbol{\varepsilon} - \alpha\Delta T\boldsymbol{I}) \quad : \quad \boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \boldsymbol{\sigma} + \alpha\Delta T\boldsymbol{I}$

11.2 Three-dimensional deformation

The vectors and tensors can be written in components with respect to a three-dimensional vector basis. For various problems in mechanics, it will be suitable to choose either a Cartesian coordinate system or a cylindrical coordinate system.

11.2.1 Cartesian components

The governing equations are written in components w.r.t. a Cartesian vector base $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$. The stresses can be represented with a Cartesian stress cube.

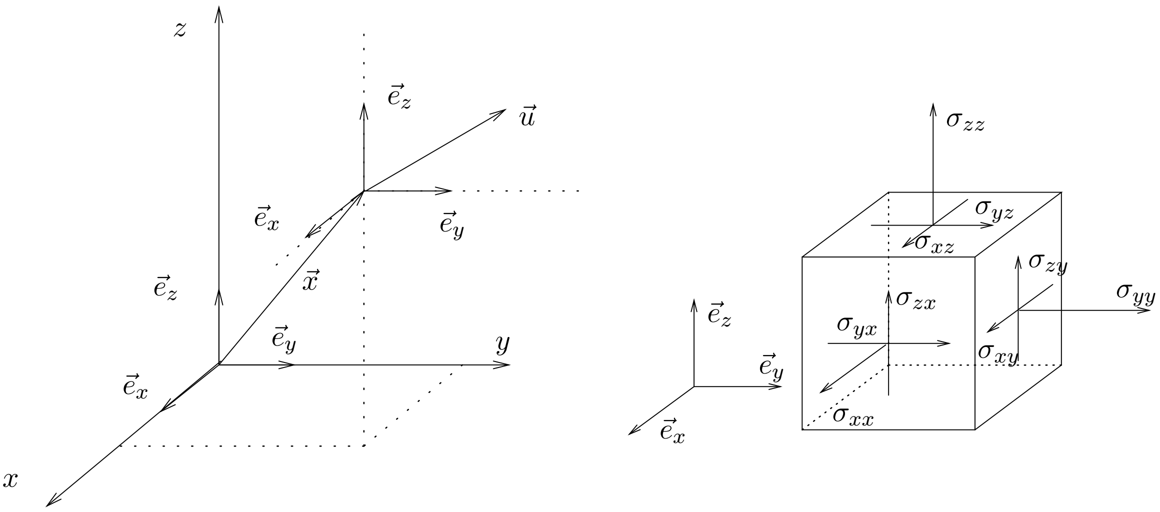


Fig. 11.1 : *Cartesian coordinate system and stress cube*

$$\underset{\sim}{x}^T = \begin{bmatrix} x & y & z \end{bmatrix} \quad ; \quad \underset{\sim}{\nabla}^T = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \quad ; \quad \underset{\sim}{u}^T = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{x,x} & u_{x,y}+u_{y,x} & u_{x,z}+u_{z,x} \\ \cdots & 2u_{y,y} & u_{y,z}+u_{z,y} \\ \cdots & \cdots & 2u_{z,z} \end{bmatrix}$$

$$\begin{aligned} &2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0 \quad \rightarrow \quad \text{cyc. 2x} \\ &\varepsilon_{xx,yz} + \varepsilon_{yz,xx} - \varepsilon_{zx,xy} - \varepsilon_{xy,xz} = 0 \quad \rightarrow \quad \text{cyc. 2x} \end{aligned}$$

$$\begin{aligned} \underset{\approx}{\varepsilon}^T &= \underset{\sim}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \varepsilon_{xy} & \varepsilon_{yz} & \varepsilon_{zx} \end{bmatrix} \\ \underset{\approx}{\sigma}^T &= \underset{\sim}{\sigma}^T = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x = \rho \ddot{u}_x \qquad && (\sigma_{xy} = \sigma_{yx}) \\ &\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y = \rho \ddot{u}_y \qquad && (\sigma_{yz} = \sigma_{zy}) \\ &\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z = \rho \ddot{u}_z \qquad && (\sigma_{zx} = \sigma_{xz}) \end{aligned}$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon}\underset{\sim}{\sigma} = \underline{\underline{C}}\,\underset{\sim}{\varepsilon} \quad ; \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}\underset{\sim}{\varepsilon} = \underline{\underline{S}}\,\underset{\sim}{\sigma}$$

11.2.2 Cylindrical components

The governing equations are written in components w.r.t. a cylindrical vector base $\{\vec{e}_r(\theta), \vec{e}_t(\theta), \vec{e}_z\}$, with :

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_t \qquad \text{and} \qquad \frac{d\vec{e}_t}{d\theta} = -\vec{e}_r$$

The stress components can be represented with a cylindrical stress cube.

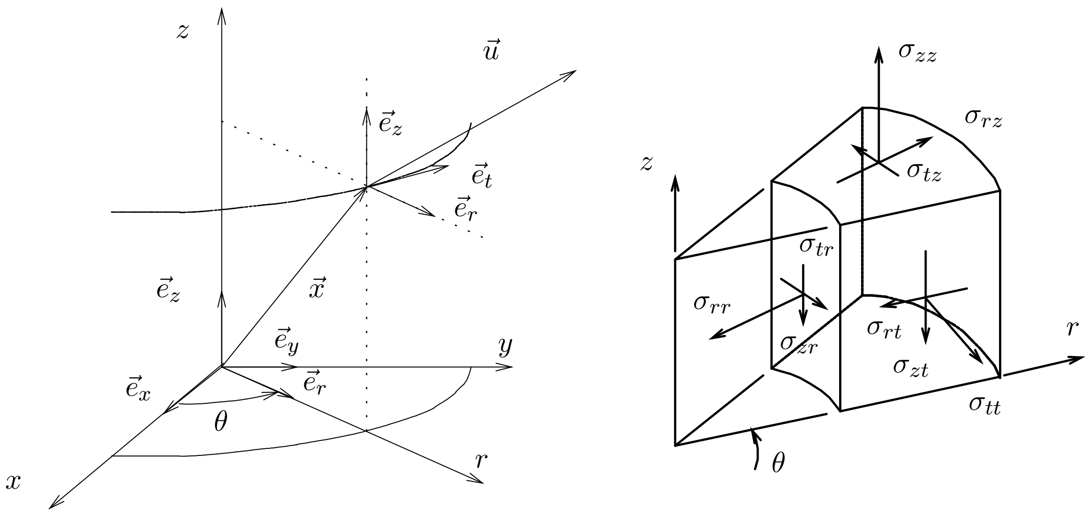


Fig. 11.2 : *Cylindrical coordinate system and stress cube*

$$\tilde{x}^T = \begin{bmatrix} r & \theta & z \end{bmatrix} \quad ; \quad \tilde{\nabla}^T = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \end{bmatrix} \quad ; \quad \tilde{u}^T = \begin{bmatrix} u_r & u_t & u_z \end{bmatrix}$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \cdots & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ \cdots & \cdots & 2u_{z,z} \end{bmatrix}$$

$$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$
$$\varepsilon_{rr,tz} + \varepsilon_{tz,rr} - \varepsilon_{zr,rt} - \varepsilon_{rt,rz} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$

$$\tilde{\tilde{\varepsilon}}^T = \tilde{\varepsilon}^T = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{tt} & \varepsilon_{zz} & \varepsilon_{rt} & \varepsilon_{tz} & \varepsilon_{zr} \end{bmatrix}$$
$$\tilde{\tilde{\sigma}}^T = \tilde{\sigma}^T = \begin{bmatrix} \sigma_{rr} & \sigma_{tt} & \sigma_{zz} & \sigma_{rt} & \sigma_{tz} & \sigma_{zr} \end{bmatrix}$$

$$\sigma_{rr,r} + \tfrac{1}{r}\,\sigma_{rt,t} + \tfrac{1}{r}\,(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = \rho\ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$
$$\sigma_{tr,r} + \tfrac{1}{r}\,\sigma_{tt,t} + \tfrac{1}{r}\,(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t = \rho\ddot{u}_t \qquad (\sigma_{tz} = \sigma_{zt})$$
$$\sigma_{zr,r} + \tfrac{1}{r}\,\sigma_{rt,t} + \tfrac{1}{r}\,\sigma_{zr} + \sigma_{rz,z} + \rho q_z = \rho\ddot{u}_z \qquad (\sigma_{zr} = \sigma_{rz})$$

$$\tilde{\tilde{\sigma}} = \underline{\underline{C}}\,\tilde{\tilde{\varepsilon}}\tilde{\sigma} = \underline{\underline{C}}\,\tilde{\varepsilon} \qquad ; \qquad \tilde{\tilde{\varepsilon}} = \underline{\underline{S}}\,\tilde{\tilde{\sigma}}\tilde{\tilde{\varepsilon}} = \underline{\underline{S}}\,\tilde{\sigma}$$

11.3 Material law

When deformations are small, every material will show linear elastic behavior. For orthotropic material there are 9 independent material constants. When there is more material symmetry, this number decreases. Finally, isotropic material can be characterized with only two material constants.

Be aware that we use now the strain components ε_{ij} and not the shear components γ_{ij}. In an earlier chapter, the parameters for orthotropic, transversally isotropic and isotropic material were rewritten in terms of engineering parameters: Young's moduli and Poisson's ratio's.

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2K & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L & 0 \\ 0 & 0 & 0 & 0 & 0 & 2M \end{bmatrix} \rightarrow \underline{\underline{S}} = \underline{\underline{C}}^{-1} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}k & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}l & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}m \end{bmatrix}$$

quadratic	$B = A \; ; \; S = R \; ; \; M = L;$
transversal isotropic	$B = A \; ; \; S = R \; ; \; M = L \; ; \; K = \frac{1}{2}(A - Q)$
cubic	$C = B = A \; ; \; S = R = Q \; ; \; M = L = K \neq \frac{1}{2}(A - Q)$
isotropic	$C = B = A \; ; \; S = R = Q \; ; \; M = L = K = \frac{1}{2}(A - Q)$

11.4 Planar deformation

In many applications the loading and deformation is in one plane. The result is that the material body is in a state of plane strain or plane stress. The governing equations can than be simplified considerably.

11.4.1 Cartesian components

In a plane strain situation, deformation in one direction – here the z-direction – is suppressed. In a plane stress situation, stresses on one plane – here the plane with normal in z-direction – are zero.

Eliminating σ_{zz} for plane strain and ε_{zz} for plane stress leads to a simplified Hooke's law. Also the equilibrium equation in the z-direction is automatically satisfied and has become obsolete.

$$\left.\begin{array}{lll} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_x = u_x(x, y) \\ u_y = u_y(x, y) \end{array}\right.$$

$$\underset{\sim}{\varepsilon}^T = \underset{\sim}{\varepsilon}^T = \left[\begin{array}{ccc} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{xy} \end{array} \right] = \left[\begin{array}{ccc} u_{x,x} & u_{y,y} & \frac{1}{2}(u_{x,y} + u_{y,x}) \end{array} \right]$$

$$2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0$$

$$\underset{\sim}{\underset{\sim}{\sigma}}^T = \underset{\sim}{\sigma}^T = \left[\ \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}\ \right]$$

$$\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \qquad\qquad (\sigma_{xy} = \sigma_{yx})$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y$$

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{bmatrix}$$

11.4.2 Cylindrical components

In a plane strain situation, deformation in one direction – here the z-direction – is suppressed. In a plane stress situation, stresses on one plane – here the plane with normal in z-direction – are zero.

Eliminating σ_{zz} for plane strain and ε_{zz} for plane stress leads to a simplified Hooke's law. Also the equilibrium equation in the z-direction is automatically satisfied and has become obsolete.

$$\left.\begin{array}{lll} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta) \end{array}\right.$$

$$\underset{\sim}{\underset{\sim}{\varepsilon}}^T = \underset{\sim}{\varepsilon}^T = \left[\ \varepsilon_{rr} \quad \varepsilon_{tt} \quad \varepsilon_{rt}\ \right] = \left[\ u_{r,r} \quad \frac{1}{r}(u_r + u_{t,t}) \quad \frac{1}{2}\left(\frac{1}{r}(u_{r,t} - u_t) + u_{t,r}\right)\ \right]$$

$$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0$$

$$\underset{\sim}{\underset{\sim}{\sigma}}^T = \underset{\sim}{\sigma}^T = \left[\ \sigma_{rr} \quad \sigma_{tt} \quad \sigma_{rt}\ \right]$$

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \qquad\qquad (\sigma_{rt} = \sigma_{tr})$$

$$\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \rho q_t = \rho \ddot{u}_t$$

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{bmatrix}$$

Axi-symmetric + $u_t = 0$

When geometry and boundary conditions are such that we have $\dfrac{\partial(\ )}{\partial\theta} = (\ )_t = 0$ the situation is referred to as being axi-symmetric.

In many cases boundary conditions are such that there is no displacement of material points in tangential direction ($u_t = 0$). In that case we have $\varepsilon_{rt} = 0 \rightarrow \sigma_{rt} = 0$

$$\left.\begin{array}{lll} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r) \\ u_t = 0 \end{array}\right.$$

$$\underset{\sim}{\underset{\sim}{\varepsilon}}^T = \underset{\sim}{\varepsilon}^T = \left[\ \varepsilon_{rr} \quad \varepsilon_{tt}\ \right] = \left[\ u_{r,r} \quad \frac{1}{r}(u_r)\ \right]$$

$$\varepsilon_{rr} = u_{r,r} = (r\varepsilon_{tt})_{,r} = \varepsilon_{tt} + r\varepsilon_{tt,r}$$

$$\underset{\approx}{\sigma}^T = \underset{\sim}{\sigma}^T = \left[\begin{array}{cc} \sigma_{rr} & \sigma_{tt} \end{array} \right]$$

$$\sigma_{rr,r} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{tt}\right) + \rho q_r = \rho \ddot{u}_r$$

$$\underline{\underline{C}}_p = \left[\begin{array}{cc} A_p & Q_p \\ Q_p & B_p \end{array} \right] \qquad ; \qquad \underline{\underline{S}}_p = \left[\begin{array}{cc} a_p & q_p \\ q_p & b_p \end{array} \right]$$

11.5 Inconsistency of plane stress

Although for plane stress the out-of-plane shear stresses must be zero, they are not, when calculated afterwards from the strains. This inconsistency is inherent to the plane stress assumption. Deviations must be small to render the assumption of plane stress valid.

$$\sigma_{xz} = 2K\varepsilon_{xz} = 2Ku_{z,x} \neq 0$$
$$\sigma_{yz} = 2K\varepsilon_{yz} = 2Ku_{z,y} \neq 0$$

Chapter 12

Solution strategies

12.1 Governing equations

The deformation of a three-dimensional continuum in three-dimensional space is described by the displacement vector $\vec{u}$ of each material point. Due to the deformation, stresses arise and the stress state is characterized by the stress tensor $\boldsymbol{\sigma}$. For static problems, this tensor has to satisfy the equilibrium equations. Solving stresses from these equations is generally not possible and additional equations are needed, which relate stresses to deformation. These constitutive equations, which describe the material behavior, relate the stress tensor $\boldsymbol{\sigma}$ to the strain tensor $\boldsymbol{\varepsilon}$, which is a function of the displacement gradient tensor $(\vec{\nabla}\vec{u})$. Components of this strain tensor cannot be independent and are related by the compatibility equations.

unknown variables		
displacements	:	$\vec{u} = \vec{u}(\vec{x}) \quad \rightarrow \quad \boldsymbol{F} = \left(\vec{\nabla}_0\vec{x}\right)^T \quad \rightarrow \quad \boldsymbol{E}\ ,\ \boldsymbol{\varepsilon}$
stresses	:	$\boldsymbol{\sigma} \quad \rightarrow \quad g(\boldsymbol{\sigma}) = g(\sigma_1, \sigma_2, \sigma_3) = g_t$
equations		
compatibility	:	$\nabla^2\left\{\mathrm{tr}(\boldsymbol{\varepsilon})\right\} - \vec{\nabla}\cdot(\vec{\nabla}\cdot\boldsymbol{\varepsilon})^T = 0$
equilibrium	:	$\vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
material law	:	$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{F}) \quad \rightarrow \quad \boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \quad \rightarrow \quad \boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \boldsymbol{\sigma}$

12.2 Boundary conditions

Some of the governing equations are partial differential equations, where differentiation is done w.r.t. the spatial coordinates. These differential equations can only be solved when proper boundary conditions are specified. In each boundary point of the material body, either the displacement or the load must be prescribed. It is also possible to specify a relation between displacement and load in such a point.

When the acceleration of the material points cannot be neglected, the equilibrium equation becomes the equation of motion, with $\rho\ddot{\vec{u}}$ as its right-hand term. In that case a solution can only be determined when proper initial conditions are prescribed, i.e. initial displacement, velocity or acceleration. In this section we will assume $\ddot{\vec{u}} = \vec{0}$.

displacement	:	$\vec{u} = \vec{u}_p \quad \forall \quad \vec{x} \in A_u$
edge load	:	$\vec{p} = \vec{n} \cdot \boldsymbol{\sigma} = \vec{p}_p \quad \forall \quad \vec{x} \in A_p$

12.2.1 Saint-Venant's principle

The so-called *Saint-Venant principle* states that, if a load on a structure is replaced by a statically equivalent load, the resulting strains and stresses in the structure will only be altered near the regions where the load is applied. With this principle in mind, the real boundary conditions can often be modeled in a simplified way. Concentrated forces can for instance be replaced by distributed loads, and vice versa. Stresses and strains will only differ significantly in the neighborhood of the boundary, where the load is applied.

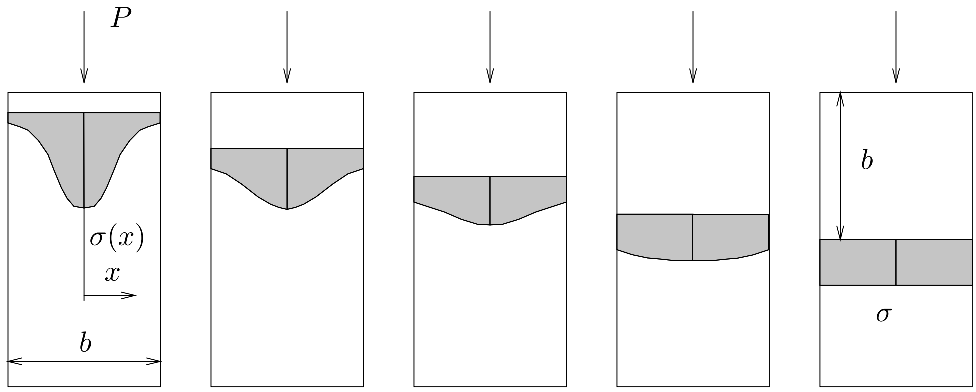


Fig. 12.1 : *Saint-Venant principle*

$$P = \int_A \sigma(x)\, dA = \sigma\, A \qquad ; \qquad A = b * t$$

12.2.2 Superposition

Under the assumption of small deformations and linear elastic material behavior, the governing equations, which must be solved to determine deformation and stresses (= solution S) are linear. When boundary conditions (fixations and loads (L)), which are needed for the solution, are also linear, the total problem is linear and the principle of superposition holds.

The principle of superposition states that the solution S for a given combined load $L = L_1 + L_2$ is the sum of the solution S_1 for load L_1 and the solution S_2 for L_2, so : $S = S_1 + S_2$.

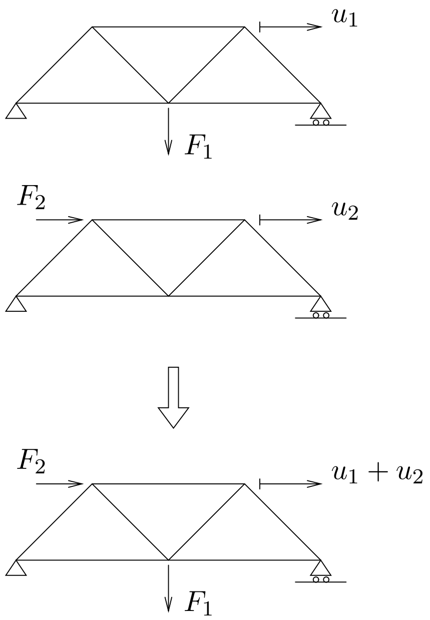


Fig. 12.2 : *Principle of superposition*

12.3 Solution : displacement method

In the *displacement method* the constitutive relation for the stress tensor is substituted in the force equilibrium equation.

Subsequently the strain tensor is replaced by its definition in terms of the displacement gradient. This results in a differential equation in the displacement $\vec{u}$, which can be solved when proper boundary conditions are specified.

In a Cartesian coordinate system the vector/tensor formulation can be replaced by index notation. It is elaborated here for the case of linear elasticity theory.

$$\left.\begin{array}{l} \vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \\ \boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l} \vec{\nabla}\cdot\left({}^4\boldsymbol{C} : \boldsymbol{\varepsilon}\right)^T + \rho\vec{q} = \rho\ddot{\vec{u}} \\ \boldsymbol{\varepsilon} = \frac{1}{2}\left\{\left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right)\right\} \end{array}\right\} \quad \rightarrow$$

$$\vec{\nabla}\cdot\left\{{}^4\boldsymbol{C} : \left(\vec{\nabla}\vec{u}\right)\right\}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \quad \rightarrow \quad \vec{u} \quad \rightarrow \quad \boldsymbol{\varepsilon} \quad \rightarrow \quad \boldsymbol{\sigma}$$

Cartesian index notation

$$\left.\begin{array}{l} \sigma_{ij,j} + \rho q_i = 0_i \\ \sigma_{ij} = C_{ijlk}\varepsilon_{lk} \end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l} C_{ijkl}\varepsilon_{lk,j} + \rho q_i = 0_i \\ \varepsilon_{lk} = \frac{1}{2}\left(u_{l,k} + u_{k,l}\right) \end{array}\right\} \quad \rightarrow$$

$$C_{ijkl}u_{l,kj} + \rho q_i = 0_i \quad \rightarrow \quad u_i \quad \rightarrow \quad \varepsilon_{ij} \quad \rightarrow \quad \sigma_{ij}$$

12.3.1 Planar, Cartesian : Navier equations

The displacement method is elaborated for planar deformation in a Cartesian coordinate system. Linear deformation and linear elastic material behavior is assumed. Elimination and substitution results in two partial differential equations for the two displacement components. For the sake of simplicity, we do not consider thermal loading here.

$$\left.\begin{array}{l}\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \qquad ; \qquad \sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y \\[2ex] \sigma_{xx} = A_p \varepsilon_{xx} + Q_p \varepsilon_{yy} \\ \sigma_{yy} = Q_p \varepsilon_{xx} + B_p \varepsilon_{yy} \\ \sigma_{xy} = 2K \varepsilon_{xy} \end{array}\right\}$$

$$\left.\begin{array}{l} A_p \varepsilon_{xx,x} + Q_p \varepsilon_{yy,x} + 2K \varepsilon_{xy,y} + \rho q_x = \rho \ddot{u}_x \\ 2K \varepsilon_{xy,x} + Q_p \varepsilon_{xx,y} + B_p \varepsilon_{yy,y} + \rho q_y = \rho \ddot{u}_y \end{array}\right\}$$

$$\left.\begin{array}{l} A_p u_{x,xx} + K u_{x,yy} + (Q_p + K) u_{y,yx} + \rho q_x = \rho \ddot{u}_x \\ K u_{y,xx} + B_p u_{y,yy} + (Q_p + K) u_{x,xy} + \rho q_y = \rho \ddot{u}_y \end{array}\right\}$$

12.3.2 Planar, axi-symmetric with $u_t = 0$

Many engineering problems present a rotational symmetry w.r.t. an axis. They are *axi-symmetric*. In many cases the tangential displacement is zero : $u_t = 0$. This implies that there are no shear strains and stresses.

The radial and tangential stresses are related to the radial and tangential strains by the planar material law. Material parameters are indicated as A_p, B_p and Q_p and can later be specified for a certain material and for plane strain or plane stress. With the strain-displacement relations the equation of motion can be transformed into a differential equation for the radial displacement u_r

$$\begin{array}{lcl} \text{displacements} & : & u_r = u_r(r) \quad ; \quad u_z = u_z(r,z) \\ \text{strains} & : & \varepsilon_{rr} = u_{r,r} \; ; \; \varepsilon_{tt} = \frac{1}{r} u_r \; ; \; \varepsilon_{zz} = u_{z,z} \\ \text{stresses} & : & \sigma_{tz} = 0 \quad ; \quad \sigma_{rz} \approx 0 \quad ; \quad \sigma_{tr} = 0 \\ \text{eq. of motion} & : & \sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \end{array}$$

$$\left.\begin{array}{l} \sigma_{rr} = A_p \varepsilon_{rr} + Q_p \varepsilon_{tt} - \Theta_{p1} \alpha \Delta T \\ \sigma_{tt} = Q_p \varepsilon_{rr} + B_p \varepsilon_{tt} - \Theta_{p2} \alpha \Delta T \end{array}\right\} \quad \rightarrow \quad \text{eq. of motion} \quad \rightarrow$$

$$u_{r,rr} + \frac{1}{r} u_{r,r} - \zeta^2 \frac{1}{r^2} u_r = f(r)$$

$$\text{with} \qquad \zeta = \sqrt{\frac{B_p}{A_p}}$$

$$\text{and} \qquad f(r) = \frac{\rho}{A_p} (\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p} \alpha (\Delta T)_{,r} + \frac{\Theta_{p1} - \Theta_{p2}}{A_p} \frac{1}{r} \alpha \Delta T$$

For isotropic material the coefficients A_p and B_p are the same, which implies that $\zeta = 1$ and also $\Theta_{p1} = \Theta_{p2}$.

$$\left.\begin{array}{l}\sigma_{rr,r}+\frac{1}{r}(\sigma_{rr}-\sigma_{tt})+\rho q_r=\rho\ddot{u}_r\\ \\ \left.\begin{array}{l}\sigma_{rr}=A_p\varepsilon_{rr}+Q_p\varepsilon_{tt}-\Theta_{p1}\alpha\Delta T\\ \sigma_{tt}=Q_p\varepsilon_{rr}+A_p\varepsilon_{tt}-\Theta_{p1}\alpha\Delta T\end{array}\right\}\end{array}\right\}$$

$$\left.\begin{array}{l}A_p\varepsilon_{rr,r}+Q_p\varepsilon_{tt,r}-\Theta_{p1}\alpha(\Delta T)_r+\\ \qquad\frac{1}{r}\left\{(A_p-Q_p)\varepsilon_{rr}+(Q_p-A_p)\varepsilon_{tt}\right\}+\rho q_r=\rho\ddot{u}_r\\ \varepsilon_{rr}=u_{r,r}\quad;\quad\varepsilon_{tt}=\frac{1}{r}u_r\end{array}\right\}$$

$$u_{r,rr}+\frac{1}{r}u_{r,r}-\frac{1}{r^2}u_r=\frac{\rho}{A_p}\left(\ddot{u}_r-q_r\right)+\frac{\Theta_{p1}}{A_p}\alpha(\Delta T)_{,r}=f(r)$$

12.4 Solution : stress method

In the *stress method*, the constitutive relation for the strain tensor is substituted in the compatibility equation, resulting in a partial differential equation for the stress tensor. This equation and the equilibrium equations constitute a set of coupled equations from which the stress tensor has to be solved.

For planar problems, this can be elaborated and results in the Beltrami-Mitchell equation for the stress components. It is again assmed that deformations are small and the material behavior is linearly elastic.

Solution of the stress equation(s) is done by introducing the so-called *Airy stress function*.

12.4.1 Beltrami-Mitchell equation : planar, Cartesian

The compatibility equation for planar deformation can be expressed in stress components, resulting in the Beltrami-Mitchell equation.

$$\left.\begin{array}{l}\varepsilon_{xx,yy}+\varepsilon_{yy,xx}=2\varepsilon_{xy,xy}\\ \\ \varepsilon_{xx}=a_p\sigma_{xx}+q_p\sigma_{yy}+\alpha\Delta T\\ \varepsilon_{yy}=q_p\sigma_{xx}+b_p\sigma_{yy}+\alpha\Delta T\\ \varepsilon_{xy}=\frac{1}{2}k\sigma_{xy}\end{array}\right\}\quad\rightarrow\quad\left\{\begin{array}{l}k\sigma_{xy,xy}=\\ \qquad a_p\sigma_{xx,yy}+q_p\sigma_{yy,yy}+\\ \qquad q_p\sigma_{xx,xx}+b_p\sigma_{yy,xx}\\ \qquad \alpha(\Delta T)_{,xx}+\alpha(\Delta T)_{,yy}\end{array}\right.$$

equilibrium $\qquad 2\sigma_{xy,xy}+\sigma_{xx,xx}+\sigma_{yy,yy}=-\rho q_{x,x}-\rho q_{y,y}$

Beltrami-Mitchell equation

$$(k+2q_p)\left(\sigma_{xx,xx}+\sigma_{yy,yy}\right)+2a_p\sigma_{xx,yy}+2b_p\sigma_{yy,xx}=-k\rho(q_{x,x}+q_{y,y})-2\alpha\left\{(\Delta T)_{xx}+(\Delta T)_{yy}\right\}$$

12.4.2 Airy stress function method

In the stress function method an Airy stress function ψ is introduced and the stress tensor is related to it in such a way that the tensor obeys the equilibrium equations

$$\vec{\nabla} \cdot \boldsymbol{\sigma}^c = \vec{0}$$

Using Hooke's law, the strain tensor can be expressed in the Airy function. Substitution of this $\boldsymbol{\varepsilon}(\psi)$ relation in one of the compatibility equations results in a partial differential equation for the Airy function, which can be solved with the proper boundary conditions.

In a Cartesian coordinate system the vector/tensor formulation can be replaced by index notation.

The material compliance tensor is : $\quad {}^4\boldsymbol{S} = -\dfrac{\nu}{E}\boldsymbol{I}\boldsymbol{I} + \dfrac{1+\nu}{E}\,{}^4\boldsymbol{I}^s$

$$\left.\begin{array}{l} \text{Airy stress function : } \psi(\vec{x}) \\ \boldsymbol{\sigma} = -\vec{\nabla}(\vec{\nabla}\psi) + \left(\nabla^2\psi\right)\boldsymbol{I} \\ \boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \boldsymbol{\sigma} \end{array}\right\} \rightarrow \left.\begin{array}{l} \boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \left[-\vec{\nabla}(\vec{\nabla}\psi) + (\nabla^2\psi)\boldsymbol{I}\right] \\ \nabla^2\left(\mathrm{tr}(\boldsymbol{\varepsilon})\right) - \vec{\nabla}\cdot\left(\vec{\nabla}\cdot\boldsymbol{\varepsilon}\right)^c = 0 \end{array}\right\}$$

$$\nabla^2(\nabla^2\psi) = \nabla^4\psi = 0 \quad\rightarrow\quad \psi \quad\rightarrow\quad \boldsymbol{\sigma} \quad\rightarrow\quad \boldsymbol{\varepsilon}$$

Cartesian index notation

$$\left.\begin{array}{l} \text{Airy stress function : } \psi(x_i) \\ \sigma_{ij} = -\psi_{,ij} + \delta_{ij}\psi_{,kk} \\ \varepsilon_{ij} = S_{ijlk}\,\sigma_{kl} \end{array}\right\} \rightarrow \left.\begin{array}{l} \varepsilon_{ij} = S_{ijlk}\left(-\,\psi_{,kl} + \delta_{kl}\psi_{,mm}\right) \\ \varepsilon_{ii,jj} - \varepsilon_{ij,ij} = 0 \end{array}\right\}$$

$$\psi_{,iijj} = 0 \quad\rightarrow\quad \psi \quad\rightarrow\quad \sigma_{ij} \quad\rightarrow\quad \varepsilon_{ij}$$

Planar, Cartesian

The stress function method is elaborated for planar deformation in a Cartesian coordinate system.

$$\left.\begin{array}{l} \sigma_{xx} = -\psi_{,xx} + \delta_{xx}(\psi_{,xx} + \psi_{,yy}) = \psi_{,yy} \\ \sigma_{yy} = -\psi_{,yy} + \delta_{yy}(\psi_{,xx} + \psi_{,yy}) = \psi_{,xx} \\ \sigma_{xy} = -\psi_{,xy} \\ \\ \varepsilon_{xx} = a_p\sigma_{xx} + q_p\varepsilon_{yy} \\ \varepsilon_{yy} = q_p\sigma_{xx} + b_p\sigma_{yy} \\ \varepsilon_{xy} = \frac{1}{2}k\sigma_{xy} \end{array}\right\} \rightarrow \left.\begin{array}{l} \varepsilon_{xx} = a_p\psi_{,yy} + q_p\psi_{,xx} \\ \varepsilon_{yy} = q_p\psi_{,yy} + b_p\psi_{,xx} \\ \varepsilon_{xy} = -\frac{1}{2}k\psi_{,xy} \\ \\ \varepsilon_{xx,yy} + \varepsilon_{yy,xx} = 2\varepsilon_{xy,xy} \end{array}\right\}$$

$$b_p\psi_{,xxxx} + a_p\psi_{,yyyy} + (2q_p + k)\psi_{,xxyy} = 0$$

isotropic $\qquad a_p = b_p = \dfrac{1}{E} \quad ; \quad q_p = \dfrac{-\nu}{E} \quad ; \quad k = \dfrac{2(1+\nu)}{E} \qquad \rightarrow$

bi-harmonic equation $\qquad \psi_{,xxxx} + \psi_{,yyyy} + 2\psi_{,xxyy} = 0$

Planar, cylindrical

In a cylindrical coordinate system, the bi-harmonic equation can be derived by transformation.

gradient operator $\quad \vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_t \frac{1}{r} \frac{\partial}{\partial \theta}$

Laplace operator $\quad \nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad \rightarrow \text{2D} \rightarrow$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

bi-harmonic equation $\quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) = 0$

stress components $\quad \sigma_{rr} = \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \quad ; \quad \sigma_{tt} = \frac{\partial^2 \psi}{\partial r^2}$

$$\sigma_{rt} = \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r} \partial \theta = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right)$$

12.5 Weighted residual formulation

Unknown variables have to be solved from the combined set of equilibrium equations and constitutive equations. Some of the equilibrium equations are partial differential equations. For the general case of large deformations and nonlinear material behavior, the equations are nonlinear. It is obvious that only for academic and very simple cases, analytic solutions exist. For more practical problems, approximate solutions must be determined with a numerical technique, of which the finite element method is widely used and will be considered here.

Application of the finite element method in continuum mechanics requires the reformulation of the equilibrium equations. They are transformed from differential equations to an integral equation, the so-called *weighted residual integral.*

First, we formulate the weighted residual integral for linear problems, so for small deformation and linear elastic material behavior. The finite element method is then explained for this case. Examples of plane stress, plane strain and axisymmetric problems will be calculated with a Matlab program.

Subsequently, we formulate the weighted residual integral for nonlinear problems, where the iterative solution procedure has to be applied. Finite element analyses can be done again with a Matlab program.

12.5.1 Three-dimensional deformation

For an approximation, the equilibrium equation is not satisfied exactly in each material point. The error can be "smeared out" over the material volume, using a *weighting function* $\vec{w}(\vec{x})$.

When the weighted residual integral is satisfied for each allowable weighting function $\vec{w}$, the equilibrium equation is satisfied in each point of the material.

$$
\begin{array}{lll}
\text{equilibrium equation} & \vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q} = \vec{0} & \forall\ \vec{x}\in V \\
\text{approximation } \rightarrow \text{ residual} & \vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q} = \vec{\Delta}(\vec{x}) \neq \vec{0} & \forall\ \vec{x}\in V \\
\text{weighted residual} & \displaystyle\int_V \vec{w}(\vec{x})\cdot\vec{\Delta}(\vec{x})\,dV = \int_V \vec{w}\cdot\left[\vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q}\right]dV &
\end{array}
$$

$$
\int_V \vec{w}\cdot\left[\vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q}\right]dV = 0 \qquad \forall\quad \vec{w}(\vec{x}) \qquad \leftrightarrow \qquad \vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q} = \vec{0} \qquad \forall\ \vec{x}\in V
$$

In the weighted residual integral, one term contains the divergence of the stress tensor. This means that the integral can only be evaluated, when the derivatives of the stresses are continuous over the domain of integration. This requirement can be relaxed by applying partial integration to the term with the stress divergence. The result is the so-called weak formulation of the weighted residual integral.

Gauss theorem is used to transfer the volume integral with the term $\vec{\nabla}.(\ )$ to a surface integral. Also $\vec{p} = \boldsymbol{\sigma}\cdot\vec{n} = \vec{n}\cdot\boldsymbol{\sigma}^c$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^c$ is used.

$$
\left.
\begin{array}{l}
\displaystyle\int_V \vec{w}\cdot\left[\vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q}\right]dV = 0 \\
\vec{\nabla}\cdot(\boldsymbol{\sigma}^T\cdot\vec{w}) = (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}^T + \vec{w}\cdot(\vec{\nabla}\cdot\boldsymbol{\sigma}^T)
\end{array}
\right\} \rightarrow
$$

$$
\left.
\begin{array}{l}
\displaystyle\int_V \left[\vec{\nabla}\cdot(\boldsymbol{\sigma}^T\cdot\vec{w}) - (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}^T + \vec{w}\cdot\rho\vec{q}\right]dV = 0 \qquad \forall\ \vec{w} \\
\displaystyle\int_V \vec{\nabla}\cdot(\boldsymbol{\sigma}^T\cdot\vec{w}) = \int_V \vec{n}\cdot\boldsymbol{\sigma}^T\cdot\vec{w}\,dA = \int_A \vec{w}\cdot\vec{p}\,dA
\end{array}
\right\} \rightarrow
$$

$$
\int_V (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}\,dV = \int_V \vec{w}\cdot\rho\vec{q}\,dV + \int_A \vec{w}\cdot\vec{p}\,dA \qquad \forall\ \vec{w}
$$

$$
\int_V (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}\,dV = f_e(\vec{w}) \qquad \forall\ \vec{w}
$$

12.5.2 Linear elastic formulation

When deformation and rotations are small, the deformation is geometrically linear. The deformed state is almost equal to the undeformed state. This implies that integration can be carried out over the undeformed volume V_0 and the undeformed area A_0.

The material behavior is described by Hooke's law, which can be substituted in the weighted residual integral, according to the displacement solution method.

The weighted residual integral is now completely expressed in the displacement $\vec{u}$. Approximate solutions can be determined with the finite element method.

$$\int_{V_0} (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}\, dV_0 = \int_{V_0} \vec{w}\cdot\rho\vec{q}\, dV_0 + \int_{A_0} \vec{w}\cdot\vec{p}\, dA_0 = f_{e0}(\vec{w}) \qquad \forall\ \vec{w}$$

$$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} = {}^4\boldsymbol{C} : \tfrac{1}{2}\left\{(\vec{\nabla}_0\vec{u}) + (\vec{\nabla}_0\vec{u})^T\right\} = {}^4\boldsymbol{C} : (\vec{\nabla}_0\vec{u})$$

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^T : {}^4\boldsymbol{C} : (\vec{\nabla}_0\vec{u})\, dV_0 = \int_{V_0} \vec{w}\cdot\rho\vec{q}\, dV_0 + \int_{A_0} \vec{w}\cdot\vec{p}\, dA_0 = f_{e0}(\vec{w}) \qquad \forall\ \vec{w}$$

This tensor relation can be written in matrix/column notation. We use the column notation of the vector gradient as it was introduced in section 4.2.3 : $\boldsymbol{L}_{0w} = (\vec{\nabla}_0\vec{w})^c \rightarrow \underset{\approx}{L}_{0w}$ and $\boldsymbol{L}_{0u} = (\vec{\nabla}_0\vec{u})^c \rightarrow \underset{\approx}{L}_{0u}$ where $\underset{\approx}{L}$ is the column with the derivatives w.r.t. the coordinates.

$$\int_{V_0} \left(\underset{\approx}{L}_{0w}\right)_t^T \underline{\underline{C}} \left(\underset{\approx}{L}_{0u}\right)_t dV_0 = f_{e0}(\underset{\sim}{w}) \qquad \forall\ \underset{\sim}{w}$$

12.5.3 Total Lagrange formulation

When deformations are large – geometrically nonlinear –, the current volume of the material is unknown, which means that the weighted residual integral can not be evaluated. Transformation of this integral is always possible. Besides the integral also the gradient operator must be transformed. The configuration, which is the target of the transformation is the *reference configuration*.

The first thing we can think of is a transformation to the undeformed configuration t_0. This transformation results in the *Total Lagrange* formulation. The second Piola-Kirchhoff stress tensor is mostly used in this case to represent the stress state.

$$\int_V (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma}\, dV = f_e(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

transformation to undeformed configuration t_0

$$\begin{aligned} &\vec{\nabla} = \boldsymbol{F}^{-c}\cdot\vec{\nabla}_0 \quad \rightarrow \quad (\vec{\nabla}\vec{w})^c = (\vec{\nabla}_0\vec{w})^c\cdot\boldsymbol{F}^{-1} \\ &dV = \det(\boldsymbol{F})dV_0 = J dV_0 \end{aligned}$$

weighted residual integral

$$\left.\begin{aligned} &\int_{V_0} (\vec{\nabla}_0\vec{w})^c\cdot\boldsymbol{F}^{-1} : \boldsymbol{\sigma} J\, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \\ &\boldsymbol{P} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-c} \end{aligned}\right\} \rightarrow$$

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^c : (\boldsymbol{P}\cdot\boldsymbol{F}^c)\, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

Iterative solution process

In the Total Lagrange formulation the weighted residual integral is transformed from the current configuration C_c to the initial undeformed configuration C_0. Unknown variables in the integral are the total deformation tensor $\boldsymbol{F}$ and the 2nd-Piola-Kirchhoff stress tensor $\boldsymbol{P}$.

To describe the essential steps of the iteration procedure, it is assumed that an approximate state C_c^* is determined with values for $\boldsymbol{F}^*$ and $\boldsymbol{P}^*$. The unknown current values are written as $\boldsymbol{F} = \boldsymbol{F}^* + \delta\boldsymbol{F}$ and $\boldsymbol{P} = \boldsymbol{P}^* + \delta\boldsymbol{P}$, where $\delta(.)$ indicates the difference between C_c^* and C_c. The iterative change of the deformation tensor $\delta\boldsymbol{F}$ can be expressed in the iterative displacement $\delta\vec{x} = \vec{u}$.

$$\left.\begin{array}{l}
\displaystyle\int_{V_0} (\vec{\nabla}_0\vec{w})^c : (\boldsymbol{P}\cdot\boldsymbol{F}^c)\, dV_0 = f_{e0}(\vec{w}) \qquad \forall\ \vec{w}(\vec{x}) \\[2ex]
\boldsymbol{F} = (\vec{\nabla}_0\vec{x})^c = \{\vec{\nabla}_0(\vec{x}^* + \delta\vec{x})\}^c = (\vec{\nabla}_0\vec{x}^*)^c + (\vec{\nabla}_0\delta\vec{x})^c = \boldsymbol{F}^* + \delta\boldsymbol{F} = \boldsymbol{F}^* + \boldsymbol{L}_{0u} \\
\boldsymbol{P} = \boldsymbol{P}^* + \delta\boldsymbol{P}
\end{array}\right\} \quad \rightarrow$$

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^c : (\boldsymbol{P}^* + \delta\boldsymbol{P})\cdot(\boldsymbol{F}^* + \boldsymbol{L}_{0u})^c\, dV_0 = f_{e0}(\vec{w}) \qquad \forall\ \vec{w}(\vec{x})$$

It is assumed that the iterative displacement, its gradient and the stress variation are very small and then the weighted residual integral is linearized with respect to $\vec{u}$. In analogy with $\boldsymbol{L}_{0u}$, $\boldsymbol{L}_{0w} = (\vec{\nabla}_0\vec{w})^c$ is introduced.

$$\begin{array}{l}
\displaystyle\int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* + \delta\boldsymbol{P})\cdot(\boldsymbol{F}^* + \boldsymbol{L}_{0u})^c\, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \\
\displaystyle\int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^*\cdot\boldsymbol{F}^{*c} + \boldsymbol{P}^*\cdot\boldsymbol{L}_{0u}^c + \delta\boldsymbol{P}\cdot\boldsymbol{F}^{*c})\ dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \\
\displaystyle\int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^*\cdot\boldsymbol{L}_{0u}^c + \delta\boldsymbol{P}\cdot\boldsymbol{F}^{*c})\ dV_0 = \\
\qquad\qquad\qquad \displaystyle f_{e0}(\vec{w}) - \int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^*\cdot\boldsymbol{F}^{*c})\, dV_0 = r^* \qquad \forall \quad \vec{w}(\vec{x})
\end{array}$$

Material model

The right-hand side of the iterative equation represents the residual load. To calculate r^* and the term with $\boldsymbol{P}^*$ in the left-hand integral, the stress $\boldsymbol{P}^*(t)$ must be determined from the constitutive equation. From the material model also a relation between $\delta\boldsymbol{P}$ and $\boldsymbol{L}_{0u}$ must be derived. The iterative change of the 2nd-Piola-Kirchhoff stress $\delta\boldsymbol{P}$, must be expressed in the iterative displacement $\vec{u}$ and substituted in the iterative weighted residual integral.

$$\delta\boldsymbol{P} = {}^4\boldsymbol{M} : \boldsymbol{L}_{0u} \quad \rightarrow$$

$$\int_{V_0} \boldsymbol{L}_{0w} : \left(\boldsymbol{P}^* \cdot \boldsymbol{L}_{0u}^c + ({}^4\boldsymbol{M} : \boldsymbol{L}_{0u}) \cdot \boldsymbol{F}^{*c} \right) \, dV_0 =$$

$$f_{e0}(\vec{w}) - \int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{F}^{*c}) \, dV_0 \qquad \forall \quad \vec{w}(\vec{x})$$

$$\int_{V_0} \left[\boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{L}_{0u}^c) + \boldsymbol{L}_{0w} : (\boldsymbol{F}^* \cdot {}^4\boldsymbol{M}^{lrc}) : \boldsymbol{L}_{0u}^c \right] dV_0 =$$

$$f_{e0}(\vec{w}) - \int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{F}^{*c}) \, dV_0 \qquad \forall \quad \vec{w}(\vec{x})$$

Matrix/column notation

We will now express the vectors and tensors in their components w.r.t. a basis of a coordinate system. A matrix-column notation is used, which is explained elsewhere. The asterisk $(\ )^*$ indicating an approximate value is omitted.

$$\int_{V_0} \left[\left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{P}} \left(\underset{\approx}{L}_{0u} \right)_t + \left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{F}}_{cr} \underline{\underline{M}}_{0c} \left(\underset{\approx}{L}_{0u} \right)_t \right] dV_0 =$$

$$f_{e0}(\underset{\sim}{w}) - \int_{V_0} \left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{F}}_{cr} \underset{\approx}{P} \, dV_0 = f_{e0}(\underset{\sim}{w}) - f_{i0}(\underset{\sim}{w})$$

$$\int_{V_0} \left(\underset{\approx}{L}_{0w} \right)_t^T \left[\underline{\underline{P}} + \underline{\underline{F}}_{cr} \underline{\underline{M}}_{0c} \right] \left(\underset{\approx}{L}_{0u} \right)_t dV_0 = f_{e0}(\underset{\sim}{w}) - f_{i0}(\underset{\sim}{w})$$

12.5.4 Updated Lagrange formulation

In the *Updated Lagrange* formulation the reference configuration is chosen to be the start of the current increment at t_n.

$$\int_V (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma} \, dV = f_e(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

transformation to begin increment configuration t_n

$$\begin{aligned} &\vec{\nabla} = \boldsymbol{F}_n^{-c} \cdot \vec{\nabla}_n \quad \rightarrow \quad (\vec{\nabla}\vec{w})^c = (\vec{\nabla}_n\vec{w})^c \cdot \boldsymbol{F}_n^{-1} \\ &dV = \det(\boldsymbol{F}_n) dV_n \end{aligned}$$

weighted residual integral

$$\int_{V_n} (\vec{\nabla}_n\vec{w})^c \cdot \boldsymbol{F}_n^{-1} : \boldsymbol{\sigma} \det(\boldsymbol{F}_n) \, dV_n = f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \quad \rightarrow$$

$$\int_{V_n} (\vec{\nabla}_n\vec{w})^c : (\boldsymbol{F}_n^{-1} \cdot \boldsymbol{\sigma}) \det(\boldsymbol{F}_n) \, dV_n = f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

Iterative solution process

To describe the essential steps of the iteration procedure, it is assumed that an approximate state C_c^* is determined with values for $\boldsymbol{F}_n^*$, $\boldsymbol{\sigma}^*$ and the other variables. The unknown current values are written as $\boldsymbol{F}_n = (\boldsymbol{I} + \boldsymbol{L}_u^*) \cdot \boldsymbol{F}_n^*$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}$, where $\delta(.)$ indicates the difference between C_c^* and C_c, and $\boldsymbol{L}_u^* = (\vec{\nabla}^* \vec{u})^c$, with $\vec{u} = \delta\underset{\sim}{\vec{x}}$ the iterative displacement.

$$
\left.
\begin{aligned}
&\int_{V_n} (\vec{\nabla}_n \vec{w})^c : (\boldsymbol{F}_n^{-1} \cdot \boldsymbol{\sigma}) \det(\boldsymbol{F}_n)\, dV_n = f_{en}(\vec{w}) \qquad \forall\ \vec{w}(\vec{x}) \\[2ex]
&\boldsymbol{F}_n = (\vec{\nabla}_n \vec{x})^c = \{\vec{\nabla}_n(\vec{x}^* + \delta\vec{x})\}^c = (\vec{\nabla}_n \vec{x}^*)^c + (\vec{\nabla}_n \delta\vec{x})^c \\
&\qquad = \boldsymbol{F}_n^* + \delta\boldsymbol{F}_n = \boldsymbol{F}_n^* + (\vec{\nabla}^* \delta\vec{x})^c \cdot (\vec{\nabla}_n \vec{x}^*)^c = \boldsymbol{F}_n^* + \boldsymbol{L}_u^* \cdot \boldsymbol{F}_n^* = (\boldsymbol{I} + \boldsymbol{L}_u^*) \cdot \boldsymbol{F}_n^* \\
&\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}
\end{aligned}
\right\} \quad \rightarrow
$$

$$
\begin{aligned}
\int_{V_n} (\vec{\nabla}_n \vec{w})^c : \left[(\boldsymbol{F}_n^*)^{-1} \cdot (\boldsymbol{I} + \boldsymbol{L}_u^*)^{-1} \cdot (\boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}) \ \det\{(\boldsymbol{I} + \boldsymbol{L}_u^*) \cdot \boldsymbol{F}_n^*\} \right] dV_n \\
= f_{en}(\vec{w}) \qquad \forall\ \vec{w}(\vec{x})
\end{aligned}
$$

Assuming that the iterative displacement and its gradient are very small, the weighted residual integral can be linearized with respect to $\vec{u}$. In analogy with $\boldsymbol{L}_u^*$, $\boldsymbol{L}_w^* = (\vec{\nabla}^* \vec{w})^c$ is introduced.

$$
\begin{aligned}
&(\boldsymbol{I} + \boldsymbol{L}_u^*)^{-1} \approx \boldsymbol{I} - \boldsymbol{L}_u^* \\
&\det\{(\boldsymbol{I} + \boldsymbol{L}_u^*) \cdot \boldsymbol{F}_n^*\} = \det(\boldsymbol{I} + \boldsymbol{L}_u^*) \det(\boldsymbol{F}_n^*) \approx \operatorname{tr}(\boldsymbol{I} + \boldsymbol{L}_u^*) \det(\boldsymbol{F}_n^*) = (1 + \boldsymbol{I} : \boldsymbol{L}_u^*) \det(\boldsymbol{F}_n^*)
\end{aligned}
$$

weighted residual integral

$$
\begin{aligned}
\int_{V_n} (\vec{\nabla}_n \vec{w})^c : \left[(\boldsymbol{F}_n^*)^{-1} \cdot (\boldsymbol{I} - \boldsymbol{L}_u^*) \cdot (\boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma})(1 + \boldsymbol{I} : \boldsymbol{L}_u^*) \det(\boldsymbol{F}_n^*) \right] dV_n \\
= f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})
\end{aligned}
$$

further linearisation

$$
\begin{aligned}
\int_{V^*} [\boldsymbol{L}_w^* : \boldsymbol{\sigma}^* \boldsymbol{I} : \boldsymbol{L}_u^{*c} + \boldsymbol{L}_w^* : \delta\boldsymbol{\sigma} - \boldsymbol{L}_w^* : (\boldsymbol{\sigma}^{*c} \cdot \boldsymbol{L}_u^{*c})^c]\, dV^* = \\
f_e^*(\vec{w}) - \int_{V^*} \boldsymbol{L}_w^* : \boldsymbol{\sigma}^*\, dV^* = r^* \qquad \forall \quad \vec{w}(\vec{x})
\end{aligned}
$$

Material model

The right-hand side of the iterative equation represents the residual load. To calculate r^* and two terms in the left-hand integral, the stress $\boldsymbol{\sigma}^*(t)$ must be determined from the constitutive equation.

The iterative change of the stress $\delta\boldsymbol{\sigma}$, must be expressed in the iterative displacement $\vec{u}$ and substituted in the iterative weighted residual integral.

$$\delta\boldsymbol{\sigma} = {}^4\boldsymbol{M} : \boldsymbol{L}_u^* \quad \rightarrow$$

$$\int_{V^*} \left[\boldsymbol{L}_w^* : \boldsymbol{\sigma}^*\boldsymbol{I} : \boldsymbol{L}_u^{*c} + \boldsymbol{L}_w^* : {}^4\boldsymbol{M} : \boldsymbol{L}_u^* - \boldsymbol{L}_w^* : (\boldsymbol{\sigma}^{*c} \cdot \boldsymbol{L}_u^{*c})^c\right] dV^* =$$
$$f_e^*(\vec{w}) - \int_{V^*} \boldsymbol{L}_w^* : \boldsymbol{\sigma}^* \, dV^* \qquad \forall \quad \vec{w}(\vec{x})$$

Matrix/column notation

We will now express the vectors and tensors in their components w.r.t. a basis of a coordinate system. A matrix-column notation is used, which is explained elsewhere. The asterisk ( )* indicating an approximate value is omitted.

$$\int_{V^*} \left[\left(\underset{\approx}{L}_w\right)_t^T \underset{\approx}{\sigma}\underset{\approx}{I}^T \left(\underset{\approx}{L}_u\right)_t + \left(\underset{\approx}{L}_w\right)_t^T \underline{\underline{M}} \left(\underset{\approx}{L}_u\right)_t - \left(\underset{\approx}{L}_w\right)_t^T \underline{\underline{\sigma}}_{tr} \left(\underset{\approx}{L}_u\right)_t\right] dV^* =$$
$$f_e(\underset{\sim}{w}) - \int_{V^*} \left(\underset{\approx}{L}_w\right)_t^T \underset{\approx}{\sigma} \, dV^* = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$
$$\int_{V^*} \left(\underset{\approx}{L}_w\right)_t^T \left[\underset{\approx}{\sigma}\underset{\approx}{I}^T - \underline{\underline{\sigma}}_{tr} + \underline{\underline{M}}\right] \left(\underset{\approx}{L}_u\right)_t dV^* = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$
$$\int_{V^*} \left(\underset{\approx}{L}_w\right)_t^T \left[\underline{\underline{\Sigma}} + \underline{\underline{M}}\right] \left(\underset{\approx}{L}_u\right)_t dV^* = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$

12.6 Finite element method

In Chapter 12.5 the weighted residual integral was derived.

$$\int_V \left(\underset{\approx}{L}_w\right)_t^T \left[\underline{\underline{W}}\right] \left(\underset{\approx}{L}_u\right)_t dV = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$

The unknown (iterative) dislacements in $\underset{\approx}{L}_u$ have to be determined such that this integral is satisfied for all weighting functions in $\underset{\approx}{L}_w$. Such a solution canot generally be determined exactly and by analytical means. Instead we have to resort to numerical techniques to determine an approximate solution. The finite element method is widely used for this task.

12.6.1 Discretisation

The integral over the body volume V is written as a sum of integrals over smaller volumes, which collectively constitute the whole volume. Such a small volume V^e is called an element. Subdividing the volume implies that also the surface with area A is subdivided in element surfaces (faces) with area A^e.

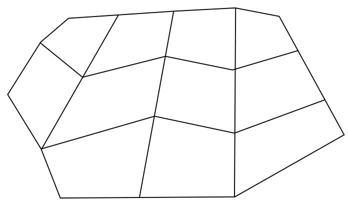


Fig. 12.3 : *Finite element discretisation*

$$\sum_e \int_{V^e} \left(\underset{\approx}{L}_w\right)_t^T \left[\underline{\underline{W}}\right] \left(\underset{\approx}{L}_u\right)_t dV^e = \sum_e f_e^e(\underset{\sim}{w}) - \sum_e f_i^e(\underset{\sim}{w}) \qquad \forall\ \underset{\sim}{w}$$

Isoparametric elements

Each point of a three-dimensional element can be identified with three local coordinates $\{\xi_1, \xi_2, \xi_3\}$. In two dimensions we need two and in one dimension only one local coordinate.

The real geometry of the element can be considered to be the result of a deformation from the original cubic, square or line element with (side) length 2. The deformation can be described with a deformation matrix, which is called the *Jacobian matrix* $\underline{J}$. The determinant of this matrix relates two infinitesimal volumes, areas or lengths of both element representations.

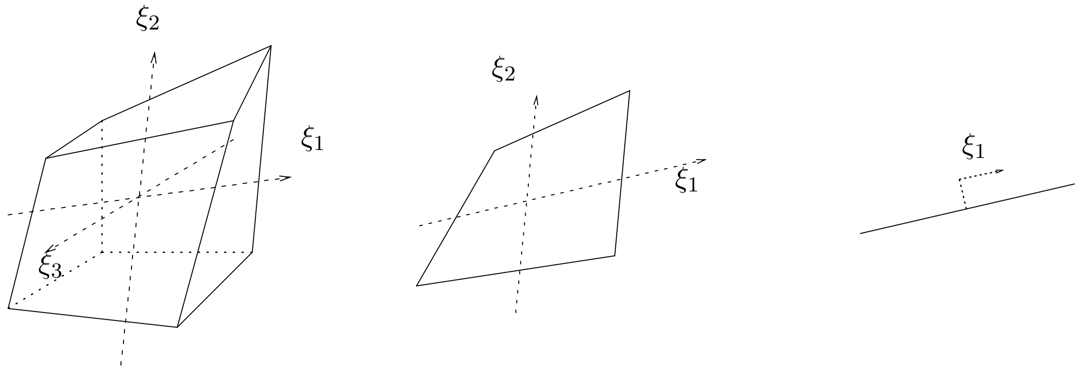


Fig. 12.4 : *Isoparametric elements*

isoparametric (local) coordinates $\quad (\xi_1, \xi_2, \xi_3) \quad ; \quad -1 \leq \xi_i \leq 1 \quad i = 1, 2, 3$

Jacobian matrix $\quad \underline{J} = \left(\underset{\sim}{\nabla}_\xi \underset{\sim}{x}^T\right)^T \quad ; \quad dV^e = \det(\underline{J})\, d\xi_1 d\xi_2 d\xi_3$

12.6.2 Interpolation

The value of the unknown quantity – here the displacement vector $\vec{u}$ or the iterative displacement vector $\delta\vec{u}$ – in an arbitrary point of the element, can be interpolated between the values of that quantity in certain fixed points of the element : the *element nodes. Interpolation functions* N are a function of the isoparametric coordinates.

The components of the vector $(\delta)\vec{u}$ are stored in a column $(\delta)\underset{\sim}{u}$. The nodal (iterative) displacement components are stored in the column $(\delta)\underset{\sim}{u}^e$. The position $\vec{x}$ of a point within the element is interpolated between the nodal point positions, the components of which are stored in the column $\underset{\sim}{x}^e$. Generally, the interpolations for position and displacement are chosen to be the same.

Besides $(\delta)\vec{u}$ and $\vec{x}$, the weighting function $\vec{w}$ also needs to be interpolated between nodal values. When this interpolation is the same as that for the displacement, the so-called *Galerkin* procedure is followed, which is generally the case for simple elements, considered here.

We consider the vector function $\vec{a}$ to be interpolated, where nep is the number of element nodes. Components a_i of $\vec{a}$ w.r.t. a global vector base, can then also be interpolated.

$$\begin{aligned}
\vec{a} &= N^1\vec{a}^1 + N^2\vec{a}^2 + \cdots + N^{nep}\vec{a}^{nep} = \underset{\sim}{N}^T\underset{\sim}{\vec{a}}^e \quad \rightarrow \\
a_i &= N^1 a_i^1 + N^2 a_i^2 + \cdots + N^{nep} a_i^{nep} = \sum_{\alpha=1}^{nep} N^\alpha a_i^\alpha = \underset{\sim}{N}^T \underset{\sim}{a}_i^e \quad \rightarrow \quad \underset{\sim}{a} = \underline{N}\,\underset{\sim}{a}^e
\end{aligned}$$

The gradient of the vector function $\vec{a}$ also has to be elaborated. The gradient is referred to as the second-order tensor $\boldsymbol{L}^c$, which can be written in components w.r.t. a vector basis. The components are stored in a column $\underset{\approx}{L}$. This column can be written as the product of the so-called *B-matrix*, which contains the derivatives of the interpolation functions, and the column with nodal components of $\vec{a}$.

$$\boldsymbol{L}^c = \vec{\nabla}\vec{a} \quad \rightarrow \quad \underset{\approx}{L}_t = \underline{B}\,\underset{\sim}{a}^e$$

12.6.3 Integration

Interpolations for both the (iterative) displacement and the weighting function and their respective derivatives are substituted in the weighted residual integrals of each element.

Calculating the element contributions implies the evaluation of an integral over the element volume V^e and the element surface A^e. This integration is done numerically, using a fixed set of nip Gauss-points, which have s specific location in the element. The value of the integrand is calculated in each Gauss-point and multiplied with a Gauss-point-specific weighting factor c^{ip} and added.

Resulting element quantities are the element stiffness matrix $\underline{K}^e$, the element external force column $\underset{\sim}{f}_e^e$ and the element internal force column $\underset{\sim}{f}_i^e$.

$$\underset{\sim}{w}^{eT}\left[\int_{V^e} \underline{B}^T\,\underline{\underline{W}}\,\underline{B}\,dV^e\right]\delta\underset{\sim}{u}^e = \underset{\sim}{w}^{eT}\underset{\sim}{f}_e^e - \underset{\sim}{w}^{eT}\underset{\sim}{f}_i^e$$

$$\underset{\sim}{w}^{eT}\left[\int_{\xi_1=-1}^{\xi_1=1}\int_{\xi_2=-1}^{\xi_2=1}\int_{\xi_3=-1}^{\xi_3=1} \underline{B}^T\left[\underline{\underline{W}}\right]\underline{B}\,\det(\underline{J})\,d\xi_1 d\xi_2 d\xi_3\right]\delta\underset{\sim}{u}^e = \underset{\sim}{w}^{eT}\underset{\sim}{f}_e^e - \underset{\sim}{w}^{eT}\underset{\sim}{f}_i^e$$

$$\underset{\sim}{w}^{eT}\,\underline{K}^e\,\delta\underset{\sim}{u}^e = \underset{\sim}{w}^{eT}\,\underset{\sim}{f}_e^e - \underset{\sim}{w}^{eT}\,\underset{\sim}{f}_i^e$$

$$\int_{V^e} g(x_1,x_2,x_3)\,dV^e = \int_{\xi_1=-1}^{1}\int_{\xi_2=-1}^{1}\int_{\xi_3=-1}^{1} f(\xi_1,\xi_2,\xi_3)\,d\xi_1 d\xi_2 d\xi_3 = \sum_{ip=1}^{nip} c^{ip}\,f(\xi_1^{ip},\xi_2^{ip},\xi_3^{ip})$$

12.6.4 Assemblation

The weighted residual contribution of all elements have to be collected into the total weighted residual integral. This means that all elements are connected or assembled. This assembling is an administrative procedure. All the element matrices and columns are placed at appropriate locations into the structural or global stiffness matrix $\underline{K}$ and the load column $\underset{\sim}{f}_e$.

Because the resulting equation has to be satisfied for all $\underset{\sim}{w}$, the nodal displacements $\underset{\sim}{u}$ have to satisfy a set of equations.

$$\begin{aligned}
&\sum_e \underset{\sim}{w}^{eT} \underline{K}^e \, \delta\underset{\sim}{u}^e = \sum_e \underset{\sim}{w}^{eT} \underset{\sim}{f}_e^e - \sum_e \underset{\sim}{w}^{eT} \underset{\sim}{f}_i^e \quad \rightarrow \\
&\underset{\sim}{w}^T \, \underline{K} \, \delta\underset{\sim}{u} = \underset{\sim}{w}^T \, \underset{\sim}{f}_e - \underset{\sim}{w}^T \, \underset{\sim}{f}_i = \underset{\sim}{w}^T \, \underset{\sim}{r} \qquad \forall \; \underset{\sim}{w} \qquad \rightarrow \\
&\underline{K} \, \delta\underset{\sim}{u} = \underset{\sim}{f}_e - \underset{\sim}{f}_i = \underset{\sim}{r}
\end{aligned}$$

12.6.5 Solution

The initial governing equations were differential equations, which obviously need boundary conditions to arrive at a unique solution. The boundary conditions are prescribed displacements or forces in certain material points. After finite element discretisation, displacements and forces can be applied in nodal points.

The set of nodal equations $\underline{K}\delta\underset{\sim}{u} = \underset{\sim}{r}$ cannot be solved yet, because the structural stiffness matrix $\underline{K}$ is singular and cannot be inverted.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots \\ k_{21} & k_{22} & k_{23} & \cdots \\ k_{31} & k_{32} & k_{33} & \cdots \\ \vdots & \vdots & \vdots & . \end{bmatrix} \begin{bmatrix} a \\ a \\ a \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \rightarrow \quad \underline{K} = \text{ singular} \quad \rightarrow \quad \det \underline{K} = 0$$

When enough boundary conditions have been applied to prevent the rigid body motion of the material body, the equations are solvable.

$$\delta\underset{\sim}{u} = \underline{K}^{-1} \, \underset{\sim}{r}$$

12.6.6 Program structure

A finite element program starts with reading data from an input file and initialization of variables and databases.

The loading is prescribed as a function of the (fictitious) time in an incremental loop. In each increment the system of nonlinear equilibrium equations is solved iteratively.

In each iteration loop the system of equations is build. In a loop over all elements, the stresses are calculated and the material stiffness is updated. The element internal nodal force column and the element stiffness matrix are assembled into the global column and matrix.

Numerical integration over the element volume (area) implies that a loop over integration points is entered. In each integration point the integrand is calculated and the result is multiplied by a weighting factor and added to the existing element value.

After taking tyings and boundary conditions into account, the unknown nodal displacements and reaction forces are calculated.

When the convergence criterion is not reached, a new iteration step is performed. After convergence output data are stored and the next incremental step is carried out.

```
read input data from input file
calculate additional variables from input data
initialize values and arrays

while load increments to be done

   for all elements
      for all integration points
         calculate contribution to initial element stiffness matrix
      end integration point loop
      assemble global stiffness matrix
   end element loop

   determine external incremental load from input

   while non-converged iteration step

      take tyings into account
      take boundary conditions into account

      calculate iterative nodal displacements
      calculate total deformation

      for all elements
         for all integration points
            calculate stresses from material behavior
            calculate material stiffness from material behavior
            calculate contribution to element internal nodal forces
            calculate contribution to element stiffness matrix
         end ntegration point loop
         assemble global stiffness matrix
         assemble global internal load column
      end element loop

      calculate residual load column
      calculate convergence norm

   end iteration step
```

```
    store data for post-processing
  end load increment
```

A more detailed description of the formulation of axi-symmetric ring elements and planar elements can be found in the appendices E E.2, E.3, F. In the next chapter, some results are presented for linear elastic problems.

12.7 Numerical solutions

In the following sections we present some problems and their numerical solutions. These solutions are determined with several finite element programs, both in-house developments in Matlab as commercial, in this case MSC.Marc/Mentat. The Matlab programs are described in more detail in the appendices (E.3,G,H). These solutions are determined with the MSC.Marc/Mentat FE-package. The numerical solutions can be compared with the analytical solutions, described in the previous section.

12.7.1 FE program `femaxi`

The Matlab program **`femaxi`** is used to analyze rings, which are subjected to various boundary conditions.

Thick-walled pressurized cylinder

The first example is the same as we have seen before: a cylinder subjected to an internal pressure. Parameter values are listed in the table below. The analytical solution was presented earlier.

isotropic	plane stress	$p_i = 100$ MPa	$p_e = 0$ MPa	
$a = 0.25$ m	$b = 0.5$ m	$h = 0.5$ m	$E = 250$ GPa	$\nu = 0.33$

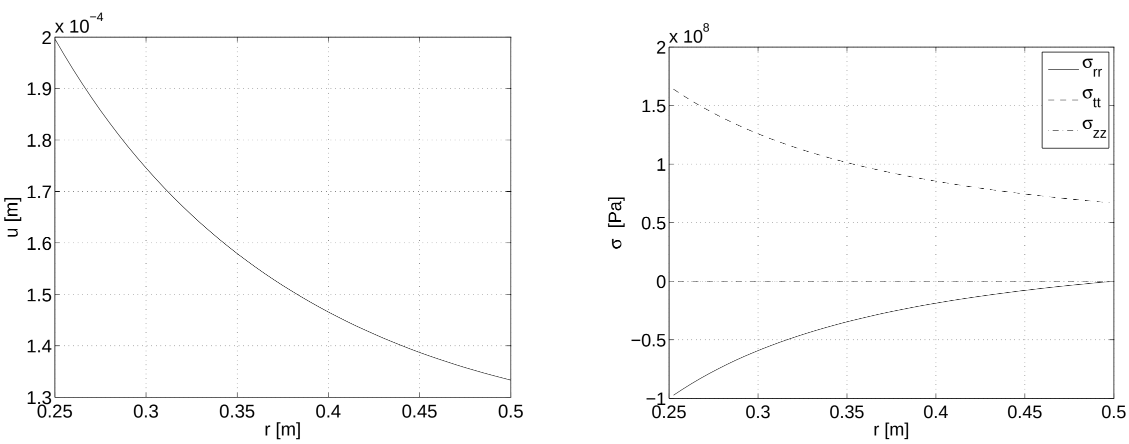


Fig. 12.5 : *Displacement and stresses in a pressurized cylinder for plane stress.*

Compound thick-walled pressurized cylinder

In this example, the disc is composed of two materials with different properties. The inner part is isotropic. The outer part is made of orthotropic material with an increased modulus in tangential direction, and reduced Poisson ratio's. In this case element type 2 (linear interpolation) has to be used, because type 1 interpolation field is not well suited for the orthotropic material behavior.

isotropic	plane stress	$p_i = 100$ MPa	$p_e = 0$ MPa
$a_1 = 0.25$ m	$a_2 = 0.375$ m	$E = 250$ GPa	$\nu = 0.33$
$a_2 = 0.375$ m	$b = 0.5$ m	$E1 = E$ GPa	$E2 = 10E$ GPa
$\nu_{12} = \nu/10$	$\nu_{32} = \nu/10$		

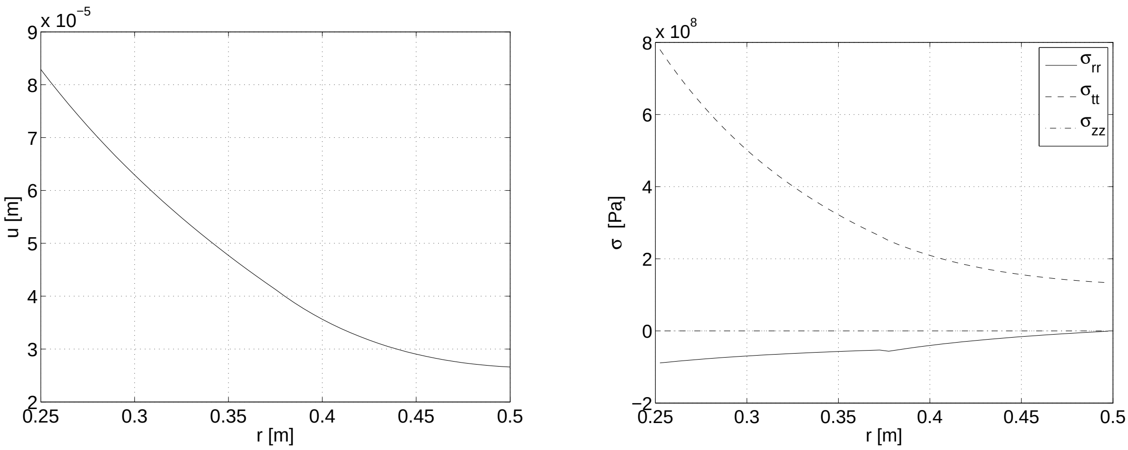


Fig. 12.6 : *Displacement and stresses in a pressurized compound cylinder for plane stress*

Rotating disc

The disc is rotated with an angular frequency $\omega = 6$ cycles/sec.

orthotropic	plane stress	$\omega = 6$ c/s		
$a = 0.2$ m	$b = 0.5$ m	$E = 200$ GPa	$\nu = 0.3$	$Gr = 7500$ kg/m^3

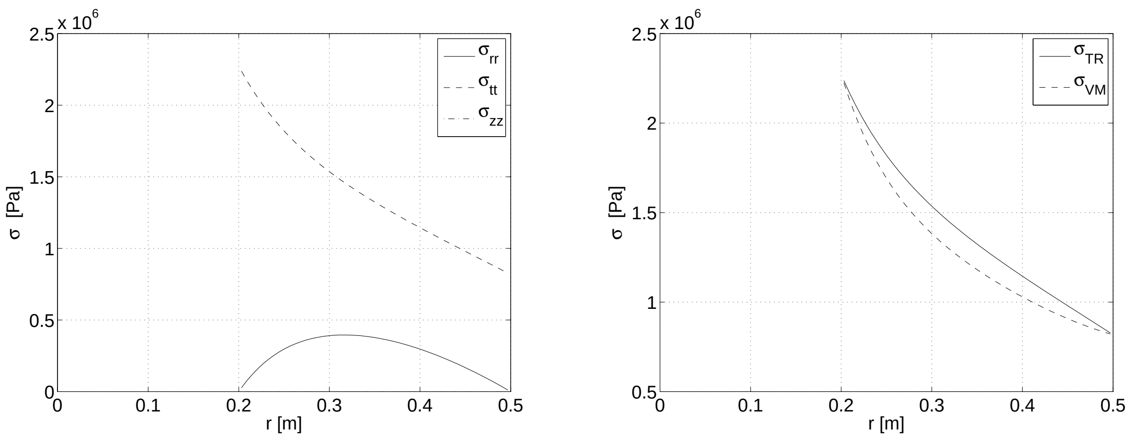


Fig. 12.7 : *Stresses in a rotating disc for plane stress.*

12.8 FE program `plaxL`

The Matlab program **`plaxL`** is used to model and analyze linear, planar and axi-symmetric problems. So deformations are small and the material behavior is linear elastic. This means that the weighted residual formulation from section 12.5.2 is used as a starting point. The program is described in detail in appendix G. An example is shown below.

12.8.1 Thick-walled pressurized cylinder: plane stress

A thick-walled open cylinder – inner radius a, outer radius b – is subjected to an internal pressure. Parameter values are listed in the table below. The figures show the element model, both undeformed and deformed., and the plot of the radial displacement over the radius.

First a quarter of the cylinder is modeled and analyzed in plane stress.

$a = 0.25$ m	$b = 0.5$ m	$h = 0.5$ m	$E = 250$ GPa	$\nu = 0.33$
$p_i = 100$ MPa	$p_e = 0$ MPa			

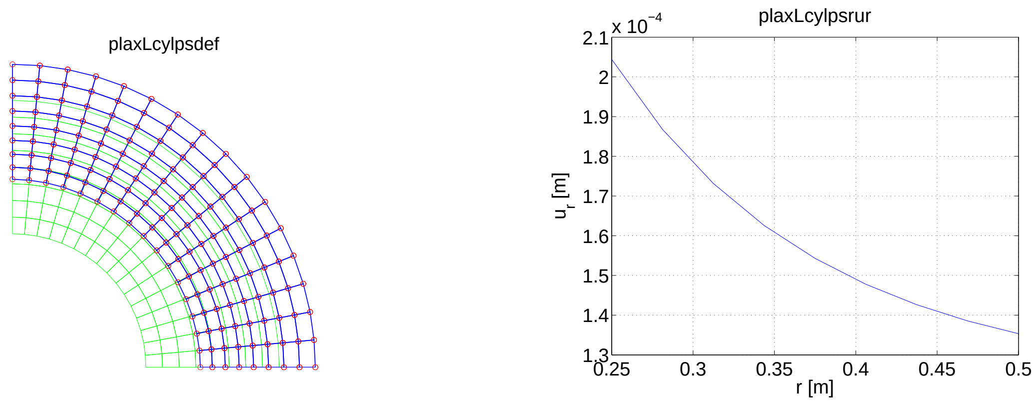
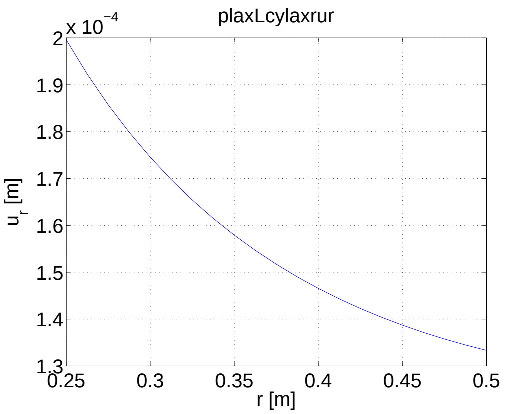


Fig. 12.8 : *Deformation (×1000) of the cylinder and radial displacement.*

The thick-walled open cylinder is now analyzed as an axi-symmetric model.



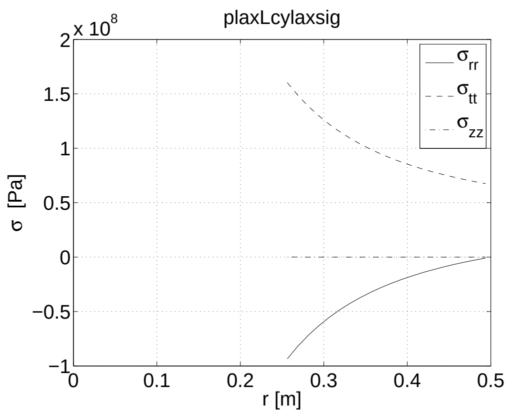


Fig. 12.9 : *Displacement and stresses in a pressurized cylinder.*

12.9 FE program `plax`

The Matlab program **`plax`** is used to model and analyze planar and axisymmetrc problems. Large deformations and rotations are allowed. Various nonlinear and time-dependent material models are implemented, of which examples will be shown in later chapters.

12.9.1 Rigid rotation

When a material body is subjected to a rigid rotation, no stresses must be generated in the material. The material model must be such that this requirement is satisfied.

In this example we rotate one element over 720^o by prescribing all nodal point displacements. A non-suitable material law, a linear relation between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, will result in high reaction forces in the nodes (figure 1 below). A correctly formulated material law, such as a linear relation between the Cauvhy stress tensor $\boldsymbol{\sigma}$ and the tensor $\frac{1}{2}(\boldsymbol{F} \cdot \boldsymbol{F}^c - \boldsymbol{I})$, will result in zero reaction forces (figure 2 below).

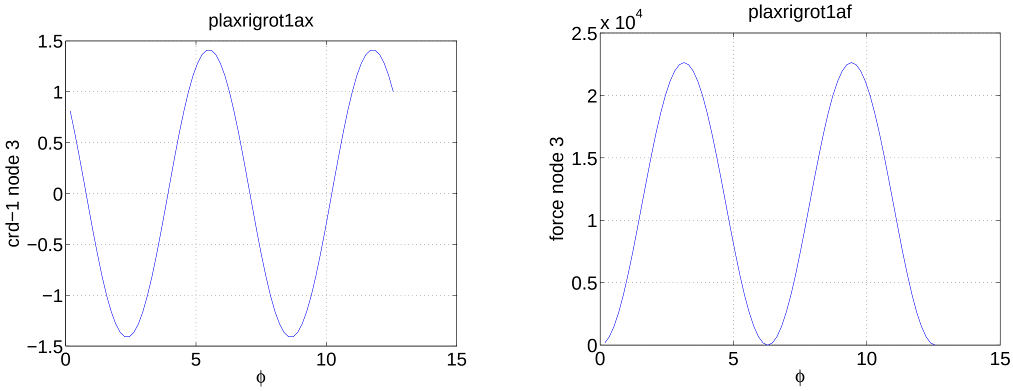


Fig. 12.10 : *Rigid rotation for non-objective elastic material model.*

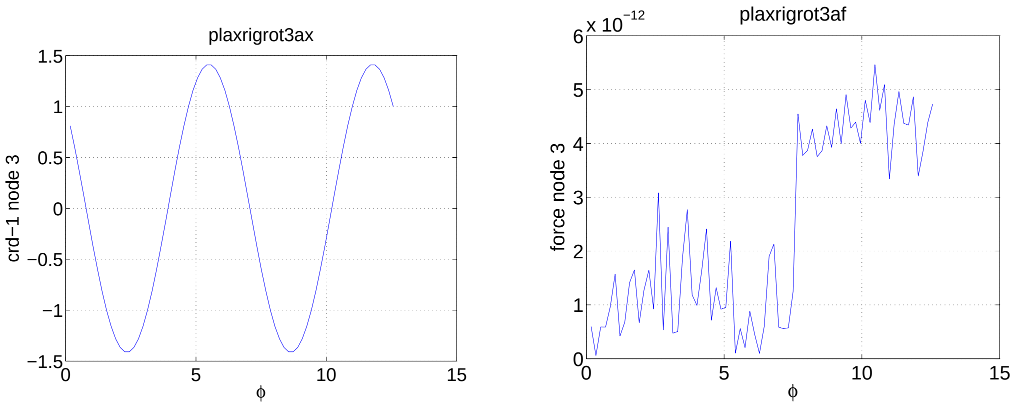


Fig. 12.11 : *Rigid rotation for objective elastic material model.*

Chapter 13

Three-dimensional material models

In this chapter we consider three-dimensional material models for various material behavior. Implementation in finite element software is the main focus, which means that the calculation of the stress and the stiffness during the iterative solution procedure is paramount.

In the following sections we consider models for elastic, elastomeric, elastoplastic, linear viscoelastic, viscoplastic and nonlinear viscoelastic materials behavior. Implementation in FEM modules is explained and simple examples are calculated.

13.1 Elastic material behavior

For elastic materials the 2nd-Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is related to the Green-Lagrange strain tensor $\boldsymbol{E}$. The Cauchy stress tensor can be written as a function of the right Cauchy-Green strain tensor $\boldsymbol{B}$. To assure the stress to be zero when there is no deformation, it is more suitable to relate the Cauchy stress to the Finger tensor $\boldsymbol{A}$.

$$
\begin{aligned}
\boldsymbol{P} &= \boldsymbol{G}(\boldsymbol{E}) && \text{with} \quad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \tfrac{1}{2}(\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}) \\
\boldsymbol{\sigma} &= J^{-1}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = J^{-1}\boldsymbol{F} \cdot \boldsymbol{G}(\boldsymbol{E}) \cdot \boldsymbol{F}^c && \text{with} \quad J = \det(\boldsymbol{F}) \\
&= \boldsymbol{K}(\boldsymbol{A}) && \text{with} \quad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B} - \boldsymbol{I}) = \tfrac{1}{2}(\boldsymbol{F} \cdot \boldsymbol{F}^c - \boldsymbol{I})
\end{aligned}
$$

13.1.1 Isotropic elastic material models

For an isotropic material a principal strain deformation of a material cube can only result in normal stresses on its faces. Using its definition it can be shown that the principal directions of the 2nd-Piola-Kirchhoff stress tensor coincide with the principal strain directions. It is easily seen that the principal directions of the Cauchy stress tensor coincide with those of the Finger tensor.

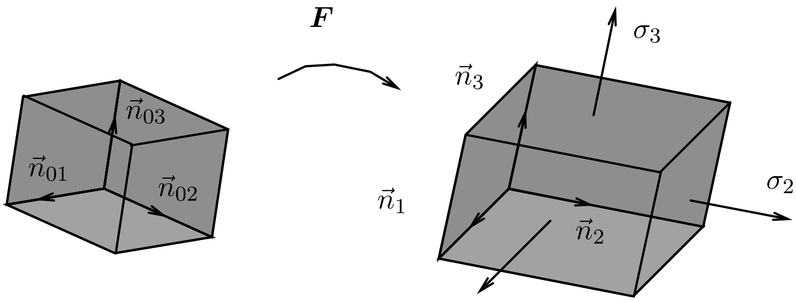


Fig. 13.12 : *Deformation in principal directions*

$$
\begin{aligned}
\boldsymbol{U} &= \lambda_1 \vec{n}_{01}\vec{n}_{01} + \lambda_2 \vec{n}_{02}\vec{n}_{02} + \lambda_3 \vec{n}_{03}\vec{n}_{03} \\
\boldsymbol{R} &= \vec{n}_1\vec{n}_{01} + \vec{n}_2\vec{n}_{02} + \vec{n}_3\vec{n}_{03} \\
\boldsymbol{F} &= \lambda_1 \vec{n}_1\vec{n}_{01} + \lambda_2 \vec{n}_2\vec{n}_{02} + \lambda_3 \vec{n}_3\vec{n}_{03} \\
\boldsymbol{P} &= J\boldsymbol{F}^{-1} \cdot (\sigma_1 \vec{n}_1\vec{n}_1 + \sigma_2 \vec{n}_2\vec{n}_2 + \sigma_3 \vec{n}_3\vec{n}_3) \cdot \boldsymbol{F}^{-c} \\
&= J\left\{\sigma_1 \lambda_1^{-2} \vec{n}_{01}\vec{n}_{01} + \sigma_2 \lambda_2^{-2} \vec{n}_{02}\vec{n}_{02} + \sigma_3 \lambda_3^{-2} \vec{n}_{03}\vec{n}_{03}\right\} \\
&= s_1 \vec{n}_{01}\vec{n}_{01} + s_2 \vec{n}_{01}\vec{n}_{01} + s_3 \vec{n}_{01}\vec{n}_{01}
\end{aligned}
$$

$\boldsymbol{P} - \boldsymbol{E}$ model

For a general isotropic material the principal directions of the 2nd-Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the Green-Lagrange strain tensor $\boldsymbol{E}$, coincide. As a result, $\boldsymbol{P}$ can be written as a polynomial function of $\boldsymbol{E}$.

Applying Cayley-Hamilton's theorem, a second-order polynomial relation remains. The coefficients α_i in this relation are not constant. For the isotropic material they are a function of the invariants of $\boldsymbol{E}$ and have to be determined experimentally.

$$\begin{aligned} \boldsymbol{P} &= s_1 \vec{n}_{01} \vec{n}_{01} + s_2 \vec{n}_{02} \vec{n}_{02} + s_3 \vec{n}_{03} \vec{n}_{03} \\ \boldsymbol{E} &= \varepsilon_1 \vec{n}_{01} \vec{n}_{01} + \varepsilon_2 \vec{n}_{02} \vec{n}_{02} + \varepsilon_3 \vec{n}_{03} \vec{n}_{03} \end{aligned}$$

$$\boldsymbol{P} = \sum s_i \vec{n}_{0i} \vec{n}_{0i} = \boldsymbol{G}(\boldsymbol{E}) = \sum G(\varepsilon_i) \vec{n}_{0i} \vec{n}_{0i} = a_0 \boldsymbol{I} + a_1 \boldsymbol{E} + a_2 \boldsymbol{E}^2 + a_3 \boldsymbol{E}^3 + ...$$

Cayley-Hamilton's theorem $\quad \boldsymbol{E}^3 = J_1(\boldsymbol{E}) \boldsymbol{E}^2 - J_2(\boldsymbol{E}) \boldsymbol{E} + J_3(\boldsymbol{E}) \boldsymbol{I}$

$$\boldsymbol{P} = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{E} + \alpha_2 \boldsymbol{E}^2 \qquad \text{with} \qquad \alpha_i = \alpha_i \left\{ J_1(\boldsymbol{E}), J_2(\boldsymbol{E}), J_3(\boldsymbol{E}) \right\}$$

$\boldsymbol{\sigma} - \boldsymbol{A}$ model

For an isotropic material the principal directions of $\boldsymbol{\sigma}$ and $\boldsymbol{A}$ coincide, which implies that $\boldsymbol{\sigma}$ can be written as a polynomial function of $\boldsymbol{A}$. Applying Cayley-Hamilton's theorem results in a second-order polynomial with coefficients depending on the invariants of $\boldsymbol{A}$.

$$\begin{aligned} \boldsymbol{\sigma} &= \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3 \\ \boldsymbol{A} &= A_1 \vec{n}_1 \vec{n}_1 + A_2 \vec{n}_2 \vec{n}_2 + A_3 \vec{n}_3 \vec{n}_3 \end{aligned}$$

$$\boldsymbol{\sigma} = \sum \sigma_i \vec{n}_i \vec{n}_i = \boldsymbol{K}(\boldsymbol{A}) = \sum K(A_i) \vec{n}_i \vec{n}_i = b_0 \boldsymbol{I} + b_1 \boldsymbol{A} + b_2 \boldsymbol{A}^2 + b_3 \boldsymbol{A}^3 + ...$$

Cayley-Hamilton's theorem $\quad \boldsymbol{A}^3 = J_1(\boldsymbol{A}) \boldsymbol{A}^2 - J_2(\boldsymbol{A}) \boldsymbol{A} + J_3(\boldsymbol{A}) \boldsymbol{I}$

$$\boldsymbol{\sigma} = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{A} + \beta_2 \boldsymbol{A}^2 \qquad \text{with} \qquad \beta_i = \beta_i \left\{ J_1(\boldsymbol{A}), J_2(\boldsymbol{A}), J_3(\boldsymbol{A}) \right\}$$

The constitutive equation for the Cauchy stress tensor can also be derived from the expression for the second Piola-Kirchhoff stress tensor, which reveals that the coefficients β_i are related to the coefficients α_i.

$$\begin{aligned} \boldsymbol{\sigma} &= J^{-1} \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = J^{-1} \boldsymbol{F} \cdot \left[\alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{E} + \alpha_2 \boldsymbol{E}^2 \right] \cdot \boldsymbol{F}^c \\ &= J^{-1} \boldsymbol{F} \cdot \left[(\alpha_0 - \tfrac{1}{2}\alpha_1 + \alpha_2) \boldsymbol{I} + (\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2) \boldsymbol{C} + \tfrac{1}{4}\alpha_2 \boldsymbol{C}^2 \right] \cdot \boldsymbol{F}^c \\ &= \{ J_3(\boldsymbol{B}) \}^{-1/2} \left[(\alpha_0 - \tfrac{1}{2}\alpha_1 + \alpha_2) \boldsymbol{B} + (\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2) \boldsymbol{B}^2 + \tfrac{1}{4}\alpha_2 \boldsymbol{B}^3 \right] \\ &\qquad\qquad \boldsymbol{B}^3 = J_1(\boldsymbol{B}) \boldsymbol{B}^2 - J_2(\boldsymbol{B}) \boldsymbol{B} + J_3(\boldsymbol{B}) \boldsymbol{I} \\ &= J_3^{-1/2} \left[(\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 + \tfrac{1}{4}\alpha_2 J_1) \boldsymbol{B}^2 + (\alpha_0 - \tfrac{1}{2}\alpha_1 + \alpha_2 - \tfrac{1}{4}\alpha_2 J_2) \boldsymbol{B} + \tfrac{1}{4}\alpha_2 J_3 \boldsymbol{I} \right] \\ &\qquad\qquad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B} - \boldsymbol{I}) \quad \rightarrow \quad \boldsymbol{B} = 2\boldsymbol{A} + \boldsymbol{I} \\ &\qquad\qquad \boldsymbol{A}^2 = \tfrac{1}{4}\boldsymbol{B}^2 - \tfrac{1}{2}\boldsymbol{B} + \tfrac{1}{4}\boldsymbol{I} \quad \rightarrow \quad \boldsymbol{B}^2 = 4\boldsymbol{A}^2 + 2\boldsymbol{B} - \boldsymbol{I} \\ &= J_3^{-1/2} \left[(2\alpha_1 - 2\alpha_2 + \alpha_2 J_1) \boldsymbol{A}^2 + (\alpha_0 + \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 J_1 - \tfrac{1}{4}\alpha_2 J_2) \boldsymbol{A} + \right. \\ &\qquad\qquad \left. (\alpha_0 + \alpha_1 - \tfrac{1}{2}\alpha_2 + \tfrac{3}{4}\alpha_2 J_1 - \tfrac{1}{4}\alpha_2 J_2 + \tfrac{1}{4}\alpha_2 J_3) \boldsymbol{I} \right] \\ &= \beta_2 \boldsymbol{A}^2 + \beta_1 \boldsymbol{A} + \beta_0 \boldsymbol{I} \end{aligned}$$

Linear elastic material models

The above isotropic elastic material models are nonlinear. The polynomial functions have a quadratic tensor term and, moreover, the coefficients are functions of the three invariants of the strain tensor. The first invariant is a linear, the second a quadratic and the third a cubic function of the tensor.

Simplification towards purely linear models is possible and allowed if it suits the experimental observations.

$\boldsymbol{P} - \boldsymbol{E}$ model

When experiments show that the relation between $\boldsymbol{P}$ and $\boldsymbol{E}$ is linear, conclusions can be drawn concerning the coefficients α_i. The coefficient of the quadratic term, α_2, must be zero. The coefficient of the linear term, α_1, must be a constant. The coefficient of the unit tensor may be a linear, isotropic function of the tensor $\boldsymbol{E}$, which means it can be written as a constant times the trace of $\boldsymbol{E}$. No constant tensor is contained in the linear models, because stress has to be zero at zero strain.

Substituting an (experimentally motivated) linear relation between $\boldsymbol{P}$ and $\boldsymbol{E}$ in the definition relation of $\boldsymbol{\sigma}$, results in a nonlinear relation between $\boldsymbol{\sigma}$ and $\boldsymbol{B}$ and vice versa.

$$\boldsymbol{P} = c_0 \mathrm{tr}(\boldsymbol{E})\boldsymbol{I} + c_1\boldsymbol{E}$$

Tensile test

For a tensile test only the axial component of $\boldsymbol{P}$ is non-zero and can be expressed in the axial and cross-sectional stretch ratios λ and μ. Because stresses perpendicular to the axial direction are zero, μ can be eliminated and the axial stress can be expressed in the axial stretch λ.

The parameters c_0 and c_1 can be expressed in the more commonly used Young's modulus E and Poisson's ratio ν.

$$\left.\begin{aligned} P &= c_0\tfrac{1}{2}(\lambda^2-1) + 2c_0\tfrac{1}{2}(\mu^2-1) + c_1\tfrac{1}{2}(\lambda^2-1) \\ 0 &= c_0\tfrac{1}{2}(\lambda^2-1) + 2c_0\tfrac{1}{2}(\mu^2-1) + c_1\tfrac{1}{2}(\mu^2-1) \end{aligned}\right\} \quad \rightarrow$$

$$\tfrac{1}{2}(\mu^2-1) = -\frac{c_0}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = -\nu\,\tfrac{1}{2}(\lambda^2-1)$$

$$P = \frac{c_1(3c_0+c_1)}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = E\,\tfrac{1}{2}(\lambda^2-1)$$

$$F = \sigma A = \frac{\lambda}{\mu^2}P\mu^2 A_0 = \lambda P A_0 = \tfrac{1}{2}\lambda(\lambda^2-1)\,E A_0$$

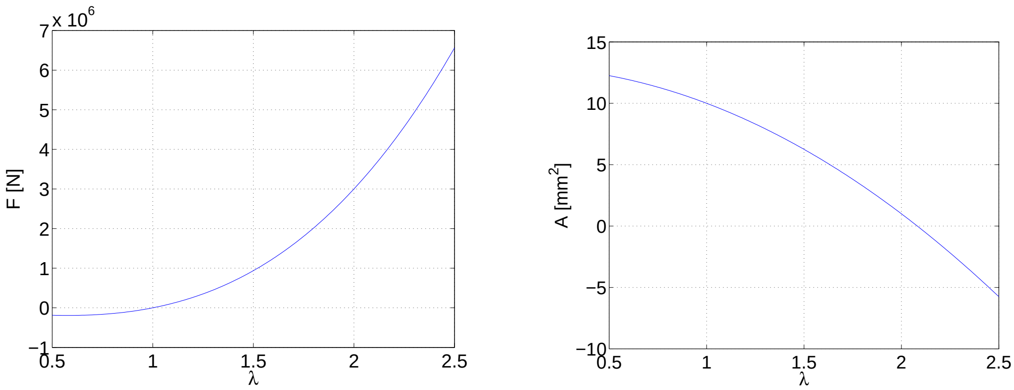


Fig. 13.13 : *Tensile test: force and cross-sectional area against stretch ratio*

Simple shear test : plane strain
For plane strain $F_{33} = 1$ holds and thus for the simple shear test det $\boldsymbol{F} = 1$. To calculate the shear and normal force, the Cauchy stress has to be derived from the material model.

$$
\begin{aligned}
&\boldsymbol{F} = \boldsymbol{I} + \gamma\vec{e}_1\vec{e}_2 \quad \rightarrow \quad J = \det(\boldsymbol{F}) = 1 \\
&\boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F} = \boldsymbol{I} + \gamma^2\vec{e}_2\vec{e}_2 + \gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
&\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \tfrac{1}{2}\gamma^2\vec{e}_2\vec{e}_2 + \tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
&\boldsymbol{P} = c_0\tfrac{1}{2}\gamma^2\boldsymbol{I} + c_1\tfrac{1}{2}\gamma^2\vec{e}_2\vec{e}_2 + c_1\tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
&\boldsymbol{\sigma} = J^{-1}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c \\
&\quad = \tfrac{1}{2}\left\{(c_0\gamma^2 + c_0\gamma^4 + c_1 2\gamma^2 + c_1\gamma^4)\vec{e}_1\vec{e}_1 + (c_0\gamma^2 + c_1\gamma^2)\vec{e}_2\vec{e}_2 + c_0\gamma^2\vec{e}_3\vec{e}_3 + \right. \\
&\qquad\quad \left.(c_0\gamma^3 + c_1\gamma + c_1\gamma^3)(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)\right\} \\
&p_n = \vec{e}_2 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \tfrac{1}{2}(c_0 + c_1)\gamma^2 \quad ; \quad p_s = \vec{e}_1 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \tfrac{1}{2}c_1\gamma + \tfrac{1}{2}(c_0 + c_1)\gamma^3 \\
&F_n = p_n d_0 w_0 \quad ; \quad F_s = p_s d_0 w_0
\end{aligned}
$$

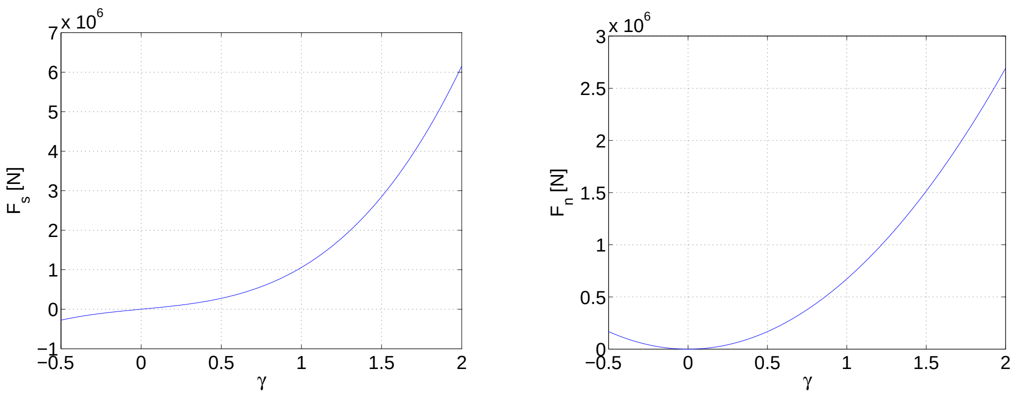


Fig. 13.14 : *Shear test plane strain: shear force and normal force against shear*

Simple shear test : plane stress
For plane strain $Gs_{33} = P_{33} = 0$ holds and thus for the simple shear test E_{33} can be expressed in E_{11} and E_{22}. To calculate the shear and normal force, the Cauchy stress has to be derived from the material model.

$$\sigma_{33} = P_{33} = 0 \quad \rightarrow \quad c_0(E_{11} + E_{22} + E_{33}) + c_1 E_{33} = 0 \quad \rightarrow \quad E_{33} = -\frac{c_0}{c_0 + c_1}(E_{11} + E_{22})$$

$$\begin{aligned}
\boldsymbol{F} &= \boldsymbol{I} + (F_{33} - 1)\vec{e}_3\vec{e}_3 + \gamma\vec{e}_1\vec{e}_2 \\
\boldsymbol{E} &= \tfrac{1}{2}(\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}) = \tfrac{1}{2}\left[\gamma^2\vec{e}_2\vec{e}_2 + \gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + \left\{2(F_{33} - 1) + (F_{33} - 1)^2\right\}\vec{e}_3\vec{e}_3\right] \\
F_{33} &= \sqrt{2E_{33} + 1} \quad \rightarrow \quad J = \det(\boldsymbol{F}) = F_{33} = \sqrt{2E_{33} + 1} \\
\boldsymbol{P} &= \frac{c_0 c_1}{c_0 + c_1}(E_{11} + E_{22}) + c_1\boldsymbol{E} \\
&= \frac{c_0 c_1}{c_0 + c_1}\tfrac{1}{2}\gamma^2\boldsymbol{I} + c_1\tfrac{1}{2}\gamma^2\vec{e}_2\vec{e}_2 + c_1\tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
\boldsymbol{\sigma} &= J^{-1}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = J^{-1}\left[\boldsymbol{P} + (\gamma P_{12} + \gamma P_{21} + \gamma^2 P_{22})\vec{e}_1\vec{e}_1 + \gamma P_{22}(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)\right] \\
p_n &= \vec{e}_2 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \sigma_{22} \quad ; \quad p_s = \vec{e}_1 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \sigma_{12} \\
F_n &= p_n d w_0 = p_n F_{33} d_0 w_0 \quad ; \quad F_s = p_s d w_0 = p_s F_{33} d_0 w_0
\end{aligned}$$

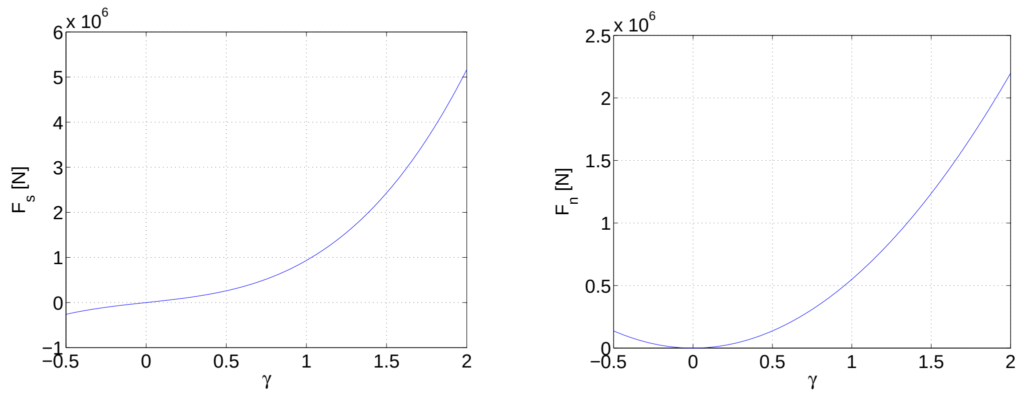


Fig. 13.15 : *Shear test plane stress: shear force and normal force against shear*

$\boldsymbol{\sigma} - \boldsymbol{A}$ model

For linear isotropic behavior the relation between $\boldsymbol{\sigma}$ and $\boldsymbol{A}$ is also characterised by two material constants.

$$\boldsymbol{\sigma} = c_0 \mathrm{tr}(\boldsymbol{A})\boldsymbol{I} + c_1\boldsymbol{A}$$

Tensile test
For a tensile test only the axial component of $\boldsymbol{\sigma}$ is non-zero and can be expressed in the axial and cross-sectional stretch ratios λ and μ. Because stresses perpendicular to the axial

direction are zero, μ can be eliminated and the axial stress can be expressed in the axial stretch λ. The parameters c_0 and c_1 can be expressed in the more commonly used Young's modulus E and Poisson's ratio ν.

$$\left.\begin{aligned} \sigma &= c_0\,\tfrac{1}{2}(\lambda^2-1) + 2c_0\,\tfrac{1}{2}(\mu^2-1) + c_1\,\tfrac{1}{2}(\lambda^2-1) \\ 0 &= c_0\,\tfrac{1}{2}(\lambda^2-1) + 2c_0\,\tfrac{1}{2}(\mu^2-1) + c_1\,\tfrac{1}{2}(\mu^2-1) \end{aligned}\right\} \quad \rightarrow$$

$$\tfrac{1}{2}(\mu^2-1) = -\frac{c_0}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = -\nu\,\tfrac{1}{2}(\lambda^2-1)$$

$$\sigma = \frac{c_0(3c_0+c_1)}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = E\,\tfrac{1}{2}(\lambda^2-1)$$

$$F = \sigma A = \sigma\mu^2 A_0 = \tfrac{1}{2}(\lambda^2-1)\{1-\nu(\lambda^2-1)\}\,EA_0$$

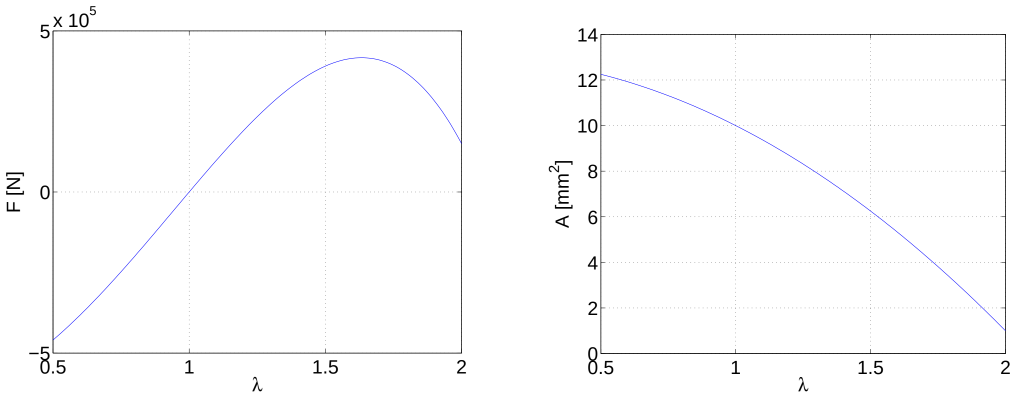


Fig. 13.16 : *Tensile test: force and cross-sectional area against stretch ratio*

Simple shear test : plane strain

A simple shear test for plane strain can also be calculated straightforwardly.

$$\begin{aligned} &\boldsymbol{F} = \boldsymbol{I} + \gamma\,\vec{e}_1\vec{e}_2 \\ &\boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^c = \boldsymbol{I} + \gamma^2\vec{e}_1\vec{e}_1 + \gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\ &\boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B}-\boldsymbol{I}) = \tfrac{1}{2}\,\gamma^2\vec{e}_1\vec{e}_1 + \tfrac{1}{2}\,\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \end{aligned}$$

$$\begin{aligned} &\boldsymbol{\sigma} = c_0\tfrac{1}{2}\gamma^2\boldsymbol{I} + c_1\tfrac{1}{2}\gamma^2\vec{e}_1\vec{e}_1 + c_1\tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\ &\sigma_{33} = c_0\tfrac{1}{2}\gamma^2 \\ &p_n = \vec{e}_2\cdot\boldsymbol{\sigma}\cdot\vec{e}_2 = c_0\tfrac{1}{2}\gamma^2 \quad ; \quad p_s = \vec{e}_1\cdot\boldsymbol{\sigma}\cdot\vec{e}_2 = c_1\tfrac{1}{2}\gamma \\ &F_n = p_n d_0 w_0 \quad ; \quad F_s = p_s d_0 w_0 \end{aligned}$$

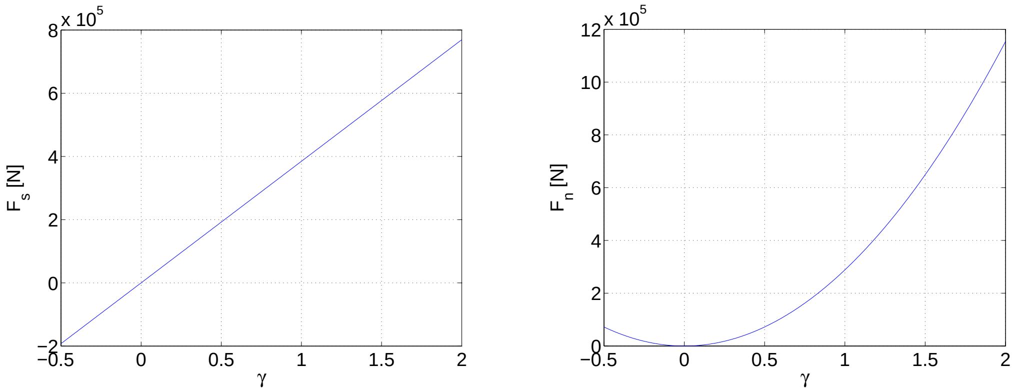


Fig. 13.17 : *Shear test plane strain: shear force and normal force against shear*

Simple shear test : plane stress
A simple shear test for plane stress needs some more consideration. For plane stress, it is assumed here that $\sigma_{33} = 0$. This assumption results in a relation for $F_{33} = \boldsymbol{F} \cdot \vec{e}_3$. For simple shear, we can than calculate the normal and shear stress. The normal and shear forces must be calculated, taking the deformed area into account.

$$
\begin{aligned}
&\sigma_{33} = c_0(A_{11} + A_{22} + A_{33}) + c_1 A_{33} = 0 \quad \rightarrow \\
&A_{33} = -\frac{c_0}{c_0 + c_1}(A_{11} + A_{22}) \quad \rightarrow \quad F_{33} = \sqrt{2A_{33} + 1} \\
&\boldsymbol{\sigma} = \frac{c_0 c_1}{c_0 + c_1}(A_{11} + A_{22})\boldsymbol{I} + c_1 \boldsymbol{A}
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{A} = \tfrac{1}{2}\gamma^2 \vec{e}_1\vec{e}_1 + \tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
&\boldsymbol{\sigma} = \frac{c_0 c_1}{c_0 + c_1}\tfrac{1}{2}\gamma^2 \boldsymbol{I} + c_1 \tfrac{1}{2}\gamma^2 \vec{e}_1\vec{e}_1 + c_1 \tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
&p_n = \vec{e}_2 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \frac{c_0 c_1}{c_0 + c_1}\tfrac{1}{2}\gamma^2 \quad ; \quad p_s = \vec{e}_1 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = c_1 \tfrac{1}{2}\gamma \\
&F_n = p_n d w_0 = p_n F_{33} d_0 w_0 \quad ; \quad F_s = p_s d w_0 = p_s F_{33} d_0 w_0
\end{aligned}
$$

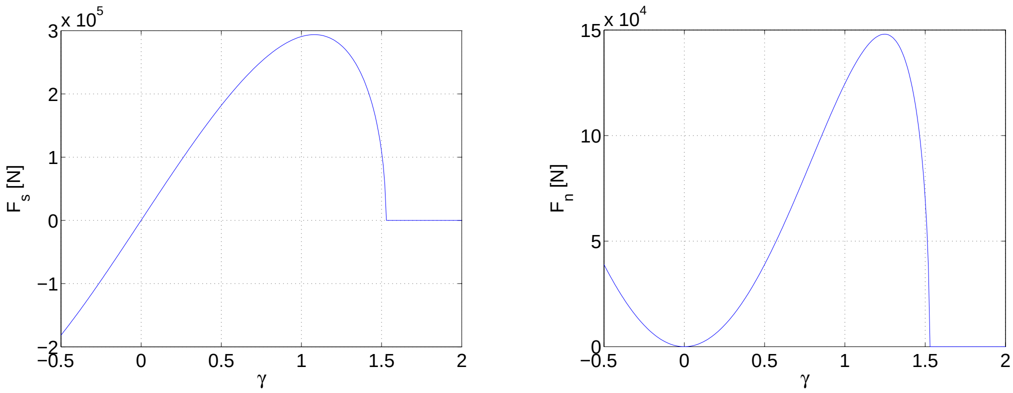


Fig. 13.18 : *Shear test plane stress: shear force and normal force against shear*

13.1.2 Hyper-elastic material models

When an explicit stored energy function is available for an elastic material, it is called hyper-elastic. The stress tensor can then be calculated as the derivative of the energy function with respect to the associated strain tensor.

When the stress-strain relation is not derived from a stored energy function, the elastic model is called hypo-elastic. For large strains such a model predicts the elastic behavior not correctly. In a closed cycle deformation loop residual stresses and elastic energy will remain. A hyper-elastic material model describes large elastic strains correctly.

The elastic energy must of course always become zero when there is no deformation. The function can be formulated with various strain tensors. The stress tensor can be derived by differentiation of the stored energy function with respect to the strain tensor.

The second Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is derived from an energy function $\phi(\boldsymbol{E})$, depending on the Green-Lagrange strain tensor. Instead of ϕ, a function $W(\boldsymbol{C})$ is usually specified. Although the stress tensor can now still be derived by differentiation – in this case W w.r.t. $\boldsymbol{C}$ – an additional requirement must be formulated or incorporated, namely that stress must be zero ($\boldsymbol{P} = \boldsymbol{O}$) when there is no deformation ($\boldsymbol{C} = \boldsymbol{I}$).

$$\phi = \phi(\boldsymbol{E}) \quad \rightarrow \quad W = W(\boldsymbol{C}) \qquad \rightarrow$$

$$\boldsymbol{P} = \frac{d\phi(d\boldsymbol{E})}{\boldsymbol{E}} = \frac{dW(\boldsymbol{C})}{d\boldsymbol{C}} : \frac{d\boldsymbol{C}}{d\boldsymbol{E}} = 2\frac{dW(\boldsymbol{C})}{d\boldsymbol{C}} = \boldsymbol{G}(\boldsymbol{E})$$

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c = \frac{2}{J}\boldsymbol{F}\cdot\frac{dW(\boldsymbol{C})}{d\boldsymbol{C}}\cdot\boldsymbol{F}^c$$

Isotropic hyper-elastic material models

For isotropic material the elastic energy function can be written as a function of the invariants of $\boldsymbol{E}$ or $\boldsymbol{C}$.

Isotropic hyper-elastic model : $\boldsymbol{P} - \boldsymbol{E}$

For isotropic material the energy function $\phi(\boldsymbol{E})$ is only depending on the invariants of the strain tensor. Again we see that $\boldsymbol{P}$ can be written as a second-order polynomial of $\boldsymbol{E}$. The coefficients α_i are now functions of the derivatives of ϕ w.r.t. the invariants of $\boldsymbol{E}$.

$$\phi = \phi(\boldsymbol{E}) = \phi\{J_1(\boldsymbol{E}), J_2(\boldsymbol{E}), J_3(\boldsymbol{E})\} \quad \rightarrow \quad \boldsymbol{P} = \frac{\partial\phi}{\partial J_1}\frac{dJ_1}{d\boldsymbol{E}} + \frac{\partial\phi}{\partial J_2}\frac{dJ_2}{d\boldsymbol{E}} + \frac{\partial\phi}{\partial J_3}\frac{dJ_3}{d\boldsymbol{E}}$$

derivatives of invariants

$$\frac{dJ_1}{d\boldsymbol{E}} = \boldsymbol{I} \quad ; \quad \frac{dJ_2}{d\boldsymbol{E}} = J_1\boldsymbol{I} - \boldsymbol{E} \quad ; \quad \frac{dJ_3}{d\boldsymbol{E}} = J_2\boldsymbol{I} - J_1\boldsymbol{E} + \boldsymbol{E}^2 \qquad \rightarrow$$

stress tensor

$$
\begin{aligned}
\boldsymbol{P} &= \left(\frac{\partial \phi}{\partial J_1} + \frac{\partial \phi}{\partial J_2} J_1 + \frac{\partial \phi}{\partial J_3} J_2 \right) \boldsymbol{I} + \left(- \frac{\partial \phi}{\partial J_2} - \frac{\partial \phi}{\partial J_3} J_1 \right) \boldsymbol{E} + \frac{\partial \phi}{\partial J_3} \boldsymbol{E}^2 \\
&= \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{E} + \alpha_2 \boldsymbol{E}^2
\end{aligned}
$$

Isotropic hyper-elastic model : $\boldsymbol{P} - \boldsymbol{C}$

For isotropic material the energy function $\phi(\boldsymbol{E})$ is only depending on the invariants of the strain tensor. Again we see that $\boldsymbol{P}$ can be written as a second-order polynomial of $\boldsymbol{E}$. The coefficients α_i are now functions of the derivatives of ϕ w.r.t. the invariants of $\boldsymbol{E}$.

$$
W = W(\boldsymbol{C}) = W\{J_1(\boldsymbol{C}), J_2(\boldsymbol{C}), J_3(\boldsymbol{C})\} \quad \rightarrow \quad \boldsymbol{P} = 2 \left(\frac{\partial W}{\partial J_1} \frac{dJ_1}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_2} \frac{dJ_2}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_3} \frac{dJ_3}{d\boldsymbol{C}} \right)
$$

derivatives of invariants

$$
\frac{dJ_1}{d\boldsymbol{C}} = \boldsymbol{I} \quad ; \quad \frac{dJ_2}{d\boldsymbol{C}} = J_1 \boldsymbol{I} - \boldsymbol{C} \quad ; \quad \frac{dJ_3}{d\boldsymbol{C}} = J_2 \boldsymbol{I} - J_1 \boldsymbol{C} + \boldsymbol{C}^2 \quad \rightarrow
$$

stress tensor

$$
\begin{aligned}
\boldsymbol{P} &= 2 \left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2} J_1 + \frac{\partial W}{\partial J_3} J_2 \right) \boldsymbol{I} + 2 \left(- \frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3} J_1 \right) \boldsymbol{C} + 2 \frac{\partial W}{\partial J_3} \boldsymbol{C}^2 \\
&= \bar{\alpha}_0 \boldsymbol{I} + \bar{\alpha}_1 \boldsymbol{E} + \bar{\alpha}_2 \boldsymbol{E}^2
\end{aligned}
$$

Isotropic hyper-elastic model : $\boldsymbol{\sigma} - \boldsymbol{A}$

The Cauchy stress tensor is a function of the 2nd-Piola-Kirchhoff stress tensor and can thus be derived from the elastic energy function $W(\boldsymbol{C})$.

$$
\begin{aligned}
\boldsymbol{\sigma} &= \frac{1}{J} \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = \frac{2}{J} \boldsymbol{F} \cdot \frac{dW(\boldsymbol{C})}{d\boldsymbol{C}} \cdot \boldsymbol{F}^c \\
&= \frac{2}{\sqrt{J_3}} \boldsymbol{F} \cdot \left(\frac{\partial W}{\partial J_1} \frac{dJ_1}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_2} \frac{dJ_2}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_3} \frac{dJ_3}{d\boldsymbol{C}} \right) \cdot \boldsymbol{F}^c \\
&= \frac{2}{\sqrt{J_3}} \boldsymbol{F} \cdot \left\{ \left(\frac{\partial W}{\partial J_1} + J_1 \frac{\partial W}{\partial J_2} + J_2 \frac{\partial W}{\partial J_3} \right) \boldsymbol{I} + \left(- \frac{\partial W}{\partial J_2} - J_1 \frac{\partial W}{\partial J_3} \right) \boldsymbol{C} + \left(\frac{\partial W}{\partial J_3} \right) \boldsymbol{C}^2 \right\} \cdot \boldsymbol{F}^c \\
&= \frac{2}{\sqrt{J_3}} \boldsymbol{F} \cdot \left(\gamma_0 \boldsymbol{I} + \gamma_1 \boldsymbol{C} + \gamma_2 \boldsymbol{C}^2 \right) \cdot \boldsymbol{F}^c = \frac{2}{\sqrt{J_3}} \left(\gamma_0 \boldsymbol{B} + \gamma_1 \boldsymbol{B}^2 + \gamma_2 \boldsymbol{B}^3 \right) \\
& \qquad\qquad \boldsymbol{B}^3 = J_1 \boldsymbol{B}^2 - J_2 \boldsymbol{B} + J_3 \boldsymbol{I} \\
&= \frac{2}{\sqrt{J_3}} \left[(\gamma_1 + \gamma_2 J_1) \boldsymbol{B}^2 + (\gamma_0 - \gamma_2 J_2) \boldsymbol{B} + (\gamma_2 J_3) \boldsymbol{I} \right] \\
& \qquad\qquad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B} - \boldsymbol{I}) \quad \rightarrow \quad \boldsymbol{B} = 2\boldsymbol{A} + \boldsymbol{I} \quad \rightarrow \quad \boldsymbol{B}^2 = 4\boldsymbol{A}^2 + 2\boldsymbol{B} - \boldsymbol{I} \\
&= \frac{2}{\sqrt{J_3}} \left[(4\gamma_1 + 4\gamma_2 J_1) \boldsymbol{A}^2 + (\gamma_0 + 2\gamma_1 + 2\gamma_2 J_1 - \gamma_2 J_2) \boldsymbol{A} + \right. \\
& \qquad\qquad\qquad \left. (\gamma_0 - \gamma_1 + \gamma_2 J_1 - \gamma_2 J_2 + \gamma_2 J_3) \boldsymbol{I} \right] \\
&= \beta_2 \boldsymbol{A}^2 + \beta_1 \boldsymbol{A} + \beta_0 \boldsymbol{I}
\end{aligned}
$$

Incompressibility

For a hyper-elastic material model the stress-strain relation is derived from an energy function $W(\boldsymbol{C})$. For an isotropic material W is a function of the invariants of $\boldsymbol{C}$. Due to the incompressibility, the energy function cannot depend on the third invariant, which has always the value 1.

From a given function $W(\boldsymbol{C})$, the 2nd-Piola-Kirchhoff stress tensor can be determined by differentiation. Subsequently the Cauchy stress tensor can be calculated from $\boldsymbol{P}$.

$$
\begin{aligned}
& J = \det(\boldsymbol{F}) = 1 \quad \rightarrow \quad \det(\boldsymbol{C}) = J_3(\boldsymbol{C}) = 1 \quad \rightarrow \quad W(\boldsymbol{C}) = W\{J_1(\boldsymbol{C}), J_2(\boldsymbol{C})\} \\
& \boldsymbol{P} = 2\left(\frac{\partial W}{\partial J_1}\frac{dJ_1}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_2}\frac{dJ_2}{d\boldsymbol{C}}\right) = 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\boldsymbol{I} - \frac{\partial W}{\partial J_2}\boldsymbol{C}\right\} \\
& \boldsymbol{\sigma} = \boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c = 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\boldsymbol{B} - \frac{\partial W}{\partial J_2}\boldsymbol{B}^2\right\}
\end{aligned}
$$

Elastic material behavior can be described by a relation between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the left Cauchy-Green strain tensor $\boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^c$. When the material is incompressible and isotropic, the deformation will not be affected by the addition of a hydrostatic stress $p\boldsymbol{I}$.

When the deformation is known, the stress cannot be determined, because the hydrostatic stress remains arbitrary. Only the so-called *extra stress tensor* $\boldsymbol{\tau}$ depends solely on $\boldsymbol{B}$ and can be calculated.

To determine the unknown hydrostatic stress $p\boldsymbol{I}$ the incompressibility condition must be used.

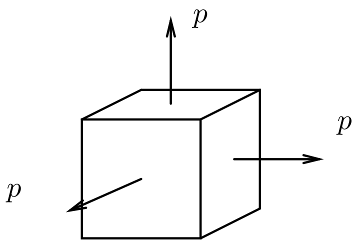


Fig. 13.19 : *Hydrostatic stress state*

$$
\begin{aligned}
\boldsymbol{\sigma} &= -\,p\boldsymbol{I} + \boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c = -\,p\boldsymbol{I} + 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\boldsymbol{B} - \frac{\partial W}{\partial J_2}\boldsymbol{B}^2\right\} \\
&= -\,p\boldsymbol{I} + \boldsymbol{\tau}
\end{aligned}
$$

13.1.3 Rivlin models

Energy functions $W(\boldsymbol{C})$ are generally written as polynomials of $(J_1 - 3)$ and $(J_2 - 3)$ such that $W = 0$ when there is no deformation ($\boldsymbol{C} = \boldsymbol{I} \rightarrow J_1 = J_2 = 3$). The invariants of $\boldsymbol{C}$ can be expressed in the principal stretch ratios λ_1, λ_2 and λ_3. The polynomial energy function $W(\boldsymbol{C})$ can then also be written as a polynomial function of these stretch ratios. This way of denoting these functions is often referred to as the Rivlin formulation.

$$W(\boldsymbol{C}) = \sum_{i=0}^{m}\sum_{j=0}^{n} C_{ij}\{J_1(\boldsymbol{C}) - 3\}^i\{J_2(\boldsymbol{C}) - 3\}^j \qquad \text{with} \quad C_{00} = 0$$

$$\begin{aligned}
J_1 &= \text{tr}(\boldsymbol{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
J_2 &= \tfrac{1}{2}\left\{\text{tr}^2(\boldsymbol{C}) - \text{tr}(\boldsymbol{C}^2)\right\} \\
&= \tfrac{1}{2}\left\{\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)^2 - \left(\lambda_1^4 + \lambda_2^4 + \lambda_3^4\right)\right\} = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\
J_3 &= \det(\boldsymbol{C}) = \lambda_1^2\lambda_2^2\lambda_3^2 = 1
\end{aligned}$$

$$W(\boldsymbol{C}) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)^i \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3\right)^j$$

Neo-Hookean model

The Neo-Hookean energy function has only one material parameter : C_{10}. The model describes the mechanical behavior of natural rubbers rather well.

$$\begin{aligned}
W &= C_{10}(J_1 - 3) \\
\boldsymbol{\sigma} &= -\,p\boldsymbol{I} + 2C_{10}\boldsymbol{B}
\end{aligned}$$

Tensile test
In a tensile test, the axial stress is σ and stresses perpendicular to the axial are zero. The hydrostatic pressure can then be eliminated and the axial stress and force can be expressed in the axial stretch ratio λ. The incompressibility condition $\lambda\mu^2 = 1$ is taken into account.

$$\begin{aligned}
&\boldsymbol{B} = \lambda^2\vec{e}_1\vec{e}_1 + \mu^2(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) = \lambda^2\vec{e}_1\vec{e}_1 + \frac{1}{\lambda}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) \\
&\sigma = -p\boldsymbol{I} + 2C_{10}\lambda^2\vec{e}_1\vec{e}_1 + 2C_{10}\frac{1}{\lambda}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) \\
&\left.\begin{aligned} \sigma &= -p + 2C_{10}\lambda^2 \\ 0 &= -p + 2C_{10}\frac{1}{\lambda} \end{aligned}\right\} \quad \rightarrow \quad \sigma = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) \\
&F = \sigma A = \sigma\mu^2 A_0 = \sigma\frac{1}{\lambda}A_0 = 2C_{10}A_0(\lambda - \frac{1}{\lambda^2})
\end{aligned}$$

Mooney-Rivlin model

The mechanical behavior of industrial rubbers cannot be captured well with the one-parameter Neo-Hookean model. Instead the Mooney-Rivlin model is often used, which has two parameters.

$$\begin{aligned}
W &= C_{10}(J_1 - 3) + C_{01}(J_2 - 3) \\
\boldsymbol{\sigma} &= -\,p\boldsymbol{I} + 2\{C_{10} + C_{01}\text{tr}(\boldsymbol{B})\}\boldsymbol{B} - 2C_{01}\boldsymbol{B}^2
\end{aligned}$$

Tensile test
In a tensile test the hydrostatic pressure can be eliminated and the axial stress and force can be expressed in the axial stretch ratio λ.

$$\boldsymbol{B} = \lambda^2 \vec{e}_1\vec{e}_1 + \mu^2(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) = \lambda^2 \vec{e}_1\vec{e}_1 + \frac{1}{\lambda}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) \quad ; \quad \mathrm{tr}(\boldsymbol{B}) = \lambda^2 + \frac{2}{\lambda}$$

$$\boldsymbol{B}^2 = \lambda^4 \vec{e}_1\vec{e}_1 + \frac{1}{\lambda^2}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)$$

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\{\lambda^2 \vec{e}_1\vec{e}_1 + \frac{1}{\lambda}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)\} - 2C_{01}\{\lambda^4 \vec{e}_1\vec{e}_1 + \frac{1}{\lambda^2}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)\}$$

$$\left.\begin{aligned} \sigma &= -p + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\lambda^2 - 2C_{01}\lambda^4 \\ 0 &= -p + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\frac{1}{\lambda} - 2C_{01}\frac{1}{\lambda^2} \end{aligned}\right\} \quad \rightarrow \quad \sigma = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) + 2C_{01}(\lambda - \frac{1}{\lambda^2})$$

$$F = \sigma A = \sigma \mu^2 A_0 = \sigma \frac{1}{\lambda} A_0 = 2A_0\{C_{10}(\lambda - \frac{1}{\lambda^2}) + C_{01}(1 - \frac{1}{\lambda^3})\}$$

Other energy functions

There are a lot of energy functions used for different elastomeric materials. They all belong to the polynomial energy functions.

3-term Mooney-Rivlin	$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3)$
Signiorini	$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{20}(J_1 - 3)^2$
Yeoh	$W = c_{10}(J_1 - 3) + c_{20}(J_1 - 3)^2 + c_{30}(J_1 - 3)^3$

2nd-order invariant model

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3) + c_{20}(J_1 - 3)^2$$

Kloaner-Segal

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{20}(J_1 - 3)^2 + c_{03}(J_2 - 3)^3$$

James, Green, Simpson (3rd-order deformation model)

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3) + c_{20}(J_1 - 3)^2 + c_{30}(J_1 - 3)^3$$

13.1.4 Ogden models

For slightly compressible materials Ogden models are used. The strain energy function is written in terms of the principal stretch ratios. The first part of the Ogden function can be shown to be purely deviatoric/ The second part accounts for the volumetric deformation. Because the volumetric behavior is characterized by a constant bulk modulus, the model is confined to slightly compressible deformation.

To describe the mechanical behavior of elastomeric materials, which show large volumetric deformations, the foam model can be used. The first part of the energy function is not purely deviatoric.

$$W = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n} J^{\frac{-\alpha_n}{3}} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3 \right) + 4.5K \left(1 - J^{\frac{1}{3}} \right)^2$$

with

μ_n	:	moduli
α_n	:	exponents
K	:	bulk modulus
J	:	volume ratio $= \det(\boldsymbol{F})$

foam model

$$W = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3 \right) + \sum_{n=1}^{N} \frac{\mu_n}{\beta_n} \left(1 - J^{\beta_n} \right)$$

13.1.5 Incremental analysis

In nonlinear analysis, the load is applied in a number of steps, the *load increments*.

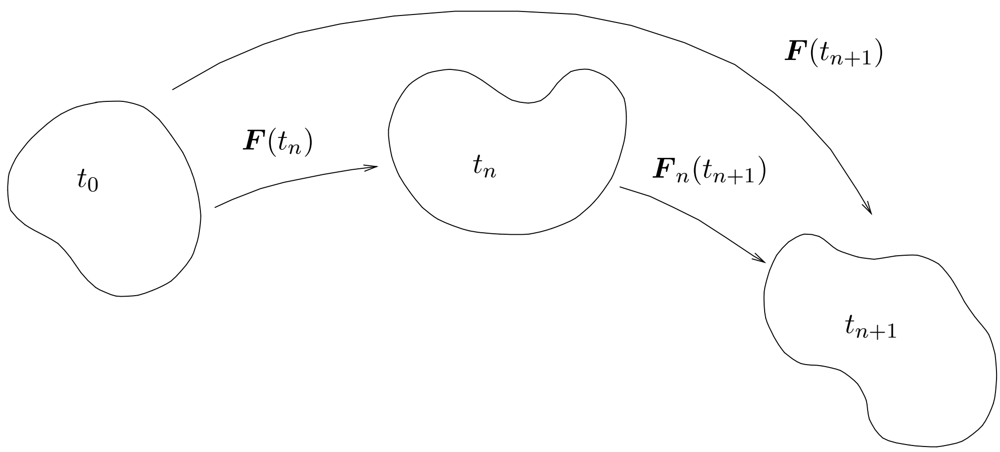


Fig. 13.20 : *Incremental deformation*

The end-increment state, i.e. deformation and stresses, must be determined such that equilibrium equations, material relations and boundary conditions are satisfied. Due to the nonlinear character of deformation and material behavior, the equations must be solved iteratively. In each iteration step both the stress and the material stiffness must be updated.

Linear P-E model

Stress update

Elastic material behavior may be described by a linear relation between the second Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the Green-Lagrange strain tensor $\boldsymbol{E}$. This relation can be derived from an elastic energy function and that is why this model is called hyper-elastic.

For a given deformation the stress in the material can be calculated directly for an elastic material. The Cauchy stress tensor $\boldsymbol{\sigma}$ can be calculated from $\boldsymbol{P}$.

$$
\begin{aligned}
\boldsymbol{P} &= c_0 \operatorname{tr}(\boldsymbol{E})\boldsymbol{I} + c_1\boldsymbol{E} & \text{with} \qquad \boldsymbol{E} &= \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \\
&= \tfrac{1}{2}c_0\boldsymbol{C} : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\boldsymbol{C} - \tfrac{1}{2}(3c_0 + c_1)\boldsymbol{I} & \text{with} \qquad \boldsymbol{C} &= \boldsymbol{F}^c \cdot \boldsymbol{F}
\end{aligned}
$$

$$
\boldsymbol{\sigma} = J^{-1}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = J^{-1}\boldsymbol{F} \cdot (\boldsymbol{P} \cdot \boldsymbol{F}^c) = J^{-1}\boldsymbol{F} \cdot (\boldsymbol{F} \cdot \boldsymbol{P}^c)^c
$$

Stiffness

In the Newton-Raphson iterative solution procedure, the variation of the stress tensor must be expressed in the iterative displacement of material points.

Starting from the $\boldsymbol{P} \sim \boldsymbol{E}$ elastic model, the relation between $\delta\boldsymbol{P}$ and $\delta\boldsymbol{F}$ is calculated. The variation of the deformation tensor $\boldsymbol{F}$ can be expressed in the gradient of the iterative displacement vector $\vec{u} = \delta\vec{x}$:

$$
\delta\boldsymbol{F} = \boldsymbol{L} \cdot \boldsymbol{F} = (\boldsymbol{F}^c \cdot \boldsymbol{L}^c)^c \qquad \text{with} \qquad \boldsymbol{L}^c = \vec{\nabla}\vec{u}
$$

Combining the variations $\delta\boldsymbol{P}$ and $\delta\boldsymbol{\sigma}$ leads to a relation between $\delta\boldsymbol{\sigma}$ and $\boldsymbol{L}$.

$$
\begin{aligned}
\delta\boldsymbol{P} &= \tfrac{1}{2}c_0\delta\boldsymbol{C} : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\delta\boldsymbol{C} \\
&\qquad \boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F} \qquad \rightarrow \qquad \delta\boldsymbol{C} = \delta\boldsymbol{F}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \delta\boldsymbol{F} \\
&= \tfrac{1}{2}c_0\left(\delta\boldsymbol{F}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \delta\boldsymbol{F}\right) : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\left(\delta\boldsymbol{F}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \delta\boldsymbol{F}\right) \\
&= c_0(\boldsymbol{F}^c \cdot \delta\boldsymbol{F}) : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\left(\delta\boldsymbol{F}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \delta\boldsymbol{F}\right) \\
&= c_0\boldsymbol{I}(\boldsymbol{F}^c : \delta\boldsymbol{F}) + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}^c \cdot \delta\boldsymbol{F})^c + (\boldsymbol{F}^c \cdot \delta\boldsymbol{F})\right\}
\end{aligned}
$$

$$
\boldsymbol{\sigma} = J^{-1}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c \quad \rightarrow
$$

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= J^{-1}\left[-\delta J\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c + \delta\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c + \boldsymbol{F} \cdot \delta\boldsymbol{P} \cdot \boldsymbol{F}^c + \boldsymbol{F} \cdot \boldsymbol{P} \cdot \delta\boldsymbol{F}^c\right] \\
&\qquad \delta J = J\operatorname{tr}(\boldsymbol{L}) = J\boldsymbol{L} : \boldsymbol{I} \quad ; \quad \delta\boldsymbol{F} = \boldsymbol{L} \cdot \boldsymbol{F} \\
&= J^{-1}\left[-(\boldsymbol{L} : \boldsymbol{I})\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c + (\boldsymbol{L} \cdot \boldsymbol{F}) \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c + \right. \\
&\qquad \left. \boldsymbol{F} \cdot \delta\boldsymbol{P} \cdot \boldsymbol{F}^c + \boldsymbol{F} \cdot \boldsymbol{P} \cdot (\boldsymbol{F}^c \cdot \boldsymbol{L}^c)\right] \\
&= -(\boldsymbol{L} : \boldsymbol{I})\boldsymbol{\sigma} + \boldsymbol{L} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \boldsymbol{L}^c + J^{-1}\boldsymbol{F} \cdot \delta\boldsymbol{P} \cdot \boldsymbol{F}^c \\
&= -\boldsymbol{\sigma}(\boldsymbol{I} : \boldsymbol{L}) + (\boldsymbol{\sigma}^c \cdot \boldsymbol{L}^c)^c + \boldsymbol{\sigma} \cdot \boldsymbol{L}^c + J^{-1}\boldsymbol{F} \cdot (\boldsymbol{F} \cdot \delta\boldsymbol{P}^c)^c
\end{aligned}
$$

Matrix/column notation

The tensorial expression is transferred to matrix/column notation.

$$
\begin{aligned}
\boldsymbol{P} &= \tfrac{1}{2}c_0\boldsymbol{C} : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\boldsymbol{C} - \tfrac{1}{2}(3c_0 + c_1)\boldsymbol{I} && \text{with} \quad \boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F} \\
\underset{\approx}{P} &= \tfrac{1}{2}c_0\underset{\approx}{C}^T\underset{\approx}{I}_t\underset{\approx}{I} + \tfrac{1}{2}c_1\underset{\approx}{C} - \tfrac{1}{2}(3c_0 + c_1)\underset{\approx}{I} && \text{with} \quad \underset{\approx}{C} = \underline{\underline{F}}_t\underset{\approx}{F}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\sigma} &= J^{-1}\boldsymbol{F} \cdot (\boldsymbol{F} \cdot \boldsymbol{P}^c)^c \\
\underset{\approx}{\sigma} &= J^{-1}\underline{\underline{F}}\left(\underline{\underline{F}}_r\underset{\approx}{P}_t\right) = J^{-1}\underline{\underline{F}}\,\underline{\underline{F}}_r\underset{\approx}{P}_t
\end{aligned}
$$

$$
\begin{aligned}
\delta\boldsymbol{P} &= c_0\boldsymbol{I}(\boldsymbol{F}^c : \delta\boldsymbol{F}) + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}^c \cdot \delta\boldsymbol{F})^c + (\boldsymbol{F}^c \cdot \delta\boldsymbol{F})\right\} \\
\delta\underset{\approx}{P} &= c_0\underset{\approx}{I}\underset{\approx}{F}_t^T\delta\underset{\approx}{F}_t + \tfrac{1}{2}c_1\left\{(\underline{\underline{F}}_t\delta\underset{\approx}{F})_r + (\underline{\underline{F}}_t\delta\underset{\approx}{F})\right\} \\
&= c_0\underset{\approx}{I}\underset{\approx}{F}_{tc}^T\delta\underset{\approx}{F} + \tfrac{1}{2}c_1\left(\underline{\underline{F}}_{tr}\delta\underset{\approx}{F} + \underline{\underline{F}}_t\delta\underset{\approx}{F}\right) \\
&= \underline{\underline{M}}_0\delta\underset{\approx}{F} = \underline{\underline{M}}_0\left(\underset{\approx}{L}_0\right)_t = \underline{\underline{M}}_0\underline{\underline{F}}_{tr}\underset{\approx}{L}_t = \underline{\underline{M}}_1\underset{\approx}{L}_t
\end{aligned}
$$

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= -\boldsymbol{\sigma}(\boldsymbol{I} : \boldsymbol{L}) + (\boldsymbol{\sigma}^c \cdot \boldsymbol{L}^c)^c + \boldsymbol{\sigma} \cdot \boldsymbol{L}^c + J^{-1}\boldsymbol{F} \cdot (\boldsymbol{F} \cdot \delta\boldsymbol{P}^c)^c \\
\delta\underset{\approx}{\sigma} &= -\underset{\approx}{\sigma}\underset{\approx}{I}^T\underset{\approx}{L}_t + \underline{\underline{\sigma}}_{tr}\underset{\approx}{L}_t + \underline{\underline{\sigma}}\underset{\approx}{L}_t + J^{-1}\underline{\underline{F}}\,\underline{\underline{F}}_r\delta\underset{\approx}{P}_t \\
&= -\underset{\approx}{\sigma}\underset{\approx}{I}^T\underset{\approx}{L}_t + \underline{\underline{\sigma}}_{tr}\underset{\approx}{L}_t + \underline{\underline{\sigma}}\underset{\approx}{L}_t + J^{-1}\underline{\underline{F}}\,\underline{\underline{F}}_{rc}\delta\underset{\approx}{P} \\
&= \left[-\underset{\approx}{\sigma}\underset{\approx}{I}^T + \underline{\underline{\sigma}}_{tr} + \underline{\underline{\sigma}} + J^{-1}\underline{\underline{F}}\,\underline{\underline{F}}_{rc}\underline{\underline{M}}_1\right]\underset{\approx}{L}_t = \underline{\underline{M}}\,\underset{\approx}{L}_t
\end{aligned}
$$

Linear s - A model

Stress update

Elastic material behavior may be described by a linear relation between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the Finger tensor $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B} - \boldsymbol{I})$ with $\boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c$. The above relation cannot be derived from an elastic energy function and is thus referred to as hypo-elastic.

$$
\begin{aligned}
\boldsymbol{\sigma} &= c_0\mathrm{tr}(\boldsymbol{A})\boldsymbol{I} + c_1\boldsymbol{A} && \text{with} \quad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B} - \boldsymbol{I}) \\
&= \tfrac{1}{2}c_0\boldsymbol{B} : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\boldsymbol{B} - \tfrac{1}{2}(3c_0 + c_1)\boldsymbol{I} && \text{with} \quad \boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c
\end{aligned}
$$

Stiffness

The variation of the Cauchy stress tensor can be related to $\delta\boldsymbol{F}$, and consequently to the gradient of the iterative displacement vector $\delta\vec{u}$.

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= \tfrac{1}{2}c_0\delta\boldsymbol{B} : \boldsymbol{II} + \tfrac{1}{2}c_1\delta\boldsymbol{B} \\
&= \tfrac{1}{2}c_0\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} : \boldsymbol{II} + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} \\
&= c_0(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c) : \boldsymbol{II} + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} \\
&= c_0\boldsymbol{IF} : \delta\boldsymbol{F}^c + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} \\
&\quad \text{with} \qquad \delta\boldsymbol{F} = \boldsymbol{L}\cdot\boldsymbol{F} = (\boldsymbol{F}^c\cdot\boldsymbol{L}^c)^c \qquad \text{and} \qquad \boldsymbol{L}^c = \vec{\nabla}\vec{u}
\end{aligned}
$$

Matrix/column notation

All tensor equations can be trasferred to matrix equations.

$$
\begin{aligned}
\boldsymbol{\sigma} &= \tfrac{1}{2}c_0\boldsymbol{B} : \boldsymbol{II} + \tfrac{1}{2}c_1\boldsymbol{B} - \tfrac{1}{2}(3c_0+c_1)\boldsymbol{I} && \text{with} \qquad \boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^c \\
\underset{\approx}{\sigma} &= \tfrac{1}{2}c_0\underset{\approx}{B}^T\underset{\approx}{I}_t\underset{\approx}{I} + \tfrac{1}{2}c_1\underset{\approx}{B} - \tfrac{1}{2}(3c_0+c_1)\underset{\approx}{I} && \text{with} \qquad \underset{\approx}{B} = \underline{\underline{F}}\,\underset{\approx}{F}_t
\end{aligned}
$$

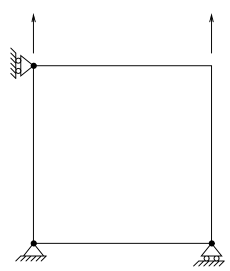
$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= c_0\boldsymbol{IF} : \delta\boldsymbol{F}^c + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} && \text{with} \qquad \delta\boldsymbol{F} = \boldsymbol{L}\cdot\boldsymbol{F} = (\boldsymbol{F}^c\cdot\boldsymbol{L}^c)^c \\
\delta\underset{\approx}{\sigma} &= c_0\underset{\approx}{I}\,\underset{\approx}{F}^T\delta\underset{\approx}{F} + \tfrac{1}{2}c_1\left\{\underline{\underline{F}}_r\,\delta\underset{\approx}{F}_t + \underline{\underline{F}}\,\delta\underset{\approx}{F}_t\right\} \\
&= \left[c_0\underset{\approx}{I}\,\underset{\approx}{F}^T + \tfrac{1}{2}c_1\left\{\underline{\underline{F}}_{rc} + \underline{\underline{F}}_c\right\}\right]\delta\underset{\approx}{F} && \text{with} \qquad \delta\underset{\approx}{F} = \left(\underline{\underline{F}}_t\,\underset{\approx}{L}_t\right)_r = \underline{\underline{F}}_{tr}\,\underset{\approx}{L}_t \\
&= \left[c_0\underset{\approx}{I}\,\underset{\approx}{F}^T\underline{\underline{F}}_{tr} + \tfrac{1}{2}c_1\left(\underline{\underline{F}}_{rc}\underline{\underline{F}}_{tr} + \underline{\underline{F}}_c\underline{\underline{F}}_{tr}\right)\right]\underset{\approx}{L}_t = \underline{\underline{M}}\,\underset{\approx}{L}_t
\end{aligned}
$$

13.1.6 Examples

A square plate is subjected to a tensile and a shear deformation. The two linear elastic models, described before, are used to model the elastic behavior. Both plane stress and plane strain states are considered.

Tensile test

A square plate or cylindrical bar is loaded uniaxially using different elastic material models. Dimensions are listed in the table. For plane stress and axisymmetry, the loading is equivalent to a tensile test.



Cartesian			
initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

cylindrical			
initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

The axial elongation is prescribed and the resulting axial force is calculated for various elastic material models. Material parameter values are $C = 100000$ MPa and $\nu = 0.3$.

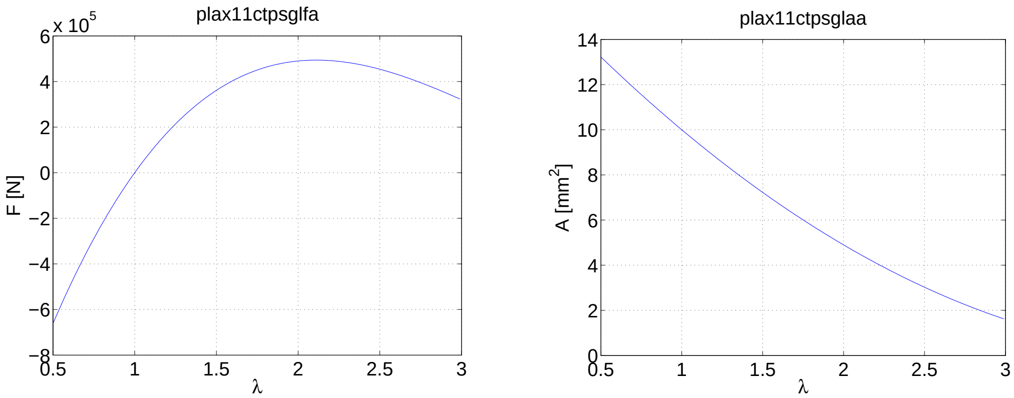


Fig. 13.21 : *Tensile force and cross-sectional area versus elongation; plane stress;* $\boldsymbol{\sigma} \sim \boldsymbol{\varepsilon}$ *model*

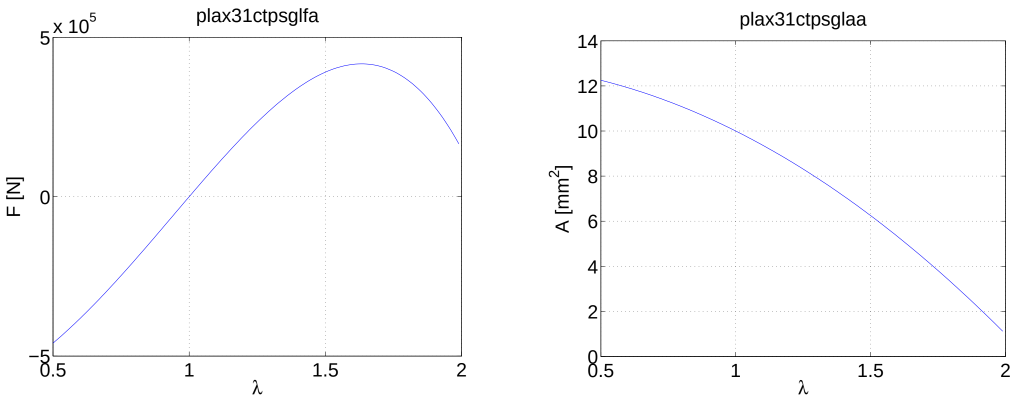


Fig. 13.22 : *Tensile force and cross-sectional area versus elongation; plane stress;* $\boldsymbol{\sigma} \sim \boldsymbol{A}$ *model*

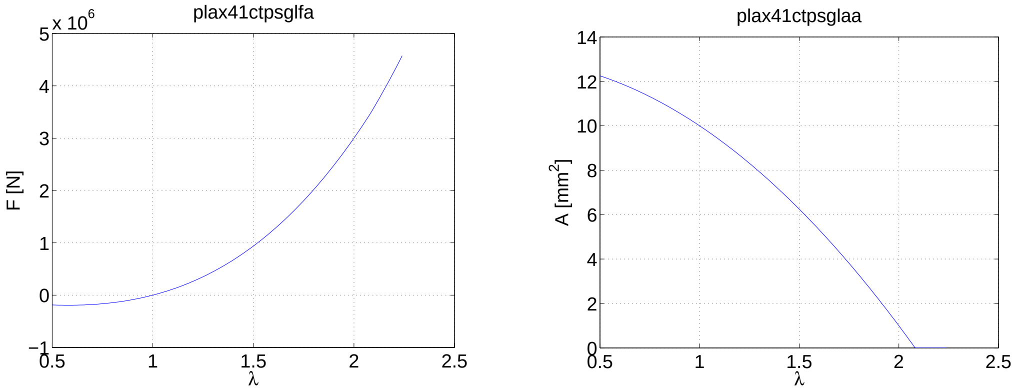


Fig. 13.23 : *Tensile force and cross-sectional area versus elongation; plane stress;* $\boldsymbol{P} \sim \boldsymbol{E}$ *model.*

The latter model is also used in a plane strain tensile test. Both Updated Lagrange and Total Lagrange formulation are used. The results are the same, which should be the case.

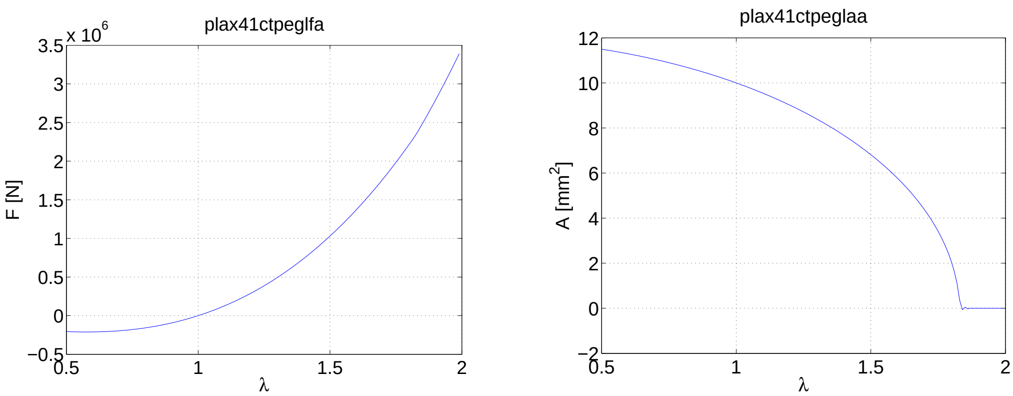


Fig. 13.24 : *Tensile force and cross-sectional area versus elongation; plane strain;* $\boldsymbol{P} \sim \boldsymbol{E}$ *model; Updated Lagrange formulation*

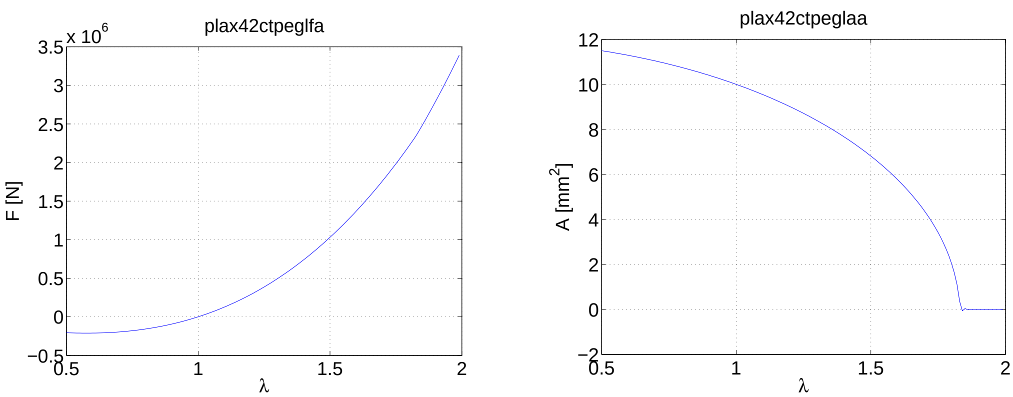
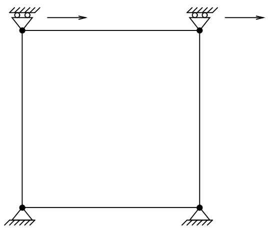


Fig. 13.25 : *Tensile force and cross-sectional area versus elongation; plane stress;* $\boldsymbol{P} \sim \boldsymbol{E}$ *model; Total Lagrange formulation*

Shear test

The simple shear test is analyzed with one element, where the horizontal displacement/force in the upper nodes is prescribed. Dimensions are listed in the table.



initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

Subsequently the material model $\boldsymbol{\sigma} \sim \boldsymbol{A}$ and $\boldsymbol{P} \sim \boldsymbol{E}$ are used in the analysis.

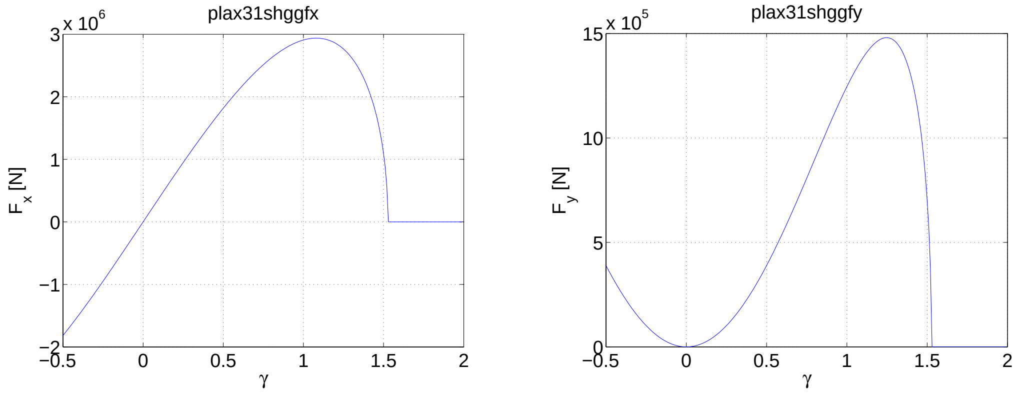


Fig. 13.26 : *Shear and normal force versus shear strain; plane stress;* $\boldsymbol{\sigma} \sim \boldsymbol{A}$ *model*

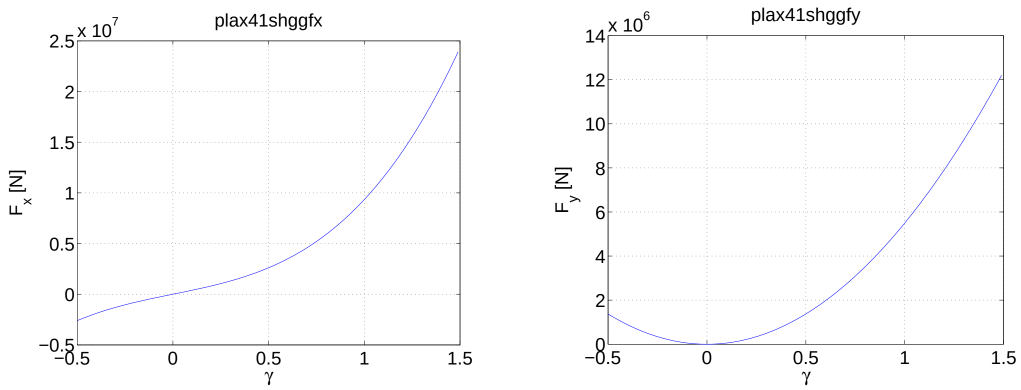


Fig. 13.27 : *Shear and normal force versus shear strain; plane stress;* $\boldsymbol{P} \sim \boldsymbol{E}$ *model*

13.2 Elastoplastic material behavior

The one-dimensional mechanical representation of an elastoplastic model consists of a spring in series with a parallel arrangement of a spring and friction slider. The series-spring represents the purely elastic part of the deformation, when stress is below the yield stress. The elastoplastic response becomes manifest when the stress exceeds the yield stress σ_y.

After yielding the total strain rate $\dot{\varepsilon}$ is the sum of the elastic strain rate $\dot{\varepsilon}_e$ and the plastic strain rate $\dot{\varepsilon}_p$. It is only for small strains that we can also add strains. The rate is a fictitious time derivative as for this material model the stress is not influenced by the strain rate.

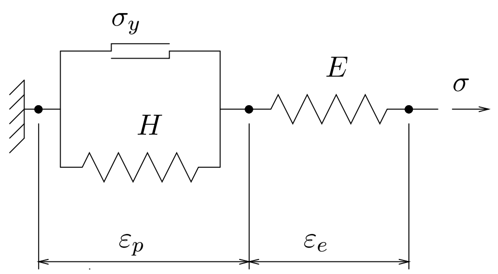


Fig. 13.28 : *Discrete model for elastoplastic material behavior*

13.2.1 Kinematics

Transformation from the undeformed configuration at time t_0 (position vector $\vec{x}_0$) to the current configuration at time t (position vector $\vec{x}$) is described by the deformation tensor $\boldsymbol{F} = (\vec{\nabla}_0 \vec{x})^c$, where $\vec{\nabla}_0$ is the gradient operator with respect to the undeformed state.

The right and left Cauchy-Green strain tensors, $\boldsymbol{C}$ and $\boldsymbol{B}$, are functions of $\boldsymbol{F}$ as is the Green-Lagrange strain tensor $\boldsymbol{E}$. The deformation rate is described by the velocity gradient tensor $\boldsymbol{L} = (\vec{\nabla}\vec{v})^c$, where $\vec{\nabla}$ is the gradient operator with respect to the current state and $\vec{v}$ is the velocity of the material volume.

The total deformation $\boldsymbol{F}$ is multiplicatively decomposed into an elastic and a plastic contribution. For the velocity gradient tensor an additive decomposition into the symmetric deformation rate tensor $\boldsymbol{D}$ and the skew-symmetric spin tensor $\boldsymbol{\Omega}$ is used. Both $\boldsymbol{D}$ and $\boldsymbol{\Omega}$ can be split into an elastic and a plastic part.

To make the decomposition unique it is commonly assumed that the plastic rotation rate during the current increment is zero, i.e. $\boldsymbol{\Omega}_p = 0$. Superimposed material rotations are thus fully represented in $\boldsymbol{F}_e$.

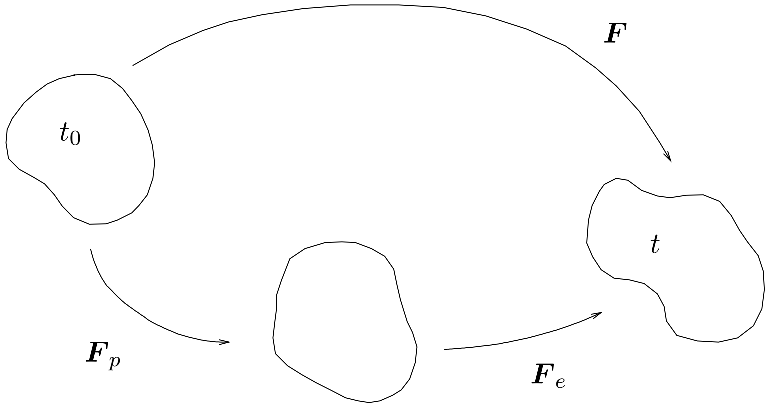


Fig. 13.29 : *Multiplicative decomposition of total deformation*

$$
\begin{aligned}
&\boldsymbol{F} = (\vec{\nabla}_0\vec{x})^c = \boldsymbol{F}_e \cdot \boldsymbol{F}_p \\
&\boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F} \qquad ; \qquad \boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c \qquad ; \qquad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \\
&\boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = (\vec{\nabla}\vec{v})^c \\
&\quad = \boldsymbol{L}_e + \boldsymbol{L}_p = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + (\boldsymbol{D}_p + \boldsymbol{\Omega}_p) = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + \boldsymbol{D}_p
\end{aligned}
$$

13.2.2 Constitutive relations

Elastic deformation

The stress is related to the elastic strain with an elastic material model. In elastoplastic deformation problems, it can often be assumed that elastic strains are small, which allows the use of a hypo-elastic generalized Hooke's law, relating the Cauchy stress tensor $\boldsymbol{\sigma}$ to the logarithmic strain tensor $\boldsymbol{\Lambda}$.

The material is assumed to be isotropic in which case the elastic material behavior is characterized by two material constants : the bulk modulus K and the shear modulus G. The fourth-order unity tensor is defined as : ${}^4\boldsymbol{I} = \vec{e}_i\vec{e}_j\delta_{il}\delta_{jk}\vec{e}_k\vec{e}_l$ and ${}^4\boldsymbol{I}^{rc}$ is its right conjugate.

The current stress state must be determined from the elastoplastic constitutive model, which is necessarily a rate formulation, i.e. a relation between a time derivative of the stress and the deformation rate. To avoid problems with large rigid rotations, the constitutive relations are formulated in invariant variables. A general invariant stress tensor $\boldsymbol{\sigma}_A$ is introduced first. It can be proved that both $\boldsymbol{\sigma}_A$ and $\dot{\boldsymbol{\sigma}}_A$ are invariant, when rigid body rotation (rotation tensor $\boldsymbol{Q}$) transforms $\boldsymbol{A}$ into $\boldsymbol{A}^*$ according to : $\boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c$.

The elastic material law can then be reformulated, such that it obeys the objectivity requirement.

$$
\left.
\begin{aligned}
&\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_e \\[1em]
&{}^4\boldsymbol{C} = c_0\boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}) = K\boldsymbol{I}\boldsymbol{I} + 2G\left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right)
\end{aligned}
\right\}
$$

invariant tensors

$$\boldsymbol{\sigma}_A = \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c = \boldsymbol{\sigma}_A^* \qquad \text{with} \quad \boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c \quad \forall \quad \boldsymbol{Q}$$
$$\dot{\boldsymbol{\sigma}}_A = \boldsymbol{A} \cdot \left\{ (\boldsymbol{A}^{-1} \cdot \dot{\boldsymbol{A}}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\boldsymbol{A}^{-1} \cdot \dot{\boldsymbol{A}})^c + \dot{\boldsymbol{\sigma}} \right\} \cdot \boldsymbol{A}^c = \boldsymbol{A} \cdot \overset{\odot}{\boldsymbol{\sigma}}_A \cdot \boldsymbol{A}^c = \dot{\boldsymbol{\sigma}}_A^*$$

objective elastic law $\qquad \overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C} : \boldsymbol{D}_e$

Yield criterion and hardening

A yield function F is used to evaluate the stress state and to check whether the deformation is purely elastic ($F < 0$) or elastoplastic ($F = 0$). The current stress state, represented by the equivalent stress $\bar{\sigma}$, is compared to a yield stress σ_y. Its initial value is σ_{y0}. This yield stress changes with plastic deformation and is therefore related to the effective plastic strain $\bar{\varepsilon}_p$. The relation between σ_y and $\bar{\varepsilon}_p$ is described by the hardening law. To decide whether elastic or elastoplastic deformation occurs, the Kuhn-Tucker relations are used.

yield criterion $\qquad F = \bar{\sigma}^2 - \sigma_y^2(\bar{\varepsilon}_p)$

effective plastic strain $\qquad \bar{\varepsilon}_p = \int\limits_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p \, d\tau$

hardening law $\qquad \sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p) \quad \text{with} \quad \dfrac{\partial \sigma_y}{\partial \bar{\varepsilon}_p} = H(\bar{\varepsilon}_p)$

Kuhn-Tucker relations $\qquad \{(F < 0) \vee (F = 0 \wedge \dot{F} < 0)\} \quad \rightarrow \quad \text{elastic}$
$\qquad \{(F = 0) \wedge (\dot{F} = 0)\} \quad \rightarrow \quad \text{elastoplastic}$

Von Mises plasticity

For the Von Mises yield criterion, the yield surface is a circular cylinder in principal stress space. The equivalent Von Mises stress can be expressed in the deviatoric stress tensor $\boldsymbol{\sigma}^d$. It is required that the disspated plastic energy per unit of time is the product of the equivalent stress and the effective or equivalent plastic strain rate :

$$\boldsymbol{\sigma} : \boldsymbol{D}_p = \bar{\sigma} \, \dot{\bar{\varepsilon}}_p$$

which leads to the definition of the effective plastic strain rate as a function of the plastic deformation rate tensor $\boldsymbol{D}_p$.

$$\bar{\sigma} = \sqrt{\tfrac{3}{2} \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d}$$

$$\dot{\bar{\varepsilon}}_p = \sqrt{\tfrac{2}{3} \boldsymbol{D}_p : \boldsymbol{D}_p}$$

$$F = \tfrac{3}{2} \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d - \sigma_y^2(\bar{\varepsilon}_p)$$

$$\begin{aligned} \dot{F} &= 2\bar{\sigma}\dot{\bar{\sigma}} - 2\sigma_y \dot{\sigma}_y = 2\bar{\sigma}\dot{\bar{\sigma}} - 2\sigma_y H \dot{\bar{\varepsilon}}_p \\ &= 3\boldsymbol{\sigma}^d : \dot{\boldsymbol{\sigma}} - 2\sigma_y H \dot{\bar{\varepsilon}}_p = 3\boldsymbol{\sigma}_A^d : \dot{\boldsymbol{\sigma}}_A - 2\sigma_y H \dot{\bar{\varepsilon}}_p = 0 \end{aligned}$$

Elastoplastic deformation

During elastoplastic deformation ($F = 0$) the plastic deformation rate $\boldsymbol{D}_p$ is related to the stress by the flow rule. For a so-called *normality* or *associative* flow rule the direction of $\boldsymbol{D}_p$ is perpendicular to the yield surface in stress space. The *length* of $\boldsymbol{D}_p$ is characterized by the plastic multiplier $\dot{\lambda}$. The normal to the yield surface can be expressed in the deviatoric stress $\boldsymbol{\sigma}^d$.

The value of the plastic multiplier $\dot{\lambda}$ can be determined from the requirement that the stress state must always reside on the yield surface during elastoplastic deformation, so : $\dot{F} = 0$. This relation is referred to as the *consistency condition*.

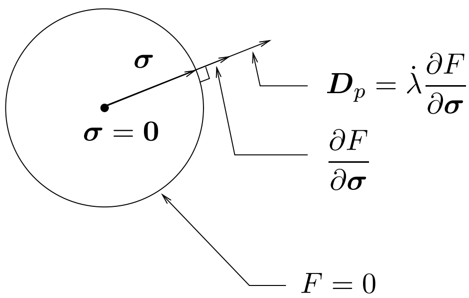


Fig. 13.30 : *Associative flow rule*

$$
\begin{aligned}
\boldsymbol{D}_p &= \dot{\lambda}\,\frac{\partial F}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\boldsymbol{a} \\
\boldsymbol{a} &= \frac{\partial F}{\partial \boldsymbol{\sigma}^d} : \frac{\partial \boldsymbol{\sigma}^d}{\partial \boldsymbol{\sigma}} \\
&= \left[3\boldsymbol{\sigma}^d : {}^4\boldsymbol{I}\right] : \frac{\partial}{\partial \boldsymbol{\sigma}}\left\{\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right\} = 3\boldsymbol{\sigma}^d : \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) = 3\boldsymbol{\sigma}^d \\
\dot{\bar{\varepsilon}}_p &= \dot{\lambda}\sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}}
\end{aligned}
$$

13.2.3 Constitutive model

The material model can be summarized as a set of constitutive relations. In accordance with $\boldsymbol{\sigma}_A$, invariant tensors $\boldsymbol{D}_A$ and $\boldsymbol{a}_A$ are defined. Also a new fourth-order material tensor ${}^4\boldsymbol{C}_A$ is introduced according to the requirement :

$$
{}^4\boldsymbol{C}_A : \boldsymbol{D}_A = \boldsymbol{A} \cdot {}^4\boldsymbol{C} : \boldsymbol{D} \cdot \boldsymbol{A}^c \qquad \forall \quad \boldsymbol{A}
$$

The set of differential equations must be integrated over the deformation history to determine the current stress $\boldsymbol{\sigma}(t)$ when the current deformation $\boldsymbol{F}(t)$ is known. It is also used to derive a relation between the variation of stress and deformation, which is an essential part of the element stiffness matrix.

$$\{(F<0)\vee(F=0\wedge\dot F<0)\} \quad\rightarrow\quad \boldsymbol{D}=\boldsymbol{D}_e \quad\rightarrow\quad \dot{\bar\varepsilon}_p=0$$
$$\overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}:\boldsymbol{D} \quad\rightarrow\quad \dot{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}_A:\boldsymbol{D}_A$$

$$\{(F=0)\wedge(\dot F=0)\} \quad\rightarrow\quad \boldsymbol{D}=\boldsymbol{D}_e+\boldsymbol{D}_p$$
$$\left.\begin{array}{l}\overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}:(\boldsymbol{D}-\dot\lambda\boldsymbol{a}) \\ 2\bar\sigma\dot{\bar\sigma}-2\sigma_y H\dot{\bar\varepsilon}_p=0\end{array}\right\} \quad\rightarrow$$
$$\left.\begin{array}{l}\dot{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}_A:\left(\boldsymbol{D}_A-\dot\lambda\boldsymbol{a}_A\right) \\ 3\boldsymbol{\sigma}_A^d:\dot{\boldsymbol{\sigma}}_A-2\sigma_y H\dot\lambda\sqrt{\frac{2}{3}\boldsymbol{a}_A:\boldsymbol{a}_A}=0\end{array}\right\}$$
$$\left.\begin{array}{l}\dot{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}_A:\left(\boldsymbol{D}_A-\dot\lambda\boldsymbol{a}_A\right) \\ 3\boldsymbol{\sigma}_A^d:{}^4\boldsymbol{C}_A:\boldsymbol{D}_A-\dot\lambda\left(3\boldsymbol{\sigma}_A^d:{}^4\boldsymbol{C}_A:\boldsymbol{a}_A+2\sigma_v H\sqrt{\frac{2}{3}\boldsymbol{a}_A:\boldsymbol{a}_A}\right)=0\end{array}\right\}$$
$$\sigma_y=\sigma_y(\sigma_{y0},\bar\varepsilon_p) \quad ;$$

13.2.4 Incremental analysis

The figure shows the relevant configurations in a large strain plastic deformation process. Although the time t is used to identify various configurations, it is noted that the material behavior is considered to be time independent. The variable t is thus a pseudo-time.

Starting from the undeformed configuration at t_0 the external load is applied and the deformation leads to the current configuration t. During a numerical analysis of this deformation process the state of the material is determined at a finite number of discrete moments $t_i,\ i=0,1,..,n+1$. The period between two subsequent moments is an increment : $\Delta t_i=t_{i+1}-t_i$.

It is assumed that the analysis has brought us to $t=t_n$, the beginning of the last increment and that all relevant variables are known and satisfying all governing equations (balance laws, boundary conditions, constitutive relations). The state at the current time $t=t_{n+1}$, the end of the current increment has to be determined.

The incremental deformation is described by the deformation tensor $\boldsymbol{F}_n$. The incremental principle elongation factors and directions, λ_{ni} and $\vec n_{ni}$ $(i=1,2,3)$, respectively, with respect to the begin increment state, can be determined from $\boldsymbol{C}_n=\boldsymbol{F}_n^c\cdot\boldsymbol{F}_n$. The incremental stretch tensor $\boldsymbol{U}_n$ and logarithmic strain tensor $\boldsymbol{\Lambda}_n$ can be expressed in λ_{ni} and $\vec n_{ni}$.

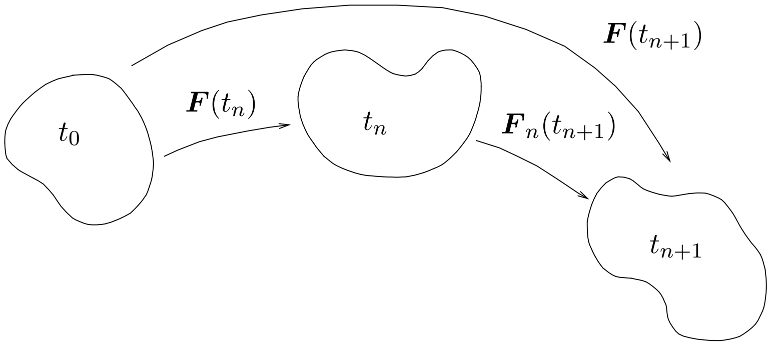


Fig. 13.31 : *Incremental deformation*

$$\boldsymbol{F}(\tau) = \boldsymbol{F}_n(\tau) \cdot \boldsymbol{F}(t_n) \quad \rightarrow \quad \boldsymbol{F}_n(\tau) = (\vec{\nabla}_n \vec{x})^c = \boldsymbol{F}(\tau) \cdot \boldsymbol{F}^{-1}(t_n)$$

$$\boldsymbol{D} = \tfrac{1}{2}\left(\dot{\boldsymbol{F}}_n \cdot \boldsymbol{F}_n^{-1} + \boldsymbol{F}_n^{-c} \cdot \dot{\boldsymbol{F}}_n^c\right) = \tfrac{1}{2}\boldsymbol{R}_n \cdot \left(\dot{\boldsymbol{U}}_n \cdot \boldsymbol{U}_n^{-1} + \boldsymbol{U}_n^{-1} \cdot \dot{\boldsymbol{U}}_n\right) \cdot \boldsymbol{R}_n^c$$

$$\boldsymbol{\Omega} = \tfrac{1}{2}\left\{\dot{\boldsymbol{F}}_n \cdot \boldsymbol{F}_n^{-1} - \boldsymbol{F}_n^{-c} \cdot \dot{\boldsymbol{F}}_n^c\right\} = \dot{\boldsymbol{R}}_n \cdot \boldsymbol{R}_n^c + \tfrac{1}{2}\boldsymbol{R}_n \cdot \left(\dot{\boldsymbol{U}}_n \cdot \boldsymbol{U}_n^{-1} - \boldsymbol{U}_n^{-1} \cdot \dot{\boldsymbol{U}}_n\right) \cdot \boldsymbol{R}_n^c$$

$$\boldsymbol{U}_n = \sum_{i=1}^{3} \lambda_{ni} \vec{n}_{ni} \vec{n}_{ni} \quad ; \quad \boldsymbol{\Lambda}_n = \sum_{i=1}^{3} \ln(\lambda_{ni}) \vec{n}_{ni} \vec{n}_{ni}$$

Elastic stress predictor

The stress integration procedure is always started with the calculation of an elastic stress predictor. It is assumed that the increment is fully elastic and that the begin-increment elasticity tensor can be used to calculate the rotation neutralized Cauchy stress tensor. Subsequently the elastic Cauchy stress tensor is calculated and used to evaluate the yield criterion with two possible outcomes :

1. the increment is indeed fully elastic,
2. the yield criterion is violated which implies that during the increment further elastoplastic deformation has taken place.

elastic trial stress $\quad \boldsymbol{\sigma}_e = \boldsymbol{\sigma}(t_n) + {}^4\boldsymbol{C} : (\boldsymbol{\Lambda} - \boldsymbol{\Lambda}(t_n))$

yield criterion $\quad F = \tfrac{3}{2}\boldsymbol{\sigma}_e^d : \boldsymbol{\sigma}_e^d - \sigma_y^2(\sigma_{y0}, \bar{\varepsilon}_p(t_n))$

$$F \leq 0 \quad \rightarrow \quad \text{elastic increment}$$

$$F > 0 \quad \rightarrow \quad \text{elastoplastic increment}$$

matrix/column notation

$$\underline{\underline{C}} = K \underset{\sim}{I} \underset{\sim}{I}^T + 2G\left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\sim}{I}\underset{\sim}{I}^T\right) \quad ; \quad \underline{\Lambda}_n \quad \rightarrow \quad \underset{\sim}{\Lambda}_n$$

$$\underset{\sim}{\sigma}_{D_e} = \underset{\sim}{\sigma}(t_n) + \underline{\underline{C}}_c \underset{\sim}{\Lambda}_n \quad \rightarrow \quad \underline{\sigma}_{D_e} \quad \rightarrow \quad \underline{\sigma}_e = \underline{R}_n \, \underline{\sigma}_{D_e} \, \underline{R}_n^T$$

$$F = \tfrac{3}{2}\left(\underset{\sim}{\sigma}_{D_{tr}}\right)^T \left(\underset{\sim}{\sigma}_{D_{tr}}\right) - \sigma_y^2(\bar{\varepsilon}_p)$$

Elastic increment

When the increment is fully elastic the end-increment Cauchy stress equals the calculated elastic Cauchy stress. As no plastic deformation has occurred during the increment, the effective plastic strain and the yield stress have not changed.

$$\boldsymbol{\sigma}(t_{n+1}) = \boldsymbol{\sigma}_e \qquad ; \qquad \Delta\lambda = 0$$
$$\bar{\varepsilon}_p(t_{n+1}) = \bar{\varepsilon}_p(t_n) \qquad ; \qquad \sigma_y(t_{n+1}) = \sigma_y(t_n)$$

Elastoplastic increment

During the increment $\Delta t = t_{n+1} - t_n$ the stress evolution equations must be solved. Before this is possible the invariant tensors $(\ )_A$ must be specified. It is also necessary to make some assumptions about the incremental deformation. Because the rigid rotation during the increment is not uniquely known, rotation neutralized quantities are used. This implies the specification of the invariant tensors by choosing $\boldsymbol{A} = \boldsymbol{R}_n^c$ resulting in Dienes tensors and Dienes rates. The complete elastoplastic model can now be formulated in rotation neutralized quantities $\boldsymbol{\sigma}_D$, $\boldsymbol{D}_D$ and $\boldsymbol{a}_D$.

$$\left.\begin{array}{l} \boldsymbol{\sigma}_D = \boldsymbol{R}_n^c \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R}_n \quad \rightarrow \quad \dot{\boldsymbol{\sigma}}_D = \boldsymbol{R}_n^c \cdot \overset{\odot}{\boldsymbol{\sigma}}_D \cdot \boldsymbol{R}_n \\ \boldsymbol{D}_D = \boldsymbol{R}_n^c \cdot \boldsymbol{D} \cdot \boldsymbol{R}_n = \frac{1}{2}\left(\dot{\boldsymbol{U}}_n \cdot \boldsymbol{U}_n^{-1} + \boldsymbol{U}_n^{-1} \cdot \dot{\boldsymbol{U}}_n\right) \end{array}\right\}$$

$$\left.\begin{array}{l} \dot{\boldsymbol{\sigma}}_D = {}^4\boldsymbol{C}_D : \left(\boldsymbol{D}_D - \dot{\lambda}\boldsymbol{a}_D\right) \\ 3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C}_D : \boldsymbol{D}_D - \dot{\lambda}\left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C}_D : \boldsymbol{a}_D + 2\sigma_y H\sqrt{\frac{2}{3}\boldsymbol{a}_D : \boldsymbol{a}_D}\right) = 0 \end{array}\right\}$$

Rotation neutralized elastoplastic increment

It is assumed that there is no rigid body rotation during the increment. All rigid body rotation will be taken into account at the end-increment time t_{n+1}. The integrated stress tensor is the so-called rotation neutralized stress tensor $\boldsymbol{\sigma}_D$.

When it is also assumed that the incremental principal strain directions are constant during the increment, the tensors $\dot{\boldsymbol{U}}_n$ and $\boldsymbol{U}_n^{-1}$ are commuting. With this assumption, the constitutive equations for the rotation neutralized Dienes stress $\boldsymbol{\sigma}_D$ can now be used for integration.

$$t_n \le \tau < t_{n+1} \qquad : \qquad \boldsymbol{R}_n = I \quad ; \quad \boldsymbol{D}_D = \boldsymbol{D} \quad ; \quad \boldsymbol{a}_D = \boldsymbol{a} \quad ; \quad {}^4\boldsymbol{C}_D = {}^4\boldsymbol{C}$$

$$\tau = t_{n+1} \qquad : \qquad \boldsymbol{R}_n(t_{n+1}) = \boldsymbol{F}(t_{n+1}) \cdot \boldsymbol{U}^{-1}(t_{n+1})$$

$$\boldsymbol{U}_n(\tau) = \sum_{i=1}^{3} \lambda_{ni}(\tau)\vec{n}_{ni}(t_n)\vec{n}_{ni}(t_n)$$
$$\boldsymbol{D} = \dot{\boldsymbol{U}}_n \cdot \boldsymbol{U}_n^{-1} = \sum_{i=1}^{3}\left(\frac{\dot{\lambda}_{ni}(\tau)}{\lambda_{ni}(\tau)}\right)\vec{n}_{ni}(t_n)\vec{n}_{ni}(t_n) = \dot{\boldsymbol{\Lambda}}_n$$

constitutive equations

$$\left.\begin{aligned}
&\dot{\boldsymbol{\sigma}}_D = {}^4\boldsymbol{C} : \left\{\dot{\boldsymbol{\Lambda}}_n - \dot{\lambda}\boldsymbol{a}\right\} \\
&3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \dot{\boldsymbol{\Lambda}}_n - \dot{\lambda}\left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H\sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}}\right) = 0
\end{aligned}\right\}$$

During the increment $\Delta t = t_{n+1} - t_n$ the stress evolution equations are integrated using an implicit Euler integration scheme.

The derivative of the incremental logarithmic strain tensor is the end-increment value divided by the time increment, because $\boldsymbol{\Lambda}_n(t_n) = \boldsymbol{O}$.

$$\left.\begin{aligned}
&\boldsymbol{\sigma}_D = \boldsymbol{\sigma}_D(t_n) + {}^4\boldsymbol{C} : (\boldsymbol{\Lambda}_n - \Delta\lambda\boldsymbol{a}) \\
&3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n - \Delta\lambda\left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H\sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}}\right) = 0
\end{aligned}\right\}$$

Iterative stress update

The set of coupled nonlinear equations is solved iteratively following a Newton-Raphson procedure. The derivative of $\boldsymbol{a}$ is :

$$\frac{\partial\boldsymbol{a}}{\partial\boldsymbol{\sigma}_D} = \frac{\partial\boldsymbol{a}}{\partial\boldsymbol{\sigma}_D^d} : \frac{\partial\boldsymbol{\sigma}_D^d}{\partial\boldsymbol{\sigma}_D} = \frac{\partial\boldsymbol{a}}{\partial\boldsymbol{\sigma}_D^d} : \left({}^4\boldsymbol{I} - \tfrac{1}{3}\,\boldsymbol{II}\right) = 3\,{}^4\boldsymbol{I}$$

From the coupled set of iterative equations $\delta\boldsymbol{\sigma}_D$ and $\delta\lambda$ leading to new values of $\boldsymbol{\sigma}_D$ and $\Delta\lambda$. The iteration process is stopped when the residuals $\boldsymbol{s}_1$ and s_2 are small enough.

For a plane stress situation, the deformation tensor must be adapted during the stress update procedure. This implies that the elastic trial stress will change as well. Excluding plane stress situations, the elastic trial stress is constant in the stress update procedure, so $\delta\boldsymbol{\sigma}_{D_e} = \boldsymbol{O}$.

$$\left.\begin{aligned}
&{}^4\boldsymbol{R} : \delta\boldsymbol{\sigma}_D + \boldsymbol{t}\delta\lambda = -\boldsymbol{s}_1 \\
&\boldsymbol{u} : \delta\boldsymbol{\sigma}_D + v\delta\lambda = -s_2
\end{aligned}\right\}$$

$$\begin{aligned}
{}^4\boldsymbol{R} &= {}^4\boldsymbol{I} + 3\Delta\lambda\,{}^4\boldsymbol{C} : {}^4\boldsymbol{I} \\
\boldsymbol{t} &= {}^4\boldsymbol{C} : \boldsymbol{a} \\
\boldsymbol{u} &= \left(3\,{}^4\boldsymbol{C} - \boldsymbol{II} : {}^4\boldsymbol{C}\right) : \boldsymbol{\Lambda}_n - \Delta\lambda\left\{\left(3\,{}^4\boldsymbol{C} - \boldsymbol{II} : {}^4\boldsymbol{C}\right) : \boldsymbol{a} + 4\sigma_y H\left(\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}\right)^{-\frac{1}{2}}\boldsymbol{a} : {}^4\boldsymbol{I}\right\} \\
v &= 3\,{}^4\boldsymbol{C} : \boldsymbol{a} : \boldsymbol{\sigma}_D^d + 2\sigma_y H\sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}} \\
\boldsymbol{s}_1 &= \boldsymbol{\sigma}_D - \boldsymbol{\sigma}_D(t_n) - {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n + \Delta\lambda\,{}^4\boldsymbol{C} : \boldsymbol{a} \\
s_2 &= 3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n - \Delta\lambda\left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H\sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}}\right)
\end{aligned}$$

Stiffness

To evaluate the iterative Updated Lagrange weighted residual equation not only the Cauchy stress $\boldsymbol{\sigma}$, but also the relation between the stress variation $\delta\boldsymbol{\sigma}$ and $\boldsymbol{L}_u = (\vec{\nabla}\vec{u})^c$ has to be known, i.e. $\delta\boldsymbol{\sigma} = {}^4\boldsymbol{M} : \boldsymbol{L}_u$.

The consistent stiffness tensor ${}^4\boldsymbol{M}$, eventually leads to the consistent stiffness matrix. It must be derived from the coupled nonlinear equations for $\boldsymbol{\sigma}$ and $\Delta\lambda$. Iterative changes (variations) of $\delta\boldsymbol{\sigma}$ and $\delta\lambda$ can be derived.

To simplify notation we omit again the upper index i, which indicates the iteration step number.

$$\left.\begin{aligned}
&\boldsymbol{\sigma}_D - \boldsymbol{\sigma}_D(t_n) - {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n + \Delta\lambda\, {}^4\boldsymbol{C} : \boldsymbol{a} = 0 \\
&3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n - \Delta\lambda \left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}} \right) = 0
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\delta\boldsymbol{\sigma}_D = \boldsymbol{\sigma}_D(t_n) + {}^4\boldsymbol{C} : \delta\boldsymbol{\Lambda}_n - \delta\lambda\, {}^4\boldsymbol{C} : \boldsymbol{a} - \Delta\lambda\, {}^4\boldsymbol{C} : \delta\boldsymbol{a} = 0 \\
&3\delta\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n + 3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \delta\boldsymbol{\Lambda}_n - \\
&\qquad \delta\lambda \left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}} \right) - \\
&\qquad \Delta\lambda \left(3\delta\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \delta\boldsymbol{a} + \right. \\
&\qquad\qquad \left. 2\delta\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}} + 2\sigma_y H \tfrac{1}{2} [\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}]^{-1/2} \tfrac{4}{3}\boldsymbol{a} : \delta\boldsymbol{a} \right) = 0
\end{aligned}\right\}$$

13.3 Linear viscoelastic material behavior

The modeling of linear viscoelastic material behavior is based on the principles of superposition and proportionality. Current stress and strain are given by a Boltzmann integral over the strain or stress history. Fourth-order relaxation (${}^4\boldsymbol{C}$) and creep (${}^4\boldsymbol{S}$) tensors relate stress to strain and vice versa.

Experiments show that long past history has less impact on the current stress than recent history. This fading memory property motivates the use of Prony series for ${}^4\boldsymbol{C}$ and ${}^4\boldsymbol{S}$. In the one-dimensional case they represent the behavior of discrete spring-dashpot models.

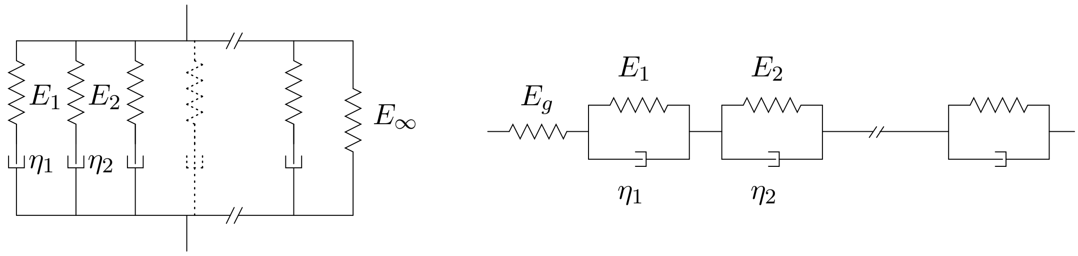


Fig. 13.32 : *Generalized Maxwell and Kelvin model*

$$\boldsymbol{\sigma}(t) = \int\limits_{\tau=0}^{t} {}^4\boldsymbol{C}(t-\tau) : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \qquad ; \qquad \boldsymbol{\varepsilon}(t) = \int\limits_{\tau=0}^{t} {}^4\boldsymbol{S}(t-\tau) : \dot{\boldsymbol{\sigma}}(\tau)\, d\tau$$

$${}^4\boldsymbol{C}(t) = {}^4\boldsymbol{C}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i e^{-\frac{t}{\tau_i}} \qquad ; \qquad {}^4\boldsymbol{S}(t) = {}^4\boldsymbol{S}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{S}_i \left\{1 - e^{-\frac{t}{\tau_i}}\right\}$$

13.3.1 Constitutive model

We now focus attention on the calculation of the current stress $\boldsymbol{\sigma}(t)$, because this is of importance in a numerical procedure like the finite element method. The hereditary integral is evaluated after substitution of the Prony series for ${}^4\boldsymbol{C}(t)$.

Using the Prony series expression for ${}^4\boldsymbol{C}(t)$ and assuming the initial strain to be zero ($\boldsymbol{\varepsilon}(\tau = 0) = \boldsymbol{O}$), an expression for $\boldsymbol{\sigma}(t)$ can be derived.

$$\left.\begin{aligned} \boldsymbol{\sigma}(t) &= \int\limits_{\tau=0}^{t} {}^4\boldsymbol{C}(t-\tau) : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\ {}^4\boldsymbol{C}(t) &= {}^4\boldsymbol{C}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i\, e^{-\frac{t}{\tau_i}} \end{aligned}\right\} \quad \rightarrow$$

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \int\limits_{\tau=0}^{t} \left[{}^4\boldsymbol{C}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i e^{-\frac{t-\tau}{\tau_i}} \right] : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau = {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i : \int\limits_{\tau=0}^{t} e^{-\frac{t-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\ &= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t) \end{aligned}$$

13.3.2 Incremental analysis

It is immediately clear that calculation of the stress involves the evaluation of a (large) number of integrals over the complete time history. For this reason the deformation time period is subdivided into a discrete number of time increments.

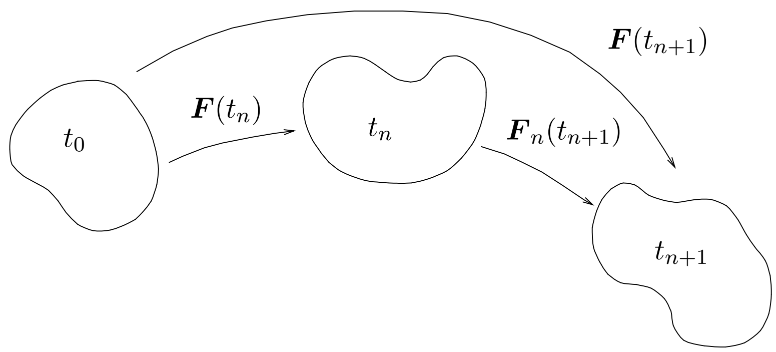


Fig. 13.33 : *Incremental deformation*

In the numerical analysis of the time dependent behavior, the total time interval $[0, t]$ is discretized :

$$[0, t] \quad \rightarrow \quad [t_1 = 0, t_2, t_3, .., t_n, t_{n+1} = t]$$

The timespan between two discrete moments in the time interval is a time increment. It is assumed that these increments are of equal length.

$$\Delta t = t_{i+1} - t_i \qquad ; \quad i = 1, ..., n$$

It is assumed that the strain is a linear function of time in each time increment.

$$\boldsymbol{\varepsilon}(\tau) = \boldsymbol{\varepsilon}(t_n) + (\tau - t_n)\frac{\Delta \boldsymbol{\varepsilon}}{\Delta t} \quad \rightarrow \quad \dot{\boldsymbol{\varepsilon}}(\tau) = \frac{\Delta \boldsymbol{\varepsilon}}{\Delta t}$$

Stress update

The hereditary integral is split in an integral over $[0, t_n]$ and an integral over the last or current increment $[t_n, t_{n+1} = t]$. Here we consider only the i-th term of the series : $\boldsymbol{\sigma}_i(t)$.

$$\begin{aligned}
\boldsymbol{\sigma}_i(t) &= {}^4\boldsymbol{C}_i : \int\limits_{\tau=0}^{t} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau = {}^4\boldsymbol{C}_i : \left[\int\limits_{\tau=0}^{t_n} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau + \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau\right] \\
&= {}^4\boldsymbol{C}_i : \left[e^{-\frac{\Delta t}{\tau_i}} \int\limits_{\tau=0}^{t_n} e^{-\frac{t_n-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau + \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau\right] \\
&= e^{-\frac{\Delta t}{\tau_i}}\,{}^4\boldsymbol{C}_i : \int\limits_{\tau=0}^{t_n} e^{-\frac{t_n-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau + {}^4\boldsymbol{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau
\end{aligned}$$

$$= e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau$$

The stress $\boldsymbol{\sigma}_i(t_n)$ is known from the previous increment. Calculation of $\Delta\boldsymbol{\sigma}_i(t)$ can be done analytically because it has been assumed that the strain is a linear function of time in each time increment.

$$\begin{aligned} \sigma_i(t) &= e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t}\, d\tau = e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\, d\tau\, \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t} \\ &= e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t} \end{aligned}$$

Calculating the current stress does not mean that the Boltzmann integral has to be evaluated over the total deformation history. When results are stored properly we can easily update the stress $\boldsymbol{\sigma}(t)$.

$$\begin{aligned} \boldsymbol{\sigma}(t) &= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t) \\ &= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t} \right] \end{aligned}$$

Stiffness

The variation of $\boldsymbol{\sigma}(t)$ results in the consistent material stiffness tensor.

$$\begin{aligned} \delta\boldsymbol{\sigma} &= \left[{}^4\boldsymbol{C}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i \, \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \right] : \delta\boldsymbol{\varepsilon} \\ &= {}^4\boldsymbol{M} : \delta\boldsymbol{\varepsilon} \end{aligned}$$

13.3.3 Isotropic material

For an isotropic material the mechanical behavior is the same in each material direction and is characterized by two material parameters, the Lamé coefficients λ and μ. The elastic stiffness tensor ${}^4\boldsymbol{C}$ can then be written as :

$${}^4\boldsymbol{C} = \lambda \boldsymbol{I}\boldsymbol{I} + 2\mu\, {}^4\boldsymbol{I}^s$$

where the fourth-order tensors $\boldsymbol{I}\boldsymbol{I}$ and ${}^4\boldsymbol{I}^s$ have the following index equivalents :

$$\begin{aligned} &\boldsymbol{I}\boldsymbol{I} \quad \rightarrow \quad \delta_{ij}\delta_{kl} \\ &2\,{}^4\boldsymbol{I}^s = {}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc} \quad \rightarrow \quad \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \end{aligned}$$

Using the above expression for ${}^4\boldsymbol{C}$ the hydrostatic and deviatoric parts of the stress tensor can be decoupled and expressed in the hydrostatic and deviatoric strain tensor, respectively.

Instead of the Lamé coefficients other elastic material parameters are often used : Young's modulus E, Poisson's ratio ν, shear modulus G and bulk modulus K. These parameters are related as only two independent material parameters exist.

$$\begin{aligned}
\boldsymbol{\sigma} &= {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \\
&= \left[\lambda \boldsymbol{I}\boldsymbol{I} + 2\mu\, {}^4\boldsymbol{I}^s\right] : \boldsymbol{\varepsilon} = \left[\lambda \boldsymbol{I}\boldsymbol{I} + \mu\left({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}\right)\right] : \boldsymbol{\varepsilon} = \lambda \boldsymbol{I}\,\mathrm{tr}(\boldsymbol{\varepsilon}) + 2\mu\, \boldsymbol{\varepsilon} \\
&= (3\lambda + 2\mu)\,\tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\, \boldsymbol{\varepsilon}^d = (3\lambda + 2\mu)\, \boldsymbol{\varepsilon}^h + 2\mu\, \boldsymbol{\varepsilon}^d = 3K\, \boldsymbol{\varepsilon}^h + 2G\, \boldsymbol{\varepsilon}^d \\
&= \boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d
\end{aligned}$$

$$K = \tfrac{1}{3}\,(3\lambda + 2\mu) = \frac{E}{3(1-2\nu)} \quad ; \quad \mu = G = \frac{E}{2(1+\nu)} \quad ; \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

For a viscoelastic isotropic material the stress tensor is also split into an hydrostatic and a deviatoric part. In analogy with the elastic model, time dependent bulk and shear moduli are used, which are expressed in a Prony series.

$$\begin{aligned}
\boldsymbol{\sigma}(t) &= \boldsymbol{\sigma}^h(t) + \boldsymbol{\sigma}^d(t) \\
&= 3 \int_{\tau=0}^{t} K(t-\tau)\, \frac{d}{d\tau}\left\{\boldsymbol{\varepsilon}^h(\tau)\right\}\, d\tau + 2 \int_{\tau=0}^{t} G(t-\tau)\, \frac{d}{d\tau}\left\{\boldsymbol{\varepsilon}^d(\tau)\right\}\, d\tau \\
K(t) &= K_\infty + \sum_{i=1}^{n} K_i\, e^{-\frac{t}{\tau_i}} = \frac{1}{3(1-2\nu)}\left[E_\infty + \sum_{i=1}^{n} E_i\, e^{-\frac{t}{\tau_i}}\right] \\
G(t) &= G_\infty + \sum_{i=1}^{n} G_i\, e^{-\frac{t}{\tau_i}} = \frac{1}{2(1+\nu)}\left[E_\infty + \sum_{i=1}^{n} E_i\, e^{-\frac{t}{\tau_i}}\right]
\end{aligned}$$

Stress update

Discretising the total time interval $[0,t]$ in equal time increments $\Delta t = t_{i+1} - t_i;\ i = 1..n$ allows an efficient calculation of the stress where an integral only has to be evaluated over the current increment, which moreover can be done rather straightforwardly when it is assumed that the incremental strain rate is constant (= linear incremental strain).

$$\begin{aligned}
\boldsymbol{\sigma}(t) &= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t) \\
&= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N}\left[e^{-\frac{\Delta t}{\tau_i}}\, \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \tau_i\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\frac{\Delta \boldsymbol{\varepsilon}}{\Delta t}\right] \\
&= 3K_\infty\, \Delta\boldsymbol{\varepsilon}^h + 2G_\infty\, \Delta\boldsymbol{\varepsilon}^d + \sum_{i=1}^{N}\left[e^{-\frac{\Delta t}{\tau_i}}\, \boldsymbol{\sigma}_i(t_n) + \frac{\tau_i}{\Delta t}\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\left\{3K_i \Delta\boldsymbol{\varepsilon}^h + 2G_i \Delta\boldsymbol{\varepsilon}^d\right\}\right]
\end{aligned}$$

Stiffness

The relation between a small change in stress and a small change in strain is straightforwardly derived from the incremental stress relation.

$$\delta\boldsymbol{\sigma} = 3K_\infty \delta\boldsymbol{\varepsilon}^h + 2G_\infty \delta\boldsymbol{\varepsilon}^d + \sum_{i=1}^{N} \frac{\tau_i}{\Delta t}\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\left\{3K_i \delta\boldsymbol{\varepsilon}^h + 2G_i \delta\boldsymbol{\varepsilon}^d\right\}$$

Matrix/column notation

The relation between the incremental stress and strain tensor can be written in indices with relation to a vector basis. Components can then be stored in columns and matrices. For a two-dimensional deformation the following columns for stress and strain components are defined :

$$\underset{\sim}{\sigma}^T = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{21} \end{bmatrix} \quad ; \quad \underset{\sim}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \varepsilon_{12} & \varepsilon_{21} \end{bmatrix}$$

Hydrostatic and deviatoric stress/strain components can be related to total stress/strain components with the following matrices :

$$\underline{\underline{A}}^h = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad \underline{\underline{A}}^d = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

resulting in : $\Delta\underset{\sim}{\varepsilon}^h = \underline{\underline{A}}^h \, \Delta\underset{\sim}{\varepsilon} \quad ; \quad \Delta\underset{\sim}{\varepsilon}^d = \underline{\underline{A}}^d \, \Delta\underset{\sim}{\varepsilon}$

The stress column can then be rewritten.

$$\begin{aligned} \underset{\sim}{\sigma}(t) = &\left(3K_\infty \underline{\underline{A}}^h + 2G_\infty \underline{\underline{A}}^d\right) \Delta\underset{\sim}{\varepsilon} + \\ &\sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \underset{\sim}{\sigma}_i(t_n) + \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \left\{3K_i \underline{\underline{A}}^h + 2G_i \underline{\underline{A}}^d\right\} \right] \Delta\underset{\sim}{\varepsilon} \end{aligned}$$

$$\begin{aligned} \delta\underset{\sim}{\sigma}(t) = &\left[\left(3K_\infty \underline{\underline{A}}^h + 2G_\infty \underline{\underline{A}}^d\right) \delta\underset{\sim}{\varepsilon} + \right. \\ &\left. \sum_{i=1}^{N} \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \left(3K_i \underline{\underline{A}}^h + 2G_i \underline{\underline{A}}^d\right) \right] \delta\underset{\sim}{\varepsilon} \end{aligned}$$

Initial stiffness formulation

Some implementations of the linear viscoelastic model (e.g. MARC) are formulated in such a way that the initial moduli K_0 and G_0 are required. The initial moduli are defined as

$$K_0 = K_\infty + \sum_{i=1}^{N} K_i \qquad ; \qquad G_0 = G_\infty + \sum_{i=1}^{N} G_i$$

The relation for stress increment and stress variation can be derived easily.

$$\Delta\boldsymbol{\sigma}(t) = 3K_0\Delta\boldsymbol{\varepsilon}^h + 2G_0\Delta\boldsymbol{\varepsilon}^d - \sum_{i=1}^{N}\left[1-\left(1-e^{-\frac{\Delta t}{\tau_i}}\right)\frac{\tau_i}{\Delta t}\right]\left\{3K_i\Delta\boldsymbol{\varepsilon}^h + 2G_i\Delta\boldsymbol{\varepsilon}^d\right\} -$$
$$\sum_{i=1}^{N}\left(1-e^{-\frac{\Delta t}{\tau_i}}\right)\left\{\boldsymbol{\sigma}_i^h(t_n) + \boldsymbol{\sigma}_i^d(t_n)\right\}$$
$$\delta\boldsymbol{\sigma} = 3K_0\delta\boldsymbol{\varepsilon}^h + 2G_0\delta\boldsymbol{\varepsilon}^d - \sum_{i=1}^{N}\left[1-\left(1-e^{-\frac{\Delta t}{\tau_i}}\right)\frac{\tau_i}{\Delta t}\right]\left\{3K_i\delta\boldsymbol{\varepsilon}^h + 2G_i\delta\boldsymbol{\varepsilon}^d\right\}$$

13.3.4 Example

An axial strain step with amplitude 0.01 is prescribed on an axisymmetric tensile bar with initial cross-sectional area $A_0 = 10$ mm^2. The stress response is calculated for a 12-mode generalized Maxwell model. The modal parameters are listed in the table.

	E [MPa]	τ [s]		E [MPa]	τ [s]
1	3.0e6	3.1e-8	2	1.4e6	3.0e-7
3	3.9e6	3.0e-6	4	5.4e6	2.9e-5
5	1.3e6	2.8e-4	6	2.3e5	2.7e-3
7	7.6e4	2.6e-2	8	3.7e4	2.5e-1
9	3.3e4	2.5e+0	10	1.7e4	2.4e+1
11	8.0e3	2.3e+2	12	1.2e4	2.2e+3

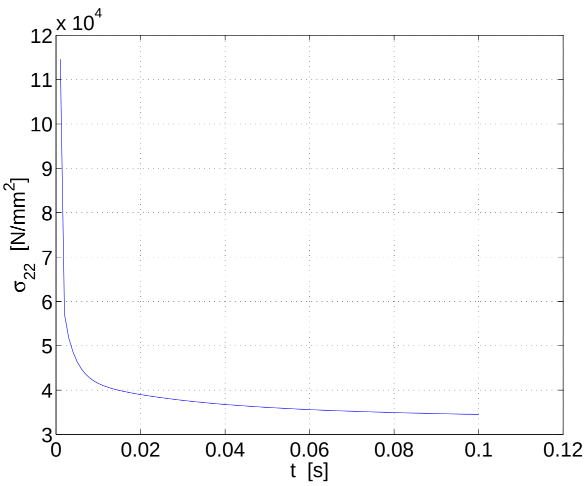


Fig. 13.34 : *Tensile stress versus time for axisymmetric element*

13.4 Viscoplastic material behavior

The one-dimensional mechanical representation of the elastoviscoplastic Perzyna model consists of a spring in series with a friction slider, a hardening spring and a linear viscous dashpot. The series-spring represents the elastic part of the material response. The viscoplastic response, represented by the hardening spring and the viscous dashpot, becomes manifest as soon as the friction slider "opens" when the stress σ exceeds a characteristic value, the yield stress σ_y.

After yielding, the total strain rate $\dot{\varepsilon}$ is the sum of the elastic strain rate $\dot{\varepsilon}_e$ and the viscoplastic strain rate $\dot{\varepsilon}_{vp}$. It is only for small strains that we can add strains.

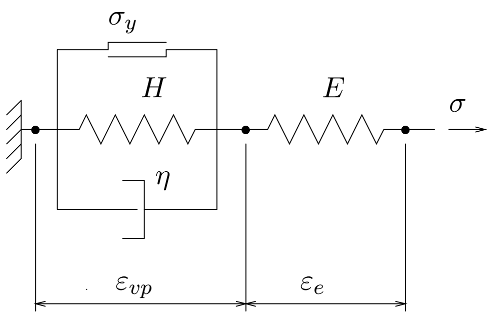


Fig. 13.35 : *Discrete model for viscoplastic material behavior*

13.4.1 Kinematics

Transformation from the undeformed configuration at time t_0 (position vector $\vec{x}_0$) to the current configuration at time t (position vector $\vec{x}$) is described by the deformation tensor $\boldsymbol{F} = (\vec{\nabla}_0 \vec{x})^c$, where $\vec{\nabla}_0$ is the gradient operator with respect to the undeformed state.

The right and left Cauchy-Green strain tensors, $\boldsymbol{C}$ and $\boldsymbol{B}$, are functions of $\boldsymbol{F}$ as is the Green-Lagrange strain tensor $\boldsymbol{E}$. Material velocity is taken into account by the deformation and rotation rate tensors $\boldsymbol{D}$ and $\boldsymbol{\Omega}$, the symmetric and skew-symmetric parts of the velocity gradient tensor $\boldsymbol{L} = (\vec{\nabla}\vec{v})^c$, where $\vec{\nabla}$ is the gradient operator with respect to the current state and $\vec{v}$ is the velocity of the material volume.

In Perzyna's model the total deformation $\boldsymbol{F}$ is decomposed multiplicatively into an elastic and a viscoplastic contribution. Regarding the kinematics, this implies the introduction of elastic and viscoplastic (rate) tensors.

To make the decomposition unique it is commonly assumed that the viscoplastic rotation rate is zero, i.e. $\boldsymbol{\Omega}_p = \boldsymbol{O}$. Superimposed material rotations are thus fully represented in $\boldsymbol{F}_e$.

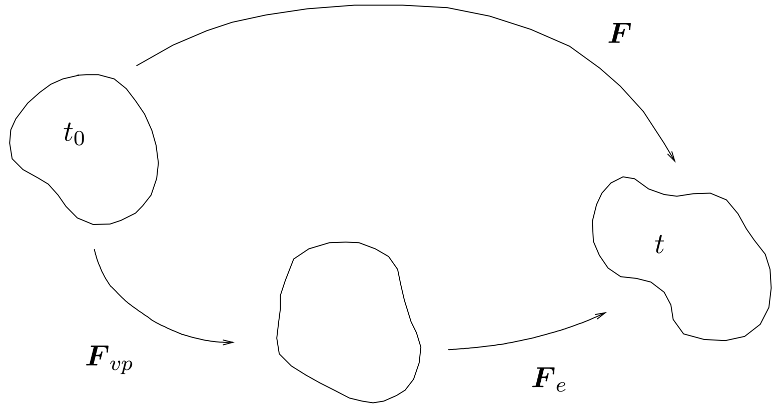


Fig. 13.36 : *Multiplicative decomposition of total deformation*

$$
\begin{aligned}
\boldsymbol{F} &= (\vec{\nabla}_0 \vec{x})^c = \boldsymbol{F}_e \cdot \boldsymbol{F}_{vp} \\
\boldsymbol{C} &= \boldsymbol{F}^c \cdot \boldsymbol{F} \qquad ; \qquad \boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c \qquad ; \qquad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \\
\boldsymbol{L} &= \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = (\vec{\nabla} \vec{v})^c \\
&= \boldsymbol{L}_e + \boldsymbol{L}_{vp} = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + (\boldsymbol{D}_{vp} + \boldsymbol{\Omega}_{vp}) = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + \boldsymbol{D}_{vp}
\end{aligned}
$$

13.4.2 Constitutive relations

Elastic deformation

The stress is related to the elastic strain. Because we want to describe large elastic strains, the elastic behavior must be described with a hyper-elastic model. In that case it is assumed that an elastic strain energy function exists, which can be used to calculate the stress. The 2nd-Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is related to the Green-Lagrange strain tensor $\boldsymbol{E}$. The current stress state is characterized by the Kirchhoff stress $\boldsymbol{\tau} = \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c$.

An elastic energy function is chosen, which characterizes isotropic, compressible material behavior. The fourth-order material tensor is completely determined by the volume ratio $J = \det(\boldsymbol{F})$ and by the constant Lamé coefficients λ and μ, which are related to Young's modulus and Poisson's ratio.

$$
\left.
\begin{aligned}
&\boldsymbol{P} = \frac{\partial W(\boldsymbol{E}_e)}{\partial \boldsymbol{E}_e} = 2\,\frac{\partial W}{\partial \boldsymbol{C}_e} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-c} \quad \rightarrow \quad \dot{\boldsymbol{P}} = 2\frac{\partial^2 W}{\partial \boldsymbol{C}^2} : \dot{\boldsymbol{C}} \\[2ex]
&W(\lambda_1, \lambda_2, \lambda_3) = \tfrac{1}{2}\,\mu \left\{ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\ln(J) \right\} + \tfrac{1}{2}\,\lambda \left\{ \ln(J) \right\}^2 \\
&\text{with} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad ; \qquad \mu = \frac{E}{2(1+\nu)}
\end{aligned}
\right\}
$$

Yield criterion and hardening

A yield function F is used to evaluate the stress state and to check whether the deformation is purely elastic ($F < 0$) or viscoplastic ($F \geq 0$). The current stress state, represented by the equivalent or effective Kirchhoff stress $\bar{\tau}$, is compared to the current yield stress τ_y, which increases from its initial value τ_{y0} due to plastic deformation and is therefore related to the effective viscoplastic strain $\bar{\varepsilon}_{vp}$. The relation between τ_y and $\bar{\varepsilon}_{vp}$ is described by the hardening rule. To decide whether elastic or viscoplastic deformation occurs, the Kuhn-Tucker relations are used.

yield criterion	$F = \bar{\tau} - \tau_y(\bar{\varepsilon}_{vp})$		
effective viscoplastic strain	$\bar{\varepsilon}_{vp} = \int\limits_{\tau=0}^{t} \dot{\bar{\varepsilon}}_{vp}\, d\tau$		
hardening law	$\tau_y = \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp})$	with	$\dfrac{\partial \tau_y}{\partial \bar{\varepsilon}_p} = H(\bar{\varepsilon}_p)$
Kuhn-Tucker relations	$F < 0$	$\rightarrow$	elastic deformation
	$F \geq 0$	$\rightarrow$	viscoplastic deformation

Von Mises plasticity

For the Von Mises yield criterion, the yield surface is a circular cylinder in principal stress space. The equivalent Von Mises stress can be expressed in the deviatoric stress tensor $\boldsymbol{\tau}^d$. It is required that the disspated viscoplastic energy per unit of time is the product of the equivalent stress and the effective or equivalent plastic strain rate :

$$\boldsymbol{\tau} : \boldsymbol{D}_{vp} = \bar{\tau}\, \dot{\bar{\varepsilon}}_{vp}$$

which leads to the definition of the effective plastic strain rate as a function of the viscoplastic deformation rate tensor $\boldsymbol{D}_{vp}$.

$$\bar{\tau} = \sqrt{\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d}$$

$$\dot{\bar{\varepsilon}}_{vp} = \sqrt{\tfrac{2}{3}\,\boldsymbol{D}_{vp} : \boldsymbol{D}_{vp}}$$

$$F = \sqrt{\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d} - \tau_y(\bar{\varepsilon}_{vp})$$

Viscoplastic deformation

During viscoplastic deformation the *direction* of the viscoplastic strain rate is defined by the commonly used *normality* or *associative* flow rule : the viscoplastic strain rate is directed normal to the yield surface in stress space. The *length* of $\boldsymbol{D}_{vp}$ is characterized by the rate of the viscoplastic multiplier $\dot{\lambda}$. The normal to the yield surface can be expressed in the deviatoric stress $\boldsymbol{\tau}^d$. The time-derivative of $\boldsymbol{C}_{vp}$ can be related to $\boldsymbol{D}_{vp}$.

In contrast to elastoplastic models, stress states outside the yield surface can exist, which explains why these viscoplastic models are often called *over-stress models*.

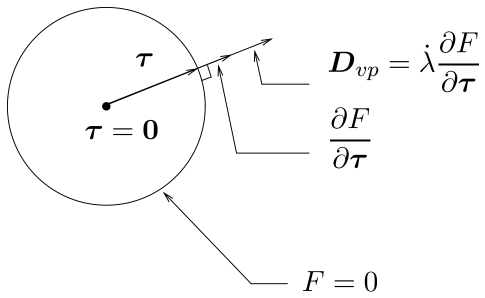


Fig. 13.37 : *Associative flow rule*

$$
\begin{aligned}
\boldsymbol{D}_{vp} &= \dot{\lambda}\,\frac{\partial F}{\partial \boldsymbol{\tau}} = \dot{\lambda}\boldsymbol{a} \quad \rightarrow \quad \dot{\boldsymbol{C}}_{vp} = 2\,\boldsymbol{F}^c \cdot \boldsymbol{D}_{vp} \cdot \boldsymbol{F} = 2\dot{\lambda}\,\boldsymbol{F}^c \cdot \boldsymbol{a} \cdot \boldsymbol{F} \\
\boldsymbol{a} &= \frac{\partial F}{\partial \boldsymbol{\tau}^d} : \frac{\partial \boldsymbol{\tau}^d}{\partial \boldsymbol{\tau}} = \left[\tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2} \boldsymbol{\tau}^d : {}^4\boldsymbol{I}\right] : \left[\frac{\partial}{\partial \boldsymbol{\tau}}\left\{\boldsymbol{\tau} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}\right\}\right] \\
&= \tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2} \boldsymbol{\tau}^d : \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) = \tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2} \boldsymbol{\tau}^d = \tfrac{3}{2}\,\frac{1}{\bar{\tau}}\,\boldsymbol{\tau}^d \\
\dot{\bar{\varepsilon}}_{vp} &= \dot{\lambda}\sqrt{\tfrac{2}{3}\,\boldsymbol{a} : \boldsymbol{a}}
\end{aligned}
$$

13.4.3 Constitutive model

The material model can be summarized as a set of constitutive equations. During viscoplastic deformation the rate of the viscoplastic multiplier is related to an over-stress function $\phi(F)$ by a *fluidity parameter* γ and a *rate-sensitivity parameter* N, which has to satisfy $N \geq 1$ to make $\phi(F)$ convex.

The set of equations must be solved to determine the stress when (an approximation of) the deformation is known. From the same set of equations a relation between the variation of stress and deformation is derived.

$$
\begin{aligned}
F < 0 \quad &\rightarrow \quad \dot{\boldsymbol{C}} = \dot{\boldsymbol{C}}_e \\
&\qquad \dot{\boldsymbol{P}} = 2\,\frac{\partial^2 W}{\partial \boldsymbol{C}^2} : \dot{\boldsymbol{C}} \qquad ; \qquad \dot{\boldsymbol{C}}_{vp} = \boldsymbol{O} \qquad ; \qquad \dot{\bar{\varepsilon}}_{vp} = 0
\end{aligned}
$$

$$
\begin{aligned}
F \geq 0 \quad &\rightarrow \quad \dot{\boldsymbol{C}} = \dot{\boldsymbol{C}}_e + \dot{\boldsymbol{C}}_{vp} \quad \rightarrow \\
&\qquad \left.\begin{aligned}
\dot{\boldsymbol{P}} &= 2\,\frac{\partial^2 W}{\partial \boldsymbol{C}^2} : \left(\dot{\boldsymbol{C}} - 2\,\boldsymbol{F}^c \cdot \dot{\lambda}\boldsymbol{a} \cdot \boldsymbol{F}\right) \\
\dot{\lambda} &= \gamma\phi(F) = \gamma\left(\frac{F}{\tau_{y0}}\right)^N
\end{aligned}\right\} \\
&\qquad \tau_y = \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp}) \qquad ; \qquad \dot{\bar{\varepsilon}}_{vp} = \dot{\lambda}\sqrt{\tfrac{2}{3}\,\boldsymbol{a} : \boldsymbol{a}}
\end{aligned}
$$

13.4.4 Incremental analysis

The figure shows the relevant configurations in a large strain viscoplastic deformation process.

Starting from the undeformed configuration at time t_0 the external load is employed and the deformation leads to the current configuration at time t. During a numerical analysis of this deformation process the state of the material is determined at a finite number of discrete moments $t_i,\ i = 0, 1, .., n+1$. The period between two subsequent moments is an increment : $\Delta t_i = t_{i+1} - t_i$. The increments are assumed to be of equal length.

It is assumed that the analysis has brought us to $t = t_n$, the beginning of the current increment and that all relevant variables are known and satisfying all governing equations (balance laws, boundary conditions, constitutive relations). The state at the current time $t = t_{n+1}$, the end of the current increment has to be determined.

The transformation during the current increment is described by the deformation tensor $\boldsymbol{F}_n(\tau)$, where τ indicates a moment in time during the last (= current) increment : $t_n \leq \tau \leq t_{n+1}$.

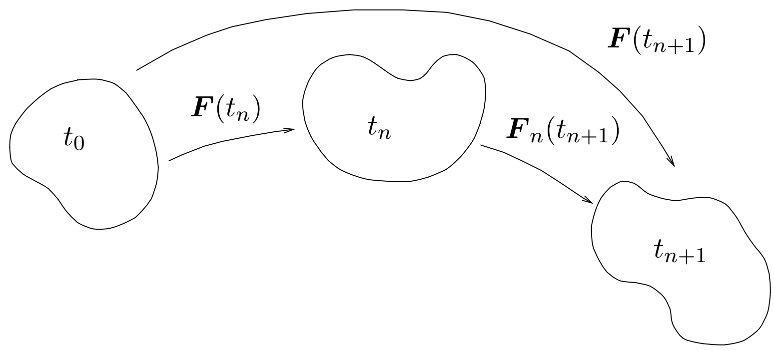


Fig. 13.38 : *Incremental deformation*

$$\boldsymbol{F}(\tau) = \boldsymbol{F}_n(\tau) \cdot \boldsymbol{F}(t_n) \quad \rightarrow \quad \boldsymbol{F}_n(\tau) = \boldsymbol{F}(\tau) \cdot \boldsymbol{F}^{-1}(t_n)$$

$$\boldsymbol{F}_n = (\vec{\nabla}_n \vec{x})^c = \boldsymbol{R}_n \cdot \boldsymbol{U}_n \quad ; \quad J_n = \det(\boldsymbol{F}_n) \quad ; \quad \vec{\nabla} = \boldsymbol{F}_n^{-c} \cdot \vec{\nabla}_n$$

$$\boldsymbol{D} = \tfrac{1}{2} \left\{ (\vec{\nabla}\vec{v})^c + (\vec{\nabla}\vec{v}) \right\} = \tfrac{1}{2} \left(\dot{\boldsymbol{F}}_n \cdot \boldsymbol{F}_n^{-1} + \boldsymbol{F}_n^{-c} \cdot \dot{\boldsymbol{F}}_n^c \right)$$

Elastic stress predictor

The first step in evaluating the end-increment stress is the calculation of the elastic stress predictor. As a first assumption the current increment is taken to be purely elastic, so $\Delta\lambda = 0$. The elastic trial stress is used to evaluate the yield condition and to see if the assumption of elastic deformation holds. There are two possibilities :

1. the increment is indeed fully elastic,

2. the yield criterion is violated which implies that during the increment further elastoviscoplastic deformation has taken place.

elastic trial stress $\quad \boldsymbol{P}_e = \boldsymbol{P}_n + 2\,\dfrac{\partial^2 W}{\partial \boldsymbol{G}^2} : (\boldsymbol{C} - \boldsymbol{C}(t_n)) \quad \rightarrow \quad \boldsymbol{\tau}_e = \boldsymbol{F}\cdot\boldsymbol{P}_e\cdot\boldsymbol{F}^c$

yield criterion $\quad F = \sqrt{\tfrac{3}{2}\,(\boldsymbol{\tau}_e)^d : (\boldsymbol{\tau}_e)^d} - \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp}(t_n))$

$$\begin{array}{lll} F < 0 & \rightarrow & \text{elastic increment} \\ F \geq 0 & \rightarrow & \text{elastoviscoplastic increment} \end{array}$$

matrix/column notation

$$\underset{\approx}{\tau}_e = \underset{\approx}{A} + \underline{\underline{H}}_c\, \underset{\approx}{e}_n$$

$$F = \sqrt{\tfrac{3}{2}\left(\underset{\approx}{\tau}_e\right)^T \left(\underset{\approx}{\tau}_e\right)_t} - \zeta(\kappa)$$

$$\text{with} \quad \left\{ \begin{array}{l} \underline{\underline{H}} = 2\,\{\mu - \lambda\ln(J)\}\,\underline{\underline{I}} + \lambda\underset{\approx}{I}\underset{\approx}{I}^T \\ \underline{e}_n = \tfrac{1}{2}\left(\underline{I} - \underline{F}_n^{-T}\underline{F}_n^{-1}\right) \rightarrow \underset{\approx}{e}_n \\ \underline{A} = \underline{F}_n\,\underline{\tau}(t_n)\underline{F}_n^T \rightarrow \underset{\approx}{A} \end{array} \right.$$

Elastic increment

When it is concluded that the current increment is purely elastic, the end-increment or current stress equals the calculated elastic trial stress. Viscoplastic strain does not need updating and is thus also known.

$$\begin{array}{lcl} \boldsymbol{\tau}(t_{n+1}) = \boldsymbol{\tau}_e & ; & \Delta\lambda = 0 \\ \bar{\varepsilon}_{vp}(t_{n+1}) = \bar{\varepsilon}_{vp}(t_n) & ; & \tau_y(t_{n+1}) = \tau_y(t_n) \end{array}$$

Viscoplastic increment

During the increment $\Delta t = t_{n+1} - t_n$ the stress evolution equations are integrated using an implicit Euler integration scheme.

$$\left.\begin{array}{l} \dot{\boldsymbol{P}} = 2\,\dfrac{\partial^2 W}{\partial \boldsymbol{C}^2} : (\dot{\boldsymbol{C}} - 2\,\boldsymbol{F}^c\cdot\dot{\lambda}\boldsymbol{a}\cdot\boldsymbol{F}) \\ \dot{\lambda} = \gamma\phi(F) = \gamma\left(\dfrac{F}{\tau_{y0}}\right)^N \end{array}\right\}$$

$$\left.\begin{array}{l} \boldsymbol{P} = \boldsymbol{P}(t_n) + 2\,\dfrac{\partial^2 W}{\partial \boldsymbol{C}^2} : \{\boldsymbol{C} - \boldsymbol{C}(t_n) - 2\,\boldsymbol{F}^c\cdot\Delta\lambda\,\boldsymbol{a}\cdot\boldsymbol{F}\} \\ \Delta\lambda = \Delta t\gamma\phi(F) \end{array}\right\}$$

The current or end-increment time $t = t_{n+1}$ is not indicated further. Constitutive equations are reformulated in the Kirchhoff stress tensor $\boldsymbol{\tau}$, using $\boldsymbol{\tau} = \boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c$, the incremental deformation tensor $\boldsymbol{F}_n$ and the Almansi strain tensor $\boldsymbol{e}_n$. A fourth-order elastic material tensor ${}^4\boldsymbol{H}$ is introduced and can be calculated for the Neo-Hookean elastic energy function W.

$$\left.\begin{array}{l} \boldsymbol{F}^{-1}\cdot\boldsymbol{\tau}\cdot\boldsymbol{F}^{-c} = \boldsymbol{F}^{-1}(t_n)\cdot\boldsymbol{\tau}(t_n)\cdot\boldsymbol{F}^{-c}(t_n) + 2\dfrac{\partial^2 W}{\partial \boldsymbol{C}^2} : \{\boldsymbol{C} - \boldsymbol{C}(t_n) - 2\,\boldsymbol{F}^c\cdot\Delta\lambda\,\boldsymbol{a}\cdot\boldsymbol{F}\} \\ \Delta\lambda = \Delta t\,\gamma\phi(F) \end{array}\right\}$$

$$\boldsymbol{F}_n = \boldsymbol{F}\cdot\boldsymbol{F}^{-1}(t_n) \quad \rightarrow \quad \boldsymbol{C} - \boldsymbol{C}(t_n) = \boldsymbol{F}^c\cdot(\boldsymbol{I} - \boldsymbol{F}_n^{-c}\cdot\boldsymbol{F}_n^{-1})\cdot\boldsymbol{F} = 2\,\boldsymbol{F}^c\cdot\boldsymbol{e}_n\cdot\boldsymbol{F}$$

$$\left.\begin{array}{l} \boldsymbol{\tau} = \boldsymbol{F}_n\cdot\boldsymbol{\tau}(t_n)\cdot\boldsymbol{F}_n^c + 4\boldsymbol{F}\cdot\dfrac{\partial^2 W}{\partial \boldsymbol{C}^2} : \boldsymbol{F}^c\cdot\,(\boldsymbol{e}_n - \Delta\lambda\,\boldsymbol{a})\,\cdot\boldsymbol{F}\cdot\boldsymbol{F}^c \\ \Delta\lambda = \Delta t\,\gamma\phi(F) \end{array}\right\}$$

$${}^4\boldsymbol{H} = 4\,\boldsymbol{F}\cdot\left(\boldsymbol{F}\cdot\frac{\partial^2 W}{\partial \boldsymbol{C}^2}\cdot\boldsymbol{F}^c\right)^{lc,rc}\cdot\boldsymbol{F}^c = 2\{\mu - \lambda\ln(J)\}\,{}^4\boldsymbol{I}^{rc} + \lambda\boldsymbol{I}\boldsymbol{I}$$

$$\left.\begin{array}{l} \boldsymbol{\tau} = \boldsymbol{F}_n\cdot\boldsymbol{\tau}(t_n)\cdot\boldsymbol{F}_n^c + {}^4\boldsymbol{H} : (\boldsymbol{e}_n - \Delta\lambda\,\boldsymbol{a}) = \boldsymbol{\tau}_e - \Delta\lambda\,{}^4\boldsymbol{H} : \boldsymbol{a} \\ \Delta\lambda = \Delta t\,\gamma\phi(F) \end{array}\right\}$$

Iterative stress update

The coupled set of equations is solved iteratively following a Newton-Raphson procedure. In the stress update procedure it may be necessary to take into account the change in the elastic trial stress and deformation. This is the case in a plane stress situation. Both δJ and $\delta\boldsymbol{\tau}_{tr}$ can then be expressed in $\delta\boldsymbol{\tau}$ and $\delta\lambda$. New variables (J_1, $\boldsymbol{J}_2$, $\boldsymbol{M}_1$, ${}^4\boldsymbol{M}_2$) are introduced, which can be specified explicitly later.

From the coupled set of iterative equations $\delta\boldsymbol{\tau}$ and $\delta\lambda$ can be solved, whereupon new (better) values of $\boldsymbol{\tau}$ and $\Delta\lambda$ are determined. The iteration process is stopped when the residuals $\boldsymbol{s}_1$ and s_2 are small enough.

When the iteration process has converged, the current values of $\boldsymbol{\tau}$ and $\Delta\lambda$ are known. Then the Cauchy stress $\boldsymbol{\sigma}$ and the viscoplastic deformation rate $\boldsymbol{D}_{vp}$ can be determined. The latter is used to calculate the effective viscoplastic strain $\bar{\varepsilon}_{vp}$. Subsequently the yield stress is updated according to the hardening rule.

$$\left.\begin{array}{l} \delta\boldsymbol{\tau} - \delta\boldsymbol{\tau}_e + {}^4\boldsymbol{H} : \boldsymbol{a}\,\delta\lambda + \Delta\lambda\,\delta{}^4\boldsymbol{H} : \boldsymbol{a} + \Delta\lambda\,{}^4\boldsymbol{H} : \delta\boldsymbol{a} = -\boldsymbol{s}_1 \\ \delta\lambda - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\boldsymbol{a} : \delta\boldsymbol{\tau} - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\left(\dfrac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right)\delta\lambda = -s_2 \end{array}\right\}$$

$$\text{with} \quad \begin{cases} \delta\boldsymbol{\tau}_e = \boldsymbol{M}_1 \delta\lambda + {}^4\boldsymbol{M}_2 : \delta\boldsymbol{\tau} \\ \delta^4\boldsymbol{H} = \left(\dfrac{\partial\, {}^4\boldsymbol{H}}{\partial J}\right) \delta J = {}^4\boldsymbol{c}\, \delta J \\ \delta\boldsymbol{a} = \left(\dfrac{\partial\boldsymbol{a}}{\partial\boldsymbol{\tau}}\right) : \delta\boldsymbol{\tau} = {}^4\boldsymbol{b} : \delta\boldsymbol{\tau} \\ \delta J = J_1 \delta\lambda + \boldsymbol{J}_2 : \delta\boldsymbol{\tau} \end{cases}$$

This can be rewritten with some abbreviations.

$$\left.\begin{aligned} {}^4\boldsymbol{R} : \delta\boldsymbol{\tau} + \boldsymbol{t}\, \delta\lambda &= -\boldsymbol{s}_1 \\ \boldsymbol{u} : \delta\boldsymbol{\tau} + v\, \delta\lambda &= -s_2 \end{aligned}\right\}$$

$$\begin{aligned} {}^4\boldsymbol{R} &= {}^4\boldsymbol{I} + \Delta\lambda\, {}^4\boldsymbol{H} : {}^4\boldsymbol{b} \\ \boldsymbol{t} &= {}^4\boldsymbol{H} : \boldsymbol{a} \\ \boldsymbol{u} &= -\,\Delta t\, \gamma \left(\frac{\partial\phi}{\partial F}\right) \boldsymbol{a} \\ v &= 1 - \Delta t\, \gamma \left(\frac{\partial\phi}{\partial F}\right) \left(\frac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right) \\ \boldsymbol{s}_1 &= \boldsymbol{\tau} - \boldsymbol{\tau}_e + \Delta\lambda\, {}^4\boldsymbol{H} : \boldsymbol{a} \\ s_2 &= \Delta\lambda - \Delta t\, \gamma\, \phi(F) \end{aligned}$$

Derivatives

The variations of various variables are determined by differentiation.

The hardening law relates the current yield stress to the equivalent viscoplastic strain. To describe the intrinsic softening followed by hardening, the relation between τ_y and $\bar{\varepsilon}_{vp}$ is taken to be a polynomial of 7th-order. Coefficients are fitted onto experimental data.

$$\tau_y = \tau_{y0} + h\bar{\varepsilon}_{vp} + a\bar{\varepsilon}_{vp}^2 + b\bar{\varepsilon}_{vp}^3 + c\bar{\varepsilon}_{vp}^4 + d\bar{\varepsilon}_{vp}^7$$

$$\begin{aligned} \frac{\partial\, {}^4\boldsymbol{H}}{\partial J} &= -2\lambda \frac{1}{J}\, {}^4\boldsymbol{I} = {}^4\boldsymbol{c} \quad \rightarrow \quad \underline{\underline{c}} = -2\lambda \frac{1}{J}\, \underline{\underline{I}} \\ \left(\frac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right) &= \frac{\partial}{\partial\bar{\varepsilon}_{vp}} \left(-\tau_y(\bar{\varepsilon}_{vp})\right) = -h - 2a\bar{\varepsilon}_{vp} - 3b\bar{\varepsilon}_{vp}^2 - 4c\bar{\varepsilon}_{vp}^3 - 7d\bar{\varepsilon}_{vp}^6 \\ \left(\frac{\partial\phi}{\partial F}\right) &= \frac{\partial}{\partial F} \left\{ \left(\frac{F}{\tau_{y0}}\right)^N \right\} = \frac{N}{\tau_{y0}} \left(\frac{F(\underline{\tau}, \bar{\varepsilon}_{vp})}{\tau_{y0}}\right)^{N-1} \\ \frac{\partial\boldsymbol{a}}{\partial\boldsymbol{\tau}} &= \frac{\partial\boldsymbol{a}}{\partial\boldsymbol{\tau}^d} : \frac{\partial\boldsymbol{\tau}^d}{\partial\boldsymbol{\tau}} \end{aligned}$$

$$\begin{aligned} \frac{\partial\boldsymbol{a}}{\partial\boldsymbol{\tau}^d} &= \frac{\partial}{\partial\boldsymbol{\tau}^d} \left\{ \tfrac{3}{2}\bar{\tau}^{-1}\boldsymbol{\tau}^d \right\} = \tfrac{3}{2} \left(-\bar{\tau}^{-2} \frac{\partial\bar{\tau}}{\partial\boldsymbol{\tau}^d} \right) \boldsymbol{\tau}^d + \tfrac{3}{2}\bar{\tau}^{-1}\, {}^4\boldsymbol{I} \\ \frac{\partial\boldsymbol{\tau}^d}{\partial\boldsymbol{\tau}} &= \frac{\partial}{\partial\boldsymbol{\tau}} \left\{ \boldsymbol{\tau} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I} \right\} = {}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I} \end{aligned}$$

$$\frac{\partial \bar{\tau}}{\partial \boldsymbol{\tau}^d} = \frac{\partial}{\partial \boldsymbol{\tau}^d} \left\{ \left(\tfrac{3}{2} \boldsymbol{\tau}^d : \boldsymbol{\tau}^d \right)^{1/2} \right\} = \tfrac{3}{2} \bar{\tau}^{-1} \boldsymbol{\tau}^d = \boldsymbol{a}$$

$$= \left(-\bar{\tau}^{-1} \boldsymbol{aa} + \tfrac{3}{2} \bar{\tau}^{-1}\, {}^4\boldsymbol{I} \right) : \left({}^4\boldsymbol{I} - \tfrac{1}{3} \boldsymbol{II} \right)$$

$$= -\bar{\tau}^{-1} \boldsymbol{aa} + \tfrac{3}{2} \bar{\tau}^{-1}\, {}^4\boldsymbol{I} - \tfrac{1}{2} \bar{\tau}^{-1}\, {}^4\boldsymbol{I} : \boldsymbol{II} = {}^4\boldsymbol{b} \quad \rightarrow$$

$$\underline{\underline{b}} = -\bar{\tau}^{-1} \underset{\approx}{a}\, \underset{\approx}{a}^T + \tfrac{3}{2} \bar{\tau}^{-1} \underline{\underline{I}} - \tfrac{1}{2} \bar{\tau}^{-1} \underset{\approx}{I}\, \underset{\approx}{I}^T = \bar{\tau}^{-1} \left(-\underset{\approx}{a}\, \underset{\approx}{a}^T + \tfrac{3}{2} \underline{\underline{I}} - \tfrac{1}{2} \underset{\approx}{I}\, \underset{\approx}{I}^T \right)$$

Stiffness

To evaluate the iterative Updated Lagrange weighted residual equation, not only the Cauchy stress $\boldsymbol{\sigma}$, but also the relation between the stress variation $\delta\boldsymbol{\sigma}$ and $\boldsymbol{L}_u = (\vec{\nabla}\vec{u})^c$ has to be known, i.e. $\delta\boldsymbol{\sigma} = {}^4\boldsymbol{M} : \boldsymbol{L}_u$.

The consistent stiffness tensor ${}^4\boldsymbol{M}$, eventually leads to the consistent stiffness matrix. It must be derived from the coupled nonlinear equations for $\boldsymbol{\tau}$ and $\Delta\lambda$. Iterative changes (variations) of $\delta\boldsymbol{\tau}$ and $\delta\lambda$ can be derived. To simplify notation we omit again the upper index i, which indicates the iteration step number.

To arrive at a relation between $\delta\boldsymbol{\tau}$ and $\delta\boldsymbol{F}_n$ some new tensors are introduced which can be specified later, when a coordinate system is chosen.

$$\left.\begin{aligned} &\boldsymbol{\tau} = \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + {}^4\boldsymbol{H} : \boldsymbol{e}_n - \Delta\lambda\, {}^4\boldsymbol{H} : \boldsymbol{a} \\ &\Delta\lambda = \Delta t\, \gamma\, \phi(F) \end{aligned}\right\} \quad \rightarrow$$

$$\left.\begin{aligned} \delta\boldsymbol{\tau} =& \delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c + \delta\, {}^4\boldsymbol{H} : (\boldsymbol{e}_n - \Delta\lambda\, \boldsymbol{a}) + \\ & {}^4\boldsymbol{H} : \delta\boldsymbol{e}_n - {}^4\boldsymbol{H} : \boldsymbol{a}\, \delta\lambda - \Delta\lambda\, {}^4\boldsymbol{H} : \left(\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}} \right) : \delta\boldsymbol{\tau} \\ \delta\lambda =& \left[\left\{ \Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \right\} / \left\{ 1 - \Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}} \right) \right\} \right] \boldsymbol{a} : \delta\boldsymbol{\tau} = c_1 \boldsymbol{a} : \delta\boldsymbol{\tau} \end{aligned}\right\} \quad \rightarrow$$

$$\left\{ {}^4\boldsymbol{I} + \Delta\lambda\, {}^4\boldsymbol{H} : \left(\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}} \right) + c_1\, {}^4\boldsymbol{H} : \boldsymbol{aa} \right\} : \delta\boldsymbol{\tau} =$$

$$\delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c + \delta\, {}^4\boldsymbol{H} : (\boldsymbol{e}_n - \Delta\lambda\, \boldsymbol{a}) + {}^4\boldsymbol{H} : \delta\boldsymbol{e}_n$$

$${}^4\boldsymbol{V} : \delta\boldsymbol{\tau} = {}^4\boldsymbol{E} : \delta\boldsymbol{F}_n \quad \rightarrow \quad \delta\boldsymbol{\tau} = {}^4\boldsymbol{V}^{-1} : {}^4\boldsymbol{E} : \delta\boldsymbol{F}_n$$

The additional tensors are calculated below.

$$\delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c = {}^4\boldsymbol{T} : \delta\boldsymbol{F}_n$$

$$J = \det(\boldsymbol{F}_n) = \det(\boldsymbol{F}_n + \delta\boldsymbol{F}_n) = J(1 + \boldsymbol{F}_n^{-1} : \delta\boldsymbol{F}_n) \quad \rightarrow \quad \delta J = J\, \boldsymbol{F}_n^{-1} : \delta\boldsymbol{F}_n$$

$$\delta\, {}^4\boldsymbol{H} = \left(\frac{\partial\, {}^4\boldsymbol{H}}{\partial J} \right) \delta J = \left(\frac{\partial\, {}^4\boldsymbol{H}}{\partial J} \right) \left(J \boldsymbol{F}_n^{-1} : \delta\boldsymbol{F}_n \right)$$

$$\left.\begin{aligned} &\delta\boldsymbol{e}_n = -\tfrac{1}{2} \delta\boldsymbol{F}_n^{-c} \cdot \boldsymbol{F}_n^{-1} - \tfrac{1}{2} \boldsymbol{F}_n^{-c} \cdot \delta\boldsymbol{F}_n^{-1} = -\, {}^4\boldsymbol{A}_1 : \delta\boldsymbol{F}_n^{-1} \\ &\delta\boldsymbol{F}_n^{-1} = -\boldsymbol{F}_n^{-1} \cdot \delta\boldsymbol{F}_n \cdot \boldsymbol{F}_n^{-1} = -\, {}^4\boldsymbol{A}_2 : \delta\boldsymbol{F}_n \end{aligned}\right\} \quad \rightarrow$$

$$\delta\boldsymbol{e}_n = ({}^4\boldsymbol{A}_1 : {}^4\boldsymbol{A}_2) : \delta\boldsymbol{F}_n = {}^4\boldsymbol{P} : \delta\boldsymbol{F}_n$$

Using the definition $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ a relation between $\delta\boldsymbol{\tau}$ and $\delta\boldsymbol{F}_n$ can be derived, which can be transformed to $\delta\boldsymbol{\sigma} = {}^4\boldsymbol{M} : \boldsymbol{L}_u$.

$$
\begin{aligned}
&\boldsymbol{\tau} = J\boldsymbol{\sigma} \quad \rightarrow \quad \boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{\tau} \quad \rightarrow \\
&\delta\boldsymbol{\sigma} = \frac{1}{J}\left(\delta\boldsymbol{\tau} - \boldsymbol{\sigma}\delta J\right) = \frac{1}{J}\left\{{}^4\boldsymbol{V}^{-1} : {}^4\boldsymbol{E} - \boldsymbol{\sigma} J \boldsymbol{F}_n^{-1}\right\} : \delta\boldsymbol{F}_n = {}^4\boldsymbol{C} : \delta\boldsymbol{F}_n \\
&\quad = {}^4\boldsymbol{C} : \left\{\boldsymbol{F}^{-c}(t_n)\cdot\delta\boldsymbol{F}^c\right\}^c = {}^4\boldsymbol{C} : \left\{\boldsymbol{F}^{-c}(t_n)\cdot\boldsymbol{F}^c\cdot\boldsymbol{L}_u^c\right\}^c \\
&\quad = {}^4\boldsymbol{M} : \boldsymbol{L}_u
\end{aligned}
$$

Matrix/column notation

The matrix/column notation for the consistent stiffness matrix is derived.

$$
\begin{aligned}
&\delta\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \delta\boldsymbol{F}_n && \rightarrow \quad \delta\underset{\sim}{\sigma} = \underline{\underline{C}}\,\delta\underset{\sim}{F_n}{}_t \\
&\delta\boldsymbol{F}_n = \left(\boldsymbol{F}^{-c}(t_n)\cdot\delta\boldsymbol{F}^c\right)^c && \rightarrow \quad \delta\underset{\sim}{F_n} = \left(\underline{\underline{F}}_t^{-1}(t_n)\delta\underset{\sim}{F}{}_t\right)_t \rightarrow \delta\underset{\sim}{F_n}{}_t = \underline{\underline{F}}_t^{-1}(t_n)\delta\underset{\sim}{F}{}_t \\
&\delta\boldsymbol{F}^c = \boldsymbol{F}^c\cdot\boldsymbol{L}_u^c && \rightarrow \quad \delta\underset{\sim}{F}{}_t = \underline{\underline{F}}_t\,\underset{\sim}{L_u}{}_t
\end{aligned}
$$

$$
\delta\underset{\sim}{\sigma} = \left[\underline{\underline{C}}\,\underline{\underline{F}}_t^{-1}(t_n)\underline{\underline{F}}_t\right]\underset{\sim}{L_u}{}_t = \underline{\underline{M}}\,\underset{\sim}{L_u}{}_t
$$

$$
\begin{aligned}
&\underline{\underline{M}} = \underline{\underline{C}}\,\underline{\underline{F}}_t^{-1}(t_n)\underline{\underline{F}}_t \\
&\underline{\underline{C}} = \frac{1}{J}\left(\underline{\underline{V}}^{-1}\,\underline{\underline{E}}_r - J\underset{\sim}{\sigma}\underset{\sim}{F_n}{}^{-T}\right) \\
&\underline{\underline{V}} = \underline{\underline{I}} + \Delta\lambda\,\underline{\underline{H}}_c\underline{\underline{b}} + c_1\underline{\underline{H}}_c\underset{\sim}{a}\underset{\sim}{a}^T \\
&\underline{\underline{E}} = \underline{\underline{T}} - 2\lambda\underline{\underline{I}}\left(\underset{\sim}{e_n} - \Delta\lambda\,\underset{\sim}{a}\right)\left(\underset{\sim}{F_n}{}^{-1}\right)^T + \underline{\underline{H}}_c\underline{\underline{P}}
\end{aligned}
$$

Plane strain

For plane strain some terms in the stress update equations vanish. During viscoplastic deformation the volume will not change, so $\delta J = 0$. Also, the elastic trial stress will remain as it is, i.e. $\delta\boldsymbol{\tau}_{tr} = \mathbf{0}$.

$$
\begin{aligned}
\delta J &= J_1\delta\lambda + \boldsymbol{J}_2 : \delta\boldsymbol{\tau} = 0 \\
\delta\boldsymbol{\tau}_{tr} &= \boldsymbol{M}_1\delta\lambda + {}^4\boldsymbol{M}_2 : \delta\boldsymbol{\tau} = \boldsymbol{O}
\end{aligned}
$$

Iterative stress update

$$
\left.
\begin{aligned}
&{}^4\boldsymbol{R} : \delta\boldsymbol{\tau} + \boldsymbol{t}\delta\lambda = -\boldsymbol{s}_1 \\[1ex]
&\boldsymbol{u} : \delta\boldsymbol{\tau} + v\delta\lambda = -s_2
\end{aligned}
\right\}
$$

$$
\begin{array}{lll}
{}^4\boldsymbol{R} = {}^4\boldsymbol{I} + \Delta\lambda\,{}^4\boldsymbol{H} : {}^4\boldsymbol{b} & ; & \boldsymbol{t} = {}^4\boldsymbol{H} : \boldsymbol{a} \\
\boldsymbol{u} = -\,\Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\boldsymbol{a} & ; & v = 1 - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\left(\dfrac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right) \\
\boldsymbol{s}_1 = \boldsymbol{\tau} - \boldsymbol{\tau}_{tr} + \Delta\lambda\,{}^4\boldsymbol{H} : \boldsymbol{a} & ; & s_2 = \Delta\lambda - \Delta t\,\gamma\,\phi(F)
\end{array}
$$

Matrix/column notation

It is assumed that there is no deformation in the x_3-direction ($u_3 = 0$), which results in the plane strain deformation in the (x_1x_2)-plane. The plane strain case can be derived rather straightforward from the three-dimensional formulation.

$$
\begin{bmatrix} \underline{\underline{R}}_c & \underset{\sim}{t} \\ \underset{\sim}{u}_t^T & v \end{bmatrix}
\begin{bmatrix} \delta\underset{\sim}{\tau} \\ \delta\lambda \end{bmatrix}
= -\begin{bmatrix} \underset{\sim}{s}_1 \\ s_2 \end{bmatrix}
$$

$$
\begin{array}{lll}
\underline{\underline{R}} = \underline{\underline{I}} + \Delta\lambda\,\underline{\underline{H}}\,\underline{\underline{b}}_t & ; & \underset{\sim}{t} = \underline{\underline{H}}\,\underset{\sim}{a}_t \\
\underset{\sim}{u} = -\Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\underset{\sim}{a} & ; & v = 1 - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\left(\dfrac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right) \\
\underset{\sim}{s}_1 = \underset{\sim}{\tau} - \underset{\sim}{\tau}_{tr} + \Delta\lambda\,\underline{\underline{H}}\,\underset{\sim}{a}_t & ; & s_2 = \Delta\lambda - \Delta t\,\gamma\phi(F)
\end{array}
$$

Stiffness

The plane strain stiffness in tensorial notation is analogous to the three-dimensional relation.

$$
\delta\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \delta\boldsymbol{F}_n = \frac{1}{J}\left\{ {}^4\boldsymbol{V}^{-1} : {}^4\boldsymbol{E} - \boldsymbol{\sigma}J\boldsymbol{F}_n^{-1}\right\} : \delta\boldsymbol{F}_n
$$

$$
\begin{aligned}
{}^4\boldsymbol{V} &= \left\{ {}^4\boldsymbol{I} + \Delta\lambda\,{}^4\boldsymbol{H} : {}^4\boldsymbol{b} + c_1\,{}^4\boldsymbol{H} : \boldsymbol{a}\boldsymbol{a}\right\} \\
{}^4\boldsymbol{E} &= \left\{ {}^4\boldsymbol{T} + {}^4\boldsymbol{c} : (\boldsymbol{e}_n - \Delta\lambda\,\boldsymbol{a})J\boldsymbol{F}_n^{-1} + {}^4\boldsymbol{H} : {}^4\boldsymbol{P}\right\} \\
\delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c &= {}^4\boldsymbol{T} : \delta\boldsymbol{F}_n \\
\delta\boldsymbol{e}_n &= {}^4\boldsymbol{P} : \delta\boldsymbol{F}_n
\end{aligned}
$$

Matrix/column notation

Matrix/column notation of the consistent stiffness matrix for plain strain deformation.

$$
\delta\underset{\sim}{\sigma} = \underline{\underline{C}}\left(\delta\underset{\sim}{F}_n\right)_t = \left[\frac{1}{J}\left\{\underline{\underline{V}}^{-1}\underline{\underline{E}}_r - \underset{\sim}{\sigma}J\underset{\sim}{F}_n^{-T}\right\}\right]\left(\delta\underset{\sim}{F}_n\right)_t
$$

$$
\begin{aligned}
\underline{\underline{V}} &= \underline{\underline{I}} + \Delta\lambda\,\underline{\underline{H}}_c\underline{\underline{b}} + c_1\underline{\underline{H}}_c\underset{\sim}{a}\underset{\sim}{a}^T \\
\underline{\underline{E}} &= \underline{\underline{T}} + 2\lambda\underline{\underline{I}}\left(\underset{\sim}{e} - \Delta\lambda\underset{\sim}{a}\right)J\underset{\sim}{F}_n^{-T} + \underline{\underline{H}}_c\underline{\underline{P}}
\end{aligned}
$$

Plane stress

For plane stress we have to take into account the variation of the trial stress and the deformation.

Iterative stress update

Again the system of equations to be solved can be written with some abbreviations.

$$\left.\begin{aligned} &{}^4\boldsymbol{R} : \delta\boldsymbol{\tau} + \boldsymbol{t}\delta\lambda = -\boldsymbol{s}_1 \\ &\boldsymbol{u} : \delta\boldsymbol{\tau} + v\delta\lambda = -s_2 \end{aligned}\right\}$$

$${}^4\boldsymbol{R} = {}^4\boldsymbol{I} - {}^4\boldsymbol{M}_2 + \Delta\lambda\,{}^4\boldsymbol{C} : \boldsymbol{a}\boldsymbol{J}_2 + \Delta\lambda\,{}^4\boldsymbol{H} : {}^4\boldsymbol{b}\;; \qquad \boldsymbol{t} = -\boldsymbol{M}_1 + \Delta\lambda\,{}^4\boldsymbol{C} : \boldsymbol{a}J_1 + {}^4\boldsymbol{H} : \boldsymbol{a}$$

$$\boldsymbol{u} = -\,\Delta t\,\gamma\left(\frac{\partial\phi}{\partial F}\right)\boldsymbol{a} \quad ; \qquad v = 1 - \Delta t\,\gamma\left(\frac{\partial\phi}{\partial F}\right)\left(\frac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right)$$

$$\boldsymbol{s}_1 = \boldsymbol{\tau} - \boldsymbol{\tau}_{trial} + \Delta\lambda\,{}^4\boldsymbol{H} : \boldsymbol{a} \quad ; \qquad s_2 = \Delta\lambda - \Delta t\,\gamma\,\phi(F)$$

Matrix/column notation

Introduction of a suitable (problem dependent !) coordinate system leads to the transformation of vectors and tensors into their components, which are stored in columns and matrices.

$$\begin{bmatrix} \underline{\underline{R}}_c & \underset{\approx}{t} \\ \underset{\approx}{u}_t^T & v \end{bmatrix} \begin{bmatrix} \delta\underset{\approx}{\tau} \\ \delta\lambda \end{bmatrix} = - \begin{bmatrix} \underset{\approx}{s}_1 \\ s_2 \end{bmatrix}$$

$$\underline{\underline{R}} = \underline{\underline{I}} - \underline{\underline{M}}_2 + \Delta\lambda\underline{\underline{C}}\,\underset{\approx}{a}_r\underset{\approx}{J}_2^T + \Delta\lambda\underline{\underline{H}}\,\underline{\underline{b}}_r \quad ; \qquad \underset{\approx}{t} = -\underline{\underline{M}}_1 + \Delta\lambda\underline{\underline{C}}\underset{\approx}{a}_t J_1 + \underline{\underline{H}}\underset{\approx}{a}_t$$

$$\underset{\approx}{u} = -\,\Delta t\,\gamma\left(\frac{\partial\phi}{\partial F}\right)\underset{\approx}{a} \quad ; \qquad v = 1 - \Delta t\,\gamma\left(\frac{\partial\phi}{\partial F}\right)\left(\frac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right)$$

$$\underset{\approx}{s}_1 = \underset{\approx}{\tau} - \underset{\approx}{\tau}_{tr} + \Delta\lambda\underline{\underline{H}}\underset{\approx}{a}_t \quad ; \qquad s_2 = \Delta\lambda - \Delta t\gamma\phi(F)$$

Stiffness

The plane stress stiffness in tensorial notation is analogous to the three-dimensional relation.

$$\delta\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \delta\boldsymbol{F}_n = \frac{1}{J}\left\{{}^4\boldsymbol{V}^{-1} : {}^4\boldsymbol{E} - \boldsymbol{\sigma}J\boldsymbol{F}_n^{-1}\right\} : \delta\boldsymbol{F}_n$$

with

$${}^4\boldsymbol{V} = \left\{{}^4\boldsymbol{I} + \Delta\lambda\,{}^4\boldsymbol{H} : \left(\frac{\partial\boldsymbol{a}}{\partial\boldsymbol{\tau}}\right) + c_1\,{}^4\boldsymbol{H} : \boldsymbol{a}\boldsymbol{a}\right\}$$

$${}^4\boldsymbol{E} = \left\{{}^4\boldsymbol{T} + \left(\frac{\partial\,{}^4\boldsymbol{H}}{\partial J}\right) : (\boldsymbol{e} - \Delta\lambda\,\boldsymbol{a})J\boldsymbol{F}_n^{-1} + {}^4\boldsymbol{H} : {}^4\boldsymbol{P}\right\}$$

$$\delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c = {}^4\boldsymbol{T} : \delta\boldsymbol{F}_n$$

Matrix-column notation

With the assumption that $\tau_{13} = \tau_{23} = \tau_{33} = 0$, the three-dimensional formulation reduces to that for two-dimensional plane stress deformation in the (x_1x_2)-plane. Columns with relevant components of stress and deformation rate are :

$$\underset{\sim}{\tau} = \left[\begin{array}{cccc} \tau_{11} & \tau_{22} & \tau_{12} & \tau_{21} \end{array} \right]^T$$
$$\underset{\sim}{D} = \left[\begin{array}{cccc} D_{11} & D_{22} & D_{12} & D_{21} \end{array} \right]^T$$

During the plane stress return mapping we have

$$\delta F_{11} = \delta F_{22} = \delta F_{12} = \delta F_{21} = 0 \qquad \text{and} \qquad \delta \underset{\sim}{\tau}^{trial} = \underset{\sim}{0}$$

As deformation in x_3-direction is allowed, δJ can be expressed in δF_{33} :

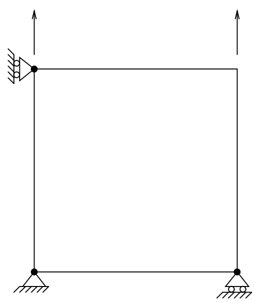
$$\delta J = (F_{11}F_{22} - F_{12}F_{21})\delta F_{33} = J_1 \delta\lambda + \underset{\sim}{J}_2^T \delta \underset{\sim}{\tau}$$

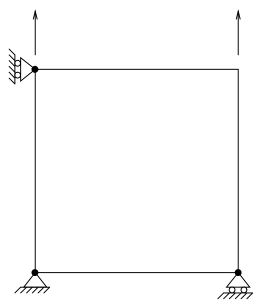
which results in the set of iterative equations for $\delta \underset{\sim}{\tau}$ and $\delta\lambda$.

13.4.5 Examples

Tensile test

A square plate or cylindrical bar is loaded uniaxially. Dimensions are listed in the table.





initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

Tensile test at various strain rates

The Perzyna model parameter values for polycarbonate (PC) are used and listed in the table. The axial elongation is prescribed as a linear function of time with a constant elongation rate. The tensile bar is axisymmetric with initial cross-sectional area $A_0 = 10$ mm^2. The axial stress and force are shown in the figure as a function of the elongation.

E	1800	MPa	ν	0.37	-
σ_{y0}	37	MPa	H	-200	MPa
γ	0.001	1/s	N	3	-
a	500	MPa	b	700	MPa
c	800	MPa	d	30000	MPa

elongation rate $\quad \dfrac{\dot{\Delta l}}{h_0} = \{0.01, 0.1, 1\}\ \mathrm{s}^{-1}$

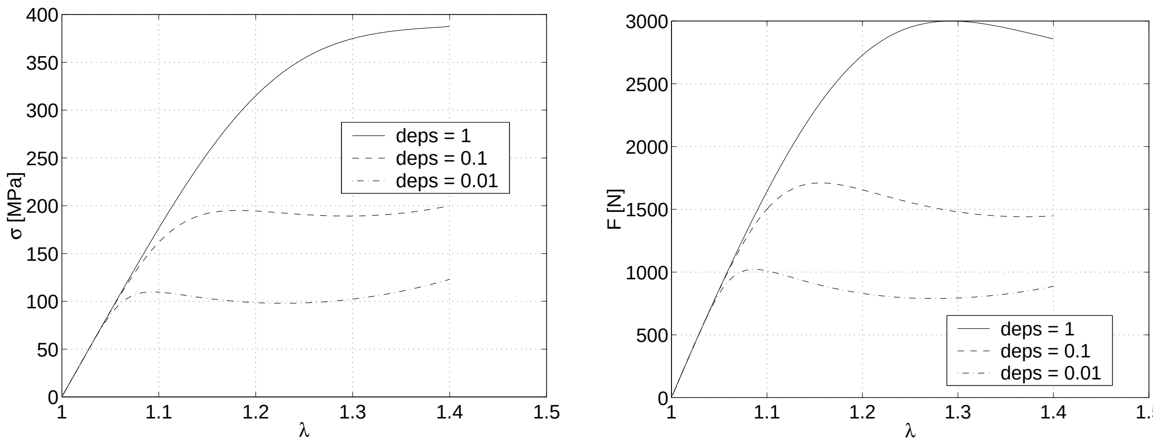
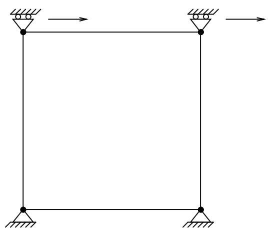


Fig. 13.39 : *Axial stress and force versus elongation for PC.*

Shear test

The simple shear test is analyzed with one element, where the horizontal displacement in the upper nodes is prescribed. Because there are no unknown degrees of freedom, the stiffness matrix is not used. The shear force is calculated for polycarbonate (PC). The prescribed strain rate is constant.



initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

strain rate $\quad \dot{\gamma} = \dfrac{\dot{u}}{h_0} = 0.01\ \mathrm{s}^{-1}$

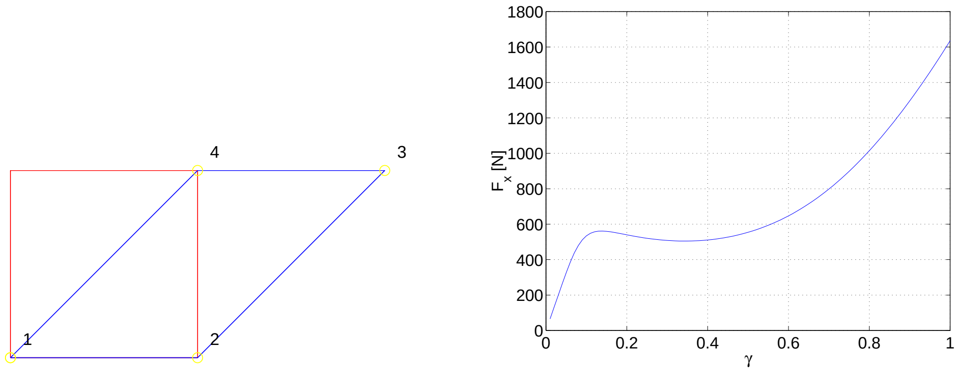


Fig. 13.40 : *Shear force versus shear strain for plane strain*

13.5 Nonlinear viscoelastic material behavior

The one-dimensional mechanical representation of the nonlinear viscoelastic (Leonov) model consists of a hardening spring in parallel with a Maxwell model, of which the viscosity is a nonlinear function of the stress.

For some materials the viscosity is decreased using a damage parameter, to describe intrinsic softening. Hardening at higher strains is described by the parallel spring.

In the model with hardening, the Cauchy stress σ is additively decomposed in an effective or driving stress s and a hardening stress w. This decomposition reflects the contribution of secondary interactions between polymer chains and that of the entangled polymer network.

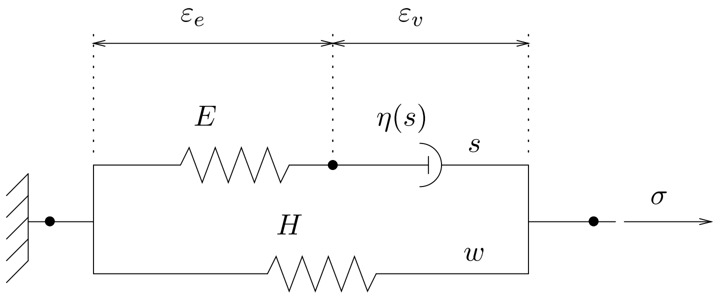


Fig. 13.41 : *Model for nonlinear viscoelastic behavior*

$$\sigma = s + w$$

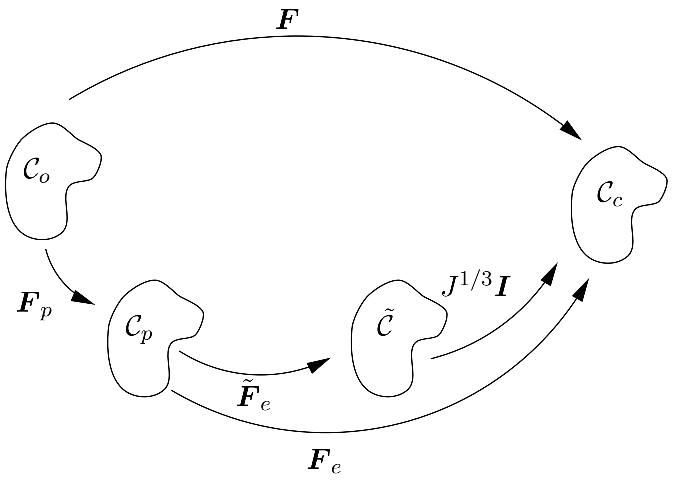
13.5.1 Kinematics

The deformation tensor $\boldsymbol{F}$ is multiplicatively decomposed into an elastic ($\boldsymbol{F}_e$) and a plastic ($\boldsymbol{F}_p$) contribution : $\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_p$. This decomposition follows from the postulate of a stress-free plastic intermediate configuration $\mathcal{C}_p$. As the decomposition is not unique with respect to rotational contributions, an extra assumption will later be needed regarding the rotations.

It is assumed that during plastic deformation the volume change is zero, i.e. $J_p = \det(\boldsymbol{F}_p) = 1$ and thus $J = \det(\boldsymbol{F}) = \det(\boldsymbol{F}_e)$. The elastic volume deformation is decoupled from the isochoric distortional deformation by the definition of the tensor $\tilde{\boldsymbol{F}}_e$ according to $\tilde{\boldsymbol{F}}_e = J^{-1/3}\boldsymbol{F}_e$.

The left Cauchy-Green strain tensor $\boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c$ is used as a strain measure. Its volume invariant elastic part is given by $\tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}_e \cdot \tilde{\boldsymbol{F}}_e^c$. The velocity gradient tensor $\boldsymbol{L} = (\vec{\nabla}\vec{v})^c = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$ can be written as the sum of the symmetric deformation rate tensor $\boldsymbol{D}$ and the skew-symmetric spin tensor $\boldsymbol{\Omega}$: $\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{\Omega}$. Using the decomposition of $\boldsymbol{F}$, we can split $\boldsymbol{L}$ in an elastic and a plastic part. This leads to associated tensors $\boldsymbol{D}_e$, $\boldsymbol{D}_p$, $\boldsymbol{\Omega}_e$ and $\boldsymbol{\Omega}_p$.

To make the decomposition of $\boldsymbol{F}$ unique, $\boldsymbol{\Omega}_p$ is chosen equal to the null tensor. It has been shown by e.g. Boyce, that this specific choice regarding rotational contributions has no significant influence on the overall stress-strain behavior.



$$
\begin{aligned}
\boldsymbol{F} &= (\vec{\nabla}_0\vec{x})^c = \boldsymbol{F}_e \cdot \boldsymbol{F}_p = J^{1/3}\boldsymbol{I} \cdot \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p \\
\boldsymbol{C} &= \boldsymbol{F}^c \cdot \boldsymbol{F} \qquad ; \qquad \boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c \quad \rightarrow \quad \tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}_e \cdot \tilde{\boldsymbol{F}}_e^c \\
\boldsymbol{L} &= \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = (\vec{\nabla}\vec{v})^c \\
&= \boldsymbol{L}_e + \boldsymbol{L}_p = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + (\boldsymbol{D}_p + \boldsymbol{\Omega}_p) = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + \boldsymbol{D}_p
\end{aligned}
$$

13.5.2 Constitutive relations

Stress decomposition

The deviatoric part of the driving stress, $\boldsymbol{s}^d$, is related to isochoric elastic deformation $\tilde{\boldsymbol{B}}_e^d$ through a generalized Hookean relation. The hydrostatic part $\boldsymbol{s}^h = -p\boldsymbol{I}$ (p = hydrostatic material pressure) is related to the volumetric deformation. Hardening is modeled according to Gaussian chain statistics as this model is applicable to a large number of thermoplastic polymers, both amorphous and semi-crystalline, up to very high extension ratios.

Material parameters are : the shear modulus G, the bulk modulus κ and the hardening modulus H. For the elastic part to be hyper-elastic, the shear modulus G should be replaced by $\frac{G}{J}$.

$$
\begin{aligned}
&\boldsymbol{\sigma} = \boldsymbol{s} + \boldsymbol{w} = \boldsymbol{s}^d + \boldsymbol{s}^h + \boldsymbol{w} \\
&\boldsymbol{s} = G\tilde{\boldsymbol{B}}_e^d + \kappa(J-1)\boldsymbol{I} \quad ; \quad \boldsymbol{w} = H\tilde{\boldsymbol{B}}^d
\end{aligned}
$$

Elastic deformation

As the model describes time- and history-dependent behavior, the elastic strain must be updated by integration of appropriate evolution equations for $\tilde{\boldsymbol{B}}_e$. The expression for $\dot{\tilde{\boldsymbol{B}}}_e$ can be derived starting from $\tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}_e \cdot \tilde{\boldsymbol{F}}_e^c$ and using the decomposition $\tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p$ and the assumption $\boldsymbol{\Omega}_p = \boldsymbol{O}$.

$$
\begin{aligned}
\tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}_e \cdot \tilde{\boldsymbol{F}}_e^c \quad &\rightarrow \quad \dot{\tilde{\boldsymbol{B}}}_e = \dot{\tilde{\boldsymbol{F}}}_e \cdot \tilde{\boldsymbol{F}}_e^c + \tilde{\boldsymbol{F}}_e \cdot \dot{\tilde{\boldsymbol{F}}}_e^c \\
&\tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p \quad \rightarrow \quad \tilde{\boldsymbol{F}}_e = \tilde{\boldsymbol{F}} \cdot \boldsymbol{F}_p^{-1} \quad \rightarrow \quad \dot{\tilde{\boldsymbol{F}}}_e = \dot{\tilde{\boldsymbol{F}}} \cdot \boldsymbol{F}_p^{-1} + \tilde{\boldsymbol{F}} \cdot \dot{\boldsymbol{F}}_p^{-1} \\
\dot{\tilde{\boldsymbol{B}}}_e &= \left(\dot{\tilde{\boldsymbol{F}}} \cdot \boldsymbol{F}_p^{-1} + \tilde{\boldsymbol{F}} \cdot \dot{\boldsymbol{F}}_p^{-1} \right) \cdot \tilde{\boldsymbol{F}}_e^c + \tilde{\boldsymbol{F}}_e \cdot \left(\boldsymbol{F}_p^{-c} \cdot \dot{\tilde{\boldsymbol{F}}}^c + \dot{\boldsymbol{F}}_p^{-c} \cdot \tilde{\boldsymbol{F}}^c \right) \\
&= \left(\dot{\tilde{\boldsymbol{F}}} \cdot \boldsymbol{F}_p^{-1} \cdot \tilde{\boldsymbol{F}}_e^{-1} + \tilde{\boldsymbol{F}} \cdot \dot{\boldsymbol{F}}_p^{-1} \cdot \tilde{\boldsymbol{F}}_e^{-1} \right) \cdot \tilde{\boldsymbol{B}}_e + \\
&\qquad \tilde{\boldsymbol{B}}_e \cdot \left(\tilde{\boldsymbol{F}}_e^{-c} \cdot \boldsymbol{F}_p^{-c} \cdot \dot{\tilde{\boldsymbol{F}}}^c + \tilde{\boldsymbol{F}}_e^{-c} \cdot \dot{\boldsymbol{F}}_p^{-c} \cdot \tilde{\boldsymbol{F}}^c \right) \\
&= \left(\tilde{\boldsymbol{L}} + \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p \cdot \dot{\boldsymbol{F}}_p^{-1} \tilde{\boldsymbol{F}}_e^{-1} \right) \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{B}}_e \cdot \left(\tilde{\boldsymbol{L}}^c + \tilde{\boldsymbol{F}}_e^{-c} \cdot \dot{\boldsymbol{F}}_p^{-c} \cdot \boldsymbol{F}_p^c \cdot \tilde{\boldsymbol{F}}_e^c \right) \\
&\qquad \boldsymbol{F}_p \cdot \boldsymbol{F}_p^{-1} = \boldsymbol{I} \quad \rightarrow \quad \boldsymbol{F}_p \cdot \dot{\boldsymbol{F}}_p^{-1} = -\dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \quad \rightarrow \\
&= (\tilde{\boldsymbol{L}} - \boldsymbol{D}_p) \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{B}}_e \cdot (\tilde{\boldsymbol{L}}^c - \boldsymbol{D}_p)
\end{aligned}
$$

Viscoplastic deformation

The viscoplastic deformation rate $\boldsymbol{D}_p$ is related to the deviatoric stress $\boldsymbol{s}^d$, through the viscosity, which is a nonlinear function of the equivalent stress $\bar{s}$, the hydrostatic pressure p, the absolute temperature T and the damage parameter D.

For polymers the Eyring viscosity function is successfully used and for metal alloys the viscosity function of Bodner-Partom ([**?**],[**?**],[**?**]).

For the equivalent deviatoric stress $\bar{s}$ is the Von Mises definition is used.

$$
\boldsymbol{D}_p = \frac{1}{2\eta} \boldsymbol{s}^d
$$

$$
\begin{aligned}
&\eta = \eta(\bar{s}, p, T, D) \\
&\bar{s} = \sqrt{\tfrac{3}{2} \boldsymbol{s}^d : \boldsymbol{s}^d} \\
&p = \kappa(J-1)\boldsymbol{I}
\end{aligned}
$$

Eyring viscosity

For polymer materials the plastic deformation rate tensor $\boldsymbol{D}_p$ is related to the deviatoric stress $\boldsymbol{s}^d$ by an Eyring viscosity η. This is a function of the equivalent Von Mises stress $\bar{s}$, the hydrostatic stress p and the absolute temperature T. In the model, presented here, the viscosity is depending on an intrinsic softening quantity D, determined by an evolution equation, which has to be solved with the other constitutive relations.

Material parameters are :

A_0	time constant
ΔH	activation energy
R	universal gas constant
p	hydrostatic pressure
μ	parameter describing pressure dependence
V	shear activation volume
D_∞	saturation value of D

$$\eta = \frac{A\bar{s}}{\sqrt{3}\sinh\left(\dfrac{\bar{s}}{\sqrt{3}\tau_0}\right)}$$

$$\bar{s} = \sqrt{\tfrac{3}{2}\boldsymbol{s}^d : \boldsymbol{s}^d}$$

$$A = A_0 \exp\left[\frac{\Delta H}{RT} + \frac{\mu p}{\tau_0} - D\right]$$

$$\tau_0 = \frac{RT}{V} \qquad ; \qquad p = -\tfrac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})$$

$$\dot{D} = h\left(1 - \frac{D}{D_\infty}\right)\frac{\bar{s}}{\sqrt{6}\,\eta} \qquad ; \quad D \in [0, D_\infty]$$

Bodner-Partom viscosity

To describe viscoplastic behavior of metals, the plastic deformation rate tensor $\boldsymbol{D}_p$ is related to the deviatoric stress $\boldsymbol{s}^d$ by a Bodner-Partom viscosity η. This is a function of the equivalent Von Mises stress $\bar{\sigma}$ and Z, the resistance to plastic flow. Γ_0 is a constant which reflects the smoothness of the transition from the elastic to the viscoplastic response and n characterizes the rate sensitivity of the viscoplastic response. The plastic flow resistance Z depends on the equivalent plastic strain $\bar{\varepsilon}_p$. Its lower and upper bounds are Z_0 and Z_1.

The Bodner-Partom model corresponds to isotropic hardening.

$$\eta = \frac{\bar{s}}{\sqrt{12\Gamma_0}}\exp\left[\tfrac{1}{2}\left(\frac{Z}{\bar{\sigma}}\right)^{2n}\right]$$

$$\bar{s} = \sqrt{\tfrac{3}{2}\boldsymbol{s}^d : \boldsymbol{s}^d}$$

$$Z = Z_1 + (Z_0 - Z_1)e^{-m\bar{\varepsilon}_p}$$

$$\dot{\bar{\varepsilon}}_p = \sqrt{\tfrac{2}{3}\boldsymbol{D}_p : \boldsymbol{D}_p} \qquad \rightarrow \qquad \bar{\varepsilon}_p$$

Plastic strain rate

The current value of $\tilde{\boldsymbol{B}}_e(t)$ can be determined by integration of $\dot{\tilde{\boldsymbol{B}}}_e$. However, the integrand $\dot{\tilde{\boldsymbol{B}}}_e$ is not objective, so that rigid body rotations will influence the results, which is of course not allowed. The problem of non-objectivity of $\dot{\tilde{\boldsymbol{B}}}_e$ can be circumvented by using an evolution equation for the Cauchy-Green plastic strain tensor $\boldsymbol{C}_p$, which is invariant.

Starting from $\tilde{\boldsymbol{F}}$ an expression for $\dot{\boldsymbol{C}}_p$ can be derived, containing $\dot{\tilde{\boldsymbol{B}}}_e$. With the earlier derived expression for $\dot{\tilde{\boldsymbol{B}}}_e$, $\dot{\boldsymbol{C}}_p$ can be expressed in $\tilde{\boldsymbol{B}}_e$ and $\boldsymbol{D}_p$. The relation between $\boldsymbol{D}_p$ and $\tilde{\boldsymbol{B}}_e^d$ allows $\dot{\boldsymbol{C}}_p$ to be related to $\tilde{\boldsymbol{B}}_e$ and $\boldsymbol{C}_p$. This equation states that the direction of the plastic strain rate is defined by the directional tensor $\boldsymbol{A}$, while the plastic strain rate magnitude is governed by the characteristic plastic deformation rate Γ.

The plastic strain rate is invariant for rigid body rotations. It is shown in literature that this formulation with a plastic predictor, can be used to apply an implicit, robustly stable and efficient time integration procedure.

$$\left.\begin{array}{l} \left.\begin{array}{l} \tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p \quad \rightarrow \quad \boldsymbol{C}_p = \boldsymbol{F}_p^c \cdot \boldsymbol{F}_p = \tilde{\boldsymbol{F}}^c \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \tilde{\boldsymbol{F}} \quad \rightarrow \\ \dot{\boldsymbol{C}}_p = \tilde{\boldsymbol{F}}^c \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \left[\tilde{\boldsymbol{B}}_e \cdot \tilde{\boldsymbol{L}}^c + \tilde{\boldsymbol{B}}_e \cdot \dot{\tilde{\boldsymbol{B}}}_e^{-1} \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{L}} \cdot \tilde{\boldsymbol{B}}_e \right] \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \tilde{\boldsymbol{F}} \end{array}\right\} \\ \\ \left.\begin{array}{l} \dot{\tilde{\boldsymbol{B}}}_e = (\tilde{\boldsymbol{L}} - \boldsymbol{D}_p) \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{B}}_e \cdot (\tilde{\boldsymbol{L}}^c - \boldsymbol{D}_p) \quad \rightarrow \\ \tilde{\boldsymbol{B}}_e \cdot \dot{\tilde{\boldsymbol{B}}}_e^{-1} = -\tilde{\boldsymbol{L}} - \tilde{\boldsymbol{B}}_e \cdot \tilde{\boldsymbol{L}}^c \cdot \tilde{\boldsymbol{B}}_e^{-1} + \boldsymbol{D}_p + \tilde{\boldsymbol{B}}_e \cdot \boldsymbol{D}_p \cdot \tilde{\boldsymbol{B}}_e^{-1} \end{array}\right\} \end{array}\right\} \rightarrow$$

$$\dot{\boldsymbol{C}}_p = \tilde{\boldsymbol{F}}^c \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \left[\boldsymbol{D}_p \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{B}}_e \cdot \boldsymbol{D}_p \right] \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \tilde{\boldsymbol{F}}$$

$$\text{with} \quad \boldsymbol{D}_p = \frac{1}{2\eta} \boldsymbol{s}^d = \frac{G}{2\eta} \tilde{\boldsymbol{B}}_e^d \quad \rightarrow$$

$$= \frac{G}{\eta} \left(\tilde{\boldsymbol{C}} - \tfrac{1}{3} \operatorname{tr}(\tilde{\boldsymbol{B}}_e) \boldsymbol{C}_p \right) = \Gamma \left(\tilde{\boldsymbol{C}} - \frac{1}{\alpha} \boldsymbol{C}_p \right) = \Gamma \boldsymbol{A}$$

13.5.3 Constitutive model

The material model can be summarized as a set of constitutive equations. The differential equations must be integrated to determine the current elastic strain and stress. Also the variation of he stress must be derived from the constitutive model, representing the current stiffness.

$$\begin{array}{l} J = \det(\boldsymbol{F}) \quad \rightarrow \quad \tilde{\boldsymbol{F}} = J^{-1/3} \boldsymbol{F} \quad \rightarrow \quad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}}^c \quad \rightarrow \quad \boldsymbol{w} = H \tilde{\boldsymbol{B}}^d \\ p = \kappa(J-1) \quad \rightarrow \quad \boldsymbol{s}^h = p\boldsymbol{I} \\ \left.\begin{array}{l} \dot{\boldsymbol{C}}_p = \dfrac{G}{\eta} \left(\tilde{\boldsymbol{C}} - \tfrac{1}{3} \operatorname{tr}(\tilde{\boldsymbol{B}}_e) \, \boldsymbol{C}_p \right) \\ \tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \tilde{\boldsymbol{F}}^c \end{array}\right\} \quad \rightarrow \quad \boldsymbol{s}^d = G \tilde{\boldsymbol{B}}_e^d \quad \rightarrow \quad \bar{s} = \sqrt{\tfrac{3}{2} \boldsymbol{s}^d : \boldsymbol{s}^d} \end{array}$$

$$\boldsymbol{\sigma} = \boldsymbol{s}^d + \boldsymbol{s}^h + \boldsymbol{w}$$

13.5.4 Incremental analysis

The plastic strain $\boldsymbol{C}_p$ at the current time t must be determined by integration of the differential equation for $\dot{\boldsymbol{C}}_p(\tau)$. In an incremental procedure the total deformation period is subdivided into a number of sequential time increments : $\Delta t = t_{i+1} - t_i$; $i = 0 \cdots n$. A solution for the governing equations is determined for the discrete end-increment times, starting from the known state – with known values of all variables – at the begin-increment time. This implies that the differential equation for $\dot{\boldsymbol{C}}_p$ has to be solved for the last increment $t_n \rightarrow t_{n+1}$ assuming that $\boldsymbol{C}_p(t_n)$ is known. For simplicity we skip the indication of the current end-increment time t_{n+1}.

We now focus attention on the last increment $[t_n, t_{n+1}]$. It is assumed that at time t_n the configuration $\mathcal{C}_n$ is completely known and all equations are satisfied. The begin-increment state $\mathcal{C}_n$ at $\tau = t_n$ is taken as the reference configuration for deformation variables, which is known as the Updated Lagrange procedure.

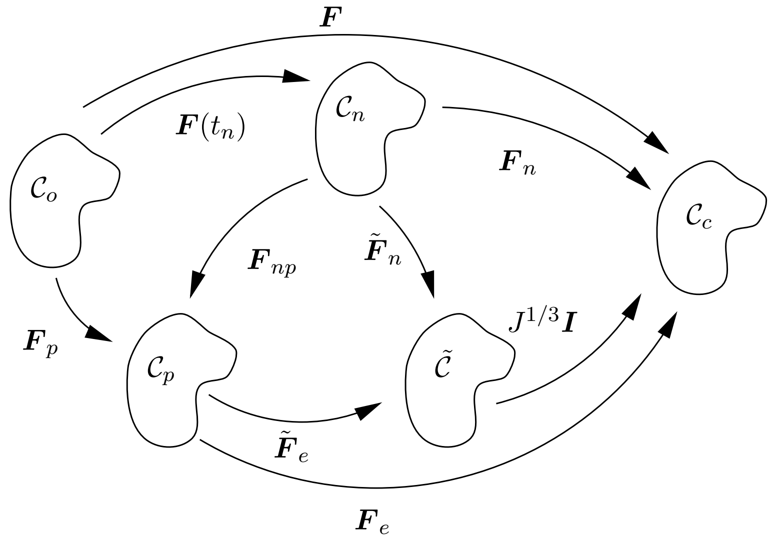


Fig. 13.42 : *Incremental deformation*

$$
\begin{aligned}
&\boldsymbol{F}(\tau) = \boldsymbol{F}_n(\tau) \cdot \boldsymbol{F}(t_n) \quad \rightarrow \quad \boldsymbol{F}_n(\tau) = \boldsymbol{F}(\tau) \cdot \boldsymbol{F}^{-1}(t_n) \\
&\tilde{\boldsymbol{F}}(\tau) = \tilde{\boldsymbol{F}}_n(\tau) \cdot \tilde{\boldsymbol{F}}(t_n) \\
&\boldsymbol{F}_n = \left(\vec{\nabla}_n \vec{x}\right)^c = \boldsymbol{R}_n \cdot \boldsymbol{U}_n
\end{aligned}
$$

Incremental plastic strain

Using the multiplicative decomposition, an expression for $\boldsymbol{C}_{p_n}(\tau)$ can be derived. It contains the tensor $\bar{\tilde{\boldsymbol{B}}}_{e_n}^{-1}$ which is the rotation neutralized version of $\tilde{\boldsymbol{B}}_e^{-1}$:

$$
\bar{\tilde{\boldsymbol{B}}}_{e_n}^{-1} = \boldsymbol{R}_n^c \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \boldsymbol{R}_n
$$

where $\boldsymbol{R}_n$ is the incremental rotation tensor.

$$
\begin{aligned}
\boldsymbol{C}_p(\tau) &= \boldsymbol{F}_p^c(\tau) \cdot \boldsymbol{F}_p(\tau) = \tilde{\boldsymbol{F}}^c(\tau) \cdot \tilde{\boldsymbol{B}}_e^{-1}(\tau) \cdot \tilde{\boldsymbol{F}}(\tau) \\
&\qquad\qquad \text{with} \quad \tilde{\boldsymbol{F}}(\tau) = \tilde{\boldsymbol{F}}_n(\tau) \cdot \tilde{\boldsymbol{F}}(t_n) \quad \rightarrow \\
&= \tilde{\boldsymbol{F}}^c(t_n) \cdot \left[\tilde{\boldsymbol{F}}_n^c(\tau) \cdot \tilde{\boldsymbol{B}}_e^{-1}(\tau) \cdot \tilde{\boldsymbol{F}}_n(\tau)\right] \cdot \tilde{\boldsymbol{F}}(t_n) \\
&= \tilde{\boldsymbol{F}}^c(t_n) \cdot \boldsymbol{C}_{p_n}(\tau) \cdot \tilde{\boldsymbol{F}}(t_n)
\end{aligned}
$$

incremental rotation neutralized plastic strain

$$
\begin{aligned}
\boldsymbol{C}_{p_n}(\tau) &= \tilde{\boldsymbol{F}}_n^c(\tau) \cdot \tilde{\boldsymbol{B}}_e^{-1}(\tau) \cdot \tilde{\boldsymbol{F}}_n(\tau) \\
&= \tilde{\boldsymbol{U}}_n(\tau) \cdot \left[\boldsymbol{R}_n^c(\tau) \cdot \tilde{\boldsymbol{B}}_e^{-1}(\tau) \cdot \boldsymbol{R}_n(\tau) \right] \cdot \tilde{\boldsymbol{U}}_n(\tau) \\
&= \tilde{\boldsymbol{U}}_n(\tau) \cdot \bar{\tilde{\boldsymbol{B}}}_{e_n}^{-1}(\tau) \cdot \tilde{\boldsymbol{U}}_n(\tau)
\end{aligned}
$$

Constitutive equations

With the incremental procedure the constitutive model is formulated in the incremental variables.

$$
J = \det(\boldsymbol{F}) \quad \rightarrow \quad \tilde{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F} \quad \rightarrow \quad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}}^c \quad \rightarrow \quad \boldsymbol{w} = H\tilde{\boldsymbol{B}}^d
$$

$$
p = \kappa(J-1) \quad \rightarrow \quad \boldsymbol{s}^h = p\boldsymbol{I}
$$

$$
\left.
\begin{aligned}
&\dot{\boldsymbol{C}}_{p_n} = \frac{G}{\eta}\left(\tilde{\boldsymbol{C}}_n - \tfrac{1}{3}\mathrm{tr}\left(\bar{\tilde{\boldsymbol{B}}}_{e_n}\right)\boldsymbol{C}_{p_n}\right) \\
&\bar{\tilde{\boldsymbol{B}}}_{e_n} = \tilde{\boldsymbol{U}}_n \cdot \boldsymbol{C}_{p_n}^{-1} \cdot \tilde{\boldsymbol{U}}_n^c \rightarrow \tilde{\boldsymbol{B}}_e = \boldsymbol{R}_n \cdot \bar{\tilde{\boldsymbol{B}}}_{e_n} \cdot \boldsymbol{R}_n^c \\
&\dot{D} = h\left(1 - \frac{D}{D_\infty}\right)\frac{\bar{s}}{\sqrt{6}\,\eta} \\
&\eta = \eta(\bar{s}, p, T, D)
\end{aligned}
\right\} \rightarrow \boldsymbol{s}^d = G\tilde{\boldsymbol{B}}_e^d \rightarrow \bar{s} = \sqrt{\tfrac{3}{2}\boldsymbol{s}^d : \boldsymbol{s}^d}
$$

$$
\boldsymbol{\sigma} = \boldsymbol{s}^d + \boldsymbol{s}^h + \boldsymbol{w}
$$

Stress update

The incremental plastic strain rate $\dot{\boldsymbol{C}}_{p_n}(\tau)$ can be integrated over the last increment $t_n \rightarrow t_{n+1}$ to determine $\boldsymbol{C}_{p_n}(t_{n+1})$. An implicit backward Euler integration scheme is used. With $\tilde{\boldsymbol{U}}_n(t_n) = \boldsymbol{I}$ we have $\boldsymbol{C}_{p_n}(t_n) = \bar{\tilde{\boldsymbol{B}}}_{e_n}^{-1}(t_n) = \tilde{\boldsymbol{B}}_{e_n}^{-1}(t_n)$.

The scalar λ is the so-called elasticity scalar, a state variable indicating the proportion of incremental elastic/plastic strains with respect to the incremental total strains ($\lambda = 1$, fully elastic increment, and $\lambda = 0$, fully plastic increment). This parameter depends on η and thus on $\boldsymbol{s}$ and $\boldsymbol{C}_p$. The isochoric elastic strain $\tilde{\boldsymbol{B}}_e$ can be calculated from $\boldsymbol{C}_{p_n}$ and $\tilde{\boldsymbol{F}}_n$.

$$
\begin{aligned}
&\dot{\boldsymbol{C}}_{p_n}(\tau) = \mathit{\Gamma}(\tau)\left[\tilde{\boldsymbol{C}}_n(\tau) - \frac{1}{\bar{\alpha}_n(\tau)}\boldsymbol{C}_{p_n}(\tau)\right] \quad ; \quad \frac{1}{\bar{\alpha}_n} = \tfrac{1}{3}\mathrm{tr}\left(\bar{\tilde{\boldsymbol{B}}}_{e_n}\right) \\
&\frac{1}{\mathit{\Delta} t}\left[\boldsymbol{C}_{p_n} - \boldsymbol{C}_{p_n}(t_n)\right] = \mathit{\Gamma}\left[\tilde{\boldsymbol{C}}_n - \frac{1}{\bar{\alpha}_n}\boldsymbol{C}_{p_n}\right] \quad \rightarrow \\
&\boldsymbol{C}_{p_n} = \frac{\bar{\alpha}_n\,\mathit{\Delta} t\,\mathit{\Gamma}}{\bar{\alpha}_n + \mathit{\Delta} t\,\mathit{\Gamma}}\tilde{\boldsymbol{C}}_n + \frac{\bar{\alpha}_n}{\bar{\alpha}_n + \mathit{\Delta} t\,\mathit{\Gamma}}\boldsymbol{C}_{p_n}(t_n) \quad \rightarrow \\
&\boldsymbol{C}_{p_n} = \bar{\alpha}_n(1-\lambda)\tilde{\boldsymbol{C}}_n + \lambda\boldsymbol{C}_{p_n}(t_n) \quad ; \quad \lambda = \frac{\bar{\alpha}_n}{\bar{\alpha}_n + \mathit{\Delta} t\,\mathit{\Gamma}} = \text{elasticity parameter}
\end{aligned}
$$

$$
\tilde{\boldsymbol{B}}_e = \boldsymbol{R}_n \cdot \bar{\tilde{\boldsymbol{B}}}_{e_n} \cdot \boldsymbol{R}_n^c = \tilde{\boldsymbol{F}}_n \cdot \boldsymbol{C}_{p_n}^{-1} \cdot \tilde{\boldsymbol{F}}_n^c
$$

Sub-incremental plastic strain update

The differential equation for the incremental plastic strain can be integrated more accurately by subdividing the current increment $\Delta t = t_{n+1} - t_n$ in a number (ns) of sub-increments $\delta t = \Delta t/ns$. The known iterative approximation for the end-increment deformation $(\tilde{\boldsymbol{F}}_n \rightarrow \tilde{\boldsymbol{C}}_n)$ is also subdivided and subsequently values for $\boldsymbol{C}^j_{p_n}$ are determined with a backward Euler integration scheme.

The incremental rotation is not taken into account during this procedure but incorporated afterward at the end-increment time. It is also assumed that the principal strain directions do not change during the integration procedure.

The sub-incremental integration scheme results in a more accurate determination of $\boldsymbol{C}_{p_n}$ and thus $\boldsymbol{\sigma}$. It allows for larger incremental time steps.

Be aware that the final $\lambda^j = \lambda^{ns+1}$ is not the elasticity parameter λ introduced earlier, indicating the elastic part of the increment. This λ must be calculated without using sub-increments or just according to

$$\lambda = \frac{\bar{\alpha}_n}{\bar{\alpha}_n + \Delta t \Gamma} = \frac{1}{1 + \Delta t \Gamma}$$

where we assumed $\bar{\alpha}_n = 1$.

$$\dot{\boldsymbol{C}}_{p_n}(\tau) = \Gamma(\tau)\left[\tilde{\boldsymbol{C}}_n(\tau) - \frac{1}{\bar{\alpha}_n(\tau)}\boldsymbol{C}_{p_n}(\tau)\right] \quad ; \quad \frac{1}{\bar{\alpha}_n} = \tfrac{1}{3}\mathrm{tr}\left(\bar{\tilde{\boldsymbol{B}}}_{e_n}\right)$$

$$\left.\begin{array}{llll} \text{sub-incremental deformation :} & j = 1 \cdots ns+1 & & \\ j = 1 \quad : \quad \tau = t_n & ; & j = ns+1 \quad : & \tau = t_{n+1} \\ \delta t = \Delta t/ns \quad ; & \delta\tilde{\boldsymbol{C}}_n = \left\{\tilde{\boldsymbol{C}}_n\right\}^{1/ns} & ; & \tilde{\boldsymbol{C}}^j_n = \left\{\delta\tilde{\boldsymbol{C}}_n\right\}^j \end{array}\right\}$$

$$\frac{1}{\delta t}\left[\boldsymbol{C}^j_{p_n} - \boldsymbol{C}^{j-1}_{p_n}\right] = \Gamma^j \left[\tilde{\boldsymbol{C}}^j_n - \frac{1}{\bar{\alpha}^j_n}\boldsymbol{C}^j_{p_n}\right] \quad \rightarrow$$

$$\boldsymbol{C}^j_{p_n} = \frac{\bar{\alpha}^j_n\,\delta t\,\Gamma^j}{\bar{\alpha}^j_n + \delta t\,\Gamma^j}\tilde{\boldsymbol{C}}^j_n + \frac{\bar{\alpha}^j_n}{\bar{\alpha}^j_n + \delta t\,\Gamma^j}\boldsymbol{C}^{j-1}_{p_n} \quad \rightarrow$$

$$\boldsymbol{C}^j_{p_n} = \bar{\alpha}^j_n(1-\lambda^j)\tilde{\boldsymbol{C}}^j_n + \lambda^j\boldsymbol{C}^{j-1}_{p_n} \quad ; \quad \lambda^j = \frac{\bar{\alpha}^j_n}{\bar{\alpha}^j_n + \delta t\,\Gamma^j}$$

incremental plastic strain $\qquad \boldsymbol{C}_{p_n} = \boldsymbol{C}_{p_n}(t_{n+1}) = \boldsymbol{C}^{ns+1}_{p_n}$

total isochoric elastic strain $\qquad \tilde{\boldsymbol{B}}_e = \boldsymbol{R}_n \cdot \bar{\tilde{\boldsymbol{B}}}_{e_n} \cdot \boldsymbol{R}^c_n = \tilde{\boldsymbol{F}}_n \cdot \boldsymbol{C}^{-1}_{p_n} \cdot \tilde{\boldsymbol{F}}^c_n$

Iterative scalar variable update

The current plastic strain depends on two scalar variables : the elasticity parameter λ and the softening parameter D. These are a function of the stress $\boldsymbol{\sigma}$, which implies that the integration has to be carried out iteratively. A Newton-Raphson iterative procedure is employed and the resulting equation system involves partial derivatives of λ and D, which can be calculated rather straightforwardly.

After convergence of the iterative process the (sub)incremental plastic strain and stress is known, but beware that these are only approximations for the real end-increment values. The update procedure is part of the iterative procedure which has to be repeated until convergence is reached.

$$
\begin{array}{lcl}
\lambda = 1/(1+\Delta t\Gamma) & \rightarrow & f(\lambda, D) = \lambda(1+\Delta t\Gamma) = 1 \\
\frac{1}{\Delta t}\left\{D - D(t_n)\right\} = \dot{D} & \rightarrow & g(\lambda, D) = D - \Delta t\dot{D} = D(t_n)
\end{array}
$$

Newton-Raphson iterative solution procedure

$$
\begin{bmatrix} \dfrac{\partial f}{\partial \lambda} & \dfrac{\partial f}{\partial D} \\ \dfrac{\partial g}{\partial \lambda} & \dfrac{\partial g}{\partial D} \end{bmatrix}
\begin{bmatrix} \delta\lambda \\ \delta D \end{bmatrix}
=
\begin{bmatrix} 1 - f^* \\ D(t_n) - g^* \end{bmatrix}
=
\begin{bmatrix} r^*_\lambda \\ r^*_D \end{bmatrix}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial \lambda} &= 1 + \Delta t\Gamma + \lambda\Delta t\,\frac{\partial \Gamma}{\partial \lambda} = 1 + \Delta t\Gamma - \lambda\Delta t\frac{G}{\eta^2}\,\frac{\partial \eta}{\partial \bar{\sigma}}\,\frac{\partial \bar{\sigma}}{\partial \lambda} \\
&= 1 + \Delta t\Gamma - \lambda\Delta t\frac{G}{\eta^2}\left[\eta\left(\frac{1}{\bar{\sigma}} - \frac{1}{\sqrt{3}\tau_0}\right)\right]\bar{\sigma} \\
\frac{\partial f}{\partial D} &= \lambda\Delta t\,\frac{\partial \Gamma}{\partial D} = -\lambda\Delta t\frac{G}{\eta^2}\,\frac{\partial \eta}{\partial D} = \lambda\Delta t\frac{G}{\eta^2}\,\eta = \lambda\Delta t\Gamma \\
\frac{\partial g}{\partial \lambda} &= -\Delta t\,\frac{\partial \dot{D}}{\partial \lambda} = -\Delta t\,\frac{\partial \dot{D}}{\partial \bar{\sigma}}\,\frac{\partial \bar{\sigma}}{\partial \lambda} = -\Delta t\,\left[\frac{\dot{D}}{\sqrt{3}\tau_0}\right]\bar{\sigma} \\
\frac{\partial g}{\partial D} &= 1 - \Delta t\,\frac{\partial \dot{D}}{\partial D} = 1 - \Delta t\,\left[\dot{D} - \frac{h\bar{\sigma}}{\sqrt{6}D_\infty\eta}\right]
\end{aligned}
$$

Matrix/column notation

The tensors and vectors in the presented mathematics can be written in components w.r.t. a vector basis. The components are stored in columns and matrices and the tensor formulations are transferred into matrix/column formulations which can be implemented rather straightforwardly in a computer code.

$$
J = \det(\underline{F}) \quad \rightarrow \quad \underline{\tilde{F}} = J^{-1/3}\underline{F} \quad \rightarrow \quad \underline{\tilde{B}} = \underline{\tilde{F}}\,\underline{\tilde{F}}^T \quad \rightarrow \quad \underline{w} = H\underline{\tilde{B}}^d
$$

$$
p = \kappa(J-1) \quad \rightarrow \quad \underline{s}^h = p\underline{I}
$$

$$\left.\begin{array}{l}\left.\begin{array}{l}\lambda = 1/(1+\Delta t \mathit{\Gamma}) \\[2ex] \frac{1}{\Delta t}\left\{D - D(t_n)\right\} = \dot{D}\end{array}\right\} \;\rightarrow\; \lambda, D \\[2ex] \underline{C}_{pn} = (1-\lambda)\underline{\tilde{C}}_n + \lambda \underline{C}_{pn}(t_n) \\[2ex] \underline{\bar{\tilde{B}}}_{en} = \underline{\tilde{U}}_n\, \underline{C}_{pn}^{-1}\, \underline{\tilde{U}}_n^T \\[2ex] \underline{\bar{s}}^d = G\underline{\bar{\tilde{B}}}_{en} \;\rightarrow\; \bar{s} = \sqrt{\frac{3}{2}\mathrm{tr}(\underline{\bar{s}}^d\,\underline{\bar{s}}^d)} \\[2ex] \eta = \eta(\bar{s}, p, T, D)\end{array}\right\} \rightarrow \underline{\bar{\tilde{B}}}_{en} \rightarrow \left\{\begin{array}{l}\underline{\tilde{B}}_e = \underline{R}_n\, \underline{\bar{\tilde{B}}}_{en}\, \underline{R}_n^T \rightarrow \\[2ex] \underline{s}^d = G\underline{\tilde{B}}_e^d\end{array}\right.$$

$$\underline{\sigma} = \underline{s}^d + \underline{s}^h + \underline{w}$$

Stiffness

The stress is related to the elastic isochoric strain $\tilde{\boldsymbol{B}}_e$, the volume change J and the total isochoric strain $\tilde{\boldsymbol{B}}$. Each of the three quantities will be considered separately and relations between their variations and $\delta\boldsymbol{F}$ will be derived.

The consistent material stiffness tensor relates the iterative change of the Cauchy stress tensor $\delta\boldsymbol{\sigma}$ to the iterative displacement $\delta\vec{u}$. In the derivation of this relation it is assumed that approximate end-increment values of all relevant variables are known. ${}^4\boldsymbol{S}_d$, ${}^4\boldsymbol{S}_h$ and ${}^4\boldsymbol{H}$ are properly defined fourth-order tensors.

$$\left.\begin{array}{l}\boldsymbol{\sigma} = \boldsymbol{s}^d + \boldsymbol{s}^h + \boldsymbol{w} = G\tilde{\boldsymbol{B}}_e^d + \kappa\boldsymbol{I}(J-1) + H\tilde{\boldsymbol{B}}^d \\[2ex] \quad\left.\begin{array}{l}\tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}\cdot\boldsymbol{C}_p^{-1}\cdot\tilde{\boldsymbol{F}}^c \\ \boldsymbol{C}_p = (1-\lambda)\tilde{\boldsymbol{C}} + \lambda\boldsymbol{C}_p(t_n) \\ \tilde{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F}\end{array}\right\}\end{array}\right\}$$

$$\begin{aligned}\delta\boldsymbol{\sigma} &= \delta\boldsymbol{s}^d + \delta\boldsymbol{s}^h + \delta\boldsymbol{w} \\ &= G\,\delta\tilde{\boldsymbol{B}}_e^d + \kappa\,\boldsymbol{I}\delta J + H\,\delta\tilde{\boldsymbol{B}}^d = \left({}^4\boldsymbol{S}_d + {}^4\boldsymbol{S}_h + {}^4\boldsymbol{H}\right) : \delta\boldsymbol{F} \\ &= {}^4\boldsymbol{S} : \delta\boldsymbol{F} = {}^4\boldsymbol{S}^{rc} : \delta\boldsymbol{F}^c \quad \text{with} \quad \delta\boldsymbol{F}^c = \vec{\nabla}_0\vec{u} = \boldsymbol{F}^c\cdot\vec{\nabla}\vec{u} = \boldsymbol{F}^c\cdot\boldsymbol{L}_u^c \\ &= {}^4\boldsymbol{S}^{rc} : (\boldsymbol{F}^c\cdot\boldsymbol{L}_u^c) \\ &= {}^4\boldsymbol{M} : \boldsymbol{L}_u^c\end{aligned}$$

Elastic strain variation

The elastic strain $\tilde{\boldsymbol{B}}_e$ must be calculated from the total deformation $\tilde{\boldsymbol{F}}$ and the plastic strain $\boldsymbol{C}_p$. Its variation is related to $\delta\tilde{\boldsymbol{F}}$ and $\delta\boldsymbol{C}_p$ which will be considered separately.

$$\begin{aligned}\tilde{\boldsymbol{B}}_e &= \tilde{\boldsymbol{F}}\cdot\boldsymbol{C}_p^{-1}\cdot\tilde{\boldsymbol{F}}^c \\ \delta\tilde{\boldsymbol{B}}_e &= \delta\tilde{\boldsymbol{F}}\cdot\boldsymbol{C}_p^{-1}\cdot\tilde{\boldsymbol{F}}^c - \tilde{\boldsymbol{F}}\cdot\boldsymbol{C}_p^{-1}\cdot\delta\boldsymbol{C}_p\cdot\boldsymbol{C}_p^{-1}\cdot\tilde{\boldsymbol{F}}^c + \tilde{\boldsymbol{F}}\cdot\boldsymbol{C}_p^{-1}\cdot\delta\tilde{\boldsymbol{F}}^c\end{aligned}$$

$$= \left(\tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-c} \cdot \delta\tilde{\boldsymbol{F}}^c\right)^c - \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \left(\tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-c} \cdot \delta\boldsymbol{C}_p^c\right)^c + \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \delta\tilde{\boldsymbol{F}}^c$$
$$= \left(\boldsymbol{M}^{(1)} \cdot \delta\tilde{\boldsymbol{F}}^c\right)^c - \boldsymbol{M}^{(2)} \cdot \left(\boldsymbol{M}^{(1)} \cdot \delta\boldsymbol{C}_p^c\right)^c + \boldsymbol{M}^{(2)} \cdot \delta\tilde{\boldsymbol{F}}^c$$

$$\tilde{\boldsymbol{B}}_e^d = \tilde{\boldsymbol{B}}_e - \tfrac{1}{3}\mathrm{tr}(\tilde{\boldsymbol{B}}_e)\boldsymbol{I} = \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) : \tilde{\boldsymbol{B}}_e$$
$$\delta\tilde{\boldsymbol{B}}_e^d = \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) : \delta\tilde{\boldsymbol{B}}_e$$

Plastic strain variation

The variation of the plastic strain $\boldsymbol{C}_p$ is related to $\delta\tilde{\boldsymbol{F}}$ (via $\delta\tilde{\boldsymbol{C}}$) and $\delta\lambda$. These variations will be considered separately.

$$\begin{aligned}
\boldsymbol{C}_p &= (1-\lambda)\tilde{\boldsymbol{C}} + \lambda\boldsymbol{C}_p(t_n) \\
\delta\boldsymbol{C}_p &= (1-\lambda)\delta\tilde{\boldsymbol{C}} + \left(\boldsymbol{C}_p(t_n) - \tilde{\boldsymbol{C}}\right)\delta\lambda \\
&= (1-\lambda)\left(\delta\tilde{\boldsymbol{F}}^c \cdot \tilde{\boldsymbol{F}} + \tilde{\boldsymbol{F}}^c \cdot \delta\tilde{\boldsymbol{F}}\right) + \left(\boldsymbol{C}_p(t_n) - \tilde{\boldsymbol{C}}\right)\delta\lambda \\
&= (1-\lambda)\left[\left(\tilde{\boldsymbol{F}}^c \cdot \delta\tilde{\boldsymbol{F}}\right)^c + \tilde{\boldsymbol{F}}^c \cdot \delta\tilde{\boldsymbol{F}}\right] + \left(\boldsymbol{C}_p(t_n) - \tilde{\boldsymbol{C}}\right)\delta\lambda
\end{aligned}$$

Deformation tensor variation

The variation of the isochoric deformation tensor $\tilde{\boldsymbol{F}}$ can be expressed in the variation of the total deformation tensor $\boldsymbol{F}$. The volume ratio J is assumed to be constant in this variation.

$$\begin{aligned}
\tilde{\boldsymbol{F}} &= J^{-1/3}\boldsymbol{F} \\
\delta\tilde{\boldsymbol{F}} &= -\tfrac{1}{6}J^{-1/3}\boldsymbol{F}\boldsymbol{I} : \left(\delta\boldsymbol{F} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F}^{-c} \cdot \delta\boldsymbol{F}^c\right) + J^{-1/3}\delta\boldsymbol{F} \\
&= -\tfrac{1}{3}J^{-1/3}\boldsymbol{F}\left(\boldsymbol{F}^{-c} : \delta\boldsymbol{F}^c\right) + J^{-1/3}\delta\boldsymbol{F}
\end{aligned}$$

Elasticity scalar variation

The variation $\delta\lambda$ of the elasticity parameter λ can be expressed in $\delta\tilde{\boldsymbol{B}}_e$ and $\delta\boldsymbol{F}$, starting from

$$\lambda = \frac{1}{1 + \Delta t\Gamma} = \frac{\eta}{\eta + G\Delta t} \quad \rightarrow \quad \delta\lambda = \frac{\lambda\Delta t\Gamma}{G\Delta t + \eta}\,\delta\eta$$

The variation $\delta\eta$ can be written as :

$$\delta\eta = \frac{\partial\eta}{\partial\bar{\sigma}}\,\delta\bar{\sigma} + \frac{\partial\eta}{\partial p}\,\delta p + \frac{\partial\eta}{\partial D}\,\frac{\partial D}{\partial\bar{\sigma}}\,\delta\bar{\sigma} + \frac{\partial\eta}{\partial D}\,\frac{\partial D}{\partial p}\,\delta p$$
$$\delta\bar{\sigma} = \frac{3G^2}{2\bar{\sigma}}\,\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e^d = \frac{3G^2}{2\bar{\sigma}}\,\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e$$
$$\delta p = -\kappa J\mathrm{tr}(\delta\boldsymbol{F}) = -\kappa J\boldsymbol{I} : \delta\boldsymbol{F}$$
$$= \frac{3G^2}{2\bar{\sigma}}\left(\frac{\partial\eta}{\partial\bar{\sigma}} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial\bar{\sigma}}\right)\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e - \kappa J\left(\frac{\partial\eta}{\partial p} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial p}\right)\boldsymbol{I} : \delta\boldsymbol{F}$$
$$= h_1\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e + h_2\boldsymbol{I} : \delta\boldsymbol{F}$$

A number of partial derivatives must be calculated to determine h_1 and h_2.

$$\delta\lambda = \frac{\lambda \Delta t \Gamma}{G\Delta t + \eta}\,\delta\eta = l_1 \tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e + l_2 \boldsymbol{I} : \delta\boldsymbol{F}$$

$$l_1 = \frac{\lambda \Delta t\, \Gamma\, h_1}{\Delta t\, G + \eta} \quad ; \quad l_2 = \frac{l_1 h_2}{h_1}$$

$$h_1 = \frac{3G^2}{2\bar{\sigma}}\left(\frac{\partial\eta}{\partial\bar{\sigma}} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial\bar{\sigma}}\right) \quad ; \quad h_2 = -\kappa J\left(\frac{\partial\eta}{\partial p} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial p}\right)$$

$$\frac{\partial\eta}{\partial\bar{\sigma}} = \eta\left(\frac{1}{\bar{\sigma}} - \frac{1}{\sqrt{3}\tau_0}\right) \quad ; \quad \frac{\partial\eta}{\partial p} = \frac{\eta\mu}{\tau_0} \quad ; \quad \frac{\partial\eta}{\partial D} = -\eta$$

$$\frac{\partial D}{\partial\bar{\sigma}} = \frac{\Delta t\,\frac{\partial\dot{D}}{\partial\bar{\sigma}}}{1 - \Delta t\,\frac{\partial\dot{D}}{\partial D}} \quad ; \quad \frac{\partial D}{\partial p} = \frac{\Delta t\,\frac{\partial\dot{D}}{\partial p}}{1 - \Delta t\,\frac{\partial\dot{D}}{\partial D}}$$

$$\frac{\partial\dot{D}}{\partial\bar{\sigma}} = \frac{\dot{D}}{\sqrt{3}\tau_0} \quad ; \quad \frac{\partial\dot{D}}{\partial p} = -\frac{\dot{D}\mu}{\tau_0} \quad ; \quad \frac{\partial\dot{D}}{\partial D} = \dot{D} - \frac{h\bar{\sigma}}{\sqrt{6}\,D_\infty\,\eta}$$

$$\text{with} \qquad \dot{D} = h\left(1 - \frac{D}{D_\infty}\right)\frac{\bar{\sigma}}{\sqrt{6}\,\eta}$$

Deviatoric stress variation

The variation of the deviatoric stress tensor is related to $\delta\tilde{\boldsymbol{B}}_e^d$ and subsequently to $\delta\boldsymbol{F}$:

$$\delta\boldsymbol{s}^d = G\,\delta\tilde{\boldsymbol{B}}_e^d = {}^4\boldsymbol{S}_d : \delta\boldsymbol{F}$$

Hydrostatic stress variation

The variation of the hydrostatic stress $\boldsymbol{s}^h$ is related to the variation of the volume factor J. The latter can be related to the variation of $\boldsymbol{F}$, resulting in a relation between $\delta\boldsymbol{s}^h$ and $\delta\boldsymbol{F}$.

$$\delta\boldsymbol{s}^h = \kappa\,\boldsymbol{I}\delta J = {}^4\boldsymbol{S}_h : \delta\boldsymbol{F}$$

$$\begin{aligned}
\dot{J} &= J\,\mathrm{tr}(\boldsymbol{D}) = J\,\tfrac{1}{2}\,\mathrm{tr}\left\{\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} + \left(\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}\right)^c\right\} \quad \rightarrow \\
\delta J &= \tfrac{1}{2}J\,\mathrm{tr}\left(\delta\boldsymbol{F}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}^{-c}\cdot\delta\boldsymbol{F}^c\right) = \tfrac{1}{2}J\left(\boldsymbol{F}^{-c} : \delta\boldsymbol{F}^c\right) + \tfrac{1}{2}J\left(\boldsymbol{F}^{-c} : \delta\boldsymbol{F}^c\right) \\
&= J\,\boldsymbol{F}^{-c} : \delta\boldsymbol{F}^c = J\,\boldsymbol{F}^{-1} : \delta\boldsymbol{F}
\end{aligned}$$

Hardening stress variation

The hardening stress $\boldsymbol{w}$ is related to the deviatoric total volume invariant strain $\tilde{\boldsymbol{B}}^d$. The variation $\delta\boldsymbol{w}$ can be related to $\delta\boldsymbol{F}$.

$$\delta\boldsymbol{w} = H\,\delta\tilde{\boldsymbol{B}}^d = {}^4\boldsymbol{H} : \delta\boldsymbol{F}$$

$$\begin{aligned}
\tilde{\boldsymbol{B}} &= \tilde{\boldsymbol{F}}\cdot\tilde{\boldsymbol{F}}^c \\
\delta\tilde{\boldsymbol{B}} &= \delta\tilde{\boldsymbol{F}}\cdot\tilde{\boldsymbol{F}}^c + \tilde{\boldsymbol{F}}\cdot\delta\tilde{\boldsymbol{F}}^c
\end{aligned}$$

$$\begin{aligned}
\tilde{\boldsymbol{B}}^d &= \tilde{\boldsymbol{B}} - \tfrac{1}{3}\mathrm{tr}(\tilde{\boldsymbol{B}})\boldsymbol{I} = \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) : \tilde{\boldsymbol{B}} \\
\delta\tilde{\boldsymbol{B}}^d &= \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) : \left\{\left(\tilde{\boldsymbol{F}}\cdot\delta\tilde{\boldsymbol{F}}^c\right)^c + \tilde{\boldsymbol{F}}\cdot\delta\tilde{\boldsymbol{F}}^c\right\}
\end{aligned}$$

Consistent material stiffness tensor

The variation of the Cauchy stress $\delta\boldsymbol{\sigma}$ is related to the variation of the deformation tensor $\delta\boldsymbol{F}$. In the iterative weighted residual equation $\delta\boldsymbol{\sigma}$ must be related to the gradient of the iterative displacement $\boldsymbol{L}_u = (\vec{\nabla}\vec{u})^c = (\vec{\nabla}\delta\vec{x})^c$. The resulting fourth-order tensor ${}^4\boldsymbol{M}$ is the consistent material stiffness tensor.

The components of $\delta\boldsymbol{\sigma}$ (in column $\delta\underset{\sim}{\sigma}$) and $\boldsymbol{L}_u$ (in column $\underset{\approx}{L}_u$) are related by the consistent stiffness matrix $\underline{\underline{M}}$.

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= \delta\boldsymbol{s}^d + \delta\boldsymbol{s}^h + \delta\boldsymbol{w} \\
&= \left({}^4\boldsymbol{S}_d + {}^4\boldsymbol{S}_h + {}^4\boldsymbol{H}\right) : \delta\boldsymbol{F} = {}^4\boldsymbol{S} : \delta\boldsymbol{F} = {}^4\boldsymbol{S}^{rc} : \delta\boldsymbol{F}^c \\
&\qquad\qquad \text{with} \quad \delta\boldsymbol{F}^c = \vec{\nabla}_0\vec{u} = \boldsymbol{F}^c \cdot \vec{\nabla}\vec{u} = \boldsymbol{F}^c \cdot \boldsymbol{L}_u^c \quad \rightarrow \\
&= {}^4\boldsymbol{S}^{rc} : (\boldsymbol{F}^c \cdot \boldsymbol{L}_u^c) = {}^4\boldsymbol{M} : \boldsymbol{L}_u^c
\end{aligned}
$$

Matrix/column notation

$$
\begin{aligned}
\delta\underset{\approx}{\tilde{B}}_e &= \left(\underline{\underline{M}}_{cr}^{(1)} + \underline{\underline{M}}_c^{(2)}\right)\delta\underset{\approx}{\tilde{F}} - \underline{\underline{M}}_c^{(2)}\underline{\underline{M}}_c^{(1)}\delta\underset{\approx}{C}_p \quad ; \quad \underline{\underline{M}}^{(1)} = \underline{\underline{\tilde{F}}}\,\underline{\underline{C}}_p^{-T} \;;\; \underline{\underline{M}}^{(2)} = \underline{\underline{\tilde{F}}}\,\underline{\underline{C}}_p^{-1} \\
&= \underline{\underline{A}}^{(1)}\delta\underset{\approx}{\tilde{F}} + \underline{\underline{A}}^{(2)}\delta\underset{\approx}{C}_p
\end{aligned}
$$

$$
\delta\underset{\approx}{\tilde{B}}{}_e^d = \left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\right)\delta\underset{\approx}{\tilde{B}}_e
$$

$$
\delta\underset{\approx}{C}_p = \left[(1-\lambda)\left(\underline{\underline{\tilde{F}}}_{tr} + \underline{\underline{\tilde{F}}}_t\right)\right]\delta\underset{\approx}{\tilde{F}} + \left(\underset{\approx}{C}_p(t_n) - \underset{\approx}{\tilde{C}}\right)\delta\lambda = \underline{\underline{C}}^{(1)}\delta\underset{\approx}{\tilde{F}} + \underset{\approx}{C}^{(2)}\delta\lambda
$$

$$
\delta\underset{\approx}{\tilde{F}} = \left[-\tfrac{1}{3}J^{-1/3}\underset{\approx}{F}\left(\underset{\approx}{F}^{-1}\right)_t^T + J^{-1/3}\underline{\underline{I}}\right]\delta\underset{\approx}{F} = \underline{\underline{F}}\,\delta\underset{\approx}{F}
$$

$$
\delta\lambda = = l_1\left(\underset{\approx}{\tilde{B}}{}_e^d\right)_t^T\delta\underset{\approx}{\tilde{B}}_e + l_2\underset{\approx}{I}_t^T\delta\underset{\approx}{F}
$$

Matrix/column notation

$$
\begin{aligned}
\delta\underset{\approx}{\tilde{B}}_e &= \underline{\underline{A}}^{(1)}\delta\underset{\approx}{\tilde{F}} + \underline{\underline{A}}^{(2)}\delta\underset{\approx}{C}_p = \left(\underline{\underline{A}}^{(1)} + \underline{\underline{A}}^{(2)}\underline{\underline{C}}^{(1)}\right)\delta\underset{\approx}{\tilde{F}} + \underline{\underline{A}}^{(2)}\underset{\approx}{C}^{(2)}\delta\lambda = \underline{\underline{B}}^{(1)}\delta\underset{\approx}{\tilde{F}} + \underset{\approx}{B}^{(2)}\delta\lambda \\
&= \underline{\underline{B}}^{(1)}\underline{\underline{F}}\delta\underset{\approx}{F} + l_1\underset{\approx}{B}^{(2)}\left(\underset{\approx}{\tilde{B}}{}_e^d\right)_t^T\delta\underset{\approx}{\tilde{B}}_e + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}_t^T\delta\underset{\approx}{F} \\
\delta\underset{\approx}{\tilde{B}}_e &= \left[\underline{\underline{I}} - l_1\underset{\approx}{B}^{(2)}\left(\underset{\approx}{\tilde{B}}{}_e^d\right)_t^T\right]^{-1}\left[\underline{\underline{B}}^{(1)}\underline{\underline{F}} + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}_t^T\right]\delta\underset{\approx}{F}
\end{aligned}
$$

$$
\delta\underset{\approx}{\tilde{B}}{}_e^d = \left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\right)\delta\underset{\approx}{\tilde{B}}_e = \left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\right)\left[\underline{\underline{I}} - l_1\underset{\approx}{B}^{(2)}\left(\underset{\approx}{\tilde{B}}{}_e^d\right)_t^T\right]^{-1}\left[\underline{\underline{B}}^{(1)}\underline{\underline{F}} + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}_t^T\right]\delta\underset{\approx}{F} = \underline{\underline{B}}^{(3)}\delta\underset{\approx}{F}
$$

$$
\begin{aligned}
\delta\underset{\approx}{\tilde{B}}{}^d &= \left(\underline{\underline{I}}_c - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\right)\left(\underline{\underline{\tilde{F}}}_{cr} + \underline{\underline{\tilde{F}}}_c\right)\delta\underset{\approx}{F} \\
&= \left(\underline{\underline{\tilde{F}}}_{cr} + \underline{\underline{\tilde{F}}}_c - \tfrac{2}{3}\underset{\approx}{I}\underset{\approx}{I}_t^T\underline{\underline{\tilde{F}}}_c\right)\underline{\underline{F}}\delta\underset{\approx}{F} = \underline{\underline{B}}^{(4)}\delta\underset{\approx}{F}
\end{aligned}
$$

Matrix/column notation

The components of the deviatoric, hydrostatic and hardening stress tensors are now stored in columns.

$$\delta \underset{\approx}{s}^d = G\delta \underset{\approx}{\tilde{B}}{}_e^d = G\underline{\underline{B}}^{(3)}\delta \underset{\approx}{F} = \underline{\underline{S}}_d \delta \underset{\approx}{F}$$

$$\delta \underset{\approx}{s}^h = \kappa\, \underset{\approx}{I}\delta J = \kappa J\, \underset{\approx}{I}\left(\underset{\approx}{F}^{-1}\right)_t^T \delta \underset{\approx}{F} = \underline{\underline{S}}_h \delta \underset{\approx}{F}$$

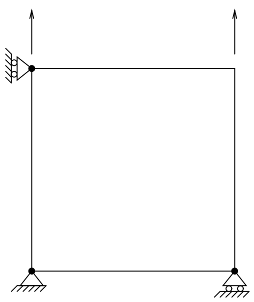
$$\delta \underset{\approx}{w} = H\,\delta \underset{\approx}{\tilde{B}}{}^d = H\,\underline{\underline{B}}^{(4)}\delta \underset{\approx}{F} = \underline{\underline{H}}\delta \underset{\approx}{F}$$

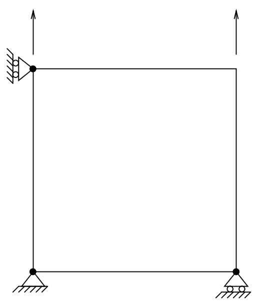
$$\begin{aligned}
\delta \underset{\approx}{\sigma} &= \delta \underset{\approx}{s}^d + \delta \underset{\approx}{s}^h + \delta \underset{\approx}{w} \\
&= \left(\underline{\underline{S}}_d + \underline{\underline{S}}_h + \underline{\underline{H}}\right)\delta \underset{\approx}{F} = \underline{\underline{S}}\delta \underset{\approx}{F} = \underline{\underline{S}}_c \delta \underset{\approx}{F}_t \quad \text{with} \quad \delta \underset{\approx}{F}_t = \underline{\underline{F}}_t \left(\underset{\approx}{L}_u\right)_t \\
&= \underline{\underline{S}}_c \underline{\underline{F}}_t \left(\underset{\approx}{L}_u\right)_t \\
&= \underline{\underline{M}}\left(\underset{\approx}{L}_u\right)_t
\end{aligned}$$

13.5.5 Examples

Tensile test

A square plate or cylindrical bar is loaded uniaxially. Dimensions are listed in the table.





initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

Viscoelastic model in tensile test

The axial elongation is prescribed with a constant elongation rate. The axial stress and force are calculated for polycarbonate (PC). Parameter values are listed in the table. The deformation is assumed to be plane strain.

E	2305	MPa	ν	0.37	-
H	29	MPa	h	270	-
D_∞	19	-	A_0	9.7573E-27	s
ΔH	2.9E5	J/mol	μ	0.06984	-
τ_0	0.72	MPa			

elongation rate $$\frac{\dot{\Delta l}}{h_0} = 0.01 \text{ s}^{-1}$$

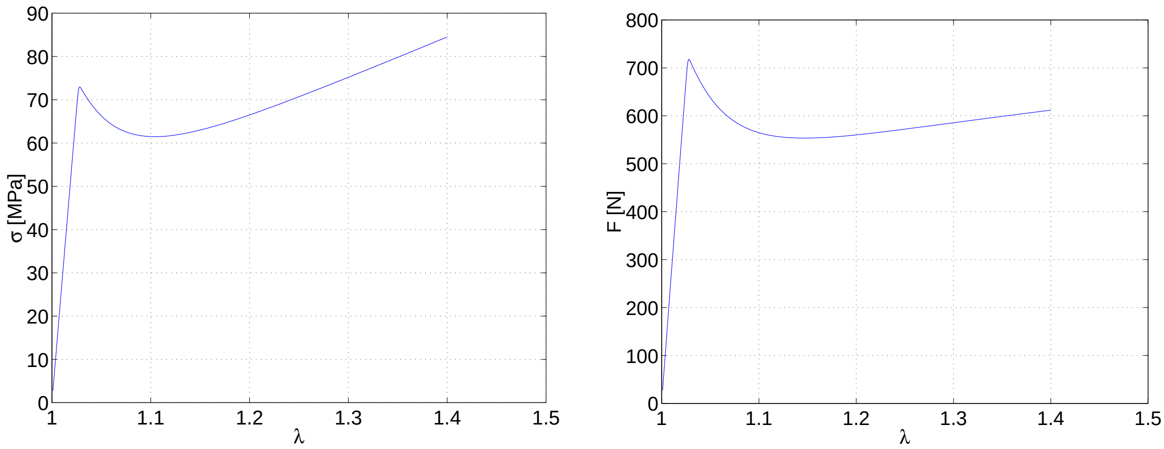
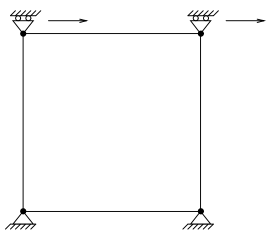


Fig. 13.43 : *Axial stress and force versus elongation*

Shear test

The simple shear test is analyzed with one element, where the horizontal displacement in the upper nodes is prescribed. Because there are no unknown degrees of freedom, the stiffness matrix is not used. Only strains, stresses and reaction forces are calculated. Material parameters for polycarbonate (PC) are used. The prescribed strain rate is constant.



initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

strain rate $$\dot{\gamma} = \frac{\dot{u}}{h_0} = 0.01 \text{ s}^{-1}$$

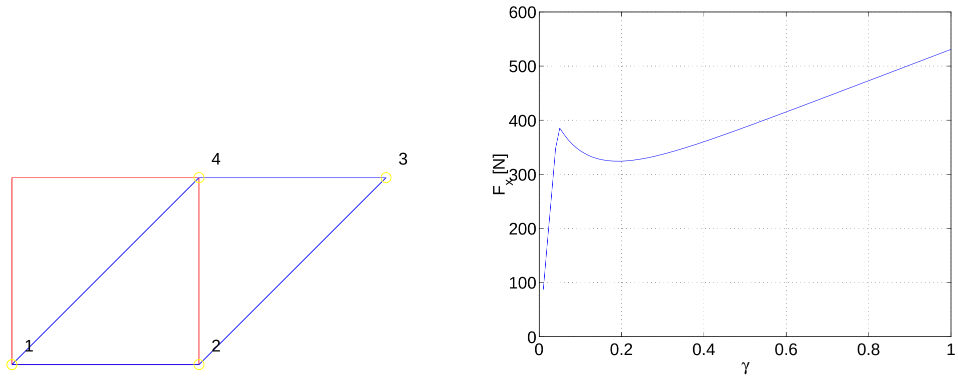


Fig. 13.44 : *Shear force versus shear strain for plane strain*

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APPENDICES

Appendix A

FE program `tr2dL`

The Matlab program `tr2dL` allows the modelling and analysis two-dimensional truss structures, where trusses are homogeneous and linear elastic. Deformation and rotations must be small, i.e. the behavior is geometrically linear.

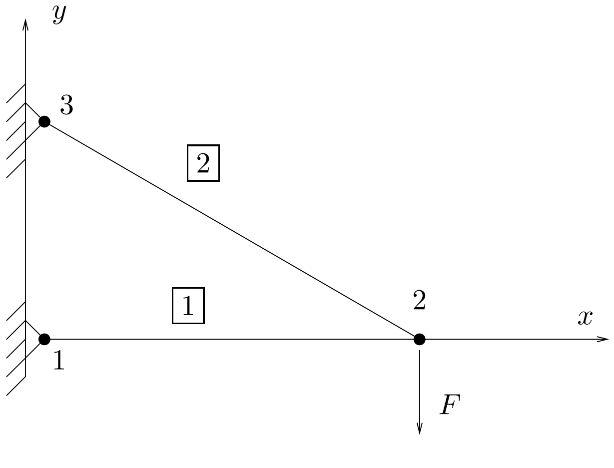
Model geometry, topology (connectivity), geometrical and material parameters, boundary conditions (prescribed displacements and point loads) and link relations (dependecies between degrees of freedom) must be available as input data.

When the analysis is finished, output date are available in the data base and various other data arrays.

In the following section an example input is presented, with explanatory comments. Finally the program source is listed and explained in more detail with included comment.

A.1 Example input file

As an example, the two-bar truss structure, shown in the figure below, will be modelled, loaded and analyzed.



Both trusses have different geometrical and material properties, which are given in the table below.

truss		1	2	
cross-sectional area	A	10	20	[mm^2]
Young's modulus	E	200	150	[GPa]
Poisson's ratio	ν	0.3	0.3	[-]

Now let us see which Matlab commands do the job. Before starting, it might be wise to close all figures and clear the Matlab work space.

```
close all; clear all;
```

First we give the coordinates of the nodes in the array "crd0". Now we have to decide on the units and in this example we choose to model everything in mm.

```
crd0 = [ 0 0; 100 0; 0 100/sqrt(3) ];
```

The connectivity of the elements is defined in the array "lok". This array has a row for each element. The first column contains the element type, which is 9 for a truss. The second row is the element group number. Typically elements with the same properties are placed in one and the same group. Because our two elements indeed have different properties, they are placed in two different groups. The third and fourth column contain the first and second node of the element.

```
lok = [ 9 1 1 2 ; 9 2 2 3 ];
```

The geometrical and material properties are provided in the array "elda" (*el*ement *da*ta). For each element group we have a row in "elda". The first column contains a zero (0), which is not important for our use. The second column contains the material identification number. For linear elastic material, which we will use here, this number is 11 (eleven). The third column contains the cross-sectional area (in mm^2). The fourth and fifth column are not used for our problems and always contain a zero (0). The sixth and seventh column contain Young's modulus and Poisson's ratio. So for our example we have :

```
elda = [
0 11 10 0 0 200000 0.3
0 11 20 0 0 150000 0.3
];
```

Boundary conditions are prescribed nodal displacements and/or prescribed nodal forces. Prescribed nodal displacements are always needed to prevent rigid body motions.

Prescribed displacements are provided in the array "pp". For each prescribed displacement component we have one row. The first column contains the node, the second column contains the direction (either 1 (= x = horizontal) or 2 (= y = vertical). The third column contains the value.

For our example we have :

```
pp = [ 1 1 0; 1 2 0; 3 1 0; 3 2 0 ];
```

The prescribed forces are given in the array "pf". Again each prescribed force component is placed on a row, with the node in the first, the direction in the second and the value in the third column.

For our example :

```
pf = [ 2 2 -100 ];
```

That is about all. The input is complete and the program can be executed to analyze the behavior.

```
tr2dL;
```

When the analysis is completed successfully, we want to see some results. First the nodal data, i.e. displacements and reaction forces. They are available in the array's "Mp" and "Mfi". Rows contain nodal data : displacement and forces in the first ($1 = x =$ horizontal) and second ($2 = y =$ vertical) directions. Just type the next commands in the Matlab shell.

```
Mp
Mfi
```

Element data, like stress and strain, are available in the data base. "eldaC". For element "e" we find the date in row "e" of "eldaC". Relevant date can be found at the following locations :

```
eldaC(e,1)  = sine of angle between axis and 1-direction
eldaC(e,2)  = cosine of angle between axis and 1-direction
eldaC(e,3)  = length
eldaC(e,4)  = cross-sectional area
eldaC(e,6)  = linear axial strain
eldaC(e,7)  = axial stress
eldaC(e,11) = axial stretch ratio
eldaC(e,12) = radial stretch ratio
eldaC(e,18) = axial force
```

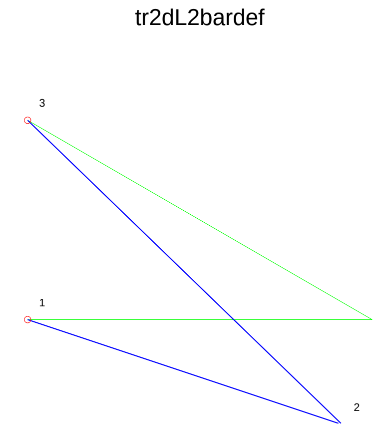
To see some results for our example just type the following in the Matlab shell :

```
eldaC(1,6)
eldaC(1,7)
eldaC(1,18)
eldaC(2,6)
eldaC(2,7)
eldaC(2,18)
```

These values can ofcourse be stored and printed in various ways.

The above input commands can also be put in one single input file. The results are shown as plot of he deformed structure, where the deformation is enlarged.

a4



A.2 Matlab program tr2dL

The program `tr2dL.m` is seeded with comments to explain variables and actions.

```
%**********************************************************************
% tr2dL : 2-dimensional linear truss element
%======================================================================
%----------------------------------------------------------------------
% Calculate some parameters from the input data.
%  nndof   :  number of nodal degrees of freedom
%  nnod    :  number of nodes
%  ndof    :  number of system degrees of freedom
%  ne      :  number of elements
%  nenod   :  number of element nodes
%  nedof   :  number of element degrees of freedom
%  negr    :  number of element groups
%  lokvg   :  location of degrees of freedom of elements in structure
%----------------------------------------------------------------------

nndof = 2;
nnod  = size(crd0,1);
ndof  = nnod * nndof;
ne    = size(lok,1);
nenod = size(lok,2)-2;
nedof = nndof * nenod;
negr  = size(elda,1);

lokvg(1:ne,:) = ...
    [ nndof*(lok(1:ne,3)-1)+1 nndof*(lok(1:ne,3)-1)+2 ...
      nndof*(lok(1:ne,4)-1)+1 nndof*(lok(1:ne,4)-1)+2 ];

%----------------------------------------------------------------------
% Calculate transformation matrix 'Trm' for local dof's, if needed.
%----------------------------------------------------------------------
```

```
Trm = eye(ndof);
if exist('tr'),
  ntr = size(tr,1);
  for itr=1:ntr
    trp = round(tr(itr,1));     tra = tr(itr,2);
    trc = cos((pi/180)*tra);  trs = sin((pi/180)*tra);
    k1  = nndof*(trp-1)+1;    k2  = nndof*(trp-1)+2;
    trm = [trc -trs ; trs trc];
    Trm([k1 k2],[k1 k2]) = trm;
  end;
else, ntr = 0; tr = []; end;

%-----------------------------------------------------------------------
% Initialization of databases
%  elpa  : element parameters
%  elda0 : initiel values
%  eldaC : current values
%  ety : element type    ;  egr : element group  ;
%  mnr : material number ;  mcl : material class ;  mty : material type
%  l0  : initial element length
%  s0  : sine of axis angle
%  c0  : cosine of axis angle
%-----------------------------------------------------------------------

for e=1:ne
  ety = lok(e,1); egr = lok(e,2);
  mnr  = elda(egr,2);    A0 = elda(egr,3);
  E    = elda(egr,6);    Gn = elda(egr,7);
  mcl  = floor(mnr/10);    mty  = rem(mnr,10);
  k1   = lok(e,3); k2 = lok(e,4);
  x10  = crd0(k1,1); y10 = crd0(k1,2); x20 = crd0(k2,1); y20 = crd0(k2,2);
  l0   = sqrt((x20-x10)*(x20-x10)+(y20-y10)*(y20-y10));
  s0   = (y20-y10)/l0;
  c0   = (x20-x10)/l0;

  elpa(e,:)  = [ety egr nenod nndof nedof];
  elda0(e,:) = [ s0 c0 l0 A0 0 0 Gn E mcl mty ];
  eldaC(e,:) = [ s0 c0 l0 A0 0 0 Gn E 0   0 ];
end; % element loop 'e'

%-----------------------------------------------------------------------
% Boundary conditions are reorganized.
% Additional arrays for later partitioning are made.
%  npdof :  number of prescribed degrees of freedom
%  npfor :  number of prescribed nodal forces
%  nudof :  number of unknown degrees of freedom
```

```
%
%  Information for partitioning the system of equations associated
%  with prescribed boundary conditions is made available in the arrays
%  ppc, ppv, pfc  and  pfv.
%----------------------------------------------------------------------

if ~exist('pp'), pp = []; ppc = []; ppv = []; end;
if ~exist('pf'), pf = []; pfc = []; pfv = []; end;

npdof = size(pp,1);
npfor = size(pf,1);
nudof = ndof - npdof;

if npdof>0
  ppc = [nndof*(round(pp(:,1))-1)+round(pp(:,2))];
  ppv = pp(:,nndof+1);
end;
if npfor>0,
  pfc = [nndof*(round(pf(:,1))-1)+round(pf(:,2))];
  pfv = pf(:,nndof+1);
end;

%  Information for partitioning the system of equations associated
%  with linked degrees of freedom is made available in the arrays
%  plc  and  prc.

if ~exist('pl'),  pl  = []; plc = []; end;
if ~exist('pr'),  pr  = []; prc = []; end;
if ~exist('lim'), lim = []; end;

npl = size(pl,1);
npr = size(pr,1);

if ~exist('lif'), lif = zeros(1,npl); end;

if npl>0
  plc = [nndof*(round(pl(:,1))-1)+round(pl(:,2))];
  prc = [nndof*(round(pr(:,1))-1)+round(pr(:,2))];
end;

%  Some extra arrays are made for later use.

pa = 1:ndof; pu = 1:ndof; prs = 1:ndof;
pu([ppc' plc']) = [];
prs([ppc' pfc' plc']) = [];

%  pe0   :  column with prescribed initial displacements
```

```
% fe0    :  array with prescribed initial forces

pe0 = zeros(ndof,1); pe0(ppc(1:npdof)) = ppv(1:npdof);
fe0 = zeros(ndof,1); fe0(pfc(1:npfor)) = pfv(1:npfor);

%-----------------------------------------------------------------------
% Initialization to zero
% pe     :  column with nodal displacements
% p      :  column with nodal displacements
% fe     :  column with external (applied) nodal forces
% fi     :  column with internal (resulting) nodal forces
% #T     :  column with transformed components
%-----------------------------------------------------------------------

pe  = zeros(ndof,1); p   = zeros(ndof,1); pT  = zeros(ndof,1);
fe  = zeros(ndof,1); fi  = zeros(ndof,1);
feT = zeros(ndof,1); fiT = zeros(ndof,1);

%-----------------------------------------------------------------------
% Loop over all elements to generate element stiffness matrix 'em'
% Assemble 'em' into structural stiffness matrix 'sm'
% ec0  : initial coordinates of element nodes
% ec   : current coordinates of element nodes
%-----------------------------------------------------------------------
sm = zeros(ndof);

for e=1:ne
  ety = elpa(e,1); egr = elpa(e,2);
  nenod = elpa(e,3); nedof = elpa(e,5);
  ec0 = crd0(lok(e,3:2+nenod),:); ec = ec0;
  em = zeros(nedof);

%-----------------------------------------------------------------------
% Element stiffness matrix

  s   = eldaC(e,1) ; c  = eldaC(e,2);
  ML  = [  c*c  c*s -c*c -c*s ;  c*s  s*s -c*s -s*s
          -c*c -c*s  c*c  c*s ; -c*s -s*s  c*s  s*s ];

%-----------------------------------------------------------------------

  l0 = elda0(e,3); A0 = elda0(e,4); E0 = elda0(e,8);

  em = (A0/l0 * E0) * ML ;
  sm(lokvg(e,1:nedof),lokvg(e,1:nedof)) = ...
    sm(lokvg(e,1:nedof),lokvg(e,1:nedof)) + em;
end; % element loop 'e'
```

```
%------------------------------------------------------------------------
% Transformation for local nodal coordinate systems
%------------------------------------------------------------------------
sm = Trm' * sm * Trm;
%------------------------------------------------------------------------
% Boundary conditions and links
%------------------------------------------------------------------------
pe = pe0; fe = fe0; rs = fe;

if npl>0, rs = rs - sm(:,plc)*lif'; end;

%------------------------------------------------------------------------
% Partitioning is done in the function                      fbibpartit.m

[sm,rs] = fbibpartit(1,sm,rs,ndof,pa,ppc,plc,prc,pe,lim);

%------------------------------------------------------------------------
% Solving the system of equations and take prescribed displacements
% and links into account.
% Update nodal point coordinates 'crd'.

sol = inv(sm)*rs; % sol = sm\rs;

pe(pu) = sol;
if npl>0, pe(plc) = lim*pe(prc) + lif'; end;

p = pe; pT = Trm * p;
crd = crd0 + reshape(pT,nndof,nnod)';

%------------------------------------------------------------------------
% Calculate stresses and strains and the internal forces 'ef'.
% Internal forces 'ef' are assembled into 'fi', the structural
% internal forces, representing the reaction forces.
%------------------------------------------------------------------------
fi = zeros(ndof,1);

for e=1:ne
  ety = elpa(e,1); egr = elpa(e,2);
  ec0 = crd0(lok(e,2+1:2+nenod),:);
  ec = crd(lok(e,2+1:2+nenod),:);
  ef = zeros(nedof,1);

%------------------------------------------------------------------------
% Element internal forces

  s   = eldaC(e,1) ; c  = eldaC(e,2);
```

```
  V    = [ -c -s c s ]';

%----------------------------------------------------------------------

  l0 = elda0(e,3); A0 = elda0(e,4);
  E0 = elda0(e,8); Gn0 = elda0(e,7);
  x1 = ec(1,1); y1 = ec(1,2); x2 = ec(2,1); y2 = ec(2,2);
  l  = sqrt((x2-x1)*(x2-x1)+(y2-y1)*(y2-y1));
  s  = (y2-y1)/l;     c  = (x2-x1)/l;
  Gl  = l/l0; Ge = Gl-1;
  Ged = -Gn0*Ge; Gm = Ged+1; A = Gm*Gm*A0; Gs = E0 * Ge; N = A * Gs;

  eldaC(e,1:7)   = [s c l A 0 Ge Gs];
  eldaC(e,11:18) = [Gl Gm Ge 0 0 Gs 0 N];

  ef = N * V;

  fi(lokvg(e,1:nedof)) = fi(lokvg(e,1:nedof)) + ef;
end;

rs = fe - fi;
fi = Trm' * fi; fiT = fi; fiT = Trm * fi; rsT = feT - fiT;

%----------------------------------------------------------------------
% Reshaping columns into matrices

Mp  = reshape(p,nndof,nnod)';    MTp = Mp;
Mfi = reshape(fi,nndof,nnod)';   Mfe = reshape(fe,nndof,nnod)';
Mrs = reshape(rs,nndof,nnod)';

if ntr>=1
MpT  = reshape(pT,nndof,nnod)';  MTpT = MpT;
MfiT = reshape(fiT,nndof,nnod)'; MfeT = reshape(feT,nndof,nnod)';
MrsT = reshape(rsT,nndof,nnod)';
end;
%----------------------------------------------------------------------

%**********************************************************************
```

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Appendix B

FE program `tr2d`

The Matlab program `tr2d` allows to model and analyze two-dimensional truss structures, where trusses are homogeneous and can behave nonlinear. Deformation and rotations can be large, i.e. the behavior is geometrically nonlinear.

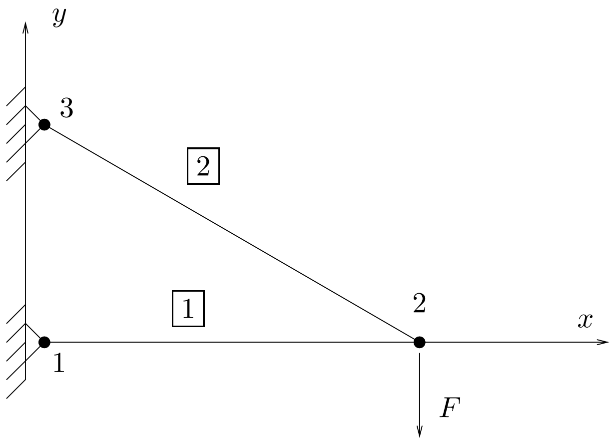
Model geometry, topology (connectivity), geometrical and material parameters, boundary conditions (prescribed displacements and point loads) and link relations (dependecies between degrees of freedom) must be available as input data. Also the history of the prescribed boundary conditions must be specified.

When the analysis is finished, output date are stored in the data base and various other data arrays.

In the following section an example input is presented, with explanatory comments. Finally the program itself is explained in more detail.

B.1 Example input file

As an example, the two-bar truss structure, shown in the figure below will be modelled, loaded and analyzed.



Both trusses have different geometrical and material properties, The material of truss 1 behaves elastcally, according to a linear relation between the axial stress σ and the linear strain

$\varepsilon = \varepsilon_l = \lambda - 1$, specified by the modulus E and the Poisson's ratio ν. Truss 2 behaves linearly elastic upto the yield stress σ_{y0} after which isotropic linear hardening occurs with hardening parameter $H = E/20$. The vertical force in node 4 is increased from 0 to -50000 N and decreases back to zero again.

The Matlab workspace is cleared and figures are closed. The file 'loadincr.m', which describes the time history of the boundary conditions, is deleted. The file 'savedata.m', which describes, which calculated values have to be saved for postprocessing, is also deleted.

```
clear all; close all;
delete('loadincr.m'); delete('savedata.m');
```

Coordinates of the nodal points and the connectivity is given in the matrices "crd0" and "lok". In the latter array, the first column contains the element type and the second column the element group.

```
crd0 = [ 0 0; 100 0; 0 100/sqrt(3) ];
lok = [ 9 1 1 2 ; 9 2 2 3 ];
```

element data are given in the array "elda", which has one row for each property group. The second column contains the material number, which has two digits. The first digit is the material class ("mcl") and the second digit is the material type ("mty"). The next classes and types are currently implemented :

mcl = 1 : elastic material
mcl = 2 : elastomeric material
mcl = 3 : elastoplastic material
mcl = 4 : linear viscoelastic material
mcl = 5 : viscoplastic material (Perzyna)
mcl = 6 : nonlinear viscoelastic material (Leonov)
mcl = 7 : elastoviscous material (creep)

For the element type you should look in the Matlab source file.

```
elda = [ 0 11 10 0 0 200000 0.3 0 0 0 ;
         0 31 20 0 0 200000 0.3 250 10000 0 ];
```

We also have to indicate which hardening law ("hm") is used and which stress update procedure ("pr", explicit or implicit).

```
hm = 'li'; pr = 'ex';
```

The boundary conditions are prescribed, first the incremental displacements, then the incremental forces.

```
pp = [ 1 1 0; 1 2 0; 3 1 0; 3 2 0; ];
pf = [ 2 2 -100 ];
```

The load history is prescribed with a call to the function file "mloin.m". In its source file, it is explained how it must be used.

```
[St,Sft,nic,GDt,tend] = mloin(0,0,400,1,'pol',[0 0 200 50 400 0]);
```

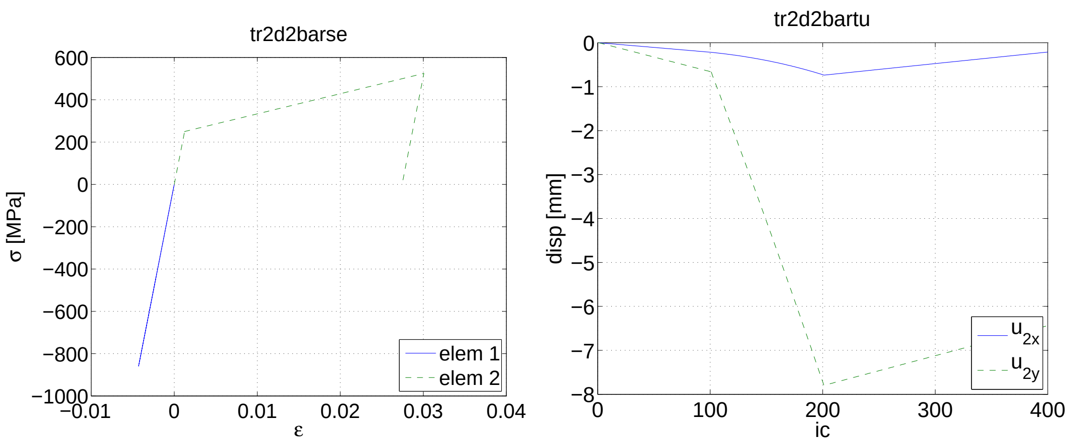
Analysis data for postprocessing must be saved. This is done by making a file "savedata.m", which is called by the program `tr2d` at the end of each increment.

```
sada=fopen('savedata.m','w');
fprintf(sada,'Sf2x(ic)=Mfi(2,1);Sf2y(ic)=Mfi(2,2); \n');
fprintf(sada,'Su2x(ic)=MTp(2,1);Su2y(ic)=MTp(2,2); \n');
fprintf(sada,'SGe1(ic)=eldaB(1,6);SGe2(ic)=eldaB(2,6); \n');
fprintf(sada,'SGs1(ic)=eldaB(1,7);SGs2(ic)=eldaB(2,7); \n');
fclose(sada);
```

Now the input is complete and the program can be called for the analysis.

```
tr2d;
```

The result is shown in the figures below.



Element stress-strain relations and displacements of node 4

B.2 Matlab program `tr2d`

The program `tr2d.m` calls a collection of command and function files, which can and should be inspected for through understanding of the procedure. The program structure is however clearly shown in the listing.

```
%**********************************************************************
% tr2d : 2-dimensional truss element
%======================================================================
tr2dchkinp;                                                % tr2dchkinp.m
fbiblcase;                                                  % fbiblcase.m
[Trm] = fbibtransbc(tr,ndof,nndof);                       % fbibtransbc.m
[elda0,eldaB,eldaC,mcl4] = tr2dinidat(ne,elgr,elda,neip,ts,lok,crd0,mm);
                                                           % tr2dinidat.m
```

```
tr2dinizer;                                                    % tr2dinizer.m

save([matf num2str(0)]);
crdB = crd0; crd = crd0;
%=========================================================================
% Incremental calculation
%=========================================================================
ic = 1; ti = 0; it = 0; slow = 1;

while ic<=nic
%-------------------------------------------------------------------------
fbibcutback;                                                   % fbibcutback.m
ti = ti + ts; it = 0; loadincr;                                  % loadincr.m
pe = peC./slow; fe = feC;
rs = fe - fi;
Dp = zeros(ndof,1); Ip = zeros(ndof,1); IpT = zeros(ndof,1);
%=========================================================================
% System matrix is assembled from element matrices
% System matrix is transformed for local nodal coord.sys.
%=========================================================================
if (ic==1 | nl==1)
%-------------------------------------------------------------------------
sm=zeros(ndof); mcl4=0; mcl42=0; mcl9=0;

for e=1:ne
  ec = crd(lok(e,3:nenod+2),:);
  [ML,MN,V,eldaC] = tr2dgeom(e,ec,eldaC);                        % tr2dgeom.m
  tr2dmat;                                                        % tr2dmat.m

  em = CL * ML + CN * MN ;                    % element stiffness matrix
  ef = (CI-CV) * V;                       % element internal load column

  sm(lokvg(e,:),lokvg(e,:)) = sm(lokvg(e,:),lokvg(e,:)) + em;
end;

sm = Trm'*sm *Trm; if ic==1, sm00=sm; end; sm0=sm;
%-------------------------------------------------------------------------
end;
%=========================================================================
% Iterative calculation
%=========================================================================
nrm = 1000; it = 1; sm0 = sm;

while (nrm>ccr) & (it<=mit)
%-------------------------------------------------------------------------
%=========================================================================
% Links and boundary conditions are taken into account
```

```
% Unknown nodal point values are solved
% Prescribed nodal values are inserted in the solution vector
%=======================================================================
%sm = sm00;                   % only used to test modified Newton-Raphson

if npl>0, rs = rs - sm(:,plc)*lif'; end;
[sm,rs] = fbibpartit(it,sm,rs,ndof,pa,ppc,plc,prc,pe,lim);% fbibpartit.m

sol = inv(sm)*rs;

p = zeros(ndof,1); p(pu) = sol;
if it==1, p(ppc) = pe(ppc); end;
if npl>0, p(plc) = lim*p(prc) + lif'; end;

Dp = p; Ip = Ip + Dp; Tp = Tp + Dp;
%=======================================================================
% Transformation dof's from local to global nodal coordinate systems
%=======================================================================
DpT = Trm * Dp; IpT = IpT + DpT; TpT = TpT + DpT;
crd = crd0 + reshape(TpT,nndof,nnod)';
%=======================================================================
% Calculate stresses and strains.
% Make system matrix and internal force vector for next step.
%=======================================================================
sm=zeros(ndof); fi=zeros(ndof,1); mcl4=0; mcl42=0; mcl9=0;

for e=1:ne
  ec = crd(lok(e,3:nenod+2),:);            % element nodal coordinates
  [ML,MN,V,eldaC] = tr2dgeom(e,ec,eldaC);                  % tr2dgeom.m
  tr2dmat;                                                  % tr2dmat.m

  em = CL * ML + CN * MN ;                   % element stiffness matrix
  ef = (CI-CV) * V;                       % element internal load column

  sm(lokvg(e,:),lokvg(e,:)) = sm(lokvg(e,:),lokvg(e,:)) + em;
  fi(lokvg(e,:)) = fi(lokvg(e,:)) + ef;
end;

sm=Trm'*sm*Trm; fi=Trm'*fi;
%=======================================================================
% Calculate residual force and convergence norm
%=======================================================================
rs = fe - fi;
nrm = fbibcnvnrm(cnm,pu,ppc,prs,Dp,Ip,rs,fi);            % fbibcnvnrm.m

it = it + 1;                      % increment the iteration step counter
```

```
fbibwr2scr;                                                % fbibwr2scr.m
%----------------------------------------------------------------------
end; %it
%======================================================================
% Transformation nodal forces from local to global nodal coord.sys.
%======================================================================
fiT = fi; feT = fe; rsT = rs;
fiT = Trm * fi; feT = Trm * fe; rsT = Trm * rs;

%======================================================================
% Update and store values
%======================================================================
fbibcol2mat1;                                            % fbibcol2mat1.m
crdB = crd; feB = fe; eldaB = eldaC; HGsB = HGsC;
savefile = [matf num2str(ic)]; savedata;                     % savedata.m

ic = ic + 1;                           % increment the increment counter
save([matf '00'],'ic');                             % save date to 'matf'

%----------------------------------------------------------------------
end; %ic

%**********************************************************************
```

Appendix C

m2cc and m2mm

The Matlab function files m̂2cc and `m2mm` put the components of a 3×3 matrix in a column and a "blown-up" matrix, respectively.

```
%**********************************************************************
function [C] = m2cc(m,s);

C = zeros(s,1);

if s==9
  C = [m(1,1); m(2,2); m(3,3);
       m(1,2); m(2,1); m(2,3); m(3,2); m(3,1); m(1,3)];
elseif s==5
  C = [m(1,1); m(2,2); m(3,3); m(1,2); m(2,1)];
elseif s==4
  C = [m(1,1); m(2,2); m(1,2); m(2,1)];
end;
%**********************************************************************
```

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```
%*******************************************************************
function [M] = m2mm(m,s);

M = zeros(s);

if s==9
   M = [ m(1,1) 0      0      0      m(1,2) 0      0      m(1,3) 0
         0      m(2,2) 0      m(2,1) 0      0      m(2,3) 0      0
         0      0      m(3,3) 0      0      m(3,2) 0      0      m(3,1)
         0      m(1,2) 0      m(1,1) 0      0      m(1,3) 0      0
         m(2,1) 0      0      0      m(2,2) 0      0      m(2,3) 0
         0      0      m(2,3) 0      0      m(2,2) 0      0      m(2,1)
         0      m(3,2) 0      m(3,1) 0      0      m(3,3) 0      0
         m(3,1) 0      0      0      m(3,2) 0      0      m(3,3) 0
         0      0      m(1,3) 0      0      m(1,2) 0      0      m(1,1) ];
elseif s==5
   M = [ m(1,1) 0      0      0      m(1,2)
         0      m(2,2) 0      m(2,1) 0
         0      0      m(3,3) 0      0
         0      m(1,2) 0      m(1,1) 0
         m(2,1) 0      0      0      m(2,2) ];
elseif s==4
   M = [ m(1,1) 0      0      m(1,2)
         0      m(2,2) m(2,1) 0
         0      m(1,2) m(1,1) 0
         m(2,1) 0      0      m(2,2) ];
end;
%*******************************************************************
```

Appendix D

Stiffness and compliance matrices

In chapter **??** the three-dimensional stiffness and compliance matrices have been derived for various materials. Increasing microstructural lattice symmetry gave rise to a reduction of the number of material constants. Starting from triclinic with no symmetry and characterized by 21 material constants, increased symmetry was seen for monoclinic (13 constants), orthotropic (9), quadratic (6), transversal isotropic (5), cubic (3) and finally, isotropic, with only 2 material constants.

In this appendix, we again present the material matrices for orthotropic, transversal isotropic and fully isotropic material. The material constants will be expressed in engineering constants, where we choose Young's moduli, Poisson's ratios and shear moduli.

In many engineering problems, the state of strain or stress is planar. Both for plane strain and plane stress, only the strain and stress components in a plane have to be related through a material law. Here we assume that this plane is the 12-plane. For plane strain we than have $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, and for plane stress $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$. The material law for these planar situations can be derived from the linear elastic three-dimensional stress-strain relation. This is done, first for the general orthotropic material law. The result is subsequently specified in engineering parameters for orthotropic, transversal isotropic and fully isotropic material.

D.1 General orthotropic material law

The general orthotropic material law is expressed by the stiffness matrix $\underline{\underline{C}}$ and/or its inverse, the compliance matrix $\underline{\underline{S}}$.

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \underline{\underline{C}}\, \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \underline{\underline{C}}^{-1}\,\underset{\sim}{\sigma} = \underline{\underline{S}}\,\underset{\sim}{\sigma}$$

The inverse of $\underline{\underline{C}}$ can be expresses in its components.

$$\underline{\underline{C}}^{-1} = \frac{1}{\Delta_c} \begin{bmatrix} BC - S^2 & -QC + RS & QS - BR & 0 & 0 & 0 \\ -QC + RS & AC - R^2 & -AS + QR & 0 & 0 & 0 \\ QS - BR & -AS + QR & AB - Q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_c(1/K) & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_c(1/L) & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_c(1/M) \end{bmatrix}$$

$$\text{with} \quad \Delta_c = ABC - AS^2 - BR^2 - CQ^2 + 2QRS$$

As will be clear later, it will mostly be easier to start with the compliance matrix and calculate the stiffness matrix by inversion.

$$\underline{\underline{S}}^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & qs - br & 0 & 0 & 0 \\ -qc + rs & ac - r^2 & -as + qr & 0 & 0 & 0 \\ qs - br & -as + qr & ab - q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_s(1/k) & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_s(1/l) & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_s(1/m) \end{bmatrix}$$

$$\text{with} \quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$$

Increasing material symmetry leads to a reduction in material parameters.

quadratic	$B = A\ ;\ S = R\ ;\ M = L;$
transversal isotropic	$B = A\ ;\ S = R\ ;\ M = L\ ;\ K = \frac{1}{2}(A - Q)$
cubic	$C = B = A\ ;\ S = R = Q\ ;\ M = L = K \neq \frac{1}{2}(A - Q)$
isotropic	$C = B = A\ ;\ S = R = Q\ ;\ M = L = K = \frac{1}{2}(A - Q)$

The planar stress-strain laws can be derived either from the stiffness matrix $\underline{\underline{C}}$ or from the compliance matrix $\underline{\underline{S}}$. The plane strain state will be denoted by the index ε and the plane stress state will be indicated with the index σ.

D.1.1 Plane strain

For a plane strain state with $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, the stress σ_{33} can be expressed in the planar strains ε_{11} and ε_{22}. The material stiffness matrix $\underline{\underline{C}}_{\varepsilon}$ can be extracted directly from $\underline{\underline{C}}$. The material compliance matrix $\underline{\underline{S}}_{\varepsilon}$ has to be derived by inversion.

$$\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0 \quad \rightarrow \quad \sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\varepsilon \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{AB - Q^2} \begin{bmatrix} B & -Q & 0 \\ -Q & A & 0 \\ 0 & 0 & \dfrac{AB - Q^2}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\varepsilon \underset{\sim}{\sigma}$$

Because the components of the three-dimensional compliance matrix $\underline{\underline{S}}$ are most conveniently expressed in Young's moduli, Poisson's ratios and shear moduli, this matrix is a good starting point to derive the planar matrices for specific cases. The plane strain stiffness matrix $\underline{\underline{C}}_\varepsilon$ must then be determined by inversion.

$$\varepsilon_{33} = 0 = r\sigma_{11} + s\sigma_{22} + c\sigma_{33} \quad \rightarrow \quad \sigma_{33} = -\frac{r}{c}\sigma_{11} - \frac{s}{c}\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} - \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \begin{bmatrix} \frac{r}{c} & \frac{s}{c} & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} ac - r^2 & qc - rs & 0 \\ qc - sr & bc - s^2 & 0 \\ 0 & 0 & kc \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\varepsilon \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & 0 \\ -qc + rs & ac - r^2 & 0 \\ 0 & 0 & \dfrac{\Delta_s}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$$

$$= \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \underline{\underline{C}}_\varepsilon \underset{\sim}{\sigma}$$

We can now derive by substitution :

$$\sigma_{33} = -\frac{1}{\Delta_s} \left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

D.1.2 Plane stress

For the plane stress state, with $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$, the two-dimensional material law can be easily derived from the three-dimensional compliance matrix $\underline{\underline{S}}_\varepsilon$. The strain ε_{33} can be directly expressed in σ_{11} and σ_{22}. The material stiffness matrix has to be derived by inversion.

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \quad \rightarrow \quad \varepsilon_{33} = r\sigma_{11} + s\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{ab-q^2} \begin{bmatrix} b & -q & 0 \\ -q & a & 0 \\ 0 & 0 & \dfrac{ab-q^2}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

We can derive by substitution :

$$\varepsilon_{33} = \frac{1}{ab-q^2} \left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

The same relations can be derived from the three-dimensional stiffness matrix $\underline{\underline{C}}$.

$$\sigma_{33} = 0 = R\varepsilon_{11} + S\varepsilon_{22} + C\varepsilon_{33} \quad \rightarrow \quad \varepsilon_{33} = -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \begin{bmatrix} R \\ S \\ 0 \end{bmatrix} \begin{bmatrix} \dfrac{R}{C} & \dfrac{S}{C} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \frac{1}{C} \begin{bmatrix} AC - R^2 & QC - RS & 0 \\ QC - SR & BC - S^2 & 0 \\ 0 & 0 & KC \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

D.1.3 Plane strain/stress

In general we can write the stiffness and compliance matrix for planar deformation as a 3×3 matrix with components, which are specified for plane strain ($p = \varepsilon$) or plane stress ($p = \sigma$).

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} - \alpha\Delta T \begin{bmatrix} \Theta_{p1} \\ \Theta_{p2} \\ 0 \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix} + \sigma\Delta T \begin{bmatrix} \theta_{p1} \\ \theta_{p2} \\ 0 \end{bmatrix}$$

The general relations presented before can be used to calculate the components of $\underline{\underline{C}}_p$ and/or $\underline{\underline{S}}_p$ when components of the three-dimensional matrices $\underline{\underline{C}}$ and/or $\underline{\underline{S}}$ are known.

In the next sections the three-dimensional and planar material matrices are presented for orthonormal, transversal isotropic and fully isotropic material.

D.2 Linear elastic orthotropic material

For an orthotropic material 9 material parameters are needed to characterize its mechanical behavior. Their names and formal definitions are :

$$
\begin{array}{lll}
\text{Young's moduli} & : & E_i = \dfrac{\partial \sigma_{ii}}{\partial \varepsilon_{ii}} \\
\text{Poisson's ratios} & : & \nu_{ij} = -\dfrac{\partial \varepsilon_{jj}}{\partial \varepsilon_{ii}} \\
\text{shear moduli} & : & G_{ij} = \dfrac{\partial \sigma_{ij}}{\partial \gamma_{ij}}
\end{array}
$$

The introduction of these parameters is easily accomplished in the compliance matrix $\underline{\underline{S}}$. Due to the symmetry of the compliance matrix $\underline{\underline{S}}$, the material parameters must obey the three Maxwell relations.

$$
\underline{\underline{S}} = \begin{bmatrix}
E_1^{-1} & -\nu_{21}E_2^{-1} & -\nu_{31}E_3^{-1} & 0 & 0 & 0 \\
-\nu_{12}E_1^{-1} & E_2^{-1} & -\nu_{32}E_3^{-1} & 0 & 0 & 0 \\
-\nu_{13}E_1^{-1} & -\nu_{23}E_2^{-1} & E_3^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{12}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{23}^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{31}^{-1}
\end{bmatrix}
$$

$$
\text{with} \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \quad ; \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3} \quad ; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1} \qquad \text{(Maxwell relations)}
$$

The stiffness matrix $\underline{\underline{C}}$ can then be derived by inversion of $\underline{\underline{S}}$.

$$
\underline{\underline{C}} = \frac{1}{\Delta_s} \begin{bmatrix}
\frac{1-\nu_{32}\nu_{23}}{E_2E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2E_3} & \frac{\nu_{21}\nu_{32}+\nu_{31}}{E_2E_3} & 0 & 0 & 0 \\
\frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1E_3} & \frac{\nu_{12}\nu_{31}+\nu_{32}}{E_1E_3} & 0 & 0 & 0 \\
\frac{\nu_{12}\nu_{23}+\nu_{13}}{E_1E_2} & \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1E_2} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta_s G_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & \Delta_s G_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta_s G_{31}
\end{bmatrix}
$$

$$
\text{with} \qquad \Delta_s = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}
$$

D.2.1 Voigt notation

In composite mechanics the so-called Voigt notation is often used, where stress and strain components are simply numbered 1 to 6. Corresponding components of the compliance (and stiffness) matrix are numbered accordingly. However, there is more to it than that. The sequence of the shear components is changed. We will not use this changed sequence in the following.

$$
\begin{aligned}
\underset{\sim}{\sigma}^T &= [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{31}] = [\sigma_1\ \sigma_2\ \sigma_3\ \sigma_6\ \sigma_4\ \sigma_5] \\
\underset{\sim}{\varepsilon}^T &= [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \gamma_{12}\ \gamma_{23}\ \gamma_{31}] = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \varepsilon_6\ \varepsilon_4\ \varepsilon_5]
\end{aligned}
$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

D.2.2 Plane strain

For plane strain the stiffness matrix can be extracted from the three-dimensional stiffness matrix. The inverse of this 3x3 matrix is the plane strain compliance matrix.

$$\sigma_{33} = \nu_{13}\frac{E_3}{E_1}\sigma_{11} + \nu_{23}\frac{E_3}{E_2}\sigma_{22}$$

$$\underline{\underline{S}}_\varepsilon = \begin{bmatrix} \frac{1-\nu_{31}\nu_{13}}{E_1} & -\frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1} & \frac{1-\nu_{32}\nu_{23}}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

$$\underline{\underline{C}}_\varepsilon = \underline{\underline{S}}_\varepsilon^{-1} = \frac{1}{\Delta_s}\begin{bmatrix} \frac{1-\nu_{32}\nu_{23}}{E_2E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2E_3} & 0 \\ \frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1E_3} & 0 \\ 0 & 0 & \Delta_s G_{12} \end{bmatrix}$$

$$\text{with} \qquad \Delta_s = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

$$\sigma_{33} = \frac{1}{\Delta_s}\left\{\frac{\nu_{12}\nu_{32}+\nu_{13}}{E_1E_2}\varepsilon_{11} + \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1E_2}\varepsilon_{22}\right\}$$

D.2.3 Plane stress

For plane stress the compliance matrix can be extracted from the three-dimensional compliance matrix. The inverse of this 3x3 matrix is the plane strain stiffness matrix.

$$\varepsilon_{33} = -\nu_{13}E_1^{-1}\sigma_{11} - \nu_{23}E_2^{-1}\sigma_{22}$$

$$\underline{\underline{S}}_\sigma = \begin{bmatrix} E_1^{-1} & -\nu_{21}E_2^{-1} & 0 \\ -\nu_{12}E_1^{-1} & E_2^{-1} & 0 \\ 0 & 0 & G_{12}^{-1} \end{bmatrix}$$

$$\underline{\underline{C}}_\sigma = \underline{\underline{S}}_\sigma^{-1} = \frac{1}{1-\nu_{21}\nu_{12}}\begin{bmatrix} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & (1-\nu_{21}\nu_{12})G_{12} \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{1}{1-\nu_{12}\nu_{21}}\{(\nu_{12}\nu_{23}+\nu_{13})\varepsilon_{11} + (\nu_{21}\nu_{13}+\nu_{23})\varepsilon_{22}\}$$

D.3 Linear elastic transversal isotropic material

Considering an transversally isotropic material with the 12-plane isotropic, the Young's modulus E_p and the Poisson's ratio ν_p in this plane can be measured. The associated shear modulus is related by $G_p = \dfrac{E_p}{2(1+\nu_p)}$. In the perpendicular direction we have the Young's modulus E_3, the shear moduli $G_{3p} = G_{p3}$ and two Poisson ratios, which are related by symmetry : $\nu_{p3}E_3 = \nu_{3p}E_p$.

$$\underline{\underline{S}} = \begin{bmatrix} E_p^{-1} & -\nu_p E_p^{-1} & -\nu_{3p}E_3^{-1} & 0 & 0 & 0 \\ -\nu_p E_p^{-1} & E_p^{-1} & -\nu_{3p}E_3^{-1} & 0 & 0 & 0 \\ -\nu_{p3}E_p^{-1} & -\nu_{p3}E_p^{-1} & E_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_p^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{p3}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{3p}^{-1} \end{bmatrix}$$

$$\text{with} \qquad \frac{\nu_{p3}}{E_p} = \frac{\nu_{3p}}{E_3}$$

$$\underline{\underline{C}} = \underline{\underline{S}}^{-1} = \frac{1}{\Delta_s}\begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_{3p}\nu_{p3}+\nu_p}{E_pE_3} & \frac{\nu_p\nu_{3p}+\nu_{3p}}{E_pE_3} & 0 & 0 & 0 \\ \frac{\nu_{p3}\nu_{3p}+\nu_p}{E_pE_3} & \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_p\nu_{3p}+\nu_{3p}}{E_pE_3} & 0 & 0 & 0 \\ \frac{\nu_p\nu_{p3}+\nu_{p3}}{E_pE_p} & \frac{\nu_p\nu_{p3}+\nu_{p3}}{E_pE_p} & \frac{1-\nu_p\nu_p}{E_pE_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_s G_p & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_s G_{p3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_s G_{3p} \end{bmatrix}$$

$$\text{with} \qquad \Delta_s = \frac{1-\nu_p\nu_p-\nu_{p3}\nu_{3p}-\nu_{3p}\nu_{p3}-\nu_p\nu_{p3}\nu_{3p}-\nu_p\nu_{3p}\nu_{p3}}{E_pE_pE_3}$$

D.3.1 Plane strain

The plane strain stiffness matrix can be extracted from the three-dimensional stiffness matrix. The inverse of this 3x3 matrix is the plane strain compliance matrix.

$$\sigma_{33} = \frac{E_3\nu_{p3}}{E_p}(\sigma_{11}+\sigma_{22}) = \nu_{3p}(\sigma_{11}+\sigma_{22})$$

$$\underline{\underline{S}}_\varepsilon = \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_p} & -\frac{\nu_{3p}\nu_{p3}+\nu_p}{E_p} & 0 \\ -\frac{\nu_{p3}\nu_{3p}+\nu_p}{E_p} & \frac{1-\nu_{3p}\nu_{p3}}{E_p} & 0 \\ 0 & 0 & \frac{1}{G_p} \end{bmatrix}$$

$$\underline{\underline{C}}_\varepsilon = \underline{\underline{S}}_\varepsilon^{-1} = \frac{1}{\Delta_s}\begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_{3p}\nu_{p3}+\nu_p}{E_pE_3} & 0 \\ \frac{\nu_{p3}\nu_{3p}+\nu_p}{E_pE_3} & \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & 0 \\ 0 & 0 & \Delta_s G_p \end{bmatrix}$$

$$\text{with} \qquad \Delta_s = \frac{1-\nu_p\nu_p-\nu_{p3}\nu_{3p}-\nu_{3p}\nu_{p3}-\nu_p\nu_{p3}\nu_{3p}-\nu_p\nu_{3p}\nu_{p3}}{E_pE_pE_3}$$

$$\sigma_{33} = \frac{1}{\Delta_s}\frac{\nu_{p3}(\nu_p+1)}{E_p^2}(\varepsilon_{11}+\varepsilon_{22})$$

D.3.2 Plane stress

For plane stress the compliance matrix can be extracted directly from the three-dimensional compliance matrix. The inverse of this 3x3 matrix is the plane strain stiffness matrix.

$$\varepsilon_{33} = -\frac{\nu_{p3}}{E_p}(\sigma_{11} + \sigma_{22})$$

$$\underline{\underline{S}}_\sigma = \begin{bmatrix} E_p^{-1} & -\nu_p E_p^{-1} & 0 \\ -\nu_p E_p^{-1} & E_p^{-1} & 0 \\ 0 & 0 & G_p^{-1} \end{bmatrix}$$

$$\underline{\underline{C}}_\sigma = \underline{\underline{S}}_\sigma^{-1} = \frac{1}{1-\nu_p\nu_p}\begin{bmatrix} E_p & \nu_p E_p & 0 \\ \nu_p E_p & E_p & 0 \\ 0 & 0 & (1-\nu_p\nu_p)G_p \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu_{p3}}{1-\nu_p}(\varepsilon_{11} + \varepsilon_{22})$$

D.4 Linear elastic isotropic material

The linear elastic material behavior can be described with the material stiffness matrix $\underline{\underline{C}}$ or the material compliance matrix $\underline{\underline{S}}$. These matrices can be written in terms of the engineering elasticity parameters E and ν.

$$\underline{\underline{S}} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{S}}^{-1} = \frac{E}{(1+\nu)(1-2\nu)}$$
$$\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

D.4.1 Plane strain

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

$$\underline{\underline{S}}_{\varepsilon} = \frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underline{\underline{C}}_{\varepsilon} = \underline{\underline{S}}_{\varepsilon}^{-1} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}\,\nu(\varepsilon_{11}+\varepsilon_{22})$$

It is immediately clear that problems will occur for $\nu = 0.5$, which is the value for incompressible material behavior.

D.4.2 Plane stress

$$\varepsilon_{33} = -\frac{\nu}{E}\,(\sigma_{11}+\sigma_{22})$$

$$\underline{\underline{S}}_{\sigma} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$\underline{\underline{C}}_{\sigma} = \underline{\underline{S}}_{\sigma}^{-1} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}\,(\varepsilon_{11}+\varepsilon_{22})$$

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Appendix E

Weighted residual formulation and FEM for axi-symmetric deformation

The equilibrium equation is transformed in a weighted residual integral according to the principle of weighted residuals. Using $dV = rd\phi dr$ and the axi-symmetry condition, the weighted residual integral becomes an integral in r only.

$$
\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r = 0 \quad \forall \quad r \quad \leftrightarrow \\
&\int_V w\{\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r\}\, dV = 0 \qquad \forall \quad w(r) \\
&2\pi t \int_{R_i}^{R_o} w\{\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r\} r\, dr = 0 \qquad \forall \quad w(r)
\end{aligned}
$$

Partial integration of the first term leads to the weak form of the weighted residual integral. The right-hand side f_e represents the contribution of the external loads.

$$
\begin{aligned}
&w\sigma_{rr,r} r = w\frac{d\sigma_{rr}}{dr} r = \frac{d}{dr}(w\sigma_{rr} r) - \frac{dw}{dr}\sigma_{rr} r - w\sigma_{rr} \qquad \rightarrow \\
&\int_{R_i}^{R_o} (w_{,r}\sigma_{rr} r + w\sigma_{tt})\, dr = \int_{R_i}^{R_o} w q_r r\, dr + [w\sigma_{rr} t]_{R_i}^{R_o} = f_e
\end{aligned}
$$

E.1 Linear elastic deformation

The material behavior is described by a linear relation between the stress and strain components. The stiffness parameters A_p, B_p and Q_p can be specified for the material symmetry and for plane stress or plane strain.

$$
\left.
\begin{aligned}
\sigma_{rr} &= A_p\varepsilon_{rr} + Q_p\varepsilon_{tt} = A_p u_{r,r} + Q_p\frac{u_r}{r} \\
\sigma_{tt} &= Q_p\varepsilon_{rr} + B_p\varepsilon_{tt} = Q_p u_{r,r} + B_p\frac{u_r}{r}
\end{aligned}
\right\} \quad \rightarrow
$$

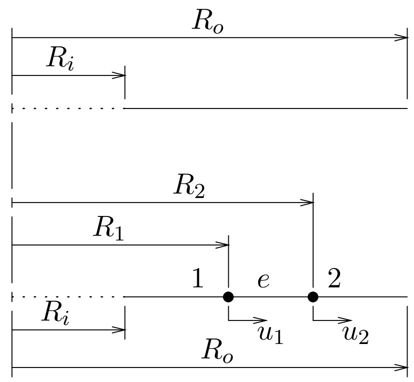
$$
\int_{R_i}^{R_o} \left\{ w_{,r}\left(A_p u_{r,r} + Q_p\frac{u_r}{r}\right) r + w\left(Q_p u_{r,r} + B_p\frac{u_r}{r}\right)\right\} dr = f_e
$$

E.2 Finite element method for an axi-symmetric ring

As always, the finite element method relies on discretisaton of the material volume, implying the weighted residual integral to be written as a sum of integrals over the indivudual elements. Unknown displacments and weighting functions are then interpolated in each element between nodal point values.

Discretisation

The disc is subdivided into ring elements, which have an inner radius R_1 and and outer radius R_2. These elements are connected in the element nodal points – in fact concentric nodal rings – and no gaps are allowed between them. The weighted residual integral can then be written as a summation of integrals over the elements.



$$\sum_{e=1}^{ne} \int_{R_1}^{R_2} \left[A_p w_{,r} u_{r,r} r + Q_p w_{,r} u_r + Q_p w u_{r,r} + B_p w \frac{1}{r} u_r \right] dr = \sum_{e=1}^{ne} f_e^e$$

Interpolation

In each element the radial displacement is written as a function of r. The coefficients are expressed in the nodal radial displacements u_1 and u_2, which leads to interpolation functions ψ_1 and ψ_2, associated with these nodes. They are a function of the radius r and are specified in section E.3. For element type 1, the interpolation of the radial displacement is in accordance with the general solution for the homogeneous equilibrium equation. For element type 2, interpolation is done linearly between the nodal displacements.

Following the Galerkin approach, the weighting function $w(r)$ is interpolated the same way as $u_r(r)$.

$$u_r = \psi_1 u_1 + \psi_2 u_2$$

$$\text{Galerkin} \quad \rightarrow \quad w = \psi_1 w_1 + \psi_2 w_2$$

The interpolation for displacement and weighting function is substituted in the weighted residual integral. Derivatives of the interpolation functions w.r.t. r, are indicated as $\psi_{i,r}$. This leads to the element stiffness matrix $\underline{K}^e$ and the column with external nodal forces $\underset{\sim}{f}^e_e$.

$$
\begin{aligned}
&\left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \int_{R_1}^{R_2} \left\{ A_p \left[\begin{array}{c} \psi_{1,r} \\ \psi_{2,r} \end{array}\right] \left[\begin{array}{cc} \psi_{1,r} & \psi_{2,r} \end{array}\right] r + Q_p \left[\begin{array}{c} \psi_{1,r} \\ \psi_{2,r} \end{array}\right] \left[\begin{array}{cc} \psi_1 & \psi_2 \end{array}\right] + \right. \\
&\qquad \left. Q_p \left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right] \left[\begin{array}{cc} \psi_{1,r} & \psi_{2,r} \end{array}\right] + B_p \left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right] \left[\begin{array}{cc} \psi_1 & \psi_2 \end{array}\right] \frac{1}{r} \right\} \left[\begin{array}{c} u_{r1} \\ u_{r2} \end{array}\right] = \left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \underset{\sim}{f}^e_e \\
&\left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \left\{ A_p \left[\begin{array}{cc} \psi_{1,r}\psi_{1,r} & \psi_{1,r}\psi_{2,r} \\ \psi_{2,r}\psi_{1,r} & \psi_{2,r}\psi_{2,r} \end{array}\right] r + Q_p \left[\begin{array}{cc} \psi_{1,r}\psi_1 & \psi_{1,r}\psi_2 \\ \psi_{2,r}\psi_1 & \psi_{2,r}\psi_2 \end{array}\right] + \right. \\
&\qquad \left. Q_p \left[\begin{array}{cc} \psi_1\psi_{1,r} & \psi_1\psi_{2,r} \\ \psi_2\psi_{1,r} & \psi_2\psi_{2,r} \end{array}\right] + B_p \left[\begin{array}{cc} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{array}\right] \frac{1}{r} \right\} dr \left[\begin{array}{c} u_{r1} \\ u_{r2} \end{array}\right] = \left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \underset{\sim}{f}^e_e \\
&\underset{\sim}{w}^{eT} \underline{K}^e \underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}^e_e
\end{aligned}
$$

Integration

The element stiffness matrix has to be build by integration of functions over the element. When the interpolation function is specified, this integration can be done analytically.

$$
\begin{aligned}
K^e_{11} &= \int_{R_i}^{R_o} \left[A_p \psi_{1,r}\psi_{1,r} r + Q_p \psi_{1,r}\psi_1 + Q_p \psi_1\psi_{1,r} + B_p \psi_1\psi_1 \frac{1}{r} \right] dr \\
K^e_{12} &= \int_{R_1}^{R_2} \left[A_p \psi_{1,r}\psi_{2,r} r + Q_p \psi_{1,r}\psi_2 + Q_p \psi_1\psi_{2,r} + B_p \psi_1\psi_2 \frac{1}{r} \right] dr \\
K^e_{21} &= \int_{R_1}^{R_2} \left[A_p \psi_{2,r}\psi_{1,r} r + Q_p \psi_{2,r}\psi_1 + Q_p \psi_2\psi_{1,r} + B_p \psi_2\psi_1 \frac{1}{r} \right] dr \\
K^e_{22} &= \int_{R_1}^{R_2} \left[A_p \psi_{2,r}\psi_{2,r} r + Q_p \psi_{2,r}\psi_2 + Q_p \psi_2\psi_{2,r} + B_p \psi_2\psi_2 \frac{1}{r} \right] dr
\end{aligned}
$$

External load

The external load is the addition of a volume load and the edge loads. The latter ones can be applied directly in the edge nodes. The volume load, however, is the result of an integration procedure over the volume of the disc.

$$
f_e = \int_{R_i}^{R_o} w q_r r \, dr + [w\sigma_{rr} r]_{R_i}^{R_o} = \sum_{e=1}^{ne} \int_{R_1}^{R_2} w q_r r \, dr + [w\sigma_{rr} r]_{R_i}^{R_o} = \sum_{e=1}^{ne} q^e_e + [w\sigma_{rr} r]_{R_i}^{R_o}
$$

The contribution of the volume load can only be integrated after specification of this volume load as a function of the radius r. It is assumed here that the volume load is a centrifugal load, for which we have : $q_r = \rho\omega^2 r$.

$$
\begin{aligned}
q^e_e &= \rho\omega^2 \int_{R_1}^{R_2} w r^2 \, dr = \left[\begin{array}{cc} w_1 & w_2 \end{array}\right] \rho\omega^2 \int_{R_1}^{R_2} \left[\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right] r^2 \, dr = \underset{\sim}{w}^{eT} \underset{\sim}{q}^e \\
f^e_e &= \underset{\sim}{w}^{eT} \underset{\sim}{q}^e + w_o \sigma_{rr}(r = R_o) R_o - w_i \sigma_{rr}(r = R_i) R_i
\end{aligned}
$$

Assembling

Because the ring elements are placed sequentially from inside to outside in the disc, assembling the global weighted residual equation is very straightforward. The requirement that it must be satisfied for all nodal values $\tilde{w}$, results in the set of equations for the nodal displacements $\tilde{u}$.

$$\tilde{w}^T \underline{K}\, \tilde{u} = \tilde{w}^T \tilde{f}_e \quad \forall \quad \tilde{w} \qquad \Rightarrow \quad \underline{K}\, \tilde{u} = \tilde{f}_e$$

Boundary conditions

In the case of the axi-symmetric disc, we do not have to prevent rigid body motion, by suppressing nodal displacements. Nodal displacement is always associated with deformation.

E.3 FE program `femaxi`

Two elelement types can be used, which have different interpolation for the radial displacement u_r.

In element type 1 the interpolation is based on the general solution of the homogeneous differential equation for u_r in the case of isotropic material behavior. In element type 2 the interpolation is a linear function of the radial coordinate.

E.3.1 Element `elt=1`

For this element, the interpolation function is chosen in accordance with the general solution.

$$\begin{aligned}
& u_r = a_1 r + a_2 \frac{1}{r} \quad \rightarrow \quad u_{r1} = a_1 R_1 + a_2 \frac{1}{R_1} \quad ; \quad u_{r2} = a_1 R_2 + a_2 \frac{1}{R_2} \quad \rightarrow \\
& u_r = \psi_1 u_{r1} + \psi_2 u_{r2} \qquad ; \qquad u_{r,r} = \psi_{1,r} u_{r1} + \psi_{2,r} u_{r2} \\
& \psi_1 = a R_1(-r + R_2^2 r^{-1}) \quad ; \quad \psi_2 = a R_2(r - R_1^2 r^{-1}) \qquad \text{with} \quad a = \frac{1}{R_2^2 - R_1^2} \\
& \psi_{1,r} = a R_1(-1 - R_2^2 r^{-2}) \quad ; \quad \psi_{2,r} = a R_2(1 + R_1^2 r^{-2})
\end{aligned}$$

$$\text{Galerkin} \quad \rightarrow \quad w = \psi_1 w_1 + \psi_2 w_2$$

Element stiffness matrix

Interpolation functions and their derivatives are substituted in the integrals of the element stifness matrix and subsequently integrated. Mind that $K^e_{21} = K^e_{12}$.

$$\begin{aligned}
K^e_{11} &= \left[A_p \psi_{1,r} \psi_{1,r} r + Q_p \psi_{1,r} \psi_1 + Q_p \psi_1 \psi_{1,r} + B_p \psi_1 \psi_1 \frac{1}{r} \right] dr \\
&= a^2 R_1^2 \left[\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) - \tfrac{1}{2}(A_p - 2Q_p + B_p) R_2^4 (R_2^{-2} - R_1^{-2}) + \right. \\
&\qquad \left. 2(A_p - B_p) R_2^2 \ln\left(\frac{R_2}{R_1}\right) \right]
\end{aligned}$$

$$
\begin{aligned}
K^e_{12} &= \int_{R_1}^{R_2} \left[A_p \psi_{1,r}\psi_{2,r} r + Q_p \psi_{1,r}\psi_2 + Q_p \psi_1 \psi_{2,r} + B_p \psi_1 \psi_2 \frac{1}{r} \right] dr \\
&= a^2 R_1 R_2 \left[-\tfrac{1}{2}(A_p - 2Q_p + B_p)(R_2^2 - R_1^2) + \tfrac{1}{2}(A_p - 2Q_p + B_p) R_1^2 R_2^2 (R_2^{-2} - R_1^{-2}) - \right. \\
&\qquad \left. (A_p - B_p)(R_2^2 - R_1^2) \ln\left(\frac{R_2}{R_1}\right) \right] \\
K^e_{22} &= \int_{R_1}^{R_2} \left[A_p \psi_{2,r}\psi_{2,r} r + Q_p \psi_{2,r}\psi_2 + Q_p \psi_2 \psi_{2,r} + B_p \psi_2 \psi_2 \frac{1}{r} \right] dr \\
&= a^2 R_2^2 \left[\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) - \tfrac{1}{2}(A_p - 2Q_p + B_p) R_1^4 (R_2^{-2} - R_1^{-2}) + \right. \\
&\qquad \left. 2(A_p - B_p) R_1^2 \ln\left(\frac{R_2}{R_1}\right) \right]
\end{aligned}
$$

Centrifugal load

For a centrifugal load, the nodal forces are calculated by integration over the element.

$$
\begin{aligned}
\underset{\sim}{q}^e &= \rho\omega^2 \int_{R_1}^{R_2} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} r^2\, dr = \rho\omega^2 a \int_{R_1}^{R_2} \begin{bmatrix} R_1(-r^3 + R_2^2 r) \\ R_2(r^3 - R_1^2 r) \end{bmatrix} dr \\
&= \rho\omega^2 \tfrac{1}{4} \begin{bmatrix} R_1(R_1^2 + R_2^2) \\ R_2(R_1^2 + R_2^2) \end{bmatrix}
\end{aligned}
$$

E.3.2 Element `elt=2`

The second element based on a linear interpolation of the radial displacement.

$$
\begin{aligned}
& u_r = a_0 + a_1 r \quad \rightarrow \quad u_{r1} = a_0 + a_1 R_1 \quad ; \quad u_{r2} = a_0 + a_1 R_2 \quad \rightarrow \\
& u_r = \psi_1 u_{r1} + \psi_2 u_{r2} \quad ; \quad u_{r,r} = \psi_{1,r} u_1 + \psi_{2,r} u_2 \\
& \psi_1 = a(R_2 - r) \quad ; \quad \psi_2 = a(-R_1 + r) \quad ; \quad \text{with} \quad a = \frac{1}{R_2 - R_1} \\
& \psi_{1,r} = -a \quad ; \quad \psi_{2,r} = a \\
& \\
& \text{Galerkin} \quad \rightarrow \quad w = \psi_1 w_1 + w_2 \psi_2
\end{aligned}
$$

Element siffness matrix

Interpolation functions and their derivatives are substituted in the integrals of the element stifness matrix and subsequently integrated. Mind that $K^e_{21} = K^e_{12}$.

$$
\begin{aligned}
K^e_{11} &= \left[A_p \psi_{1,r}\psi_{1,r} r + Q_p \psi_{1,r}\psi_1 + Q_p \psi_1 \psi_{1,r} + B_p \psi_1 \psi_1 \frac{1}{r} \right] dr \\
&= a^2 \left[\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) - 2(Q_p + B_p) R_2 (R_2 - R_1) + B_p R_2^2 \ln\left(\frac{R_2}{R_1}\right) \right] \\
K^e_{12} &= \int_{R_1}^{R_2} \left[A_p \psi_{1,r}\psi_{2,r} r + Q_p \psi_{1,r}\psi_2 + Q_p \psi_1 \psi_{2,r} + B_p \psi_1 \psi_2 \frac{1}{r} \right] dr
\end{aligned}
$$

$$= a^2 \left[-\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) + (Q_p + B_p)(R_2^2 - R_1^2) - B_p R_1 R_2 \ln\left(\frac{R_2}{R_1}\right) \right]$$

$$K_{22}^e = \int_{R_1}^{R_2} \left[A_p \psi_{2,r}\psi_{2,r} r + Q_p \psi_{2,r}\psi_2 + Q_p \psi_2 \psi_{2,r} + B_p \psi_2 \psi_2 \frac{1}{r} \right] dr$$

$$= a^2 \left[\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) - 2(Q_p + B_p) R_1 (R_2 - R_1) + B_p R_1^2 \ln\left(\frac{R_2}{R_1}\right) \right]$$

Centrifugal load

For a centrifugal load, the nodal forces are calculated by integration over the element.

$$\underset{\sim}{q}^e = \rho\omega^2 \int_{R_1}^{R_2} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} r^2 \, dr = a\rho\omega^2 \int_{R_1}^{R_2} \begin{bmatrix} R_2 r^2 - r^3 \\ -R_1 r^2 + r^3 \end{bmatrix} dr$$

$$= a\rho\omega^2 \begin{bmatrix} \frac{1}{12}R_2^4 - \frac{1}{3}R_2 R_1^3 + \frac{1}{4}R_1^4 \\ \frac{1}{12}R_1^4 - \frac{1}{3}R_1 R_2^3 + \frac{1}{4}R_2^4 \end{bmatrix}$$

E.3.3 Matlab program `femaxi`

The program `femaxi.m` is listed below and is explained by some comments.

```
%**********************************************************************
% As input the next parameters must be provided.
% ne   :  number of elelements
% elt  :  element type
% Ri   :  inner radius
% Ro   :  outer radius
% elda :  element data
%
% Default values for some parameters are set.
% elt  :  element type
% Gr   :  density
% G    :  shear modulus
% Go   :  radial frequency [rad/s]
% pf   :  prescribed nodal forces
% pu   :  prescribed nodal displacements

if ~exist('elt'), elt = 2; end;
if ~exist('Gr'),  Gr = 1; end;
if ~exist('G'),   G = E/(2*(1+Gn)); end;
if ~exist('Go'),  Go = 0; end;
if ~exist('pf'),  pf = []; end;
if ~exist('pu'),  pu = []; end;

npf = size(pf,1); npu = size(pu,1);

% Next columns are needed later to implement boundary conditions properly.
```

```
pa = 1:ne+1; if npu>0, pa(pu(:,1)) = -pa(pu(:,1)); end;

% Nodal coordinates

crd = Ri:((Ro-Ri)/ne):Ro;

% Stiffness parameters are calculated and stored in element data base.

for e=1:ne
  E1   = elda(e,1); E2   = elda(e,2); E3   = elda(e,3);
  Gn12 = elda(e,4); Gn21 = elda(e,5);
  Gn23 = elda(e,6); Gn32 = elda(e,7);
  Gn31 = elda(e,8); Gn13 = elda(e,9);
  G12  = elda(e,10);
  if Gn12~=0 & Gn21==0, Gn21 = Gn12*(E2/E1); end;
  if Gn21~=0 & Gn12==0, Gn12 = Gn21*(E1/E2); end;
  if Gn23~=0 & Gn32==0, Gn32 = Gn23*(E3/E2); end;
  if Gn32~=0 & Gn23==0, Gn23 = Gn32*(E2/E3); end;
  if Gn31~=0 & Gn13==0, Gn13 = Gn31*(E1/E3); end;
  if Gn13~=0 & Gn31==0, Gn31 = Gn13*(E3/E1); end;
  DD = (1-Gn12*Gn21-Gn23*Gn32-Gn31*Gn13-Gn12*Gn23*Gn31-Gn21*Gn32*Gn13);
  A  = (1-Gn32*Gn23)/(DD)*E1;
  B  = (1-Gn31*Gn13)/(DD)*E2;
  C  = (1-Gn12*Gn21)/(DD)*E3;
  Q  = (Gn31*Gn23+Gn21)/(DD)*E1;
  R  = (Gn21*Gn32+Gn31)/(DD)*E1;
  S  = (Gn12*Gn31+Gn32)/(DD)*E2;
  K  = G12;

  Ae = A; Be = B; Qe = Q;
  As = (A*C - R*R)/C; Bs = (B*C - S*S)/C; Qs = (Q*C - R*S)/C;
  if     sta=='psn'
    Ap = Ae; Bp = Be; Qp = Qe;
    Zs1 = R; Zs2 = S; Ze1 = 0; Ze2 = 0;
  elseif sta=='pss'
    Ap = As; Bp = Bs; Qp = Qs;
    Zs1 = 0; Zs2 = 0; Ze1 = -(R/C); Ze2 = -(S/C);
  end;
  elda(e,11:17) = [ Ap Bp Qp Zs1 Zs2 Ze1 Ze2 ];
end;

% Initialization to zero value.

K = zeros(ne+1,ne+1); f = zeros(1,ne+1)'; ur = zeros(1,ne+1);

% Element loop for stiffness matrix.
```

```
for e=1:ne
  Ap  = elda(e,11); Bp  = elda(e,12); Qp  = elda(e,13);
  Zs1 = elda(e,14); Zs2 = elda(e,15);
  Ze1 = elda(e,16); Ze2 = elda(e,17);
  R1  = crd(e); R2 = crd(e+1);

  if elt==1

  a    = 1/(R2^2 - R1^2);
  Ke11 = (a^2)*(R1^2)* ...
         (...
           (1/2)*(Ap+2*Qp+Bp)*(R2^2-R1^2) - ...
           (1/2)*(Ap-2*Qp+Bp)*(R2^4)*(R2^(-2)-R1^(-2)) + ...
           (2)*(Ap-Bp)*(R2^2)*log(R2/R1) ...
         );
  Ke12 = (a^2)*(R1*R2)* ...
         (...
          -(1/2)*(Ap+2*Qp+Bp)*(R2^2-R1^2) + ...
           (1/2)*(Ap-2*Qp+Bp)*(R1^2)*(R2^2)*(R2^(-2)-R1^(-2)) - ...
           (Ap-Bp)*(R2^2+R1^2)*log(R2/R1) ...
         );
  Ke22 = (a^2)*(R2^2)* ...
         (...
           (1/2)*(Ap+2*Qp+Bp)*(R2^2-R1^2) - ...
           (1/2)*(Ap-2*Qp+Bp)*(R1^4)*(R2^(-2)-R1^(-2)) + ...
           (2)*(Ap-Bp)*(R1^2)*log(R2/R1) ...
         );
  Ke21 = Ke12;
%  fe1  = Gr*Go^2*a*R1*( -(1/4)*(R2^4-R1^4) + (1/2)*R2^2*(R2^2-R1^2) );
%  fe2  = Gr*Go^2*a*R2*(  (1/4)*(R2^4-R1^4) - (1/2)*R1^2*(R2^2-R1^2) );
  fe1  = Gr*Go^2*(1/4)*R1*(R2^2-R1^2);
  fe2  = Gr*Go^2*(1/4)*R2*(R2^2-R1^2);


%  A11  = a^2*R1^2*( (Ap+2*Qp+Bp) + 2*R2^2*(Ap-Bp) );
%  B11  = a^2*R1^2*R2^4*(Ap+Bp-2*Qp);
%  A12  = -a^2*R1*R2*( (Ap+2*Qp+Bp) + R1^2*(Ap-Bp) + R2^2*(Ap-Bp) );
%  B12  = -a^2*R1^3*R2^3*(Ap+Bp-2*Qp);
%  A21  = -a^2*R1*R2*( (Ap+2*Qp+Bp) + R1^2*(Ap-Bp) + R2^2*(Ap-Bp) );
%  B21  = -a^2*R1^3*R2^3*(Ap+Bp-2*Qp);
%  A22  = a^2*R2^2*( (Ap+2*Qp+Bp) + 2*R1^2*(Ap-Bp) );
%  B22  = a^2*R2^2*R1^4*(Ap+Bp-2*Qp);
%  Ke11 = A11*(1/2)*(R2^2 - R1^2) - B11*(1/2)*(R2^(-2) - R1^(-2));
%  Ke12 = A12*(1/2)*(R2^2 - R1^2) - B12*(1/2)*(R2^(-2) - R1^(-2));
%  Ke21 = A21*(1/2)*(R2^2 - R1^2) - B21*(1/2)*(R2^(-2) - R1^(-2));
%  Ke22 = A22*(1/2)*(R2^2 - R1^2) - B22*(1/2)*(R2^(-2) - R1^(-2));
%  fe1  = Gr*Go^2*(1/(4*a))*R1;
```

```
%  fe2  = Gr*Go^2*(1/(4*a))*R2;

  elseif elt==2

  a    = 1/(R2 - R1);
  Ke11 = a^2*( ...
               (1/2)*(Ap+2*Qp+Bp)*(R2^2-R1^2) - ...
               (2)*(Qp+Bp)*(R2)*(R2-R1) + ...
               (Bp)*(R2^2)*log(R2/R1) ...
             );
  Ke12 = a^2*( ...
               -(1/2)*(Ap+2*Qp+Bp)*(R2^2-R1^2) + ...
               (Qp+Bp)*(R2+R1)*(R2-R1) - ...
               (Bp)*(R1*R2)*log(R2/R1) ...
             );
  Ke22 = a^2*( ...
               (1/2)*(Ap+2*Qp+Bp)*(R2^2-R1^2) - ...
               (2)*(Qp+Bp)*(R1)*(R2-R1) + ...
               (Bp*R1^2)*log(R2/R1) ...
               );
  Ke21 = Ke12;
  fe1  = Gr*Go^2*a * ( (1/12)*R2^4-(1/3)*R2*R1^3+(1/4)*R1^4 );
  fe2  = Gr*Go^2*a * ( (1/12)*R1^4-(1/3)*R1*R2^3+(1/4)*R2^4 );

  end;

  Ke = [ Ke11 Ke12; Ke21 Ke22 ];
  fe = [ fe1 fe2 ]';
  K(e:e+1,e:e+1) = K(e:e+1,e:e+1) + Ke;
  f(e:e+1)       = f(e:e+1)       + fe;
end;

% Taking prescribed displacements to right hand side.
% Eliminating obsolete rows and columns.

if npf>0
  f(pf(:,1)) = f(pf(:,1)) + pf(:,2).*crd(pf(:,1));
end;
if npu>0
  f = f - K(:,pu(:,1))*pu(:,2);
  K(:,pu(:,1)) = []; K(pu(:,1),:) = []; f(pu(:,1)) = [];
end;

% Inversion leads to solution of unknowns.

urr = inv(K)*f;
```

```
% Inserting prescribed displacements.

k=1; for j=1:ne+1, if pa(j)>0, ur(j)=urr(k); k=k+1; end;end;
if npu>0, ur(pu(:,1)) = pu(:,2); end;

% Calculating strains and stresses.

for e=1:ne
  R1 = crd(e); R2 = crd(e+1); Rm = (1/2)*(R1+R2); crdm(e) = Rm;
  u1 = ur(e); u2 = ur(e+1);

  if elt==1

  a = 1/(R2^2 - R1^2);
  Gy1 = a*R1*(-Rm + R2^2/Rm); Gy2 = a*R2*(Rm - R1^2/Rm);
  Gy1r = a*R1*(-1 - R2^2/(Rm^2)); Gy2r = a*R2*(1 + R1^2/(Rm^2));

  elseif elt==2

  a = 1/(R2 - R1);
  Gy1 = a*(R2 - Rm); Gy2 = a*(-R1+Rm);
  Gy1r = -a; Gy2r = a;

  end;

  Gerr(e) = Gy1r*u1 + Gy2r*u2;
  Gett(e) = (1/Rm)*(Gy1*u1 + Gy2*u2);
  Gezz(e) = Ze1 * Gerr(e) + Ze2 * Gett(e);
  Gsrr(e) = Ap*Gerr(e) + Qp*Gett(e);
  Gstt(e) = Qp*Gerr(e) + Bp*Gett(e);
  Gszz(e) = Zs1 * Gerr(e) + Zs2 * Gett(e);
  GsTR(e) = max( [abs(Gsrr(e)-Gstt(e)) ...
                  abs(Gsrr(e)-Gszz(e)) abs(Gstt(e)-Gszz(e))] );
  GsVM(e) = sqrt(0.5*((Gsrr(e)-Gstt(e))^2 + ...
                      (Gstt(e)-Gszz(e))^2 + (Gszz(e)-Gsrr(e))^2));
end;

%**********************************************************************
```

Appendix F

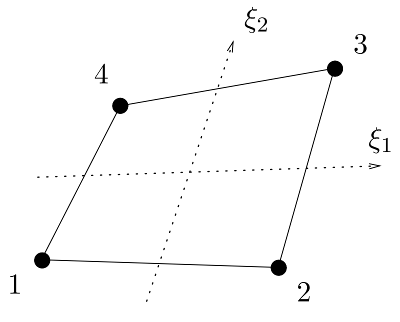
Planar elements

The 4-noded and 8-noded elements are described in the next sections of this appendix.

F.1 Four-node quadrilateral element

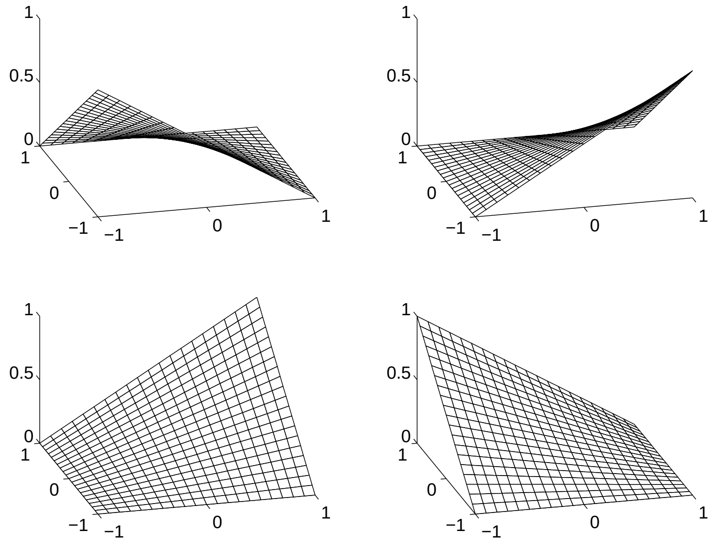
In two-dimensional finite element analysis the four-node element is used very much. In the undeformed and deformed configuration the element sides are straight lines. As its name indicates, it has four nodal points, which are located in its corners. The numbering of the nodes is anti-clockwise by convention.

The shape functions, which are used to interpolate global coordinates and displacement components and weighting function components between their respective nodal values, must be linear along an element side. In the two-dimensional plane these functions are functions of the isoparametric coordinates ξ_1 and ξ_2. These functions are not completely linear : they have a term $\xi_1\xi_2$. This implies that their derivatives are not completely constant.



Four-node quadrilateral element

$$
N^1 = \tfrac{1}{4}(1-\xi_1)(1-\xi_2) \quad ; \quad N^2 = \tfrac{1}{4}(1+\xi_1)(1-\xi_2)
$$
$$
N^3 = \tfrac{1}{4}(1+\xi_1)(1+\xi_2) \quad ; \quad N^4 = \tfrac{1}{4}(1-\xi_1)(1+\xi_2)
$$



Linear interpolation functions in 4-node element

F.1.1 Cartesian coordinate system

In a Cartesian coordinate system the displacement of every point of a quadrilateral element has two components, u_x and u_y. Both components are interpolated between the nodal displacement components, using the shape functions. The element shape and the weighting function is interpolated in the same way as the displacement.

displacement

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 & 0 \\ 0 & N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 \end{bmatrix} \begin{bmatrix} u_x^1 \\ u_y^1 \\ u_x^2 \\ u_y^2 \\ u_x^3 \\ u_y^3 \\ u_x^4 \\ u_y^4 \end{bmatrix} \rightarrow \underset{\sim}{u} = \underline{N}\, \underset{\sim}{u}^e$$

element shape $\qquad \underset{\sim}{x} = \underline{N}\, \underset{\sim}{x}^e \quad ; \quad \underset{\sim}{x}_0 = \underline{N}\, \underset{\sim}{x}_0^e$

weighting function $\qquad \underset{\sim}{w} = \underline{N}\, \underset{\sim}{w}^e$

The columns $\underset{\approx}{L}_u$ and $\underset{\approx}{L}_w$ represent the components of the gradient of the displacement and the weighting function, respectively. After interpolation of displacement and weighting function, the so-called B-matrix appears.

The B-matrix contains derivatives of the shape functions $\{N^\alpha; \alpha = 1, 2, 3, 4\}$ with respect to the Cartesian coordinates x and y. The Jacobian matrix $\underline{J}$ contains the derivatives of the Cartesian coordinates x and y with respect to the isoparametric coordinates ξ_1 and ξ_2.

$$\begin{bmatrix} u_{x,x} \\ u_{y,y} \\ u_{y,x} \\ u_{x,y} \end{bmatrix} = \begin{bmatrix} N^1_{,x} & 0 & N^2_{,x} & 0 & N^3_{,x} & 0 & N^4_{,x} & 0 \\ 0 & N^1_{,y} & 0 & N^2_{,y} & 0 & N^3_{,y} & 0 & N^4_{,y} \\ 0 & N^1_{,x} & 0 & N^2_{,x} & 0 & N^3_{,x} & 0 & N^4_{,x} \\ N^1_{,y} & 0 & N^2_{,y} & 0 & N^3_{,y} & 0 & N^4_{,y} & 0 \end{bmatrix} \begin{bmatrix} u^1_x \\ u^1_y \\ u^2_x \\ u^2_y \\ u^3_x \\ u^3_y \\ u^4_x \\ u^4_y \end{bmatrix} \rightarrow \left(\underset{\approx}{L}_u \right)_t = \underline{B}\, \underset{\sim}{u}^e$$

$$\begin{bmatrix} N^1_{,x} & N^1_{,y} \\ N^2_{,x} & N^2_{,y} \\ N^3_{,x} & N^3_{,y} \\ N^4_{,x} & N^4_{,y} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^1_{,2} \\ N^2_{,1} & N^2_{,2} \\ N^3_{,1} & N^3_{,2} \\ N^4_{,1} & N^4_{,2} \end{bmatrix} \begin{bmatrix} \xi_{1,x} & \xi_{1,y} \\ \xi_{2,x} & \xi_{2,y} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^1_{,2} \\ N^2_{,1} & N^2_{,2} \\ N^3_{,1} & N^3_{,2} \\ N^4_{,1} & N^4_{,2} \end{bmatrix} \underline{J}^{-T}$$

$$\underline{J} = \begin{bmatrix} x_{,1} & y_{,1} \\ x_{,2} & y_{,2} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^2_{,1} & N^3_{,1} & N^4_{,1} \\ N^1_{,2} & N^2_{,2} & N^3_{,2} & N^4_{,2} \end{bmatrix} \begin{bmatrix} x^1_e & y^1_e \\ x^2_e & y^2_e \\ x^3_e & y^3_e \\ x^4_e & y^4_e \end{bmatrix}$$

The deformation matrix can be calculated in each element integration point. Besides nodal point coordinates in the current state, the coordinates in the reference state must be available.

$$\underline{F} = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & 0 \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & 0 \\ 0 & 0 & F_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} \end{bmatrix} = \begin{bmatrix} N^1_{,x0} & N^2_{,x0} & N^3_{,x0} & N^4_{,x0} \\ N^1_{,y0} & N^2_{,y0} & N^3_{,y0} & N^4_{,y0} \end{bmatrix} \begin{bmatrix} x^1_e & y^1_e \\ x^2_e & y^2_e \\ x^3_e & y^3_e \\ x^4_e & y^4_e \end{bmatrix}$$

$$= \begin{bmatrix} \xi_{1,x0} & \xi_{2,x0} \\ \xi_{1,y0} & \xi_{2,y0} \end{bmatrix} \begin{bmatrix} N^1_{,1} & N^2_{,1} & N^3_{,1} & N^4_{,1} \\ N^1_{,2} & N^2_{,2} & N^3_{,2} & N^4_{,2} \end{bmatrix} \begin{bmatrix} x^1_e & y^1_e \\ x^2_e & y^2_e \\ x^3_e & y^3_e \\ x^4_e & y^4_e \end{bmatrix} = \underline{J}_0^{-1} \underline{J}$$

F.1.2 Cylindrical coordinate system

In a cylindrical coordinate system the displacement of every point of a quadrilateral element has two components, u_r and u_z. Both components are interpolated between the nodal displacement components, using the shape functions. The element shape and the weighting function is interpolated in the same way as the displacement.

displacement

$$\begin{bmatrix} u_r \\ u_z \end{bmatrix} = \begin{bmatrix} N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 & 0 \\ 0 & N^1 & 0 & N^2 & 0 & N^3 & 0 & N^4 \end{bmatrix} \begin{bmatrix} u_r^1 \\ u_z^1 \\ u_r^2 \\ u_z^2 \\ u_r^3 \\ u_z^3 \\ u_r^4 \\ u_z^4 \end{bmatrix} \rightarrow \underset{\sim}{u} = \underline{N}\,\underset{\sim}{u}_e$$

element shape $\quad r = \underset{\sim}{N}^T \underset{\sim}{r} \quad ; \quad z = \underset{\sim}{N}^T \underset{\sim}{z}$

$\quad r_0 = \underset{\sim}{N}^T \underset{\sim}{r}_0 \quad ; \quad z_0 = \underset{\sim}{N}^T \underset{\sim}{z}_0$

weighting function $\quad \underset{\sim}{w} = \underline{N}\,\underset{\sim}{w}^e$

The columns $\underset{\approx}{L}_u$ and $\underset{\approx}{L}_w$ represent the components of the gradient of the displacement and the weighting function, respectively. After interpolation of displacement and weighting function, the so-called B-matrix appears. For ease of programming, we swap the derivatives $u_{t,t}$ and $u_{z,z}$ in the column $\underset{\approx}{L}_u$ and analoguously $w_{t,t}$ and $w_{z,z}$ in the column $\underset{\approx}{L}_w$.

The B-matrix contains derivatives of the shape functions $\{N^\alpha; \alpha = 1,2,3,4\}$ with respect to the cylindrical coordinates r and z. The Jacobian matrix $\underline{J}$ contains the derivatives of the cylindrical coordinates r and z with respect to the isoparametric coordinates ξ_1 and ξ_2.

$$\begin{bmatrix} u_{r,r} \\ u_{z,z} \\ \frac{1}{r}u_r \\ u_{z,r} \\ u_{r,z} \end{bmatrix} = \begin{bmatrix} N^1_{,r} & 0 & N^2_{,r} & 0 & N^3_{,r} & 0 & N^4_{,r} & 0 \\ 0 & N^1_{,z} & 0 & N^2_{,z} & 0 & N^3_{,z} & 0 & N^4_{,z} \\ \frac{1}{r}N^1 & 0 & \frac{1}{r}N^2 & 0 & \frac{1}{r}N^3 & 0 & \frac{1}{r}N^4 & 0 \\ 0 & N^1_{,r} & 0 & N^2_{,r} & 0 & N^3_{,r} & 0 & N^4_{,r} \\ N^1_{,z} & 0 & N^2_{,z} & 0 & N^3_{,z} & 0 & N^4_{,z} & 0 \end{bmatrix} \begin{bmatrix} u_r^1 \\ u_z^1 \\ u_r^2 \\ u_z^2 \\ u_r^3 \\ u_z^3 \\ u_r^4 \\ u_z^4 \end{bmatrix} \rightarrow \left(\underset{\approx}{L}_u\right)_t = \underline{B}\underset{\sim}{u}^e$$

$$\begin{bmatrix} N^1_{,r} & N^1_{,z} \\ N^2_{,r} & N^2_{,z} \\ N^3_{,r} & N^3_{,z} \\ N^4_{,r} & N^4_{,z} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^1_{,2} \\ N^2_{,1} & N^2_{,2} \\ N^3_{,1} & N^3_{,2} \\ N^4_{,1} & N^4_{,2} \end{bmatrix} \begin{bmatrix} \xi_{1,r} & \xi_{1,z} \\ \xi_{2,r} & \xi_{2,z} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^1_{,2} \\ N^2_{,1} & N^2_{,2} \\ N^3_{,1} & N^3_{,2} \\ N^4_{,1} & N^4_{,2} \end{bmatrix} \underline{J}^{-T}$$

$$\underline{J} = \begin{bmatrix} r_{,1} & z_{,1} \\ r_{,2} & z_{,2} \end{bmatrix} = \begin{bmatrix} N^1_{,1} & N^2_{,1} & N^3_{,1} & N^4_{,1} \\ N^1_{,2} & N^2_{,2} & N^3_{,2} & N^4_{,2} \end{bmatrix} \begin{bmatrix} r_e^1 & z_e^1 \\ r_e^2 & z_e^2 \\ r_e^3 & z_e^3 \\ r_e^4 & z_e^4 \end{bmatrix}$$

The deformation matrix can be calculated in each element integration point. Besides nodal point coordinates in the current state, the coordinates in the reference state must be available.

$$
\underline{F} = \begin{bmatrix} \dfrac{\partial r}{\partial r_0} & \dfrac{\partial r}{\partial z_0} & 0 \\ \dfrac{\partial z}{\partial r_0} & \dfrac{\partial z}{\partial z_0} & 0 \\ 0 & 0 & \frac{r}{r_0} \end{bmatrix}
$$

$$
\begin{bmatrix} \dfrac{\partial r}{\partial r_0} & \dfrac{\partial z}{\partial r_0} \\ \dfrac{\partial r}{\partial z_0} & \dfrac{\partial z}{\partial z_0} \end{bmatrix} = \begin{bmatrix} N^1_{,r0} & N^2_{,r0} & N^3_{,r0} & N^4_{,r0} \\ N^1_{,z0} & N^2_{,z0} & N^3_{,z0} & N^4_{,z0} \end{bmatrix} \begin{bmatrix} r_e^1 & z_e^1 \\ r_e^2 & z_e^2 \\ r_e^3 & z_e^3 \\ r_e^4 & z_e^4 \end{bmatrix}
$$

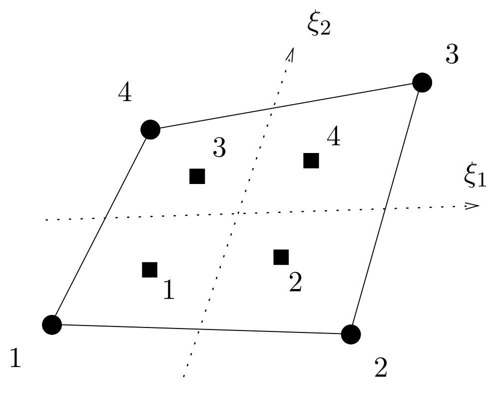
$$
= \begin{bmatrix} \xi_{1,r0} & \xi_{2,r0} \\ \xi_{1,z0} & \xi_{2,z0} \end{bmatrix} \begin{bmatrix} N^1_{,1} & N^2_{,1} & N^3_{,1} & N^4_{,1} \\ N^1_{,2} & N^2_{,2} & N^3_{,2} & N^4_{,2} \end{bmatrix} \begin{bmatrix} r_e^1 & z_e^1 \\ r_e^2 & z_e^2 \\ r_e^3 & z_e^3 \\ r_e^4 & z_e^4 \end{bmatrix} = \underline{J}_0^{-1} \underline{J}
$$

F.1.3 Numerical integration

To generate the element stiffness matrix and the residual force column, integration over the element volume (2D : area) must be carried out. With quadrilateral elements this integration cannot be done analytically, so numerical integration is necessary.

The numerical integration which we employ here is the Gauss quadrature integration. The integrand is evaluated in a number of discrete points, the integration points or Gauss points. The integration point values are multiplied by a weighting factor, ζ, after which they are added. The location of the integration points (= their isoparametric coordinates) and the value of the weighting factor are determined in such a way that a polynomial of a certain degree is integrated exactly.

The four-node quadrilateral integration point locations and weighting factor values are shown in the table. Their choice is such that a polynomial of third order in each direction is integrated exactly.



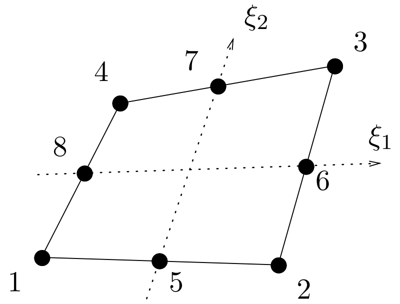
ip	ξ_1	ξ_2	ζ
1	$-\frac{1}{3}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	1
2	$\frac{1}{3}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	1
3	$-\frac{1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	1
4	$\frac{1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	1

Integration points in a 4-node element

F.2 Eight-node quadrilateral element

The eight-node element has four sides (”quadrilateral”), which are straight lines in the undeformed configuration. The nodes 1 to 4 are located in the corners (corner nodes), the nodes 5 to 8 are located in the middle of the sides (midpoint nodes). The numbering is anti-clockwise.

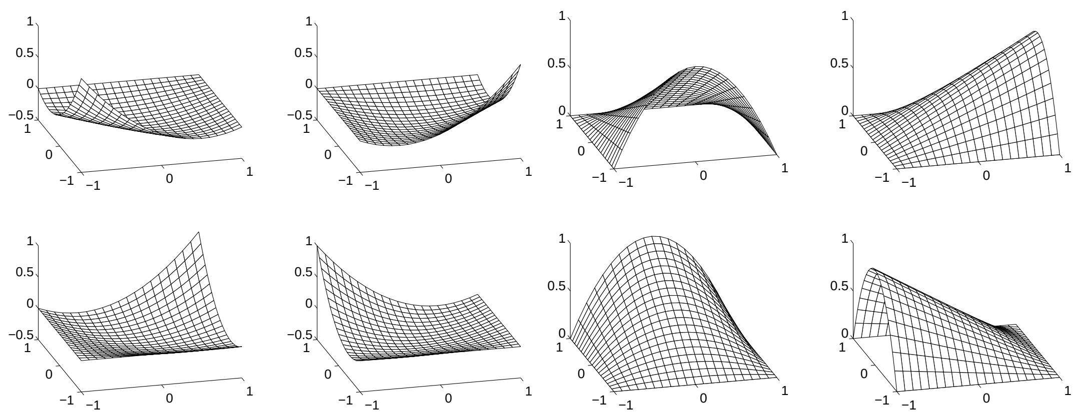
Global coordinates, displacement components and weighting function components are interpolated with shape functions which are quadratic along an element side. This implies that in the deformed configuration these sides may be parabolic. The shape functions are functions of the isoparametric coordinates ξ_1 and ξ_2.



Eight-node quadrilateral element

$$
\begin{aligned}
N^1 &= \tfrac{1}{4}(\xi_1 - 1)(\xi_2 - 1)(-\xi_1 - \xi_2 - 1) & N^5 &= \tfrac{1}{2}(\xi_1^2 - 1)(\xi_2 - 1)\\
N^2 &= \tfrac{1}{4}(\xi_1 + 1)(\xi_2 - 1)(-\xi_1 + \xi_2 + 1) & N^6 &= \tfrac{1}{2}(-\xi_1 - 1)(\xi_2^2 - 1)\\
N^3 &= \tfrac{1}{4}(\xi_1 + 1)(\xi_2 + 1)(\xi_1 + \xi_2 - 1) & N^7 &= \tfrac{1}{2}(\xi_1^2 - 1)(-\xi_2 - 1)\\
N^4 &= \tfrac{1}{4}(\xi_1 - 1)(\xi_2 + 1)(\xi_1 - \xi_2 + 1) & N^8 &= \tfrac{1}{2}(\xi_1 - 1)(\xi_2^2 - 1)
\end{aligned}
$$

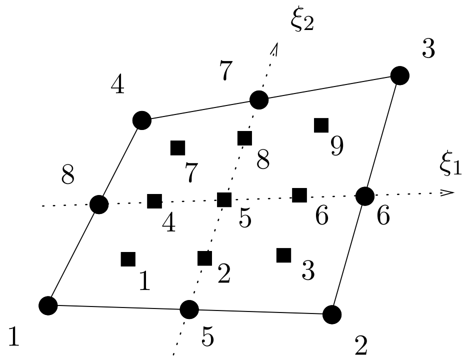
The figures show the shape functions associated with the nodal points. The first four plots show the shape functions of the corner nodes and second series of four plots shows those of the mid-side nodes.



Quadratic interpolation functions in 8-node element

F.2.1 Numerical integration

The table contains the location of the 9 integration (Gauss) points and their weighting functions for the eight-node quadrilateral element. Their choice is such that a polynomial of fifth order in each direction is integrated exactly.



Integration points in 8-node element

$a = 0.77459$; $p = 0.55556$; $q = 0.88889$

ip	ξ_1	ξ_2	ζ
1	$-a$	$-a$	$p \times p$
2	0	$-a$	$p \times q$
3	a	$-a$	$p \times p$
4	$-a$	0	$p \times q$
5	0	0	$q \times q$
6	a	0	$p \times q$
7	$-a$	a	$p \times p$
8	0	a	$p \times q$
9	a	a	$p \times p$

a46

Appendix G

FE program `plaxL`

The program `plaxL` can be used to model and analyze linear elastic deformation of planar and axisymmetric structures. The planar problems can be either plane strain or plane stress. For axisymmetric problems, the rotation axis is the global y-axis and the radial axis is the global x-axis. The half cross-section must be modelled for $x \geq 0$. Both linear 4-node quadrilateral and quadratic 8-node quadrilateral elements can be used.

G.1 Example input file

As an example, a tensile test is modelled and analyzed, both in plane stress and axisymmetricly. Because the deformation is homogeneous, only one element is used.

Figures are closed and the Matlab work space is cleared.

```
close all; clear all;
```

The coordinates of the nodes are given in the matrix "crd0". Units are chosen to be mm.

```
crd0 = [ 0 0; 100 0; 100 100; 0 100 ];
```

The location array "lok" contains information about the element type (first column), element group (second column) and connectivity (last four columns). The element type can be : 3 for plane stress, 11 for plane strain and 10 for axisymmetry. As we have only one element, there is only one element group. The element node numbers are 1, 2, 3 and 4.

```
lok = [ 3 1 1 2 3 4 ];
```

For each group we have to supply geometric and material data in the array "elda". In the subsequent columns we provide :

- 1 : random
- 2 : material number : 11 is isotropic linear material
- 3 : thickness
- 4 : unused
- 5 : unused
- 6 : Young's modulus
- 7 : Poisson's ratio

```
elda = [ 0 11 1 0 0 200000 0.3 ];
```

Prescribed displacements are provided in the array "pp". For each prescribed displacement component we have one row. The first column contains the node, the second column contains the direction (either 1 (= x = horizontal) or 2 (= y = vertical). The third column contains the value.

When we want to prescribe the elongation of 1 mm, we have :

```
pp = [ 1 1 0; 1 2 0; 2 2 0; 4 1 0 ];
pp = [ pp; 3 2 1; 4 2 1 ];
```

We could also have prescribed a tensile force of 200000 N.

```
pp = [ 1 1 0; 1 2 0; 2 2 0; 4 1 0 ];
pf = [ 3 2 100000; 4 2 100000 ];
```

The input is complete and `plaxL` can be executed to analyze the behavior.

```
plaxL;
```

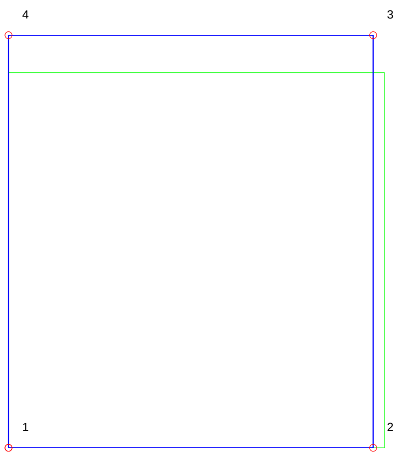
After the analysis the nodal displcacements are available in the array "Mp". Element data can be found in the database "eidaC".

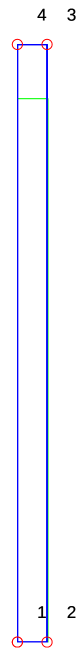
When an axisymmetric model is used, the axis is along the global y-axis, so vertical. The coordinate and location array are then :

```
crd0 = [ 0 0; sqrt(100/pi) 0; sqrt(100/pi) 100; 0 100 ];
lok = [ 10 1 1 2 3 4 ];
```

When a tensile force is applied it will be different in the node on the central axis then in the node on the outer edge.

The figure below shows the deformation in both cases.





G.2 Matlab program `plaxL`

The program `plaxL.m` is listed below and is seeded with comments to explain variables and actions.

```
%*********************************************************************
% plaxL : 2-dimensional planar/axisym.; linear
%=====================================================================

nndof = 2;
nnod  = size(crd0,1);
ndof  = nnod * nndof;
ne    = size(lok,1);
negr  = size(elda,1);

Trm = eye(ndof);
if exist('tr'),
  ntr = size(tr,1);
  for itr=1:ntr
    trp = round(tr(itr,1));     tra = tr(itr,2);
    trc = cos((pi/180)*tra);  trs = sin((pi/180)*tra);
    k1  = nndof*(trp-1)+1;    k2  = nndof*(trp-1)+2;
    trm = [trc -trs ; trs trc];
    Trm([k1 k2],[k1 k2]) = trm;
  end;
else, ntr = 0; tr = []; end;

for e=1:ne
  ety = lok(e,1); egr = lok(e,2);
  if     ety==3,  nenod=4; neip=4; vrs=2;
  elseif ety==11, nenod=4; neip=4; vrs=1;
  elseif ety==10, nenod=4; neip=4; vrs=3; end;
  nedof = nndof * nenod;

  k=1;
  for n=1:nenod
    for v=1:nndof
      lokvg(e,k) = nndof*(lok(e,2+n)-1)+v;
      k=k+1;
    end;
  end;

  elpa(e,1:7) = [ety egr nenod nndof nedof neip vrs];

  mnr = elda(egr,2);         thk = elda(egr,3);
  gf2 = elda(egr,4);         gf3 = elda(egr,5);
  mcl = floor(mnr/10);       mty = rem(mnr,10);
```

```
  if mnr==11
    E = elda(egr,6);   Gn = elda(egr,7);
  end;

  for ip=1:neip
    gip = neip*(e-1) + ip;
    eida0(gip,:) = [ ety mnr thk gf2 gf3 E Gn ];
    eidaC(gip,:) = [ ety mnr thk gf2 gf3 E Gn ];
  end;

end;

%-----------------------------------------------------------------------

if ~exist('pp'), pp = []; ppc = []; ppv = []; end;
if ~exist('pf'), pf = []; pfc = []; pfv = []; end;

npdof = size(pp,1);
npfor = size(pf,1);
nudof = ndof - npdof;

if npdof>0
  ppc = [nndof*(round(pp(:,1))-1)+round(pp(:,2))];
  ppv = pp(:,nndof+1);
end;
if npfor>0,
  pfc = [nndof*(round(pf(:,1))-1)+round(pf(:,2))];
  pfv = pf(:,nndof+1);
end;

%  Information for partitioning the system of equations associated
%  with linked degrees of freedom is made available in the arrays
%  plc  and  prc.

if ~exist('pl'),  pl  = []; plc = []; end;
if ~exist('pr'),  pr  = []; prc = []; end;
if ~exist('lim'), lim = []; end;

npl = size(pl,1);
npr = size(pr,1);

if ~exist('lif'), lif = zeros(1,npl); end;

if npl>0
  plc = [nndof*(round(pl(:,1))-1)+round(pl(:,2))];
  prc = [nndof*(round(pr(:,1))-1)+round(pr(:,2))];
end;
```

```
%  Some extra arrays are made for later use.

pa = 1:ndof; pu = 1:ndof; prs = 1:ndof;
pu([ppc' plc']) = [];
prs([ppc' pfc' plc']) = [];

%  pe0   :  column with prescribed initial displacements
%  fe0   :  array with prescribed initial forces

pe0 = zeros(ndof,1); pe0(ppc(1:npdof)) = ppv(1:npdof);
fe0 = zeros(ndof,1); fe0(pfc(1:npfor)) = pfv(1:npfor);

%-----------------------------------------------------------------------
% Initialization to zero
%  pe    :  column with nodal displacements
%  p     :  column with nodal displacements
%  fe    :  column with external (applied) nodal forces
%  fi    :  column with internal (resulting) nodal forces
%  #T    :  column with transformed components
%-----------------------------------------------------------------------

pe  = zeros(ndof,1); p   = zeros(ndof,1); pT  = zeros(ndof,1);
fe  = zeros(ndof,1); fi  = zeros(ndof,1);
feT = zeros(ndof,1); fiT = zeros(ndof,1);

%-----------------------------------------------------------------------
% Loop over all elements to generate element stiffness matrix 'em'
% Assemble 'em' into structural stiffness matrix 'sm'
%  ec0  : initial coordinates of element nodes
%  ec   : current coordinates of element nodes
%-----------------------------------------------------------------------
sm=zeros(ndof);

for e=1:ne
  ety = elpa(e,1); egr = elpa(e,2);
  nenod = elpa(e,3); nedof = elpa(e,5); neip = elpa(e,6);
  ec0 = crd0(lok(e,3:2+nenod),:); ec = ec0;
  em = zeros(nedof);

    [ksi,psi,psidksi,ipwf] = fbibfe2dq48(e,elpa(e,:));  % fbibfe2dq48.m
    vrs = elpa(e,7);

    for ip=1:neip
      gip = neip*(e-1) + ip;
%-----------------------------------------------------------------------
  if vrs==3
```

```
    r0   = psi(ip,:)*ec0(:,1);     r    = psi(ip,:)*ec(:,1);
    thk0 = r0*2*pi;                thk  = r*2*pi;
    mF33 = r/r0;                   ax   = 1;
  else
    r0   = 1;                      r    = 1;
    thk0 = eida0(gip,3);           thk  = thk0;
    mF33 = 1;                      ax   = 0;
  end;

  dpsi(:,1) = psidksi(:,2*ip-1);   dpsi(:,2) = psidksi(:,2*ip);
  jc0  = dpsi' * ec0;   jci0 = inv(jc0);
  jc   = dpsi' * ec;      jci  = inv(jc);
  dt   = det(jc);         dt0  = det(jc0);
  dfie0   = zeros(5,2*nenod);
  dpsixy0 = dpsi * jci0' ;
  dfie0(1,2*(1:nenod)-1) = dpsixy0(1:nenod,1)';
  dfie0(2,2*(1:nenod))   = dpsixy0(1:nenod,2)';
  dfie0(3,2*(1:nenod)-1) = ax.*psi(ip,1:nenod)/r0;
  dfie0(4,2*(1:nenod))   = dpsixy0(1:nenod,1)';
  dfie0(5,2*(1:nenod)-1) = dpsixy0(1:nenod,2)';
  mF  = eye(3); mF(1:2,1:2)  = jci0*jc;  mF(3,3)  = mF33;  mF  = mF';
%-----------------------------------------------------------------------
      du = zeros(5,1);
      [mmM,ccGs,ccGe,vm] = plaxelas1(eida0(gip,6:7),vrs,du);
                                                          % plaxelas1.m

      em = em + dfie0' * mmM * dfie0 * thk0 * dt0 * ipwf(ip);
    end;

  sm(lokvg(e,1:nedof),lokvg(e,1:nedof)) = ...
    sm(lokvg(e,1:nedof),lokvg(e,1:nedof)) + em;
end; % element loop 'e'

%-----------------------------------------------------------------------
% Transformation for local nodal coordinate systems
%-----------------------------------------------------------------------
sm = Trm'*sm *Trm;
%-----------------------------------------------------------------------
% Boundary conditions and links
%-----------------------------------------------------------------------
pe = pe0; fe = fe0; rs = fe;

if npl>0, rs = rs - sm(:,plc)*lif'; end;

%-----------------------------------------------------------------------
% Partitioning is done in the function                     fbibpartit.m
```

```
[sm,rs] = fbibpartit(1,sm,rs,ndof,pa,ppc,plc,prc,pe,lim);

%----------------------------------------------------------------------
% Solving the system of equations and take prescribed displacements
% and links into account.
% Update nodal point coordinates 'crd'.

sol = inv(sm)*rs; % sol = sm\rs;

pe(pu) = sol;
if npl>0, pe(plc) = lim*pe(prc) + lif'; end;

p = pe; pT = Trm * p;
crd = crd0 + reshape(pT,nndof,nnod)';

%----------------------------------------------------------------------
% Calculate stresses and strains and the internal forces 'ef'.
% Internal forces 'ef' are assembled into 'fi', the structural
% internal forces, representing the reaction forces.
%----------------------------------------------------------------------
fi = zeros(ndof,1);

for e=1:ne
  ety = elpa(e,1); egr = elpa(e,2);
  nenod = elpa(e,3); nedof = elpa(e,5); neip = elpa(e,6);
  ec0 = crd0(lok(e,2+1:2+nenod),:);
  ec = crd(lok(e,2+1:2+nenod),:);
  Tpe  = p(lokvg(e,1:nedof));
  em = zeros(nedof); ef = zeros(nedof,1);

    [ksi,psi,psidksi,ipwf] = fbibfe2dq48(e,elpa(e,:));  % fbibfe2dq48.m
    vrs = elpa(e,7);

    for ip=1:neip
      gip = neip*(e-1) + ip;
%----------------------------------------------------------------------

  if vrs==3
     r0   = psi(ip,:)*ec0(:,1);     r    = psi(ip,:)*ec(:,1);
     thk0 = r0*2*pi;                thk  = r*2*pi;
     mF33 = r/r0;                   ax   = 1;
  else
     r0   = 1;                      r    = 1;
     thk0 = eida0(gip,3);           thk  = thk0;
     mF33 = 1;                      ax   = 0;
  end;
```

```
  dpsi(:,1) = psidksi(:,2*ip-1);    dpsi(:,2) = psidksi(:,2*ip);
  jc0  = dpsi' * ec0;   jci0 = inv(jc0);
  jc   = dpsi' * ec;      jci  = inv(jc);
  dt   = det(jc);         dt0  = det(jc0);
  dfie   = zeros(5,2*nenod);
  dpsixy = dpsi * jci' ;
  dfie(1,2*(1:nenod)-1) = dpsixy(1:nenod,1)';
  dfie(2,2*(1:nenod)) = dpsixy(1:nenod,2)';
  dfie(3,2*(1:nenod)-1) = ax.*psi(ip,1:nenod)/r;
  dfie(4,2*(1:nenod)) = dpsixy(1:nenod,1)';
  dfie(5,2*(1:nenod)-1) = dpsixy(1:nenod,2)';
  dfie0   = zeros(5,2*nenod);
  dpsixy0 = dpsi * jci0' ;
  dfie0(1,2*(1:nenod)-1) = dpsixy0(1:nenod,1)';
  dfie0(2,2*(1:nenod))   = dpsixy0(1:nenod,2)';
  dfie0(3,2*(1:nenod)-1) = ax.*psi(ip,1:nenod)/r0;
  dfie0(4,2*(1:nenod))   = dpsixy0(1:nenod,1)';
  dfie0(5,2*(1:nenod)-1) = dpsixy0(1:nenod,2)';
  mF  = eye(3); mF(1:2,1:2)  = jci0*jc;  mF(3,3)  = mF33;  mF  = mF';
%-----------------------------------------------------------------------

      du = dfie0*Tpe;
      [mmM,ccGs,ccGe,vm] = plaxelas1(eidaC(gip,6:7),vrs,du);
      ccGs = ccGs'; ccGe = ccGe';
      thk = thk0 + thk0*ccGe(3);

      eidaC(gip,3)     = thk;
      eidaC(gip,17:20) = ccGe(1:4);
      eidaC(gip,21:24) = ccGs(1:4);
      eidaC(gip,25)    = dt;
      eidaC(gip,90)    = r0;
      eidaC(gip,91)    = r;

      ef = ef + dfie' * ccGs' * thk * dt * ipwf(ip);
    end;

  fi(lokvg(e,1:nedof))                  = fi(lokvg(e,1:nedof)) + ef;
end;
rs = fe - fi;
fi=Trm'*fi; fiT = fi; fiT = Trm * fi; rsT = feT - fiT;

%-----------------------------------------------------------------------
% Reshaping columns into matrices

Mp  = reshape(p,nndof,nnod)';
Mfi = reshape(fi,nndof,nnod)';   Mfe = reshape(fe,nndof,nnod)';
Mrs = reshape(rs,nndof,nnod)';
```

```
if ntr>=1
MpT = reshape(TpT,nndof,nnod)';
MfiT = reshape(fiT,nndof,nnod)'; MfeT = reshape(feT,nndof,nnod)';
MrsT = reshape(rsT,nndof,nnod)';
end;
%----------------------------------------------------------------------

%**********************************************************************
```

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Appendix H

FE program plax

The Matlab program `plax` allows to model and analyze two-dimensional planar and axisymmetric structures, where the deformation can be large and the material nonlinear..

Model geometry, topology (connectivity), geometrical and material parameters, boundary conditions (prescribed displacements and point loads) and link relations (dependecies between degrees of freedom) must be available as input data. Also the history of the prescribed boundary conditions must be specified. When the analysis is finished, output date are stored in the data base and various other data arrays.

In the following section an example input is presented, with explanatory comments. Finally the program itself is explained in more detail.

H.1 Matlab program plax

The program `plax.m` calls a collection of command and function files, which can and should be inspected for through understanding of the procedure. The program structure is however clearly shown in the listing.

```
%*******************************************************************
%  plax : 2-dimensional planar/axisym.; nonlinear
%===================================================================
plaxchkinp;                                          % plaxchkinp.m
fbiblcase;                                            % fbiblcase.m
plaxinizer;                                          % plaxinizer.m
[eida0,eidaB,eidaC,eismB,eismC,elip,neip] = ...      % plaxinidat.m
                                 plaxinidat(ne,elgr,elda,neip,GDt,mm);

if res==0, save([matf num2str(0)]); end;
crdB = crd0; crd = crd0;
%===================================================================
%  Calculate shape functions and their derivatives
%   for a 4-node element              plaxq4.m
%   for a 8-node element              plaxq8.m
%===================================================================
```

```
if     nenod==4, [ksi,psi,psidksi,ipwf,lokvg] = plaxq4(lok,ne,nndof,neip);
elseif nenod==8, [ksi,psi,psidksi,ipwf,lokvg] = plaxq8(lok,ne,nndof,neip);
end;
%======================================================================
%  Incremental calculation
%======================================================================
ic = 1; ti = 0; it = 0; slow = 1;

if res>0                                             % Restart analysis
  ic = res; load([matf num2str(ic-1)]);
  crd = crdB; eidaC = eidaB; it = 1;
end;

while ic<=nic
%----------------------------------------------------------------------
fbibcutback;                                            % fbibcutback.m
ti = ti + GDt; it = 0; loadincr;
pe = peC./slow; fe = feC;
rs = fe - fi;
Dp = zeros(ndof,1); Ip = zeros(ndof,1);

GDt = GDt0/slow;
%======================================================================
%  System matrix is assembled from element matrices
%======================================================================
if (ic==1 | nl==1 | ic==res)
%----------------------------------------------------------------------
sm = zeros(ndof);                    % structural stiffness matrix
elmalivi = 0;                        % counter of lin.viscoel. elements
gip = 0;                             % global integration point number

for e=1:ne
  ety  = elda(elgr(e),1);               mat  = elda(elgr(e),2);
  em   = zeros(nedof);                  ef   = zeros(nedof,1);
  ec0  = crd0(lok(e,3:nenod+2),:);      ecB  = crdB(lok(e,3:nenod+2),:);
  ec   = crd(lok(e,3:nenod+2),:);       Tpe  = Tp(lokvg(e,:));
  vole0  = 0; voleC = 0;
  if mat==8, elmalivi = elmalivi + 1; end;

  plaxelem; % -> ef, em                                    % plaxelem.m

  sm(lokvg(e,:),lokvg(e,:)) = sm(lokvg(e,:),lokvg(e,:)) + em;
end;

if ic==1, sm00=sm; end; sm0=sm;
%----------------------------------------------------------------------
end;
```

```
%=======================================================================
%  Iterative calculation
%=======================================================================
nrm = 1000; it = 1; sm0 = sm;

while (nrm>ccr) & (it<=mit)
%-----------------------------------------------------------------------
%=======================================================================
%  Links and boundary conditions are taken into account
%  Unknown nodal point values are solved
%  Prescribed nodal values are inserted in the solution vector
%=======================================================================
%sm = sm00;                 % only used to test modified Newton-Raphson

if npl>0, rs = rs - sm(:,plc)*lif'; end;
[smp,rsp] = fbibpartit(it,sm,rs,ndof,pa,ppc,plc,prc,pe,lim);% fbibpartit.m

sol = smp\rsp; soll=sol; % dsol = smp\(smp*sol-rsp); soll = sol-dsol;

p = zeros(ndof,1); p(pu) = soll;
if it==1, p(ppc) = pe(ppc); end;
if npl>0, p(plc) = lim*p(prc) + lif'; end;

Dp  = p;  Ip = Ip + Dp;  Tp = Tp + Dp;
crd = crd0 + reshape(Tp,nndof,nnod)';
%=======================================================================
%  Calculate stresses and strains.
%  Make system matrix and internal force vector for next step.
%=======================================================================
sm=zeros(ndof); fi=zeros(ndof,1); elmalivi=0; gip=0;

for e=1:ne
  ety  = elda(elgr(e),1);              mat  = elda(elgr(e),2);
  em   = zeros(nedof);                 ef   = zeros(nedof,1);
  ec0  = crd0(lok(e,3:nenod+2),:);     ecB  = crdB(lok(e,3:nenod+2),:);
  ec   = crd(lok(e,3:nenod+2),:);      Tpe  = Tp(lokvg(e,:));
  vole0  = 0; voleC = 0;
  if mat==8, elmalivi = elmalivi + 1; end;

  plaxelem; % -> ef, em                                    % plaxelem.m

  fi(lokvg(e,:))            = fi(lokvg(e,:)) + ef;
  sm(lokvg(e,:),lokvg(e,:)) = sm(lokvg(e,:),lokvg(e,:)) + em;
end;
%=======================================================================
%  Calculate residual force and convergence norm
%=======================================================================
```

```
rs = fe - fi;
nrm = fbibcnvnrm(cnm,pu,ppc,prs,Dp,Ip,rs,fi);            % fbibcnvnrm.m

it = it + 1;                         % increment the iteration step counter

%plaxipc;                                                     % plaxipc.m

fbibwr2scr;                                                % fbibwr2scr.m
%-----------------------------------------------------------------------
end; %it

%=======================================================================
%  Update and store values
%=======================================================================
fbibcol2mat1;                                            % fbibcol2mat1.m
crdB = crd; feB = fe; eidaB = eidaC; eismB = eismC;
savefile = [matf num2str(ic)]; savedata;                     % savedata.m

ic = ic + 1;                              % increment the increment counter
save([matf '00'],'ic');                               % save date to 'matf'

%-----------------------------------------------------------------------
end; %ic

%***********************************************************************
```