Material Models

Lecture notes - course 4A330

dr.ir. P.J.G. Schreurs

Eindhoven University of Technology
Department of Mechanical Engineering
Materials Technology
August 6, 2012

Contents

Preface

In the course "Mechanica" in the first year, attention has been given to the mechanical behavior of trusses and beams. Deformations were always assumed to be small. The material behavior was linearly elastic and described by a linear relation between stress and strain. Two material parameters were relevant : Young's modulus and Poisson's ratio.

The course "Modelleren van Mechanisch Materiaalgedrag" starts with the three-dimensional deformation of a continuum. This deformation is described by six quantities : three elongational strains and three shear strains. The deformation results in a stress state, which is described by six stress components. A general three-dimensional material model relates the six stresses to the six strains.

To simplify the discussion of various material models, attention is focused on the deformation of a truss, where only the axial stress is relevant. Because material behavior is described by nonlinear relations between axial stress and axial strain and also because deformations may be large, the equations, which describe the deformation are nonlinear. Solution of these equations has to be done with an iterative procedure.

Three different material models will be discussed. First, elastic material behavior is considered, where elastic strains can be large, as is the case for rubber material. Then, attention is focused on elastoplastic behavior, where deformation is permanent after unloading. Finally, viscoelastic material behavior is discussed, which is time- and strain rate dependent.

After the introduction of the material models for elastic, elastoplastic and viscoelastic behavior, these models are used in a finite element program to analyze the mechanical behavior of simple truss structures.

Notation

In many cases it is useful to store scalar quantities or vectors in a column or a matrix. A column with three scalars a_1, a_2, a_3 and its transposed version are denoted as follows :

$$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad ; \qquad \underset{\sim}{a}^T = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

An example of a three-by-two matrix and its transposed version is

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad ; \qquad \underline{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

First and second-order time derivatives of a function $f = f(t)$ (t = time) are denoted as

$$\dot{f} = \frac{df}{dt} \qquad ; \qquad \ddot{f} = \frac{d^2 f}{dt^2}$$

Chapter 1

Introduction

1.1 One-dimensional structural elements

In the course "Mechanica" in the first year, attention was focused on the deformation of one-dimensional structural elements under tensile, bending and torsional loads as is shown in the figure.

Internal and external loads must be in equilibrium. For statically determined structures unknown loads can be determined from equilibrium equations. When internal loads are known, the deformation of each element and thus the deformation of the structure as a whole, can be determined. This can only be done when the material behavior of the elements is known. For statically undetermined structures, the equilibrium equations cannot be solved without taking the material behavior into account.

The element behavior is described by a local equilibrium equation, a differential equation which can be solved with proper boundary conditions.

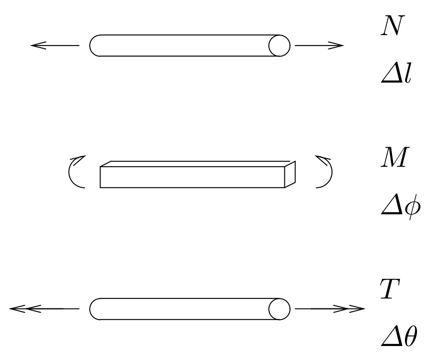


Fig. 1 : *Structural elements*

1.1.1 Tension

A truss with length L has a straight axis along the global x-axis. It is loaded by axial forces N_0, N_L and $q_h(x)$ per unit of length. The loads on a small part of length dx have to satisfy the equilibrium equation.

For linear elastic material behavior, the Young's modulus E relates the axial normal stress σ to the axial strain ε. The linear strain definition is used to relate ε to the axial displacement u. The cross-sectional force $N(x)$ can than be expressed in E, the cross-sectional area A and the strain.

Assuming that E and A are uniform, the equilibrium equation results in a second-order differential equation for u.

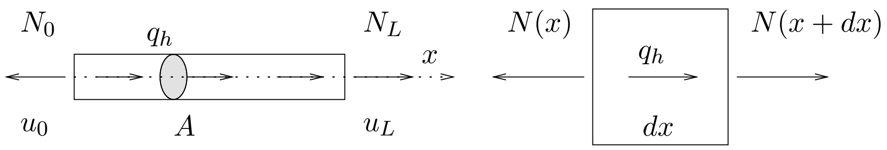


Fig. 2 : *Local equilibrium for tension*

equilibrium equation	$\dfrac{dN}{dx} + q_h = 0 \quad ; \quad N = \displaystyle\int_A \sigma \, dA$	
material behavior	$\sigma = E\varepsilon = E\dfrac{du}{dx} \quad \rightarrow \quad N = EA\dfrac{du}{dx}$	
differential equation	(E, A uniform)	$EA\dfrac{d^2u}{dx^2} + q_h = 0$

1.1.2 Pure bending

A beam with length L has a straight axis along the global x-axis. It is loaded by bending moments M_0, M_L and $q_m(x)$ per unit of length. The loads on a small part of length dx have to satisfy the equilibrium equation.

For linear elastic material behavior, the Young's modulus E relates the axial normal stress σ to the axial strain ε. The linear strain is related to the curvature, the derivative of the rotation ϕ. The cross-sectional moment $M(x)$ can than be expressed in E, the cross-sectional quadratic moment of inertia I and the curvature.

Assuming that E and I are uniform, the equilibrium equation results in a second-order differential equation for ϕ or a third-order differential equation for the normal displacement w.

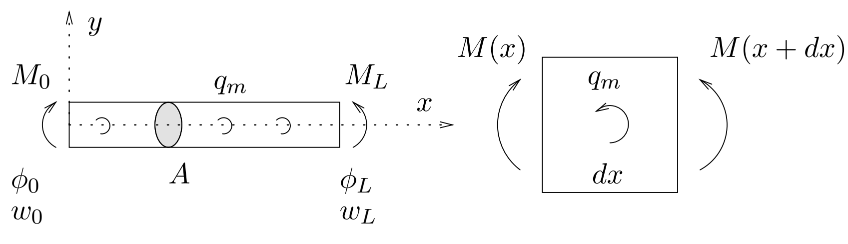


Fig. 3 : *Local equilibrium for pure bending*

equilibrium equation	$\dfrac{dM}{dx} + q_m = 0 \quad ; \quad M = -\displaystyle\int_A \sigma y \, dA$
material behavior	$\sigma = E\varepsilon = E\left(-y\dfrac{d\phi}{dx}\right) \quad \rightarrow \quad M = EI\dfrac{d\phi}{dx}$

differential equation $\quad (E,\, I$ uniform$) \quad EI\,\dfrac{d^2\phi}{dx^2} + q_m = 0 \quad ; \quad \dfrac{d\phi}{dx} = \dfrac{d^2w}{dx^2}$

1.1.3 Bending

A beam with length L has a straight axis along the global x-axis. It is loaded by bending moments M_0, M_L and $q_m(x)$ per unit of length and by shear loads D_0, D_L and $q_v(x)$ per unit of length. The loads on a small part of length dx have to satisfy two equilibrium equations, which can be combined into one.

For linear elastic material behavior, the Young's modulus E relates the axial normal stress σ to the axial strain ε. The linear strain is related to the curvature, the derivative of the rotation ϕ. The cross-sectional moment $M(x)$ can than be expressed in E, the cross-sectional quadratic moment of inertia I and the curvature.

Assuming that E and I are uniform, the equilibrium equation results in a third-order differential equation for ϕ or a fourth-order differential equation for the normal displacement w.

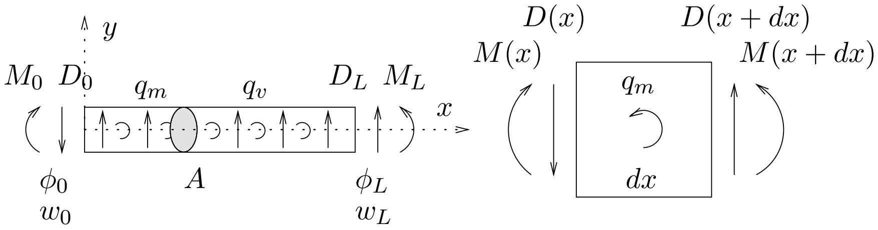


Fig. 4 : *Local equilibrium for bending*

equilibrium equations $\quad \dfrac{dM}{dx} + q_m + D = 0 \quad ; \quad \dfrac{dD}{dx} + q_v = 0$

$$M = -\int_A \sigma y\, dA \quad ; \quad D = \int_A \tau\, dA$$

material behavior $\quad \sigma = E\varepsilon = E\left(-y\,\dfrac{d\phi}{dx}\right) \quad \rightarrow \quad M = EI\,\dfrac{d\phi}{dx}$

differential equation $\quad (E,\, I$ uniform$) \quad EI\,\dfrac{d^3\phi}{dx^3} + \dfrac{dq_m}{dx} - q_v = 0 \quad ; \quad \dfrac{d\phi}{dx} = \dfrac{d^2w}{dx^2}$

1.1.4 Torsion

A torsion bar with length L has a straight axis along the global x-axis. It is loaded by torsional moments T_0, T_L and $q_t(x)$ per unit of length. The loads on a small part of length dx have to satisfy the equilibrium equation.

For linear elastic material behavior, the shear modulus G relates the shear stress τ to the shear strain γ. The shear strain is related to the derivative of the axial rotation θ. The cross-sectional moment $T(x)$ can than be expressed in G, the cross-sectional quadratic moment of inertia K and the derivative of the axial rotation θ.

Assuming that G and K are uniform, the equilibrium equation results in a second-order differential equation for θ.

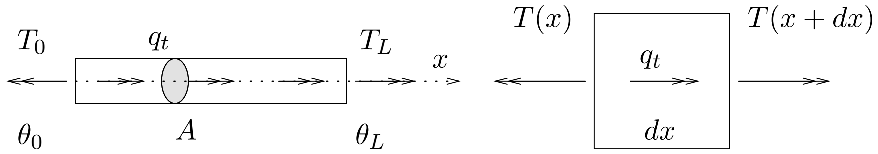


Fig. 5 : *Local equilibrium for torsion*

equilibrium equation $\qquad \dfrac{dT}{dx} + q_t = 0 \quad ; \quad T = \displaystyle\int_A \tau r \, dA$

material behavior $\qquad \tau = G\gamma = G\, r\, \dfrac{d\theta}{dx} \quad \rightarrow \quad T = GK\, \dfrac{d\theta}{dx}$

differential equation $\qquad$ (G, K uniform) $\qquad GK\, \dfrac{d^2\theta}{dx^2} + q_t = 0$

1.2 Three-dimensional, large strain deformation

The equilibrium equations which describe tension, bending and torsion of a one-dimensional bar or beam, can be solved analytically, when proper boundary conditions are specified. It is however obvious that the considered cases are very simple and often not practical: 1) deformations are infinitesimally small, 2) material behavior is linearly elastic, 3) material and geometric parameters are uniform, 4) there is only one relevant degree of freedom and one relevant load. In practice these limitations are almost never found. Most deformations are of a three-dimensional nature, characterizd by more than one deformation variables and loads. These deformations may be very large, eg. in the case of compliant materials and during forming processes. Material behavior is then modelled with nonlinear relations between stresses and strains.

In the next chapter, we will first consider three-dimensional deformation. It will become clear that we may still find reasons to study material behavior and material models in the one-dimensional case of a tensile bar.

Chapter 2

Three-dimensional deformation

2.1 Coordinate systems

To identify a point in a three-dimensional space, we need three independent coordinates.

Cartesian coordinates are distances along three perpendicular coordinate axes passing through a fixed origin (O), which are generally indicated as x, y and z.

Cylindrical coordinates comprise two distances along two perpendicular coordinate axes (r and z) passing through a fixed origin, and one angle θ indicating the rotation of the r-axis about the z-axis w.r.t. a fixed plane.

As the same spatial point can be identified uniquely by either $\{x, y, z\}$ or $\{r, \theta, z\}$, there is a relation between the coordinates systems, which is easily verified to be as follows :

$$x = r\,\cos(\theta) \quad ; \quad y = r\,\sin(\theta) \quad ; \quad z = z$$
$$r = \sqrt{x^2 + y^2} \quad ; \quad \theta = \arctan\left(\frac{y}{x}\right) \quad ; \quad z = z$$

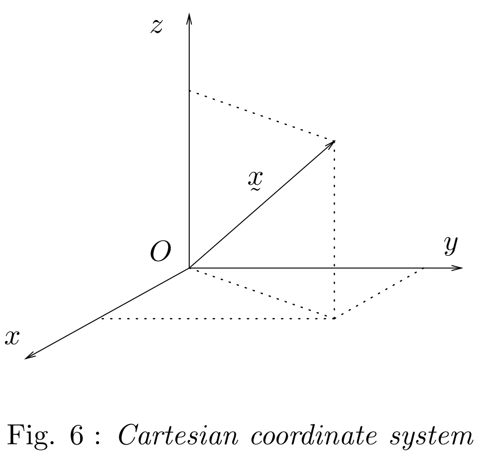


Fig. 6 : *Cartesian coordinate system*

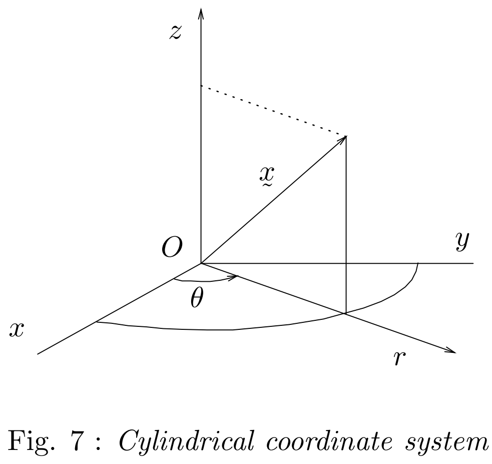


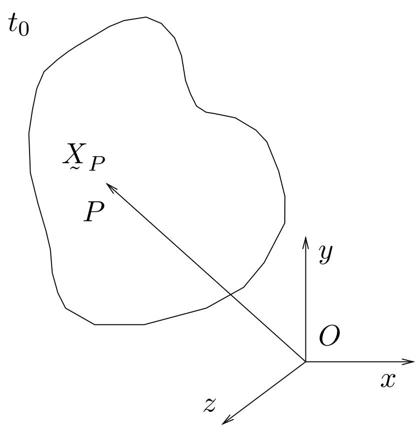
Fig. 7 : *Cylindrical coordinate system*

$$\underset{\sim}{x}^T = [\ x \quad y \quad z\ ] \qquad\qquad \underset{\sim}{x}^T = [\ r \quad \theta \quad z\ ]$$

2.2 Three-dimensional continuum

A material body consisting of a continuous set of material points, is called a continuum. In the initial state, indicated by the time t_0, the body is not subjected to any load and is considered to be undeformed.

A material point P is identified with three Cartesian coordinates, stored in the column $\underset{\sim}{X}_P$. It is of course possible to use other (e.g. cylindrical) coordinates, when this is appropriate.



$$\underset{\sim}{X}_P = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \quad \rightarrow$$

$$\underset{\sim}{X}_P^T = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}$$

Fig. 8 : *Continuum body in undeformed state*

2.3 Deformation of a three-dimensional continuum

When the continuum is loaded by volume and/or surface loads, it will deform. The deformed state is indicated by the time t.

Material point P will get a new spatial position with new coordinates, stored in the column $\underset{\sim}{x}_P$. The displacement $\underset{\sim}{u}_P$ is the difference between the current (or deformed) and initial (or undeformed) coordinates. The time derivative of the displacement is the velocity and the second time derivative the acceleration.

The deformation of the continuum is known when the displacement of each individual point is determined. This displacement also includes rigid body movement, but only relative displacements of material points are relevant for the deformation. This implies that deformation can only be defined locally by studying what happens to a small material volume at a material point P.

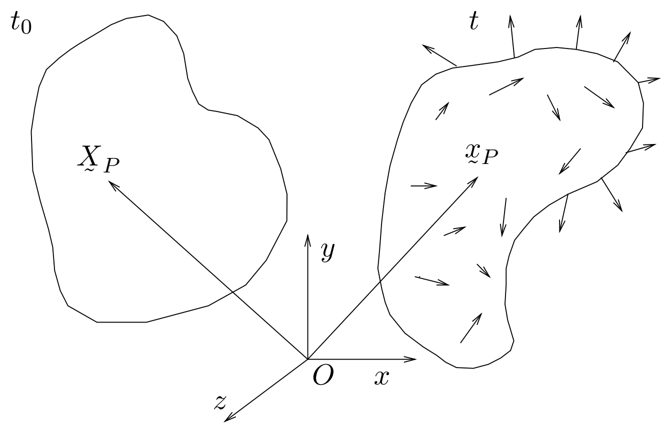


Fig. 9 : *Deformation of continuum body*

initial position point P $\qquad$ $\underset{\sim}{X}_P = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}^T$

current position point P $\qquad$ $\underset{\sim}{x}_P = \begin{bmatrix} x & y & z \end{bmatrix}^T$

displacement point P

$$\underset{\sim}{u}_P = \underset{\sim}{x}_P - \underset{\sim}{X}_P = \begin{bmatrix} (x - x_0) & (y - y_0) & (z - z_0) \end{bmatrix}^T = \begin{bmatrix} u & v & w \end{bmatrix}^T$$

velocity point P $\qquad$ $\dot{\underset{\sim}{u}}_P = \begin{bmatrix} \dot{u} & \dot{v} & \dot{w} \end{bmatrix}$

acceleration point P $\qquad$ $\ddot{\underset{\sim}{u}}_P = \begin{bmatrix} \ddot{u} & \ddot{v} & \ddot{w} \end{bmatrix}$

2.3.1 Local deformation

To characterize the deformation of the material in point P, we consider a small orthogonal volume element at point P. After deformation the material points in this volume will occupy a spatial volume which is not longer orthogonal (generally !). Because the material volume was taken to be infinitesimal small, the new volume can be assumed to have straight sides and planar faces. The total deformation is a combination of elongations of the sides and their rotation.

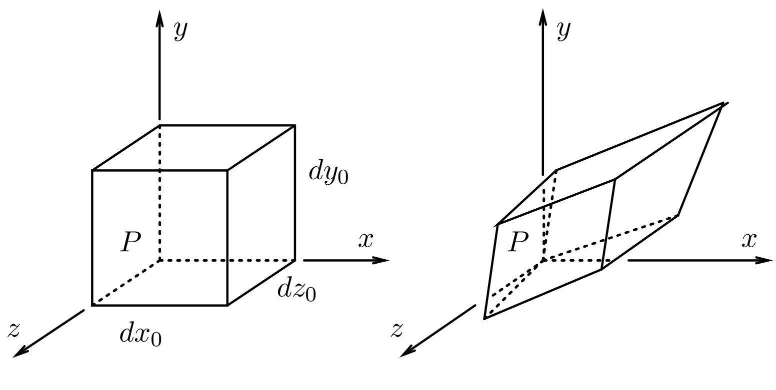


Fig. 10 : *General deformation of a small material volume*

Elongation

To study the elongation in more detail we consider a cylindrical tensile bar, having initial length l_0, which is elongated axially in x-direction the become of a length l. The resulting contraction implies that the cross-sectional area is reduced.

Two neighboring points P and Q are initially in the vertices of a small orthogonal volume, which will deform due to axial elongation and cross-sectional contraction. The elongation or stretch of its sides is characterized by three elongation factors or stretch ratios λ_{xx}, λ_{yy} and λ_{zz}. Because an elongation factor is the ratio of two lengths, it must always be positive. When deformation results in a line element to become longer, the elongation factor is larger than one. When it becomes shorter, its elongation factor is smaller than one.

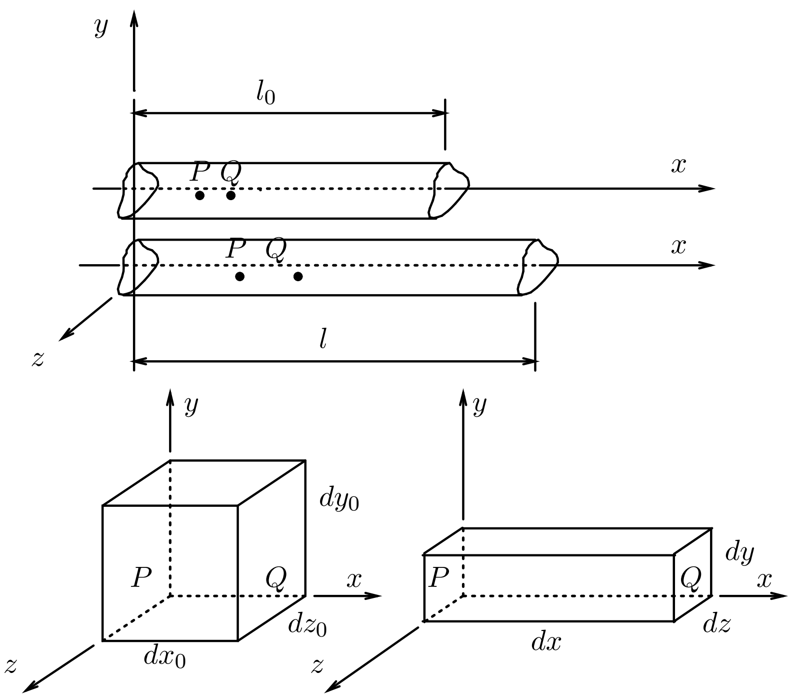


Fig. 11 : *Elongation and contraction of a material volume element in a tensile bar*

$$\lambda_{xx} = \frac{dx}{dx_0} \quad ; \quad \lambda_{yy} = \frac{dy}{dy_0} \quad ; \quad \lambda_{zz} = \frac{dz}{dz_0}$$
$$\lambda > 0 \quad ; \quad \lambda < 1 \quad \vee \quad \lambda > 1$$

Shear

To study the rotation of line elements in more detail we consider a thin-walled tube with cylindrical cross-section and its axis along the global x-axis. One end of the tube is fixed while the other rotates about the x-axis. The rotation angle $\Delta\phi$ is small.

In a point P on the z-axis, an orthogonal material volume is considered before and after deformation. Two sides of the initial volume are perpendicular. After deformation, the angle between two sides, which are initially in x- and y-direction, is θ. The shear is defined as the sine of the change of the angle. When rotations are small, this may be replaced by $\frac{\pi}{2} - \theta$.

In point P the shear is indicated as γ_{xy} because the material line elements are initially orientated in x- and y-direction. In point Q the rotation of the tube leads to the same shear, which is however indicated by γ_{yz}.

In a general three-dimensional deformation we would have three shears, which would also have a different value.

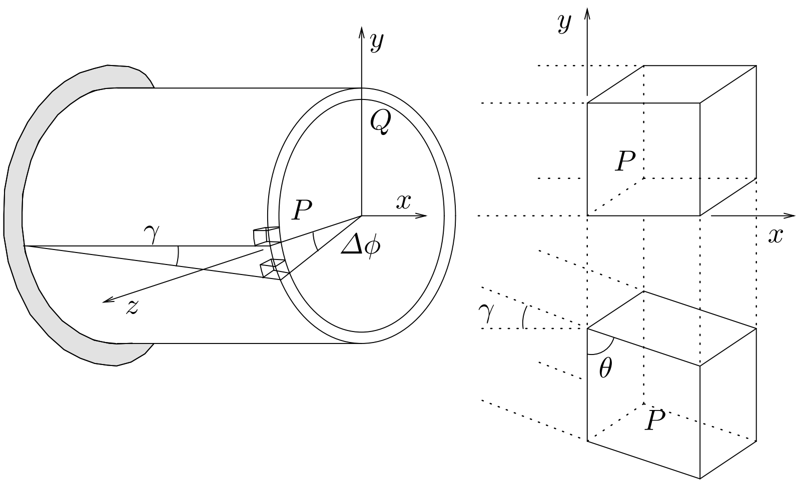


Fig. 12 : *Shear of a small material volume element in the wall of a tube*

$$\gamma_{xy} = \sin\left(\frac{\pi}{2} - \theta_{xy}\right) \quad ; \quad \gamma_{yz} = \sin\left(\frac{\pi}{2} - \theta_{yz}\right) \quad ; \quad \gamma_{zx} = \sin\left(\frac{\pi}{2} - \theta_{zx}\right)$$
$$\gamma_{yx} = \gamma_{xy} \quad ; \quad \gamma_{zy} = \gamma_{yz} \quad ; \quad \gamma_{xz} = \gamma_{zx}$$

Total deformation

The total deformation of a material volume is characterized by three stretch ratios and three shears. These variables can be stored in a column $\underset{\sim}{U}$.

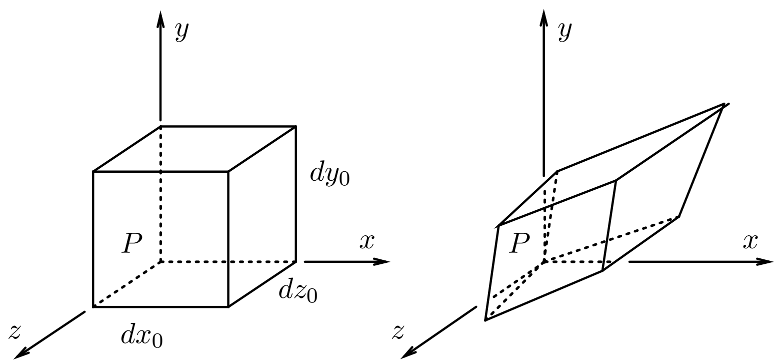


Fig. 13 : *Total deformation is elongation and shear combined*

total deformation $\underset{\sim}{U} = \left[\begin{array}{cccccc} \lambda_{xx} & \lambda_{yy} & \lambda_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{array} \right]^T$

2.3.2 Principal directions of deformation

In each point P there is exactly one orthogonal material volume, which will not show any shear during deformation from t_0 to t. Rigid rotation may occur, although this is not shown in the figure.

The directions $\{1, 2, 3\}$ of the sides of the initial orthogonal volume are called *principal directions* of deformation and associated with them are the three *principal elongation factors* λ_1, λ_2 and λ_3. For this material volume the three principal elongation factors characterize the deformation uniquely. Be aware of the fact that the principal directions change when the deformation proceeds. They are a function of the time t.

The relative volume change J is the product of the three principal elongation factors. For incompressible material there is no volume change, so the above product will have value one.

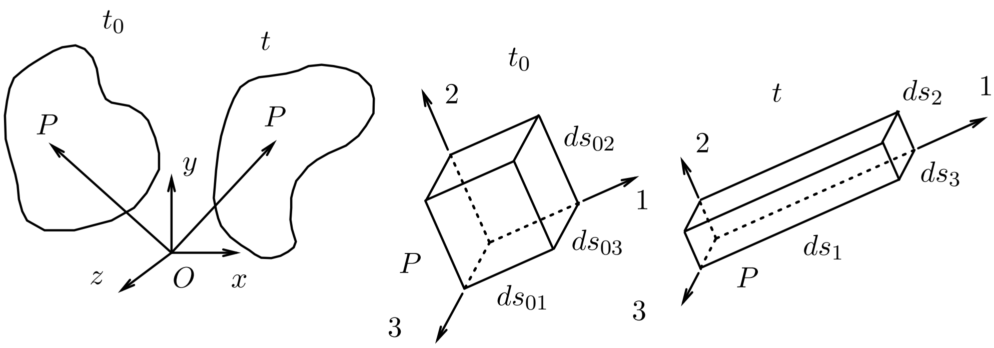


Fig. 14 : *Deformation of material cube with sides in principal directions*

$$\lambda_1 = \frac{ds_1}{ds_{01}} \quad ; \quad \lambda_2 = \frac{ds_2}{ds_{02}} \quad ; \quad \lambda_3 = \frac{ds_3}{ds_{03}}$$
$$\gamma_{12} = \gamma_{23} = \gamma_{31} = 0$$

principal deformation $\underset{\sim}{U} = \left[\begin{array}{cccccc} \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \end{array} \right]^T$

volume change $\dfrac{dV}{dV_0} = \dfrac{ds_1 ds_2 ds_3}{ds_{01} ds_{02} ds_{03}} = \lambda_1 \lambda_2 \lambda_3 = J$

if incompressible $\lambda_1 \lambda_2 \lambda_3 = 1$

2.3.3 Strains in principal directions

It is common practice to describe elongational deformation of a material line element by a strain ε, which is a function of the elongation factor $\lambda : \varepsilon = f(\lambda)$. Several strain definitions exist of which the linear, the logarithmic and the Green-Lagrange strain are the most used in solid mechanics. As follows immediately from all definitions, the strain is zero when there is no elongation and all strain definitions result in linear strain when elongation is very small :

$$f(\lambda = 1) = 0 \qquad ; \qquad \lim_{\lambda \longrightarrow 1} f(\lambda) = \lambda - 1$$

linear strains

$$\underset{\sim}{\mathcal{E}} = \left[\begin{array}{cccccc} \varepsilon_{\ell_1} & \varepsilon_{\ell_2} & \varepsilon_{\ell_3} & 0 & 0 & 0 \end{array} \right]^T = \left[\begin{array}{cccccc} \lambda_1 - 1 & \lambda_2 - 1 & \lambda_3 - 1 & 0 & 0 & 0 \end{array} \right]^T$$

logarithmic strains

$$\underset{\sim}{\mathit{\Lambda}} = \left[\begin{array}{cccccc} \varepsilon_{ln_1} & \varepsilon_{ln_2} & \varepsilon_{ln_3} & 0 & 0 & 0 \end{array} \right]^T = \left[\begin{array}{cccccc} \ln(\lambda_1) & \ln(\lambda_2) & \ln(\lambda_3) & 0 & 0 & 0 \end{array} \right]^T$$

Green-Lagrange strains

$$\underset{\sim}{E} = \left[\begin{array}{cccccc} \varepsilon_{gl_1} & \varepsilon_{gl_2} & \varepsilon_{gl_3} & 0 & 0 & 0 \end{array} \right]^T = \left[\begin{array}{cccccc} \frac{1}{2}(\lambda_1^2 - 1) & \frac{1}{2}(\lambda_2^2 - 1) & \frac{1}{2}(\lambda_3^2 - 1) & 0 & 0 & 0 \end{array} \right]^T$$

2.3.4 Deformation and strain rate

Deformation takes place in the time interval $\tau \in [t_0, t]$, where t_0 indicates the initial state and t the current state. For some materials the deformation rate is important for their behavior. It can be determined by time-derivation of $\underset{\sim}{U}$. Also the strain rate can be determined accordingly for the various strain definitions.

general deformation rate $\qquad \dot{\underset{\sim}{U}} = \left[\begin{array}{cccccc} \dot{\lambda}_{xx} & \dot{\lambda}_{yy} & \dot{\lambda}_{zz} & \dot{\gamma}_{xy} & \dot{\gamma}_{xy} & \dot{\gamma}_{xy} \end{array} \right]$

deformation rate in principal directions $\qquad \dot{\underset{\sim}{U}} = \left[\begin{array}{cccccc} \dot{\lambda}_1 & \dot{\lambda}_2 & \dot{\lambda}_3 & 0 & 0 & 0 \end{array} \right]^T$

strain rates in principal directions

$$\begin{aligned}
\dot{\underset{\sim}{\mathcal{E}}} &= \left[\begin{array}{cccccc} \dot{\lambda}_1 & \dot{\lambda}_2 & \dot{\lambda}_3 & 0 & 0 & 0 \end{array} \right]^T \\
\dot{\underset{\sim}{\mathit{\Lambda}}} &= \left[\begin{array}{cccccc} \dot{\lambda}_1/\lambda_1 & \dot{\lambda}_2/\lambda_2 & \dot{\lambda}_3/\lambda_3 & 0 & 0 & 0 \end{array} \right]^T \\
\dot{\underset{\sim}{E}} &= \left[\begin{array}{cccccc} \dot{\lambda}_1\lambda_1 & \dot{\lambda}_2\lambda_2 & \dot{\lambda}_3\lambda_3 & 0 & 0 & 0 \end{array} \right]^T
\end{aligned}$$

2.4 Stresses in a three-dimensional continuum

Deformation of the material will result in stresses, which, when too high, will cause damage.

Stresses cannot be measured directly. They can sometimes be determined by solving equilibrium equations with proper boundary conditions. Mostly, however, they are determined from the strains, which can be measured in various ways. Of course the relation between stresses and strains must then be known quantitatively. This relation is characteristic for the material under the deformation circumstances and is called the material model. The mathematical expressions for such a material model are also referred to as constitutive equations.

First we will introduce stresses.

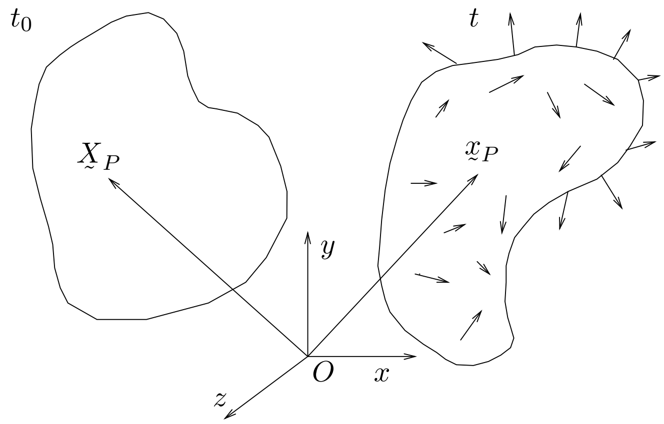


Fig. 15 : *Deformation of continuum body under external load*

Axial stress

To introduce stresses, we consider first the rather simple deformation of a tensile bar, which is subjected to an axial load. The load N in a cross-section (area A in the deformed state) is the integral of the axial stress σ over A. For homogeneous material the stress is uniform in the cross-section and is called the *true* or *Cauchy stress*. When it is assumed to be uniform in the cross-section, it is the ratio of N and A. The *engineering stress* is the ratio of N and the initial cross-sectional area A_0, which makes calculation easy, because A does not have to be known. For small deformations it is obvious that $A \approx A_0$ and thus that $\sigma \approx \sigma_n$.

The axial stress σ is visualized as the stress on the faces of a *stress cube* in a material point P. To indicate that the stress acts on the face with the normal in x-direction, the indices $(\ )_{xx}$ are used.

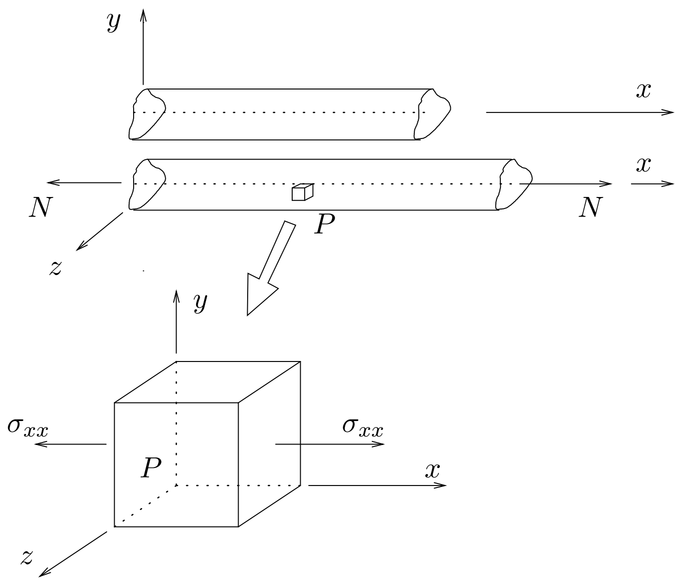


Fig. 16 : *Stresses on a small material volume in a tensile bar*

cross-sectional area (contraction !)	$A_0 \quad \rightarrow \quad A$
axial tensile force	$N = \int_A \sigma \, dA = \sigma A$
true or Cauchy stress	$\sigma = \dfrac{N}{A} = \sigma_{xx}$
engineering stress	$\sigma_n = \dfrac{N}{A_0}$

Hydrostatic stress

A hydrostatic loading of the material body results in a hydrostatic stress state in each material point P. This can again be indicated by stresses (either tensile or compressive) on a stress cube. The three stress variables, with the same value, are normal to the faces of the stress cube. They are indicated with different indices, which represent the normal direction of the associated face.

For isotropic material, such a hydrostatic stress will result in a mere volume change without a shape change.

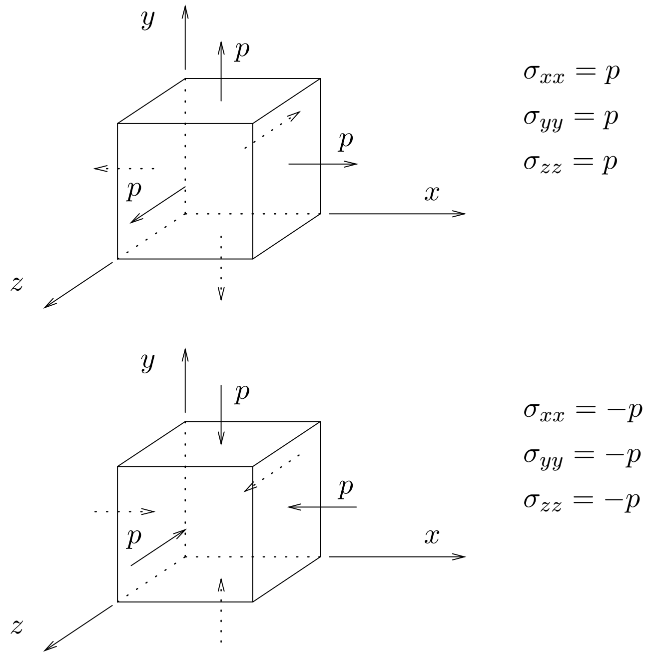


Fig. 17 : *Stresses on a material volume under hydrostatic loading*

Shear stress

The axial torsion of a thin-walled tube (radius R, wall thickness t) is the result of an axial torsional moment (torque) T. This load causes a shear stress τ in the cross-sectional wall. Although this shear stress has the same value in each point of the cross-section, the stress cube looks differently in each point because of the circumferential direction of τ. In points with $y = 0$ or $z = 0$, the stresses are denoted in such a way that the first index indicates the direction of the shear stress and the second index indicates the normal direction (here x).

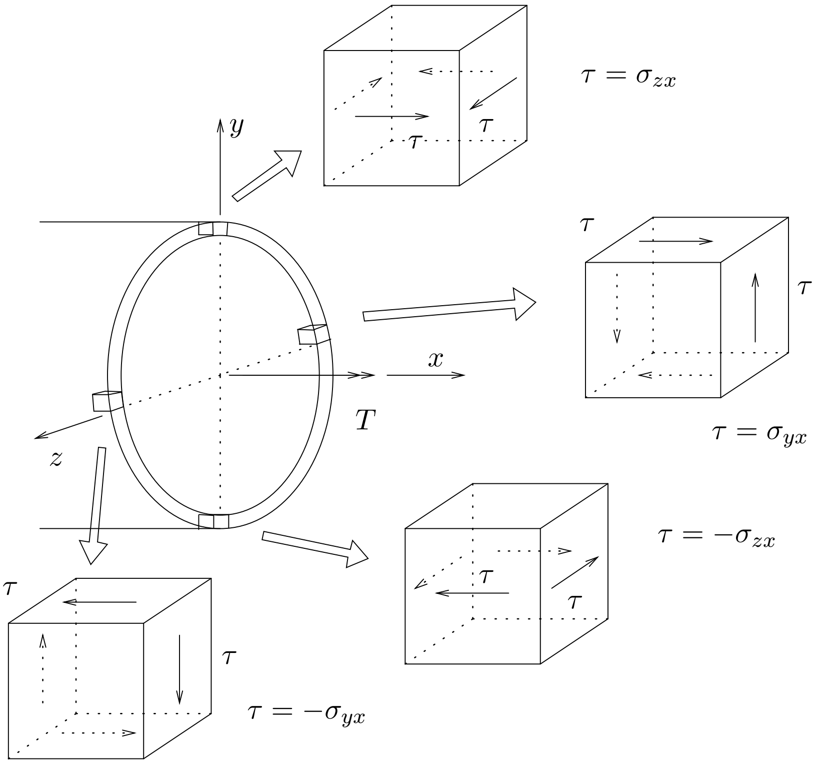


Fig. 18 : *Stresses on a small material volume in the wall of a tube under shear loading*

shear stress in cross section $\qquad \tau = \dfrac{T}{2\pi R^2 t}$

2.4.1 Stress cube

A general three-dimensional deformation will result in a three-dimensional stress state in a material point. Stresses can be indicated as stress vectors on the faces of a stress cube, which has its sides in the direction of the coordinate axes. On each face of the stress cube there is one normal stress and two shear stresses, indicated with proper indices. The 9 stress components are stored in a *stress matrix* $\underline{\sigma}$. Equilibrium of moments will learn that the stress matrix is symmetric. The 6 independent stress components can be stored in a *stress column* $\underset{\sim}{\sigma}$.

Stress cube : Cartesian

In a Cartesian coordinate system the stress components are indicated with indices $\{x, y, z\}$.

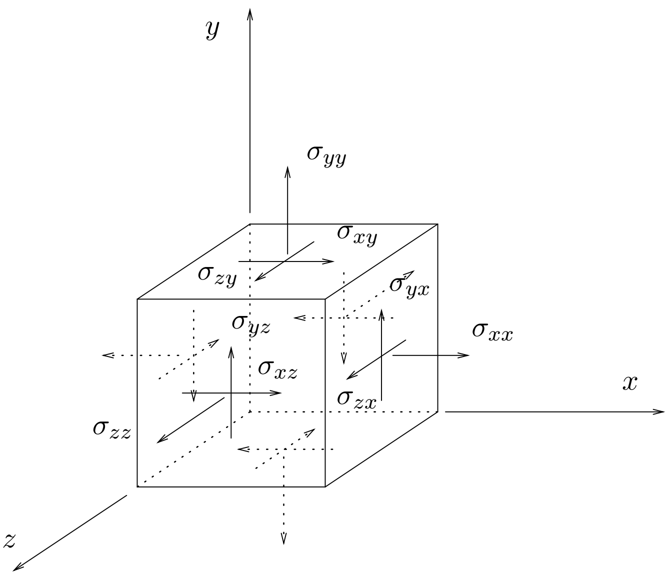


Fig. 19 : *Cartesian stress cube*

stress matrix $\quad \underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \underline{\sigma}^T$

stress column $\quad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{bmatrix}^T$

Stress cube : cylindrical

When a cylindrical coordinate system is used, the stress components are indicated with indices r, t (for θ) and z. Also, instead of a stress cube a stress "wedge" is used.

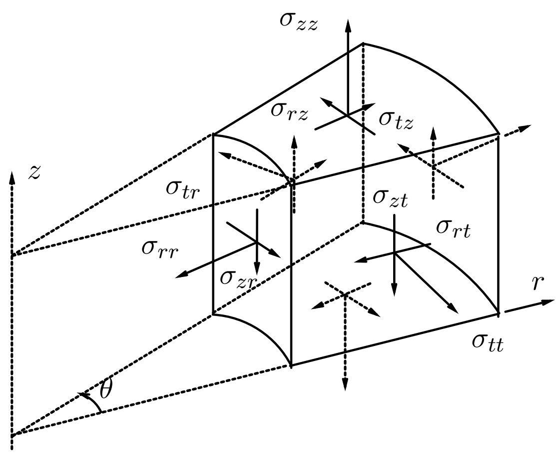


Fig. 20 : *Cylindrical stress cube*

stress matrix $\quad \underline{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{rt} & \sigma_{rz} \\ \sigma_{tr} & \sigma_{tt} & \sigma_{tz} \\ \sigma_{zr} & \sigma_{zt} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{rr} & \sigma_{tr} & \sigma_{zr} \\ \sigma_{rt} & \sigma_{tt} & \sigma_{zt} \\ \sigma_{rz} & \sigma_{tz} & \sigma_{zz} \end{bmatrix} = \underline{\sigma}^T$

stress column $\quad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{tt} & \sigma_{zz} & \sigma_{rt} & \sigma_{tz} & \sigma_{zr} \end{bmatrix}^T$

2.4.2 Principal stresses and directions

The general three-dimensional stress state is characterized by 6 independent stress components, which can be stored in a stress matrix or a stress column, and can be visualized as stresses on a stress cube.

In each point of the material there is one unique and special stress cube, which has on its faces only normal stresses. It is called the *principal stress cube* and the three normal stresses are the *principal stresses*. The axes $\{1, 2, 3\}$ along the sides of the principal stress cube are called the *principal stress directions*.

It can be demonstrated that the principal stresses are the eigenvalues of the global stress matrix and the principal stress directions are the associated eigenvectors.

The stress state in a material point is characterized completely by the three principal stresses.

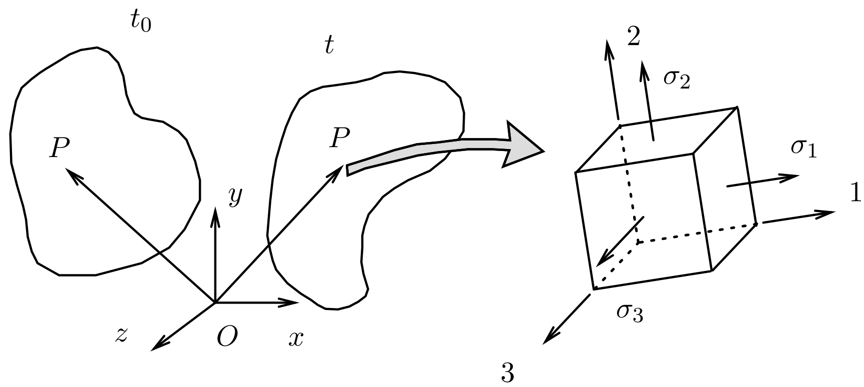


Fig. 21 : *Principal stress cube with principal stresses*

principal stress matrix $\quad \underline{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

principal stress column $\quad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & 0 & 0 & 0 \end{bmatrix}^T$

2.5 Material behavior

The relation between stress and strain depends mainly on the material characteristics, but also on loading conditions and environmental circumstances. The current stress components $\underset{\sim}{\sigma}(t)$ generally depend on the total history of stress, strain and strain rate. The relationship has to be studied experimentally and written in mathematical form. The parameters in the resulting material model must be determined by (sometimes sophisticated) experiments.

When the stress depends linearly of the strain, the 6 stress components are related to the 6 strain components by a 6×6 material stiffness matrix $\underline{C}$. Its inverse is the material compliance matrix $\underline{S}$.

Isotropic material behavior, where the relation between stress and strain is independent of the direction in the material, is completely characterized by two material variables.

constitutive equations $\quad \underset{\sim}{\sigma}(t) = \underset{\sim}{f}\left(\underset{\sim}{\sigma}(\tau), \underset{\sim}{\varepsilon}(\tau), \dot{\underset{\sim}{\varepsilon}}(\tau) \quad \middle| \quad \tau \leq t\right)$

linear behavior $\quad \underset{\sim}{\sigma} = \underline{C}\,\underset{\sim}{\varepsilon} \quad \rightarrow \quad \underset{\sim}{\varepsilon} = \underline{C}^{-1}\underset{\sim}{\sigma} = \underline{S}\,\underset{\sim}{\sigma}$

isotropic linear material

$$\underline{C} = \begin{bmatrix} A & B & B & 0 & 0 & 0 \\ B & A & B & 0 & 0 & 0 \\ B & B & A & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{A-B}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{A-B}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{A-B}{2} \end{bmatrix} \quad ; \quad \underline{S} = \begin{bmatrix} a & b & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 & 0 \\ b & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(a-b) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(a-b) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(a-b) \end{bmatrix}$$

2.5.1 Isotropic material behavior in principal directions

For isotropic material, the principal strain directions coincide with the principal stress directions. (Rigid body rotation is eliminated.) The three principal strains and strain rates are related to the three principal stresses.

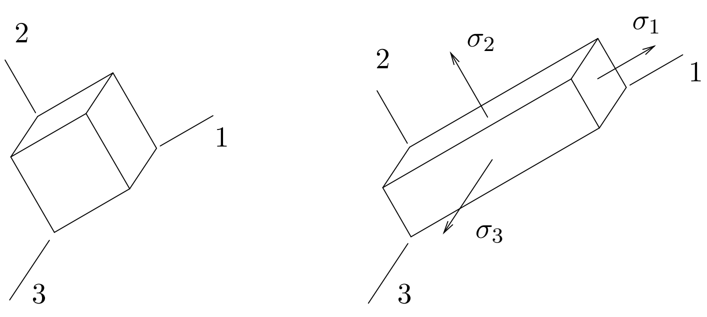


Fig. 22 : *Deformation in principal directions*

$$\underset{\sim}{\sigma}(t) = \begin{bmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \sigma_3(t) \end{bmatrix} = \underset{\sim}{f}\left(\begin{bmatrix} \sigma_1(\tau) \\ \sigma_2(\tau) \\ \sigma_3(\tau) \end{bmatrix}, \begin{bmatrix} \varepsilon_1(\tau) \\ \varepsilon_2(\tau) \\ \varepsilon_3(\tau) \end{bmatrix}, \begin{bmatrix} \dot{\varepsilon}_1(\tau) \\ \dot{\varepsilon}_2(\tau) \\ \dot{\varepsilon}_3(\tau) \end{bmatrix} \quad \middle| \quad \tau \leq t \right)$$

2.5.2 One-dimensional material behavior

For a tensile bar there is only one non-zero principal stress $\sigma_1 = \sigma$. The axial stress is a history dependent function of the axial strain ε and the axial strain rate $\dot{\varepsilon}$. For isotropic material the contraction strains ε_2 and ε_3 are identical, both indicated as ε_d and related to the axial strain ε.

The axial loading of a tensile bar will reveal the most importatnt aspects of material behavior and the material model, which describes it. Because only one stress variable is involved, the mathematics of the model is rather simple. Moreover, for al kind of three-dimensional material behavior, modelling is done in terms of equivalent variables, which are scalar representations of three-dimensional strain and stress states.

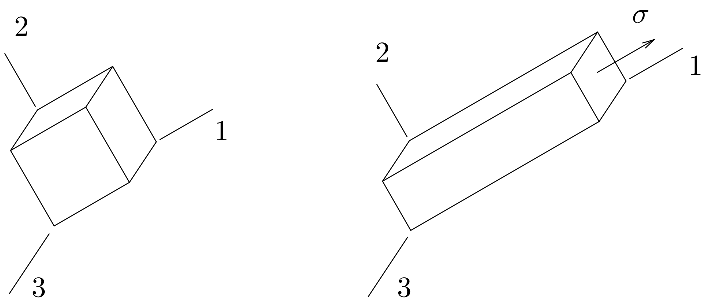


Fig. 23 : *Deformation of a tensile bar*

tensile bar	$\Rightarrow$	$\sigma_2 = \sigma_3 = 0$	;	$\sigma_1 = \sigma$
		$\varepsilon_2 = \varepsilon_3 = \varepsilon_d$	;	$\varepsilon_1 = \varepsilon$
one-dimensional material model		$\sigma(t) = f\left(\sigma(\tau), \varepsilon(\tau), \dot{\varepsilon}(\tau) \quad \vert \quad \tau \leq t\right)$		

Chapter 3

Truss structures

A truss is a mechanical element whose dimension in one direction – the truss axis – is much larger than the dimensions in each direction perpendicular to the axis. A truss structure is an assembly of trusses, which are connected mutually and to the suroundings with hinges. The truss can transfer only axial forces along its axis, so bending is not possible, and the axis must be and remain straight.

In this chapter, we first consider small elongation and rotation of a truss. The material behaves linearly elastic and the resulting equilibrium equation is linear.

Large elongations and rotations lead to a set of nonlinear equations. Moreover, the material behavior is likely to be nonlinear as well. Solution of the set of equations must be done iteratively.

3.1 Homogeneous truss

We consider a truss to be oriented with its axis along the global x-axis. Its undeformed length is l_0. The undeformed cross-sectional area has a uniform value A_0. It is assumed that the material of the truss is isotropic and homogeneous.

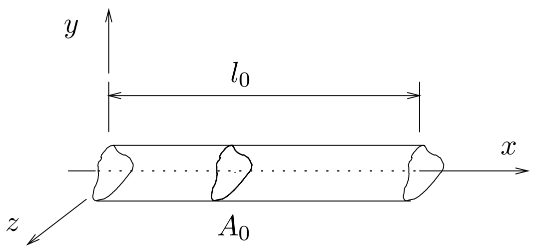


Fig. 24 : *Homogeneous truss*

3.1.1 Elongation and contraction

In the deformed state the length of the truss is l and its cross-sectional area is A. The elongation is described by the axial elongation factor λ. The change in cross-sectional area is described by the contraction μ. It is assumed that the load, which provokes the deformation, is such that the deformation is homogeneous. This means that λ and μ are the same in each point of the truss. The volume change is described by the volume ratio J.

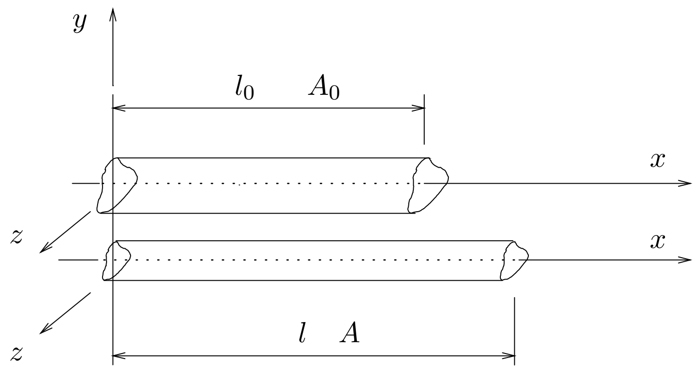


Fig. 25 : *Deformation of a homogeneous truss*

elongation factor	$\lambda = \dfrac{l}{l_0} = \dfrac{l_0 + \Delta l}{l_0} = 1 + \dfrac{\Delta l}{l_0}$
contraction	$\mu = \sqrt{\dfrac{A}{A_0}}$
volume change	$J = \dfrac{lA}{l_0 A_0} = \lambda \mu^2$

3.1.2 Stress

The deformation of the truss is caused by an external axial force N. In each cross-section (x) of the truss, an internal axial force $N(x)$ exists. With no volume load, the cross-sectional load will be the same in each cross-section. If a volume load is applied, this is not the case, but we will not consider such loading here.

The axial load is such that it causes only axial deformation and no bending. In the absence of a volume load the deformation will be homogeneous.

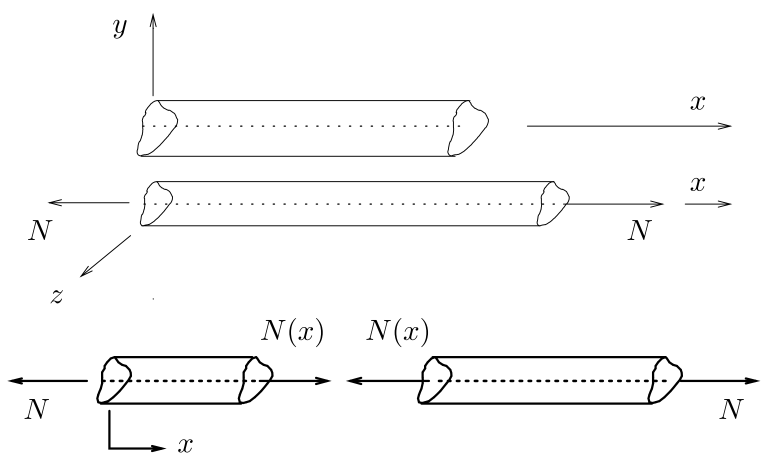


Fig. 26 : *Axial loading of a homogeneous truss*

$$N(x) = N$$

The cross-sectional force is the result of the axial cross-sectional stress. For a homogeneous material with no volume loads, the stress is uniform over the cross-section. This leads to the definition of *true stress*, being the axial force divided by the deformed (= real) cross-sectional area. In many (engineering) applications the *engineering* or *nominal stress* is used, defined as the ratio of the axial force and the undeformed cross-sectional area. True stress and engineering stress, are related by the contraction μ.

In literature a truss is sometimes called a *tie* when it carries a tensile force and a *strut* when it is loaded in compression.

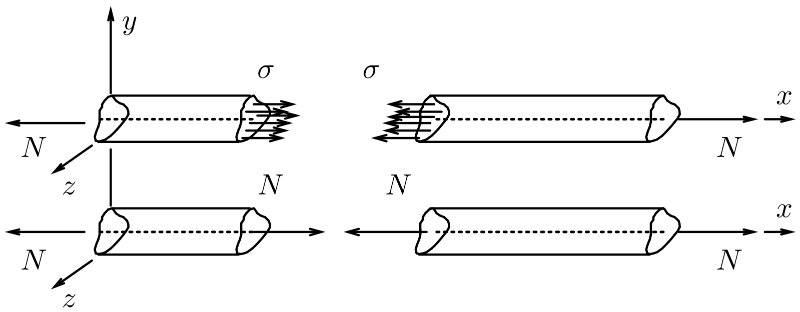


Fig. 27 : *Cross-sectional stress in an axially loaded homogeneous truss*

axial stress	$\sigma = \sigma(y, z)$
cross-sectional force	$N(x) = N = \int_A \sigma(y, z)\, dA$
stress uniform in cross-section	$N = \int_A \sigma\, dA = \sigma A$
true stress	$\sigma = \frac{N}{A}$
engineering stress	$\sigma_n = \frac{N}{A_0}$
relation	$\sigma = \frac{N}{A} = \frac{A_0}{A}\frac{N}{A_0} = \frac{1}{\mu^2}\sigma_n$

3.2 Linear deformation

When the elongation of the truss is very small, the contraction is even smaller so that the deformed cross-sectional area can be taken to be equal to the initial cross-sectional area. Consequently there is no difference between the true stress and the engineering stress.

3.2.1 Linear strain

Elongation is generally described by the strain ε. For small elongation and rotation, the linear strain is used. For the elongation, this strain is related to the elongation facor λ and for the contraction to the contraction μ.

linear strain	$\varepsilon = \varepsilon_l = \lambda - 1$
contractive linear strain	$\varepsilon_d = \mu - 1$

3.2.2 Linear elastic behavior

The linear elastic material behavior is characterized by two matarial constants: Young's modulus and Poisson's ratio. Young's modulus relates the axial stress to the axial strain. Poisson's ratio relates the contractive strain to the axial strain. For most materials Poisson's ratio is about 0.3. For small elongations this value is constant. For small deformation the volume change factor J can be expressed in the linear strain. For incompressible material $J = 1$ implying $\nu = \frac{1}{2}$.

axial stress $\sim$ strain	$\sigma = E\varepsilon$
contraction strain	$\varepsilon = \lambda - 1 \quad \rightarrow \quad \varepsilon_d = \mu - 1 = -\nu\varepsilon = -\nu(\lambda - 1)$
volume change	$J = (\varepsilon + 1)(-\nu\varepsilon + 1)^2 \approx \varepsilon(1 - 2\nu) + 1$

3.2.3 Equilibrium

We consider a truss with length l_0 and cross-sectional area A_0 with its axis along the global x-axis. One end ($x = 0$) is fixed and the other ($x = l_0$) can be displaced in x-direction only. The elongation of the truss equals this displacement u. The displacement is caused by an *external axial force* f_e. In the deformed state the length of the truss is $l = l_0 + u$ and its cross-sectional area is A. The material of the truss is homogeneous.

When the external axial force f_e is prescribed, the elongation $\Delta l = u$ of the truss can be determined by solving the equilibrium equation in point P, which states that the internal force must be equal to the external force. The *internal force* f_i is a function of the elongation, a relation which is determined by the material behavior. It represents the resistance of the truss against elongation.

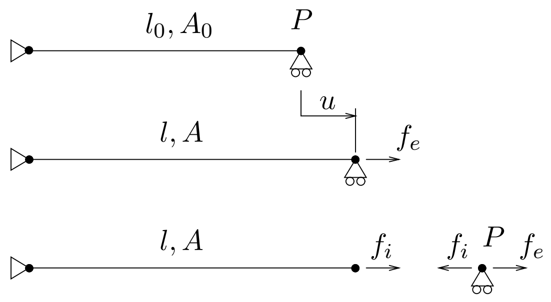


Fig. 28 : *Equilibrium of external and internal axial force*

external force	f_e
internal force	$f_i = f_i(u)$
equilibrium of point P	$f_i(u) = f_e$

When the deformation (= elongation) is very small, there is virtually no difference between the undeformed and the deformed geometry. Such deformation is referred to as being *geometrically linear*. The true axial stress $\sigma = N/A$ approximately equals the engineering stress $\sigma_n = N/A_0$, where N is the axial force.

When, moreover, the material behavior is not influenced by the deformation, as is the case for linear elastic behavior, – this is referred to as *physical linearity* – the total deformation is linear and the internal force f_i can be linearly related to the displacement u.

The equilibrium equation can be solved directly, yielding the displacement u.

3.2.4 Solution procedure

Because the relation between the external force f_e and the axial displacement u is linear, the latter can be solved directly from the equilibrium equation $f_i = f_e$, yielding the exact solution $u_{exact} = u_s$. The stiffness K of the truss depends on the Young's modulus E and on the initial geometry (A_0 and l_0).

$$
\boxed{
\begin{array}{l}
f_i = \sigma_n A_0 = E\varepsilon A_0 = \dfrac{EA_0}{l_0} u = Ku \\[2ex]
f_i = f_e \quad \rightarrow \quad Ku = f_e \quad \rightarrow \quad u = u_s = \dfrac{f_e}{K} = \dfrac{l_0}{EA_0} f_e
\end{array}
}
$$

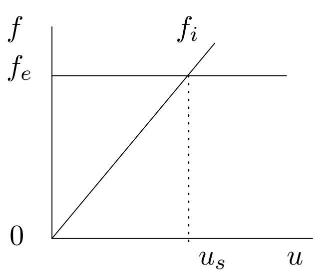


Fig. 29 : *Solution of linear equilibrium equation*

Proportionality and superposition

Two important characteristics hold for linear problems :

- the deformation is *proportional* to the load : when the external force f_e is multiplied by a factor, say α, the elongation u is also multiplied by α.

- *superposition* holds : when we determine the elongation u_1 and u_2 for two separate forces, f_{e1} and f_{e2} respectively, the elongation for the combined loading $f_{e1} + f_{e2}$ is the sum of the separate elongations : $u_1 + u_2$.

3.3 Nonlinear deformation

When deformation and/or rotation of the truss are large, various strains and stresses can be defined and related by material laws. The material behavior can be expected to be no longer linearly elastic.

3.3.1 Strains for large elongation

The deformation of the truss can be characterized uniquely by the two elongation factors λ and μ. However, it is common and useful to introduce deformation variables which are a function of the elongation factors : the strains. A wide variety of strain definitions is possible and used.

All strain definitions must obey some requirements, one of which is that they have to result in the same value for small elongations, being the value of the linear strain. When we plot the various strains as a function of the elongation factor, it is immediately clear that the strains, which are defined here, obey this requirement.

It is obvious that one and the same strain definition must be used throughout the same specimen and analysis. This implies that the contraction strain is defined analogously to the elongational strain. These strains are related by a material parameter, the Poisson's ratio ν. It is assumed, until stated otherwise, that this parameter is constant.

linear strain	$\varepsilon = \varepsilon_l = \lambda - 1$
logarithmic strain	$\varepsilon = \varepsilon_{ln} = \ln(\lambda)$
Green-Lagrange strain	$\varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$

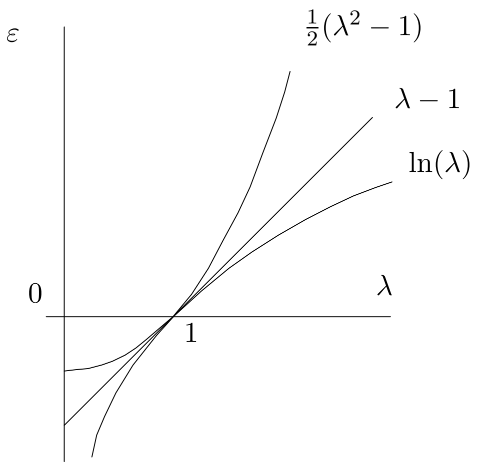


Fig. 30 : *Three strain definitions as function of the elongation factor*

Linear strain

The linear strain definition results in unrealistic contraction, when the elongation is too large. The cross-sectional area of the truss can become zero, which is of course not possible.

linear strain	$\varepsilon = \varepsilon_l = \lambda - 1 = \dfrac{\Delta l}{l_0}$
contraction strain	$\varepsilon_d = \mu - 1 = -\nu\varepsilon_l = -\nu(\lambda - 1)$

change of cross-sectional area

$$\mu = \sqrt{\frac{A}{A_0}} = 1 - \nu(\lambda - 1) \quad \rightarrow \quad A = A_0\{1 - \nu(\lambda - 1)\}^2$$

restriction of elongation

$$1 - \nu(\lambda - 1) > 0 \quad \rightarrow \quad \lambda - 1 < \frac{1}{\nu} \quad \rightarrow \quad \lambda < \frac{1 + \nu}{\nu}$$

Logarithmic strain

The logarithmic strain definition does not lead to unrealistic values for the contraction. Therefore it is very suitable to describe large deformations. [1]

logarithmic strain	$\varepsilon = \varepsilon_{ln} = \ln(\lambda)$

---

[1] $\ln x = {}^e\log(x) = y \quad \rightarrow \quad x = e^y$

contraction strain $\qquad \varepsilon_d = \ln(\mu) = -\nu \varepsilon_{ln} = -\nu \ln \lambda$

change of cross-sectional area

$$\mu = \sqrt{\frac{A}{A_0}} = e^{-\nu \varepsilon_{ln}} = e^{-\nu \ln(\lambda)} = \left[e^{\ln(\lambda)}\right]^{-\nu} = \lambda^{-\nu} \qquad \rightarrow \qquad A = A_0 \lambda^{-2\nu}$$

A deformation process may be executed in a number of steps, as is often done in forming processes. The start of a new step can be taken to be the reference state to calculate current strains. In that case the logarithmic strain is favorably used, because the subsequent strains can be added to determine the total strain w.r.t. the initial state.

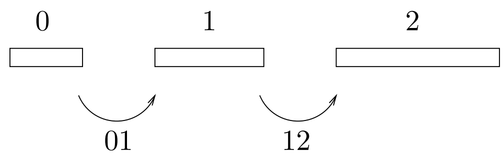


Fig. 31 : *Two-step deformation process*

$l_0 \rightarrow l_1 \qquad \varepsilon_l(01) = \frac{l_1 - l_0}{l_0}$

$\qquad\qquad \varepsilon_{ln}(01) = \ln(\frac{l_1}{l_0})$

$l_1 \rightarrow l_2 \qquad \varepsilon_l(12) = \frac{l_2 - l_1}{l_1}$

$\qquad\qquad \varepsilon_{ln}(12) = \ln(\frac{l_2}{l_1})$

$l_0 \rightarrow l_2 \qquad \varepsilon_l(02) = \frac{l_2 - l_0}{l_0} \neq \varepsilon_l(01) + \varepsilon_l(12)$

$\qquad\qquad \varepsilon_{ln}(02) = \ln(\frac{l_2}{l_0}) = \ln(\frac{l_2}{l_1}\frac{l_1}{l_0}) = \varepsilon_{ln}(01) + \varepsilon_{ln}(12)$

Green-Lagrange strain

Using the Green-Lagrange strain leads again to restrictions on the elongation to prevent the cross-sectional area to become zero.

Green-Lagrange strain $\qquad \varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$

contraction strain $\qquad \varepsilon_d = \frac{1}{2}(\mu^2 - 1) = -\nu \varepsilon_{ln} = -\nu \frac{1}{2}(\lambda^2 - 1)$

change of cross-sectional area

$$1 - \nu(\lambda^2 - 1) > 0 \quad \rightarrow \quad \lambda < \sqrt{\frac{1+\nu}{\nu}}$$

3.3.2 Mechanical power for an axially loaded truss

The figure shows a tensile bar which is elongated due to the action of an axial force F. Its undeformed cross-sectional area and length are A_0 and l_0, respectively. In the deformed configuration the cross-sectional area and length are A and l.

At constant force F an infinitesimal small increase in length is associated with a change in mechanical energy per unit of time (power) : $P = F\dot{l}$. The elongation rate $\dot{l}$ can be expressed in various strain rates.

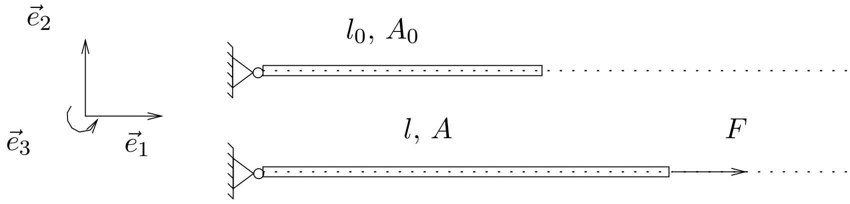


Fig. 32 : *Axial elongation of homogeneous truss*

linear strain	$\varepsilon_l = \lambda - 1$	$\rightarrow$	$\dot{\varepsilon}_l = \dot{\lambda} = \dfrac{\dot{l}}{l_0}$
logarithmic strain	$\varepsilon_{ln} = \ln(\lambda)$	$\rightarrow$	$\dot{\varepsilon}_{ln} = \dot{\lambda}\lambda^{-1} = \dfrac{\dot{l}}{l}$
Green-Lagrange strain	$\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$	$\rightarrow$	$\dot{\varepsilon}_{gl} = \dot{\lambda}\lambda = \lambda\dfrac{\dot{l}}{l_0} = \lambda^2\dfrac{\dot{l}}{l}$

$$
\begin{array}{lllllllll}
P & = & F\dot{\ell} & = & F\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A_0}A_0\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A_0}V_0\dot{\varepsilon}_l \\
P & = & F\dot{\ell} & = & F\ell\dot{\varepsilon}_{ln} & = & \dfrac{F}{A}A\ell\dot{\varepsilon}_{ln} & = & \dfrac{F}{A}V\dot{\varepsilon}_{ln} \\
P & = & F\dot{\ell} & = & F\ell_0\dot{\varepsilon}_l & = & \dfrac{F}{A}A\ell\dfrac{\ell_0}{\ell}\dot{\varepsilon}_l & = & \dfrac{F}{A}V\lambda^{-1}\dot{\varepsilon}_l \\
P & = & F\dot{\ell} & = & F\ell\lambda^{-2}\dot{\varepsilon}_{gl} & = & \dfrac{F}{A}A\ell\lambda^{-2}\dot{\varepsilon}_{gl} & = & \dfrac{F}{A}V\lambda^{-2}\dot{\varepsilon}_{gl}
\end{array}
$$

Various stress definitions automatically emerge when the mechanical power is considered in the undeformed volume $V_0 = A_0l_0$ or the current volume $V = Al$ of the tensile bar. The stresses are :

σ	:	Cauchy or true stress
σ_n	:	engineering or nominal stress
σ_{p1}	:	1st Piola-Kirchhoff stress = σ_n
σ_κ	:	Kirchhoff stress
σ_{p2}	:	2nd Piola-Kirchhoff stress

$$
\begin{array}{lclclcl}
P & = & & = & & = & V_0\sigma_n\dot{\varepsilon}_l \\
P & = & V\sigma\dot{\varepsilon}_{ln} & = & V_0(J\sigma)\dot{\varepsilon}_{ln} & = & V_0\sigma_\kappa\dot{\varepsilon}_{ln} \\
P & = & V(\sigma\lambda^{-1})\dot{\varepsilon}_l & = & V_0(J\sigma\lambda^{-1})\dot{\varepsilon}_l & = & V_0\sigma_{p1}\dot{\varepsilon}_l \\
P & = & V(\sigma\lambda^{-2})\dot{\varepsilon}_{gl} & = & V_0(J\sigma\lambda^{-2})\dot{\varepsilon}_{gl} & = & V_0\sigma_{p2}\dot{\varepsilon}_{gl}
\end{array}
$$

specific mechanical power : $P = V_0\dot{W}_0 = V\dot{W}$

$$
\begin{array}{lclclclcl}
\dot{W}_0 & = & \sigma_n\dot{\varepsilon}_l & = & \sigma_\kappa\dot{\varepsilon}_{ln} & = & \sigma_{p1}\dot{\varepsilon}_l & = & \sigma_{p2}\dot{\varepsilon}_{gl} \\
\dot{W} & = & & = & \sigma\dot{\varepsilon}_{ln} & = & \sigma\lambda^{-1}\dot{\varepsilon}_l & = & \sigma\lambda^{-2}\dot{\varepsilon}_{gl}
\end{array}
$$

3.3.3 Equilibrium

Deformations may be so large that the geometry changes considerably. This and/or nonlinear boundary conditions render the deformation problem nonlinear. Proportionality and superposition do not hold in that case. The internal force f_i is a nonlinear function of the elongation u.

Nonlinear material behavior may also result in a nonlinear function $f_i(u)$. This nonlinearity is almost always observed when deformation is large.

Solving the elongation from the equilibrium equation is only possible with an iterative solution procedure.

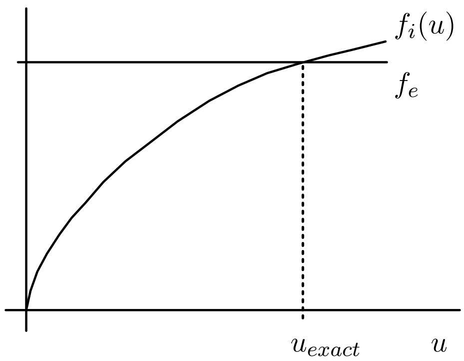


Fig. 33 : *Nonlinear internal load and constant external load*

external force	f_e
internal force	$f_i = \sigma A = f_i(u)$
equilibrium of point P	$f_i(u) = f_e$

3.3.4 Solution procedure

It is assumed that an approximate solution u^* for the unknown exact solution u_{exact} exists. (Initially $u^* = 0$ is chosen.)

The *residual load* r^* is the difference between $f(u^*)$ and f_e. For the exact solution this residual is zero. What we want the iterative solution procedure to do, is generating better approximations for the exact solution so that the residual becomes very small (ideally zero).

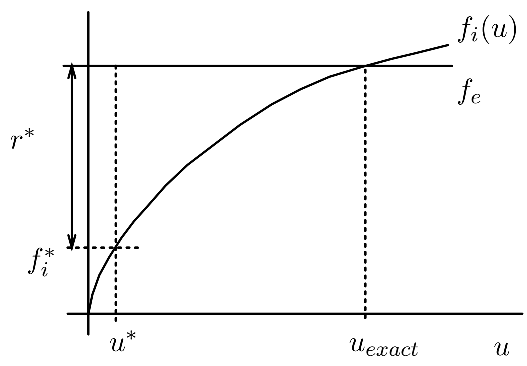


Fig. 34 : *Approximation of exact solution*

analytic solution	$f_i(u_{exact}) = f_e \quad \rightarrow \quad f_e - f_i(u_{exact}) = 0$
approximation u^*	$f_e - f_i(u^*) = r(u^*) \neq 0$
residual	$r^* = r(u^*)$

The unknown exact solution is written as the sum of the approximation and an unknown error δu. The internal force $f_i(u_{exact})$ is then written as a Taylor series expansion around u^* and linearized with respect to δu. The first derivative of f_i with respect to u is called the *tangential stiffness* K^*. Subsequently δu is solved from the linear iterative equation. The solution is called the *iterative displacement.*

$$\left.\begin{array}{l} f_i(u_{exact}) = f_e \\ u_{exact} = u^* + \delta u \end{array}\right\} \quad \rightarrow \quad f_i(u^* + \delta u) = f_e$$

$$f_i(u^*) + \left.\frac{df_i}{du}\right|_{u^*} \delta u = f_e \quad \rightarrow \quad f_i^* + K^* \delta u = f_e$$

$$K^* \, \delta u = f_e - f_i^* = r^* \quad \rightarrow \quad \delta u = \frac{1}{K^*} r^*$$

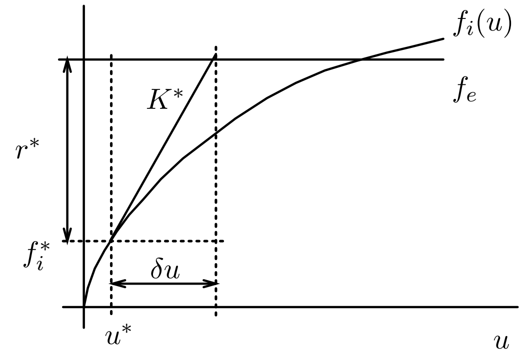


Fig. 35 : *Tangential stiffness and iterative solution*

With the iterative displacement δu a new approximate solution u^{**} can be determined by simply adding it to the known approximation.

When u^{**} is a better approximation than u^*, the iteration process is *converging.* As the exact solution is unknown, we cannot calculate the deviation of the approximation directly. There are several methods to quantify the *convergence.*

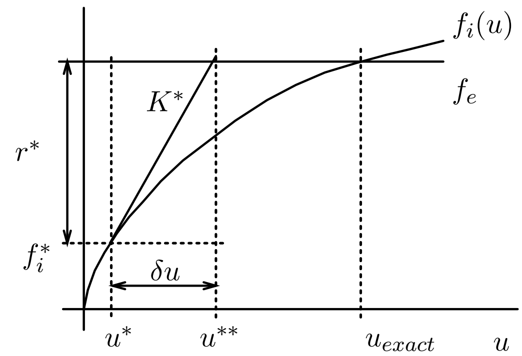


Fig. 36 : *New approximation of the exact solution*

new approximation	$u^{**} = u^* + \delta u$
error	$u_{exact} - u^{**}$
error smaller	$\rightarrow$ convergence

Convergence control

When the new approximation u^{**} is better than u^*, the residual r^{**} is smaller than r^*. If its value is not small enough, a new approximate solution is determined in a new iteration step. If its value is small enough, we are satisfied with the approximation u^{**} for the exact solution and the iteration process is terminated. To make this decision the residual is compared to a *convergence criterion* c_r. It is also possible to compare the iterative displacement δu with a convergence criterion c_u. If $\delta u < c_u$ it is assumed that the exact solution is determined close enough.

When the convergence criterion is satisfied, the displacement u will not satisfy the nodal equilibrium exactly, because the convergence limit is small but not zero. When incremental loading is applied, the difference between f_i and f_e is added to the load in the next increment, which is known as *residual load correction.*

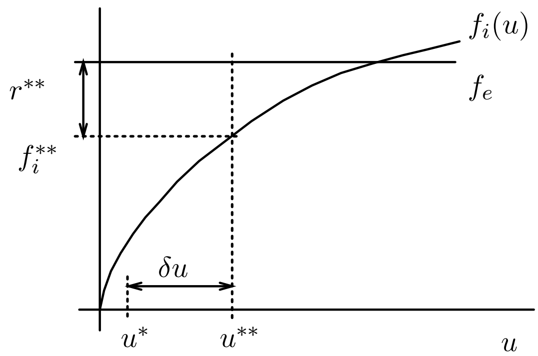


Fig. 37 : *New residual for approximate solution*

residual force	$\|r^{**}\| \leq c_r$	$\rightarrow$	stop iteration
iterative displacement	$\|\delta u\| \leq c_u$	$\rightarrow$	stop iteration

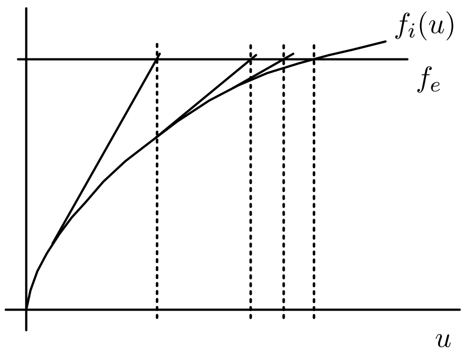


Fig. 38 : *Converging iteration process*

Residual and tangential stiffness

The residual and the tangential stiffness can be calculated from the material model, which describes the *material behavior.* It is assumed that this is a relation between the axial Cauchy stress σ and the elongation factor or stretch ratio $\lambda = \frac{l}{l_0}$: $\sigma = \sigma(\lambda)$. It is also necessary to now the relation between the cross-sectional area A and λ.

internal nodal force $\qquad f_i^* = N(\lambda^*) = A^* \sigma^*$

tangential stiffness $\qquad K^* = \left.\dfrac{\partial f_i}{\partial u}\right|_{u^*} = \left.\dfrac{\partial N(\lambda)}{\partial u}\right|_{u^*} = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{d\lambda}{du}$

geometry $\qquad \lambda = 1 + \dfrac{\Delta l}{l_0} = 1 + \dfrac{1}{l_0}\,u \quad \rightarrow \quad \dfrac{d\lambda}{du} = \dfrac{1}{l_0}$

tangential stiffness $\qquad K^* = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{\partial\lambda}{\partial u} = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{1}{l_0} = \left.\dfrac{dN}{d\lambda}\right|^* \dfrac{1}{l_0} = \dfrac{1}{l_0}\dfrac{d}{d\lambda}(\sigma A)\Big|^*$

$$\boxed{K^* = \frac{1}{l_0}\left.\frac{d\sigma}{d\lambda}\right|^* A^* + \frac{1}{l_0}\,\sigma^* \left.\frac{dA}{d\lambda}\right|^*}$$

Incremental loading

The external loading may be time-dependent. To determine the associated deformation, the time is discretized : the load is prescribed at subsequent, discrete moments in time and deformation is determined at these moments. A time interval between two discrete moments is called a *time increment* and the time dependent loading is referred to as *incremental loading.* This incremental loading is also applied for cases where the real time (seconds, hours) is not relevant, but when we want to prescribe the load gradually. One can than think of the "time" as a fictitious or virtual time.

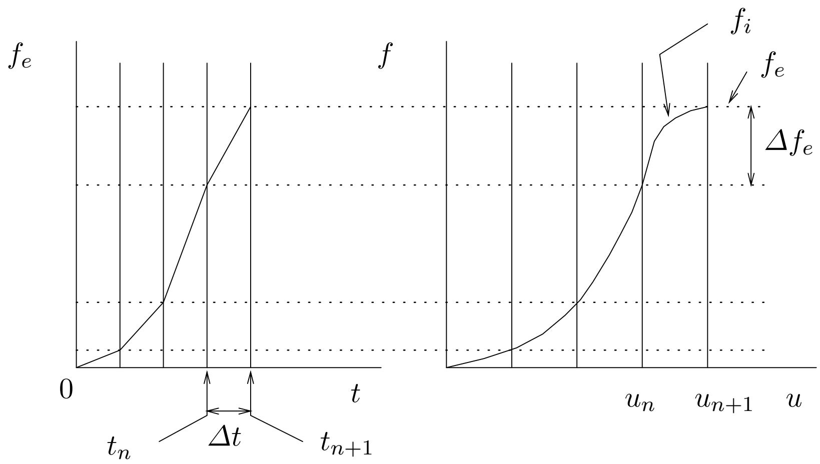


Fig. 39 : *Incremental loading*

Non-converging solution process

The iteration process is not always converging. Some illustrative examples are shown in the next figures.

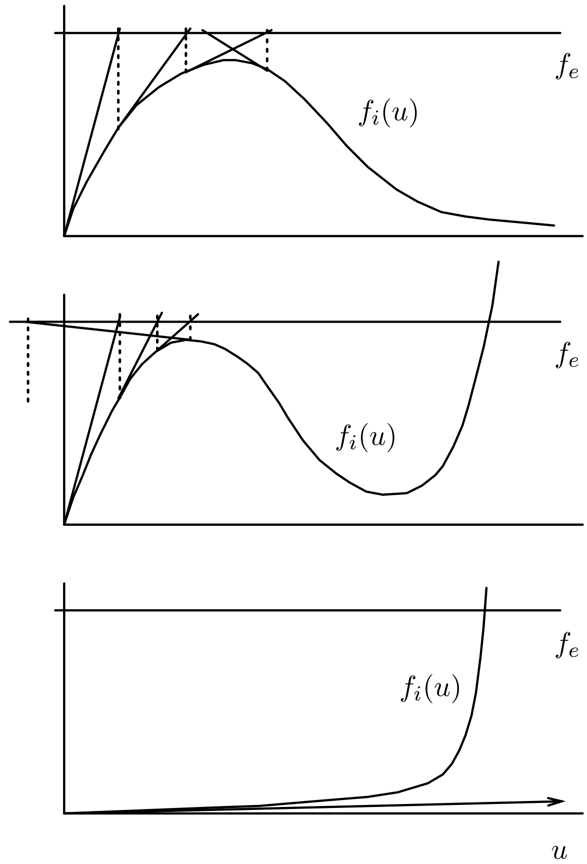


Fig. 40 : *Non-converging solution processes*

Modified Newton-Raphson procedure

Sometimes, it is possible to reach the exact solution by modifying the Newton-Raphson iteration process. The tangential stiffness is then not updated in every iteration step. Its initial value is used throughout the iterative procedure.

The figure shows a so-called "snap-through" problem, where no convergence can be reached due to a cycling *full* Newton-Raphson iteration process. With *modified* Newton-Raphson, iteration proceeds to the equilibrium $f_i = f_e$.

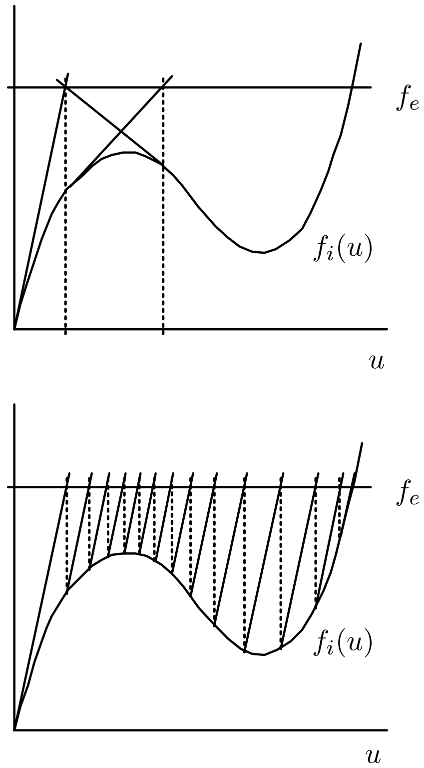


Fig. 41 : *Modified Newton-Raphson procedure*

3.4 Material behavior

Characterization of the mechanical behavior of an unknown material almost always begins with performing a tensile experiment. A stepwise change in the axial stress may be prescribed and the strain of the tensile bar can be measured and plotted as a function of time. From these plots important conclusions can be drawn concerning the material behavior.

Another way of representing the measurement data of the tensile experiment is by plotting the stress against the strain, resulting in the stress-strain curve. The relation between stress and strain may be *linear* or *nonlinear*. Also, the relation may be *history dependent*, due to changes in the material structure. Different behavior in tensile and compression may be observed.

All these features must be captured by the material model, which describes the experimentally observed behavior as accurate as possible in a range required by the application.

3.4.1 Time history plots

For *elastic* material behavior the strain follows the stress immediately and becomes zero after stress release. For *elastoplastic* material behavior the strain also follows the stress immediately, but there is permanent deformation after stress release. When the material is *viscoelastic* the strain shows time delayed response on a stress step, which indicates a time dependent behavior. When time dependent behavior is accompanied by permanent deformation, the behavior is referred to as *viscoplastic*.

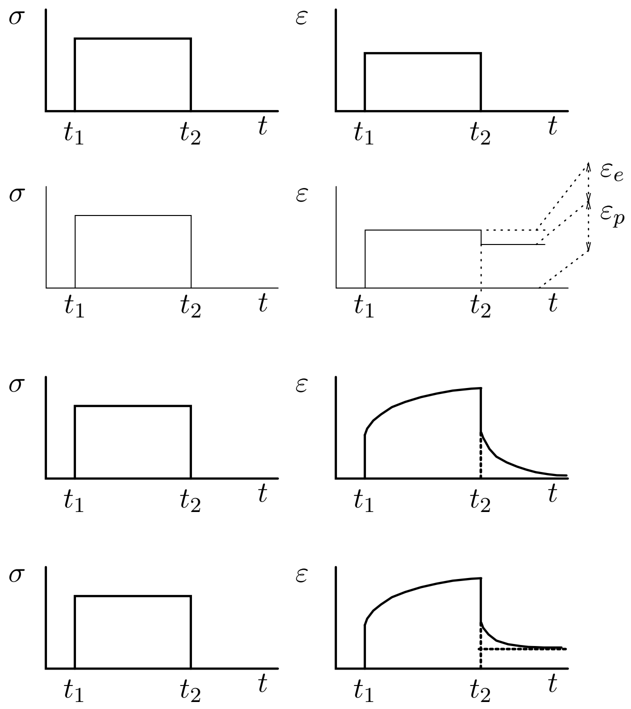


Fig. 42 : *Strain response for a stress-step for a) elastic, b) elastoplastic, c) viscoelastic and d) viscoplastic material behavior*

3.4.2 Tensile curves

Tensile curve : elastic behavior

When elastic behavior is well described by a linear relation between a stress and a strain, the elastic behavior is referred to as linear.

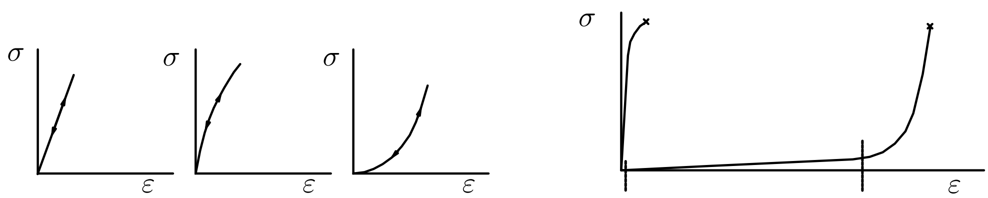


Fig. 43 : *Tensile curves for elastic material behavior*

Tensile curve : viscoelastic behavior

Viscoelastic behavior is time-dependent. The stress is a function of the strain rate. There is a phase difference between stress and strain, which results in a hysteresis loop when the loading is cycling in time.

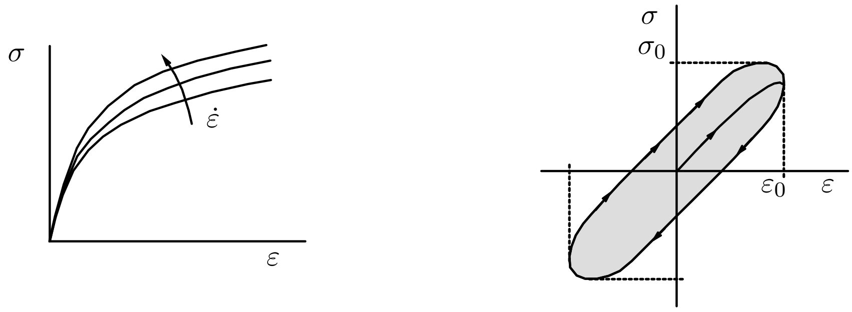


Fig. 44 : *Tensile curve and hysteresis loop for viscoelastic material behavior*

Tensile curve : elastoplastic behavior

When a material is loaded or deformed above a certain threshold, the resulting deformation will be permanent or plastic. When time (strain rate) is of no importance, the behavior is referred to as elastoplastic. Stress-strain curves may indicate different characteristics, especially when the loading is reversed from tensile to compressive.

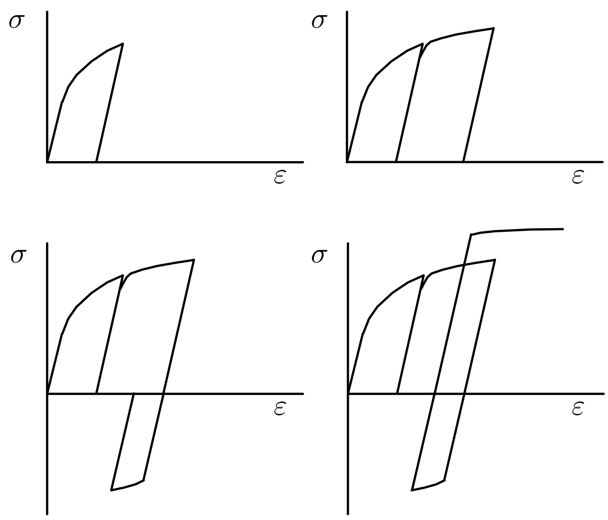


Fig. 45 : *Tensile curves for elastoplastic material behavior*

Tensile curve : viscoplastic behavior

A combination of plasticity and time-dependent phenomena is called viscoplastic behavior. This behavior is often observed for polymeric materials. For some polymers the stress reaches a maximum and subsequently drops with increasing strain. This phenomenon is referred to as *intrinsic softening*. In a tensile experiment it will provoke necking of the tensile bar.

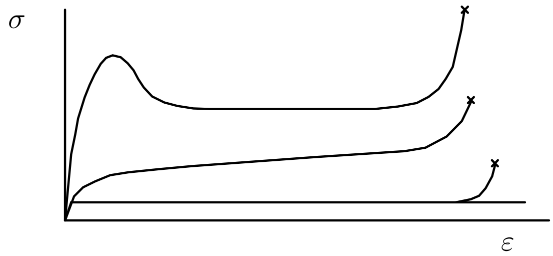


Fig. 46 : *Tensile curves for viscoplastic material behavior*

Tensile curve : damage

Structural damage influences the material properties. The onset and evolution of damage can be described with a damage model. For materials like concrete and ceramics, the onset and propagation of damage causes softening. Because damage is often associated with the initiation and growth of voids, the stress-strain curve is different for tensile and compressive loading.

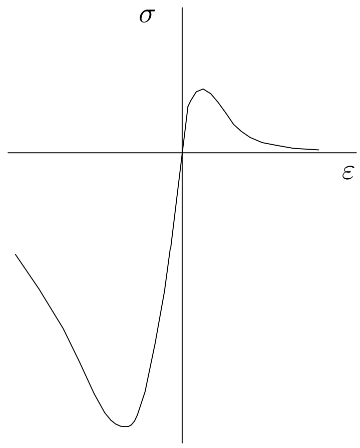


Fig. 47 : *Tensile curve for damaging material with different behavior in tension and compression*

3.4.3 Discrete material models

Material models relate stresses to deformation and possibly deformation rate. For three-dimensional continua the material model is often represented by a (large) number of coupled (differential) equations. As a simplified introduction, we will present material models first in a one-dimensional setting. The material behavior is represented by the behavior of a one-dimensional, discrete, mechanical system of springs, dashpots and friction sliders. For such a system the relation between the axial stress σ and the axial strain ε can be derived.

When the model is employed in a truss, the stress σ will be used to calculate the internal axial force.

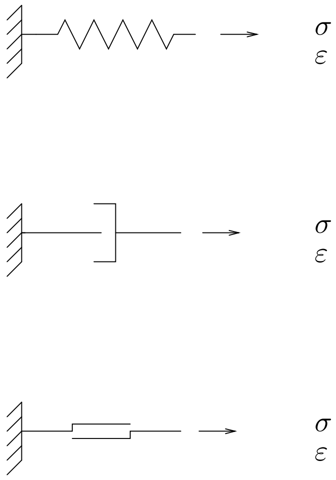


Fig. 48 : *Discrete elements : spring, dashpot and friction slider*

Chapter 4

Elastic material behavior

When a material behaves elastically, the current stress can be calculated directly from the current strain, because there is no path and/or time dependency. When the stress is released, the strain will become zero, so there is no permanent deformation at zero stress. All stored strain energy is released and there is no dissipation. For the one-dimensional case the elastic behavior is described by a relation between the stress σ and the elongation factor λ or the strain ε.

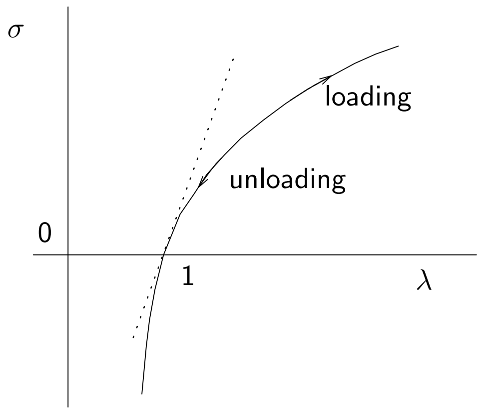


Fig. 49 : *Non-linear elastic material behavior*

Large strain elastic behavior

For large deformations, nonlinear elastic behavior can be observed in polymers, elastomeric materials (rubbers) and, on a small scale, in atomic bonds, when a tensile/compression test is carried out and the axial force F is plotted as a function of λ. In a material model we want to describe such behavior with a mathematical relation between a stress and a strain. Consideration of the stored elastic energy per unit of volume learns that each stress definition is associated with a certain strain definition, so these should be combined in a material model. However, when the observed material behavior is described accurately by another stress/strain combination, it can be used as well.

For three-dimensional models more considerations have to be taken into account. Care has to be taken that the material model does not generate stresses for large rigid body rotations of the material.

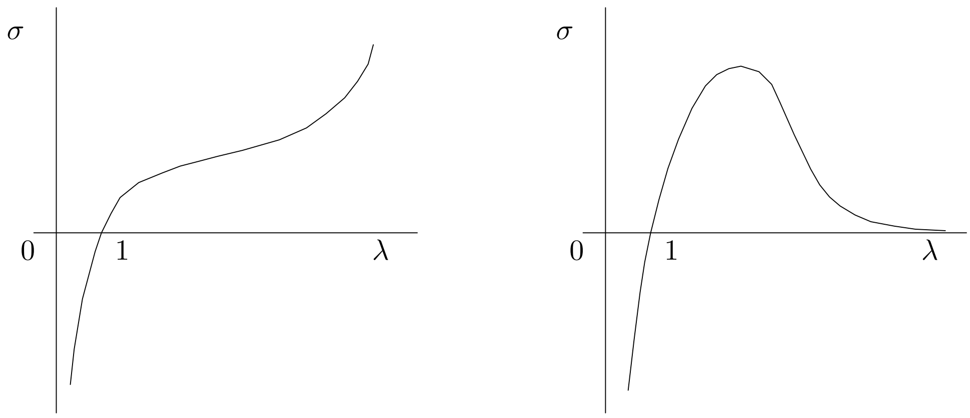


Fig. 50 : *Non-linear stress-strain relations for an elastomeric material and for an atomic bond*

Small strain elastic behavior

For small elongations, all strain definitions are the same, as are all stress definitions. The relation between stress and strain is linear and the constant material parameter is the Young's modulus.

strain	$\varepsilon = \varepsilon_{gl} = \varepsilon_{ln} = \varepsilon_l = \lambda - 1$
stress	$\sigma = \dfrac{F}{A} = \dfrac{F}{A_0} = \sigma_n$
linear elastic behavior	$\sigma = E\varepsilon = E(\lambda - 1)$
modulus	$E = \lim\limits_{\lambda \to 1} \dfrac{d\sigma}{d\lambda} = \lim\limits_{\varepsilon \to 0} \dfrac{d\sigma}{d\varepsilon}$

4.1 Elastic models

The discrete one-dimensional model for elastic material behavior is a spring. The behavior is modeled with a relation between the stress σ and the elongation factor λ or a strain ε. The material stiffness C_λ is the derivative of σ w.r.t. the stretch ratio λ. The derivative w.r.t. the strain ε results in the stiffness C_ε.

Consideration of the stored elastic energy per unit of material volume learns that, in a material model, true stress σ should be combined with logarithmic strain ε_{ln}, engineering stress σ_n with linear strain ε_l or 2nd-Piola-Kirchhoff stress σ_{p2} with Green-Lagrange strain ε_{gl}.

Experimentally observed tensile behavior can often be described with a linear relation between a certain stress and its associated strain.

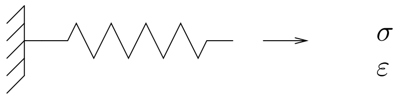


Fig. 51 : *Spring*

constitutive equation $\boxed{\sigma = \sigma(\lambda)}$

stiffness $\boxed{C_\lambda = \dfrac{d\sigma}{d\lambda} = \dfrac{d\sigma}{d\varepsilon}\dfrac{d\varepsilon}{d\lambda} = C_\varepsilon \dfrac{d\varepsilon}{d\lambda}}$

elastic models (examples) $\left\{ \begin{array}{ll} \text{linear true-log.} & \sigma = C\ln(\lambda) = C\varepsilon_{ln} \\ \text{linear eng.-lin.} & \sigma_n = C(\lambda - 1) = C\varepsilon_l \end{array} \right.$

4.2 Hyper-elastic models

Elastomeric materials (rubbers) show very large elastic deformations (elongation up to 5). The material models for these materials are therefore referred to as *hyper-elastic*. They are derived from an elastic energy function, which has to be determined experimentally. The so-called Rivlin or Mooney type of these functions are expressed in the principal elongation factors $\lambda_i, i = 1, 2, 3$. Experimental observations indicate that elastomeric materials are incompressible, so that we have $\lambda_1\lambda_2\lambda_3 = 1$.

$$\boxed{\begin{array}{l} W = \displaystyle\sum_i^n \sum_j^m C_{ij} \left(I_1 - 3\right)^i \left(I_2 - 3\right)^j \qquad \text{with} \quad C_{00} = 0 \\ I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 = \dfrac{1}{\lambda_3^2} + \dfrac{1}{\lambda_1^2} + \dfrac{1}{\lambda_2^2} \end{array}}$$

The incremental change of the elastically stored energy per unit of deformed volume, can be expressed in the principal stresses and the principal logarithmic strains.

$$\boxed{dW = \sigma_1 d\varepsilon_{ln_1} + \sigma_2 d\varepsilon_{ln_2} + \sigma_3 d\varepsilon_{ln_3}}$$

Mooney models

For incompressible materials like elastomer's (rubber) the stored elastic energy per unit of deformed volume is specified and fitted onto experimental data. Several specific energy functions are used.

Neo-Hookean	$W = C_{10}\,(I_1 - 3)$
Mooney-Rivlin	$W = C_{10}\,(I_1 - 3) + C_{01}\,(I_2 - 3)$
Signiorini	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2$
2nd-order invariant model	$W = C_{10}(I_1-3)+C_{01}(I_2-3)+C_{11}(I_1-3)(I_2-3)+C_{20}(I_1-3)^2$
Yeoh	$W = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$
Klosner-Segal	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{03}(I_2 - 3)^3$

Third-order model of James, Green and Simpson

$$\begin{aligned} W = {} & C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + \\ & C_{20}(I_1 - 3)^2 + C_{02}(I_2 - 3)^2 + C_{21}(I_1 - 3)^2(I_2 - 3) + \\ & C_{30}(I_1 - 3)^3 + C_{03}(I_2 - 3)^3 + C_{12}(I_1 - 3)(I_2 - 3)^2 \end{aligned}$$

Ogden models

For 'slightly' compressible materials the Ogden specific energy functions are used. Because the volume change is not zero, these functions depend on the volume change factor J. The second part of the energy function accounts for the volumetric deformation. Because the volumetric behavior is characterized by a constant bulk modulus, the model is confined to slightly compressible deformation.

For 'highly' compressible materials like foams, specific energy functions also exist. The first part of the energy function also describes volume change.

slightly compressible	$W = \sum_{i=1}^{N} \frac{a_i}{b_i}\left[J^{-\frac{b_i}{3}}\left(\lambda_1^{b_i} + \lambda_2^{b_i} + \lambda_3^{b_i}\right) - 3\right] + 4.5K\left(J^{\frac{1}{3}} - 1\right)^2$
highly compressible	$W = \sum_{i=1}^{N} \frac{a_i}{b_i}\left(\lambda_1^{b_i} + \lambda_2^{b_i} + \lambda_3^{b_i} - 3\right) + \sum_{i=1}^{N} \frac{a_i}{c_i}\left(1 - J^{c_i}\right)$

One-dimensional models

For tensile (or compressive) loading of a homogeneous and isotropic truss, where the axial direction is taken to be the 1-direction, we have : $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$. In this case there is only an axial stress $\sigma_1 = \sigma$, so that we have

$$dW = \sigma\, d\varepsilon_{ln} \quad \rightarrow \quad \sigma = \frac{dW}{d\varepsilon_{ln}} = \frac{dW}{d\lambda}\frac{d\lambda}{d\varepsilon_{ln}} = \frac{dW}{d\lambda}\lambda$$

The Neo-Hookean model is the simplest model as it contains only one material parameter. Axial stress σ and axial force F can be calculated easily. From statistical mechanics it is known that for an ideal rubber material the stress is :

$$\sigma = \frac{\rho R T}{M}\left(\lambda^2 - \frac{1}{\lambda}\right) \qquad \text{with} \qquad \begin{array}{lcl} \rho & : & \text{density} \\ R & : & \text{gas constant} = 8.314\ \mathrm{JK^{-1}mol^{-1}} \\ T & : & \text{absolute temperature} \\ M & : & \text{average molecular weight} \end{array}$$

Most rubber materials cannot be characterized well with the Neo-Hookean model. The more complex Mooney-Rivlin model yields better results. The stiffness C_λ is a function of the elongation factor λ. The initial stiffness E is often referred to as the modulus.

Neo-Hookean

$$\begin{aligned} W &= C_{10}\left(\lambda^2 + \frac{2}{\lambda} - 3\right) \\ \sigma &= C_{10}\left(2\lambda - \frac{2}{\lambda^2}\right)\lambda = 2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) \\ C_\lambda &= \frac{\partial \sigma}{\partial \lambda} = 2C_{10}\left(2\lambda + \frac{1}{\lambda^2}\right) \quad ; \quad E = \lim_{\lambda \to 1} \frac{\partial \sigma}{\partial \lambda} = 6C_{10} \\ F &= \sigma A = \sigma \frac{1}{\lambda} A_0 = 2C_{10} A_0 \left(\lambda - \frac{1}{\lambda^2}\right) \end{aligned}$$

Mooney-Rivlin

$$\begin{aligned} W &= C_{10}\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + C_{01}\left(\frac{1}{\lambda^2} + 2\lambda - 3\right) \\ \sigma &= 2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) + 2C_{01}\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{1}{\lambda} \\ C_\lambda &= \frac{\partial \sigma}{\partial \lambda} = 2C_{10}\left(2\lambda + \frac{1}{\lambda^2}\right) + 2C_{01}\left(1 + \frac{2}{\lambda^3}\right) \quad ; \quad E = \lim_{\lambda \to 1} \frac{\partial \sigma}{\partial \lambda} = 6(C_{10} + C_{01}) \\ F &= \sigma A = \sigma \frac{1}{\lambda} A_0 = A_0 \frac{1}{\lambda}\left[2C_{10}\left(\lambda^2 - \frac{1}{\lambda}\right) + 2C_{01}\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{1}{\lambda}\right] \end{aligned}$$

Chapter 5

Elastoplastic material behavior

Below a certain load (stress) value, the deformation of all materials will remain elastic. When the stress exceeds a limit value, plastic deformation occurs, which means that permanent elongation is observed after release of the load. At increased loading above the limit value, the stress generally increases with increasing elongation, a phenomenon referred to as *hardening*.

Reversed loading will first result in elastic deformation, but after reaching a limit value of the stress, plastic deformation will be observed again.

Looking at the stress-strain curve after a few loading reversals, it can be seen that elastoplastic material behavior is history dependent: the stress is not uniquely related to the strain; its value depends on the deformation history. The total stress-strain history must be taken into account to determine the current stress.

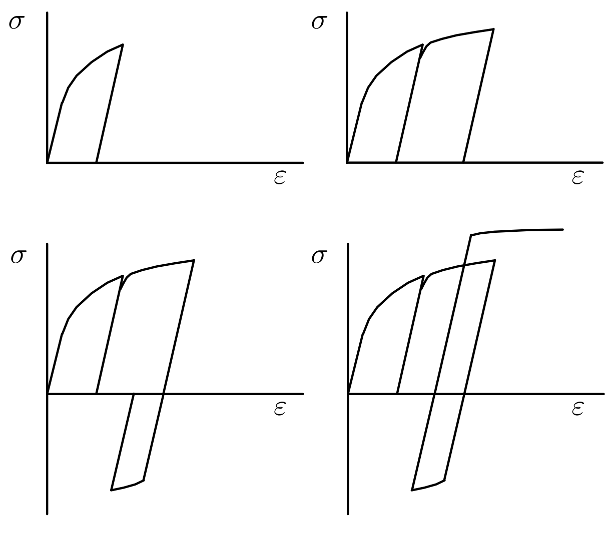


Fig. 52 : *Stress-strain curves for elastoplastic material behavior*

5.1 Uniaxial test

To investigate the characteristics of elastoplastic material behavior, a uniaxial tensile/compression test is done. Although strains during an elastoplastic deformation can be very large (eg. in forming processes), we consider only small to moderate axial strains of the tensile bar. In that case all stress and strain definitions result in (approximately) the same values and are indicated by σ and ε.

Tensile test

When a tensile bar, with undeformed length l_0 and cross-sectional area A_0, is subjected to a tensile test, the axial force F and the length l can be measured. The axial stress σ and the axial strain ε can then be calculated.

$$\boxed{\sigma = \sigma_n = \frac{F}{A_0} \qquad ; \qquad \varepsilon = \varepsilon_l = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}}$$

Until point P ($\sigma = \sigma_P$, the proportionality limit) is reached, the material behavior is assumed to be linear elastic : $\sigma = E\varepsilon$, where E is Young's modulus. When the stress exceeds the initial yield stress $\sigma_{y0} > \sigma_P$, unloading will reveal permanent (= plastic) deformation of the bar. The exact value of σ_{y0} cannot be determined so in practice σ_{y0} is taken to be the stress where a plastic strain of 0.2 % remains. In the following however, we will assume that σ_{y0} is exactly known and that $\sigma_{y0} = \sigma_P$.

The axial force and therefore the nominal stress will reach a maximum value. At that point necking of the tensile bar will be observed. The maximum nominal stress is the tensile strength σ_T. In forming processes strains can be much higher than in a tensile test, because of the compression in certain directions.

Experiments have shown that during plastic deformation the volume of the metals and metal alloys remains constant : plastic deformation is incompressible.

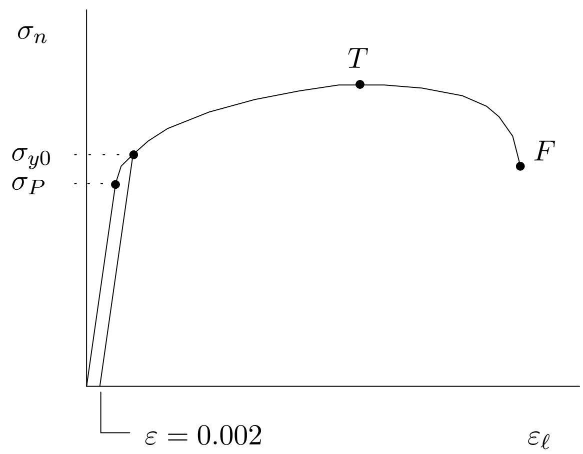


Fig. 53 : *Stress-strain curve during tensile test*

σ_P	proportional limit
$\sigma = \sigma_{y0}$	yield
σ_{y0}	initial yield stress
ε_{y0}	strain at σ_{y0} : $\varepsilon_{y0} = \sigma_{y0}/E$
$\varepsilon_{0.2}$	0.2-strain : $\varepsilon_p = 0.2\% = 0.002$
σ_T	tensile strength
σ_F	fracture strength
ε_F	fracture strain ($\approx 5\% = 0.05$ (metals))

Compression test

For metal alloys a compression test instead of a tensile test will reveal that first yield will occur at $\sigma = -\sigma_{y0}$. The initial material behavior is the same in tension and compression.

In general terms the transition from purely elastic behavior to elastoplastic behavior is determined by a yield criterion. For this one-dimensional case this criterion says that first yielding will occur when :

$$\boxed{f = \sigma^2 - \sigma_{y0}^2 = 0}$$

The function f is the yield function.

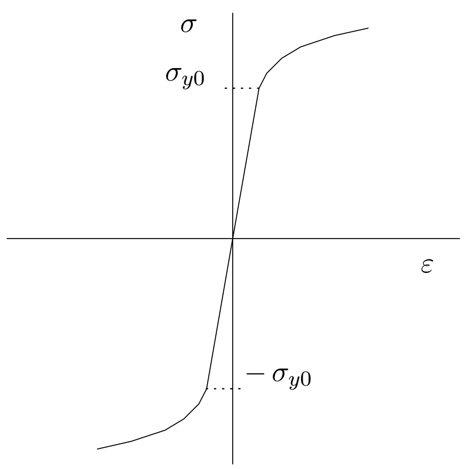


Fig. 54 : *Stress-strain curve during tensile or compression test*

Interrupted tensile test

When the axial load is released at σ_A with $\sigma_{y0} < \sigma_A < \sigma_T$, the unloading stress-strain path is elastic and characterized by the initial Young's modulus E. The permanent or plastic elongation is represented by the plastic strain ε_p. The difference between the total strain in point A and the plastic strain is the elastic strain $\varepsilon_e = \varepsilon_A - \varepsilon_p = \frac{\sigma_A}{E}$.

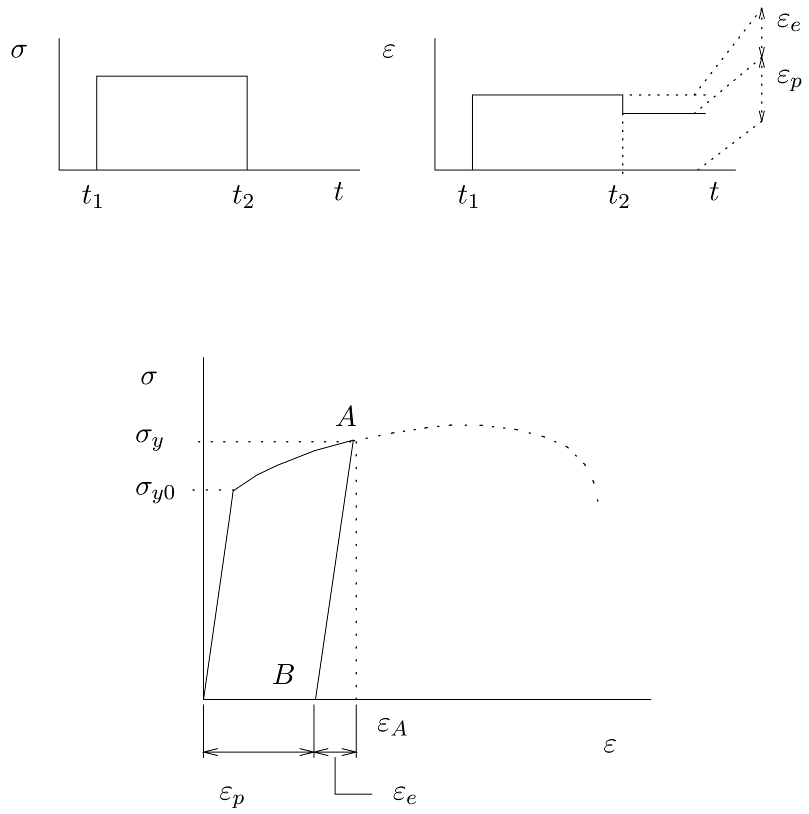


Fig. 55 : *Stress-strain curve after interrupted tensile test*

Resumed tensile test

When after unloading, the bar is again loaded with a tensile force, the elastic line BA will be transversed where $\Delta\sigma = E\Delta\varepsilon = E\Delta\varepsilon_e$ holds. The Young's modulus E is assumed to be not affected by the plastic deformation. For $\sigma \geq \sigma_A(\varepsilon \geq \varepsilon_A)$ further elastoplastic deformation takes place and the stress-strain curve will be followed as if unloading had not occurred.

At reloading the stress σ_A characterizes the transition from elastic to elastoplastic behavior and is called the *current yield stress* σ_y. As we see, plastic deformation results in an increase of the yield stress, a phenomenon which is called hardening. The deformation history, which influences the material behavior must be characterized by a history parameter, which has to be specified.

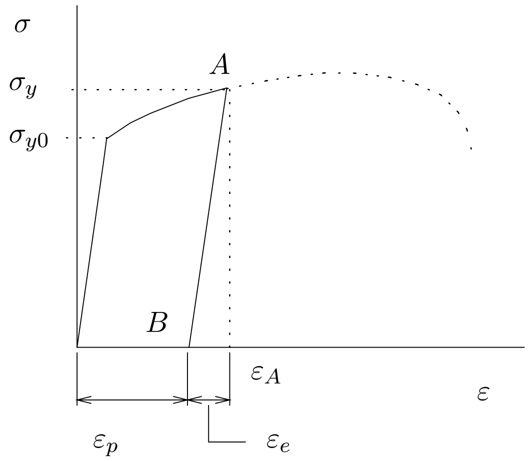


Fig. 56 : *Stress-strain curve after resumed loading during tensile test*

5.2 Hardening

To study the hardening phenomenon, the tensile bar is not reloaded in tension but in compression. Two extreme observations may be done, illustrated in the figure.

In the first case we can observe that the elastic trajectory increases in length due to plastic deformation : $AA' > Y_0Y_0'$. The elastic trajectory is symmetric about $\sigma = 0$ ($BA = BA'$). What we observe is *isotropic hardening*.

In the second case it is observed that the elastic trajectory remains of constant length : $AA' = Y_0Y_0'$. It is symmetric about the line OC ($CA = CA'$). After unloading the yield stress under compression is different than the yield stress under tension. This is called *kinematic hardening*. The stress in point C, the center of the elastic trajectory, is the *shift stress* $\sigma = q$. This phenomenon is also referred to as the *Bauschinger effect*.

Real materials will show a combination of isotropic and kinematic hardening.

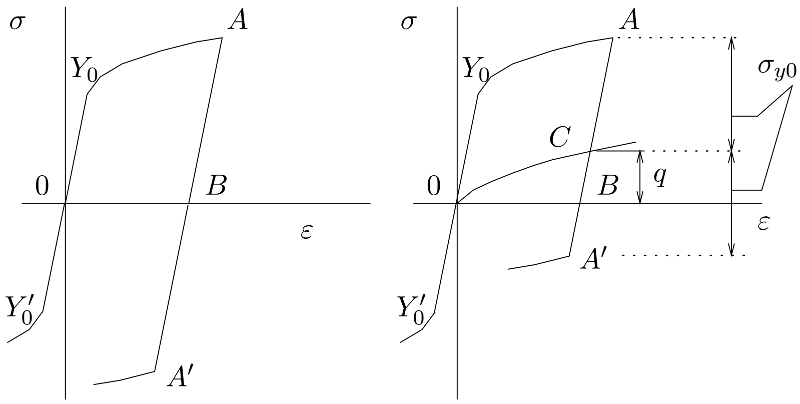


Fig. 57 : *Isotropic and kinematic hardening*

isotropic hardening

- elastic area : larger & symmetric w.r.t. $\sigma = 0$

$$\left.\begin{array}{lll} \text{tensile} & : & \sigma = \sigma_y \\ \text{compression} & : & \sigma = -\sigma_y \end{array}\right\} \quad \rightarrow \quad f = \sigma^2 - \sigma_y^2 = 0$$

kinematic hardening

- elastic area : constant & symmetric w.r.t. $\sigma = q$

$$\left.\begin{array}{lll} \text{tensile} & : & \sigma = q + \sigma_{y0} \\ \text{compression} & : & \sigma = q - \sigma_{y0} \end{array}\right\} \quad \rightarrow \quad f = (\sigma - q)^2 - \sigma_{y0}^2 = 0$$

5.2.1 Effective plastic strain

Isotropic hardening could be described by relating the yield stress σ_y to the plastic strain ε_p. However, as the figure shows, this would lead to the unrealistic conclusion that the yield stress would increase while the plastic strain decreases. To prevent this problem, the effective plastic strain $\bar{\varepsilon}_p$ is taken as the history parameter. It is a measure of the total plastic strain, be its change positive or negative, and as such cannot decrease.

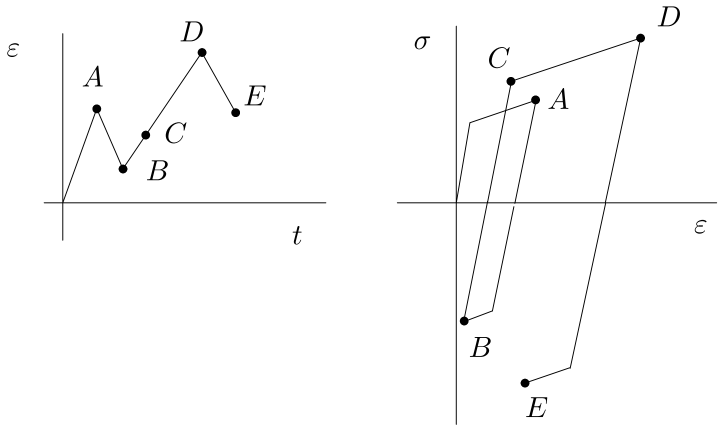


Fig. 58 : *Increasing yield stress at decreasing plastic strain*

$$\bar{\varepsilon}_p = \sum_{\varepsilon} |\Delta\varepsilon_p| = \sum_{\tau=0}^{\tau=t} \frac{|\Delta\varepsilon_p|}{\Delta t}\,\Delta t = \int_{\tau=0}^{t} |\dot{\varepsilon}_p|\, d\tau = \int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p\, d\tau$$

5.2.2 Linear hardening

Linear isotropic hardening is modeled by a linear relation between the yield stress σ_y and the effective plastic strain $\bar{\varepsilon}_p$. The characteristic material parameter is the hardening modulus H.

Purely kinematic hardening is modeled by a linear relation between the shift stress q and the plastic strain ε_p. Both variables may increase and decrease during elastoplastic deformation. The characteristic material parameter is the hardening modulus K.

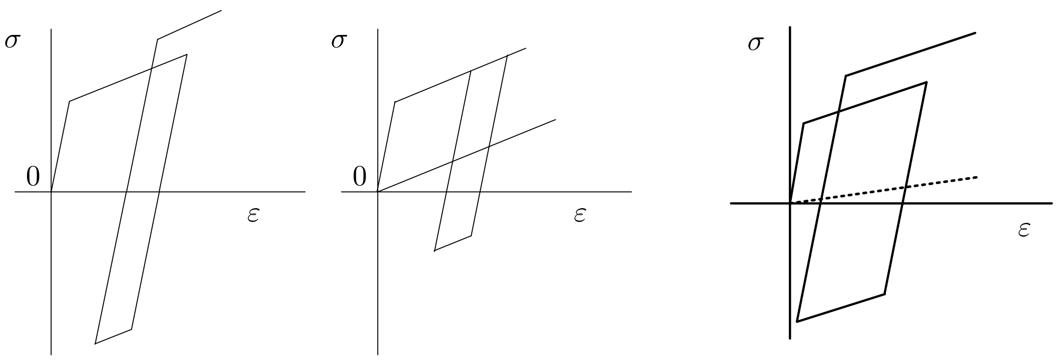


Fig. 59 : *Linear isotropic and linear kinematic hardening*

isotropic hardening

σ_y can **not** decrease	$\rightarrow$	*effective* plastic strain
hardening		$\sigma_y = \sigma_{y0} + H\,\bar{\varepsilon}_p$
yield criterion		$f = \sigma^2 - \sigma_y^2(\bar{\varepsilon}_p) = 0$

kinematic hardening

q can decrease	$\rightarrow$	no *effective* plastic strain
hardening		$q = K\,\varepsilon_p$
yield criterion		$f = \{\sigma - q(\varepsilon_p)\}^2 - \sigma_{y0}^2 = 0$

isotropic-kinematic hardening

hardening	$\sigma_y = \sigma_{y0} + H\,\bar{\varepsilon}_p \quad : \quad q = K\,\varepsilon_p$
yield criterion	$f = \{\sigma - q(\varepsilon_p)\}^2 - \sigma_y^2(\bar{\varepsilon}_p) = 0$

5.2.3 Other "hardening" models

Other post yield behavior which may be observed during a tensile test are :

- exponential hardening (Nadai model), which is often more realistic than linear hardening;

- no hardening, which is referred to as ideal plastic behavior;
- softening, in wich case the post yield stress decreases at increasing strain.

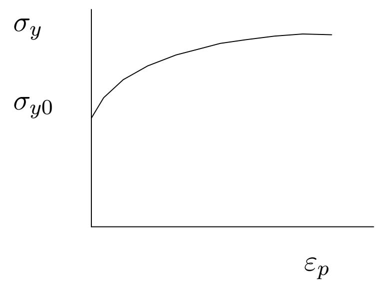


Fig. 60 : *Exponential hardening (Nadai)*

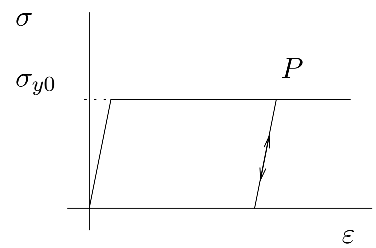


Fig. 61 : *No hardening (ideal plastic)*

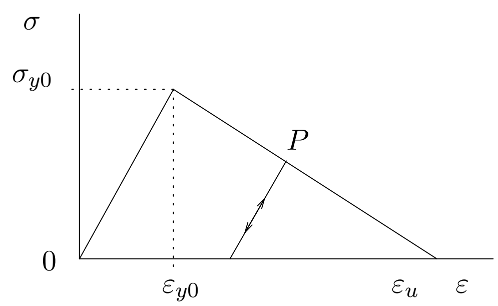


Fig. 62 : *Softening*

5.2.4 Cyclic load

A truss can be loaded with a prescribed strain $-\varepsilon_m \leq \varepsilon \leq \varepsilon_m$. It is assumed that the stress will reach values above the initial yield stress σ_{y0} and that hardening occurs.

For purely isotropic hardening the stress will increase after each load reversal and finally no further plastic deformation will take place.

For purely kinematic hardening the stress-strain path will be one single hysteresis loop, where the stress cycles, as does the strain, between two constant values $-\sigma_m \leq \sigma \leq \sigma_m$.

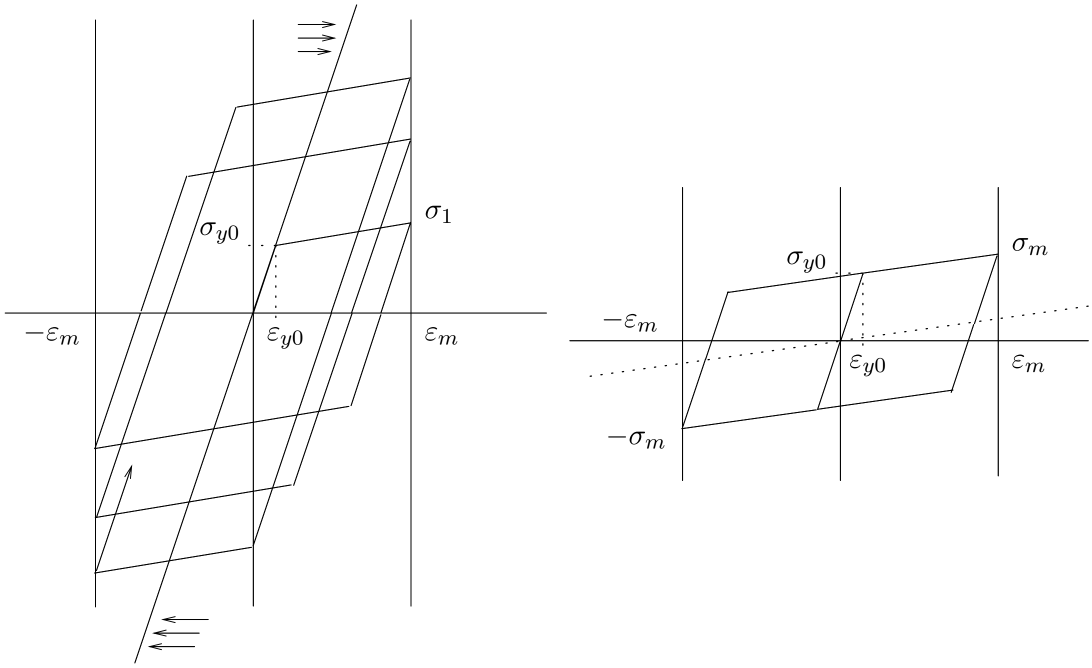


Fig. 63 : *Stress-strain curve during cyclic loading for isotropic and for kinematic hardening*

5.3 Elastoplastic model

The elastoplastic deformation characteristics can be represented by a discrete mechanical model. A friction element represents the yield limit and a hardening spring – stiffness H ($H > 0$) – provides the stiffness reduction after reaching the yield limit. The elastoplastic model describes rate-independent plasticity – there is no dashpot in the discrete model –, so the time is fictitious and "rate" is just referring to momentary change.

In an elastoplastic material model a yield criterion is used to decide at which stress level a purely elastic deformation will be followed by elastoplastic deformation. In the yield criterion f, the equivalent stress $\bar{\sigma}$ is compared to the current yield stress σ_y, which changes due to ongoing plastic deformation (= hardening or softening) as described by the hardening rule. In the one-dimensional situation considered here, the effective stress is simply the axial stress minus the shift stress q : $\bar{\sigma} = \sigma - q$.

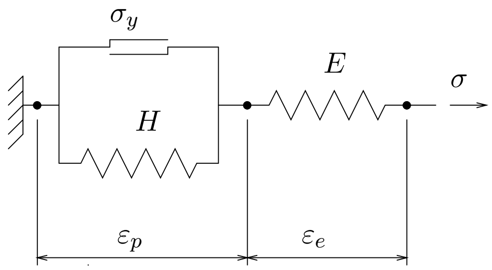


Fig. 64 : *Discrete mechanical model for elastoplastic material behavior*

constitutive relations

- $f = (\sigma - q)^2 - \sigma_y^2$
- $f < 0 \quad | \quad f = 0 \wedge \dot{f} < 0 \quad \rightarrow$ elastic
  $f = 0 \wedge \dot{f} = 0 \quad \rightarrow$ elastoplastic
- $\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p) \quad ; \quad q = q(\varepsilon_p)$
- $\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$
- $\sigma = E\varepsilon_e$
- $\bar{\varepsilon}_p = \sum_{\varepsilon} |\Delta\varepsilon_p|$

5.3.1 Material stiffness

When loading is not monotonous, it follows from the stress-strain curve that the current stress is not directly related to the current strain, but depends on the total deformation history. The material model can only be formulated as a relation between a change of the stress and a change of the strain, being the material stiffness, and as such the tangent to the stress-strain curve. This "rate" equation must then be integrated to determine the current stress, which means tracing the whole stress history.

When the current material behavior is elastic the relation between $\Delta\sigma$ and $\Delta\varepsilon$ is determined by the Young's modulus : $\Delta\sigma = E\Delta\varepsilon$. When the current deformation is elastoplastic, the material stiffness is S and we have $\Delta\sigma = S\,\Delta\varepsilon$. The elastoplastic material stiffness depends on the Young's modulus and hardening parameters. This relation will be derived for isotropic, kinematic and combined hardening. Only monotonous tensile loading is considered, but results are general.

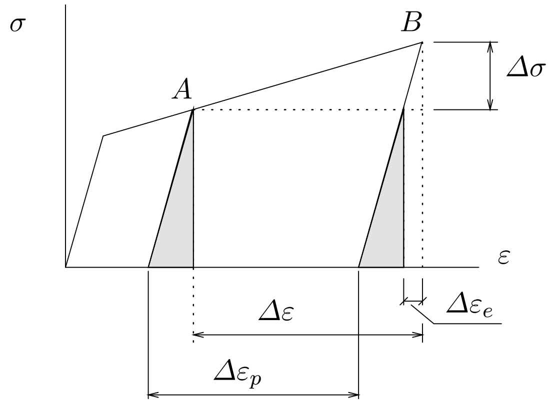


Fig. 65 : *Stress-strain curve for monotonic tensile loading*

isotropic hardening

$$\Delta\sigma = E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) = E\left(\Delta\varepsilon - \frac{\Delta\sigma_y}{H}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{H}\right)$$

constitutive equation $$\boxed{\Delta\sigma = \frac{EH}{E+H}\,\Delta\varepsilon = S\Delta\varepsilon}$$

change plastic strain $$\Delta\varepsilon_p = \frac{\Delta\sigma}{H} = \frac{E}{E+H}\,\Delta\varepsilon$$

kinematic hardening

$$\Delta\sigma = E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) = E\left(\Delta\varepsilon - \frac{\Delta q}{K}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{K}\right)$$

constitutive equation $$\boxed{\Delta\sigma = \frac{EK}{E+K}\,\Delta\varepsilon = S\Delta\varepsilon}$$

change plastic strain $$\Delta\varepsilon_p = \frac{\Delta\sigma}{K} = \frac{E}{E+K}\,\Delta\varepsilon$$

Note that the stiffness equals Young's modulus when H (or K) approaches infinity.

$$\lim_{H\to\infty} \frac{EH}{E+H} = \lim_{H\to\infty} \frac{E}{\frac{E}{H}+1} = E$$

For combined isotropic and kinematic hardening, the material stiffness can be derived in the same way as before. In the derivation tension and compression loading is considered at the same time. For the tension the (+)-sign applies and for compression the (−)-sign.

$$\left.\begin{array}{l} \Delta\sigma = E\Delta\varepsilon_e = E\,(\Delta\varepsilon - \Delta\varepsilon_p) \\[2ex] \sigma = q \pm \sigma_y \quad \rightarrow \quad \Delta\sigma = \Delta q \pm \Delta\sigma_y = K\Delta\varepsilon_p \pm H(\pm)\Delta\varepsilon_p \end{array}\right\} \quad \rightarrow$$

$$E\Delta\varepsilon - E\Delta\varepsilon_p = (K + H)\Delta\varepsilon_p \qquad \rightarrow$$

$$\Delta\varepsilon_p = \frac{E}{E + K + H}\,\Delta\varepsilon \quad \rightarrow \quad \Delta\sigma = \frac{E(K + H)}{E + K + H}\,\Delta\varepsilon$$

$$\Delta\sigma_y = (\pm)\,\frac{EH}{E + K + H}\,\Delta\varepsilon \quad \rightarrow \quad \Delta q = \frac{EK}{E + K + H}\,\Delta\varepsilon$$

Chapter 6

Viscoelastic material behavior

Viscoelastic material behavior is a combination of elastic and viscous behavior. Both cases will be illustrated first.

Linear elastic material behavior

For a linear elastic material the stress is uniquely related to the strain by the Young's modulus E [Pa]. The linear elastic truss behaves like a spring with constant stiffness k [N/m].

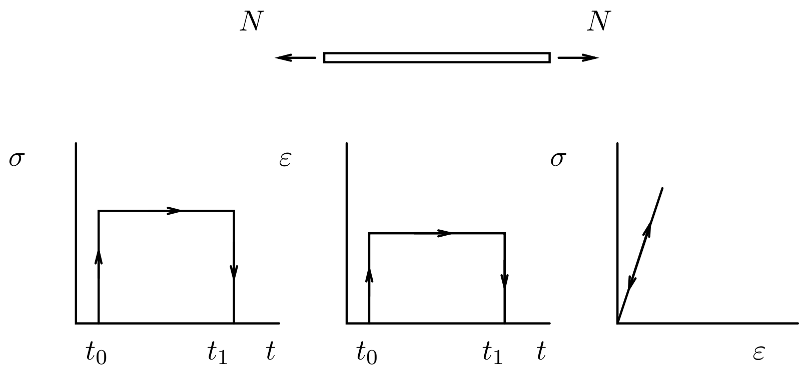


Fig. 66 : *Tensile experiment for linear elastic truss*

$$\varepsilon = \frac{1}{E}\,\sigma \quad \rightarrow \quad \sigma = E\varepsilon \quad \rightarrow \quad \boxed{N = \sigma A = EA\varepsilon = \frac{EA}{l}\,\Delta l = k\,\Delta l}$$

A linear elastic material can be subjected to a loading stress cycle. The work per unit of volume during the cycle appears to be zero indicating that there has been no dissipation. This is also obvious when looking at the stress-strain curve associated with the load cycle : the area below the curve is zero.

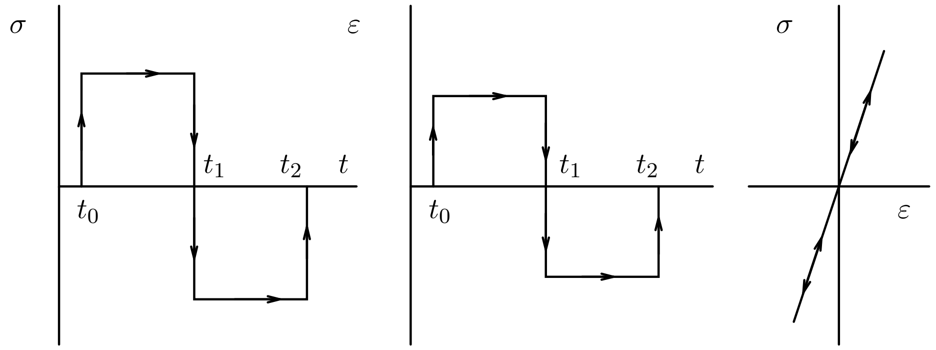


Fig. 67 : *Loading cycle applied to linear elastic truss*

$$
\begin{aligned}
U_d &= \int_{t_0}^{t_1} \sigma \, d\varepsilon + \int_{t_1}^{t_2} \sigma \, d\varepsilon = \int_{t_0}^{t_1} E\varepsilon \, d\varepsilon + \int_{t_1}^{t_2} E\varepsilon \, d\varepsilon \\
&= \tfrac{1}{2} E [\varepsilon_1^2 - \varepsilon_0^2 + \varepsilon_2^2 - \varepsilon_1^2] = 0
\end{aligned}
$$

Linear viscous material behavior

For a linear viscous material the stress is uniquely related to the strain rate by the viscosity η [Pa.s]. The linear viscous "truss" behaves like a dashpot with constant damping value b [Ns/m].

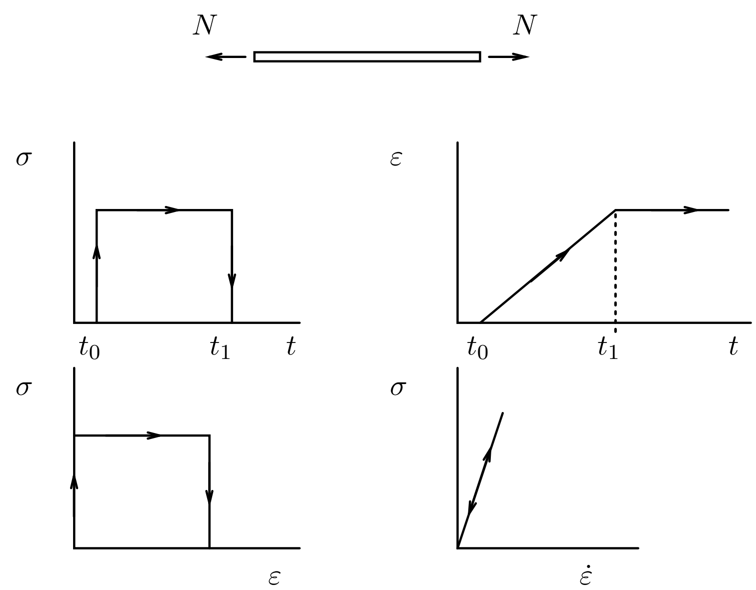


Fig. 68 : *Tensile experiment for linear viscous truss*

$$
\dot{\varepsilon} = \frac{1}{\eta}\,\sigma \quad \rightarrow \quad \sigma = \eta\dot{\varepsilon} \quad \rightarrow \quad \boxed{N = \sigma A = \eta A \dot{\varepsilon} = \frac{\eta A}{l}\,\dot{\Delta l} = b\,\dot{\Delta l}}
$$

The linear viscous material is subjected to a loading stress cycle. The work per unit of volume can be calculated and appears to be non-zero. All the work is dissipated as can be seen from the stress-strain curve : the area included by the stress-strain trajectory represents the specific dissipated energy.

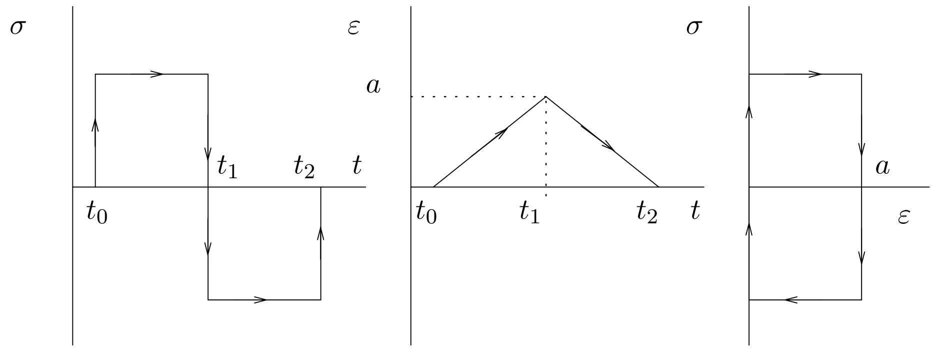


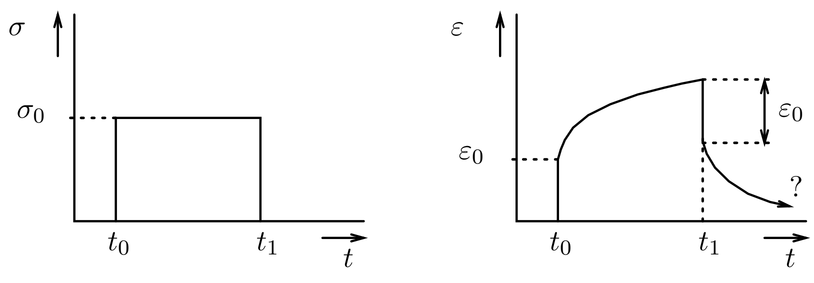
Fig. 69 : *Loading cycle applied to linear viscous truss*

$$
\begin{aligned}
U_d &= \int_{t_0}^{t_1} \sigma \, d\varepsilon + \int_{t_1}^{t_2} \sigma \, d\varepsilon = \int_{t_0}^{t_1} \eta\dot{\varepsilon} \, d\varepsilon + \int_{t_1}^{t_2} \eta\dot{\varepsilon} \, d\varepsilon = \int_{t_0}^{t_1} \eta c \, d\varepsilon - \int_{t_1}^{t_2} \eta c \, d\varepsilon \\
&= \eta c[\varepsilon_1 - \varepsilon_0 - \varepsilon_2 + \varepsilon_1] = 2\eta c a
\end{aligned}
$$

Viscoelastic material behavior

Viscoelastic material behavior is a combination of elastic and viscous behavior. Part of the deformation energy will be dissipated, while the rest is stored as reversible elastic energy. Viscoelastic behavior can be characterized by mechanical models build from springs and dashpots.

We will assume the deformation to be small, so that the choice of stress and strain definitions is irrelevant. First the characteristics of the viscoelastic material behavior will be described, based on experimental observations. To predict the behavior, viscoelastic models are needed, which will be based on the behavior of springs and dashpots.



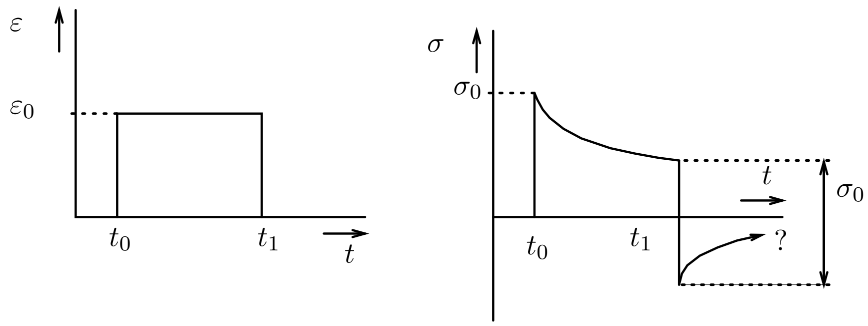


Fig. 70 : *Stress-strain curves for viscoelastic material behavior*

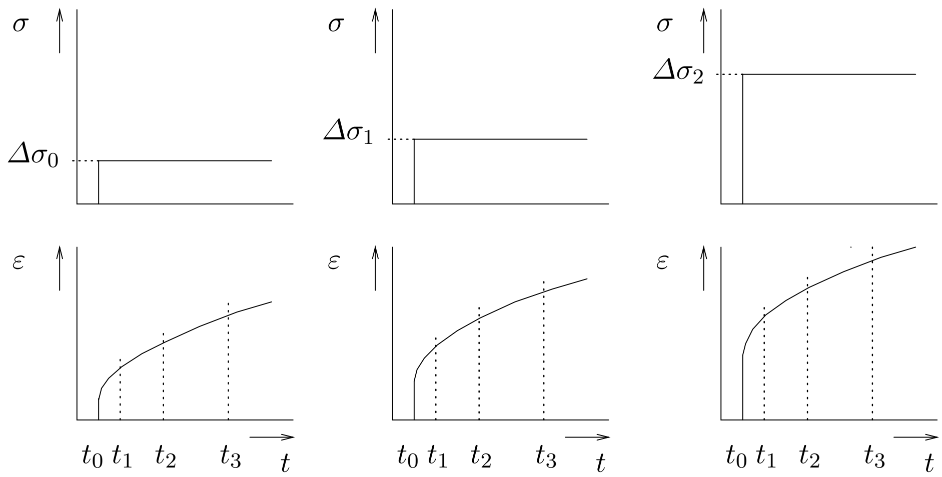
6.1 Uniaxial test

To investigate the characteristics of viscoelastic material behavior, a tensile test is carried out, where a tensile bar is loaded with stress excitations, which are prescribed as a step-function in time.

6.1.1 Proportionality

In a tensile test, a stress-step is applied and the strain is measured as a function of time. The test is repeated for increased stress amplitudes. From the measurement data, the strain values at the same time after loading are plotted against the stress amplitudes. The resulting plots are *isochrones*, because they represent the strain at the same time after loading.

For linear viscoelastic material behavior the isochrones are straight lines. This means that the strain as a function of time is *proportional* to the stress. The strain response can be written as the product of the stress amplitude $\Delta\sigma$ and a function of the time $D(t - t_0)$, whose value is zero for $t < t_0$.



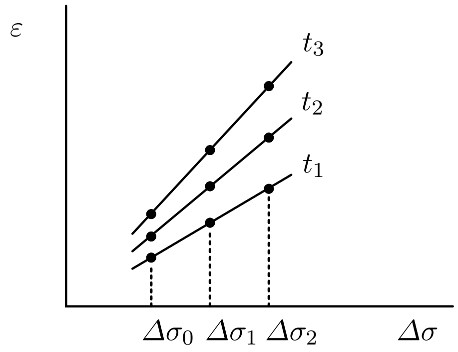


Fig. 71 : *Proportionality of strain response and stress excitation*

$$\varepsilon(t) = \Delta\sigma D(t - t_0) \qquad \text{for} \qquad \forall \quad t \geq t_0$$

6.1.2 Superposition

A tensile test is carried out three times. In the first two tests, a stress step with different amplitude is applied and the strain response is measured. Then, in the third experiment, the two stress steps are applied subsequently and again the strain response is measured.

For linear viscoelastic material behavior, the strain response in the third experiment is the sum of the separate responses in the first two experiments. This means that strain responses can be determined by *superposition*.

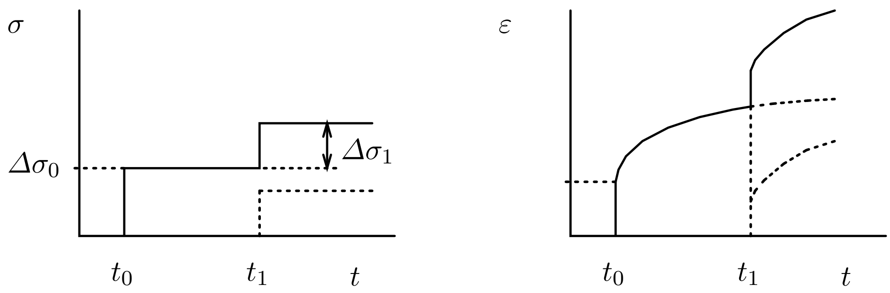


Fig. 72 : *Superposition of strain responses to two stress excitations*

separate excitations

$$t_0 \;:\; \Delta\sigma_0 \;\rightarrow\; \varepsilon(t) = \Delta\sigma_0 D(t - t_0) \qquad \text{for} \qquad t \geq t_0$$
$$t_1 \;:\; \Delta\sigma_1 \;\rightarrow\; \varepsilon(t) = \Delta\sigma_1 D(t - t_1) \qquad \text{for} \qquad t \geq t_1$$

subsequent excitations

$$t_0 \;:\; \Delta\sigma_0 \;\rightarrow$$
$$\varepsilon(t) = \Delta\sigma_0 D(t - t_0) \qquad \text{for} \qquad t_0 \leq t < t_1$$
$$t_1 \;:\; \Delta\sigma = \Delta\sigma_0 + \Delta\sigma_1 \;\rightarrow$$
$$\varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1) \qquad \text{for} \qquad t \geq t_1$$

6.2 Boltzmann integral

Linear viscoelasticity is characterized by two properties, which are explained by considering a strain response to a stress excitation.

1. proportionality : At every time the strain response is proportional to the amplitude of a constant stress step which is applied at $t = t_0$: $\varepsilon_i(t) = \Delta\sigma_i \, D(t - t_0)$ for $t \geq t_0$

   For a unit step at $t = 0$ the strain response is identified as the creep function $D(t)$.

2. superposition : The strain response to two subsequently (at time $t = t_0$ and $t = t_1$) applied constant amplitude ($\Delta\sigma_0$ and $\Delta\sigma_1$) stress steps equals the sum of the separate responses for $t > t_1$: $\varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1)$ for $t \geq t_1$

Every stress excitation can be seen as an infinite sequence of infinitesimal small stress steps. The superposition property then leads to the Boltzmann integral expressing the strain response. This integral is also called Duhamel or memory integral.

For strain excitation and stress response the same observations can be made.

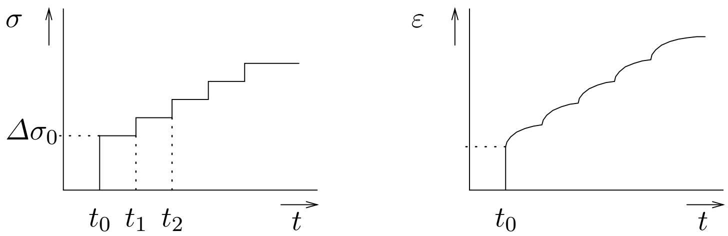


Fig. 73 : *Superposition of strain responses to subsequent stress excitations*

Boltzmann integral for strain

$$
\begin{aligned}
\varepsilon(t) &= \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1) + \Delta\sigma_2 D(t - t_2) + .. \\
&= \sum_{i=1}^{n} \Delta\sigma_i D(t - t_i) \qquad \rightarrow \qquad \text{limit } n\rightarrow\infty \qquad (t\rightarrow\tau) \\
&= \int_{\tau=t_0^-}^{t} D(t-\tau)\, d\sigma(\tau) = \int_{\tau=t_0^-}^{t} D(t-\tau) \frac{d\sigma(\tau)}{d\tau}\, d\tau
\end{aligned}
$$

$$
\boxed{\varepsilon(t) = \int_{\tau=t_0^-}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau}
$$

Boltzmann integral for stress

$$
\boxed{\sigma(t) = \int_{\tau=t_0^-}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau}
$$

6.3 Step excitations

Step excitations are important for the (experimental) characterization of viscoelastic materials. A unit step (amplitude = 1) can be described with the Heaviside function.

The derivative of the unit step function is the Dirac function or unit pulse. It has the important property that integration of the product of a function $f(\tau)$ and $\delta(\tau, t^*)$ over an interval which contains $\tau = t^*$, the "location" of the Dirac pulse, results in the value $f(t^*)$.

Heaviside function

$$\boxed{H(t,t^*) \quad \left\{ \begin{array}{ccc} t < t^* & : & H(t,t^*) = 0 \\ t > t^* & : & H(t,t^*) = 1 \end{array} \right\}}$$

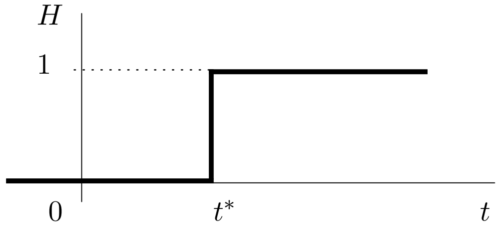


Fig. 74 : *Unit step or Heaviside function*

Dirac function

$$\boxed{\delta(t,t^*) = \frac{d}{dt}\left\{H(t,t^*)\right\}}$$

$$\int_{\tau=0}^{t>t^*} \delta(\tau,t^*)\; d\tau = 1 \quad ; \quad \int_{\tau=0}^{t>t^*} f(\tau)\delta(\tau,t^*)\; d\tau = f(t^*)$$

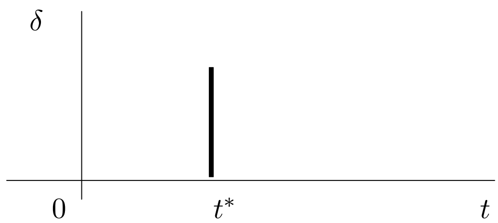


Fig. 75 : *Unit pulse or Dirac function*

6.3.1 Creep (retardation)

The strain response to a stress step excitation at $t = 0$ having an amplitude σ_0 equals $\sigma_0 D(t)$. The creep function $D(t)$ can be measured and experiments have revealed the following properties :

- $\dot{D}(t) \geq 0 \qquad \forall \quad t \geq 0$

- $\ddot{D}(t) < 0 \qquad \forall \quad t \geq 0$

The measured strain response can be used to fit a proposed model for $D(t)$.

stress step $\qquad \sigma(t) = \sigma_0 H(t,0) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0 \delta(t,0)$

strain response

$$\boxed{\varepsilon(t) = \int_{\tau=0^-}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau = \int_{\tau=0^-}^{t} D(t-\tau)\sigma_0\delta(\tau,0)\, d\tau = \sigma_0 D(t)}$$

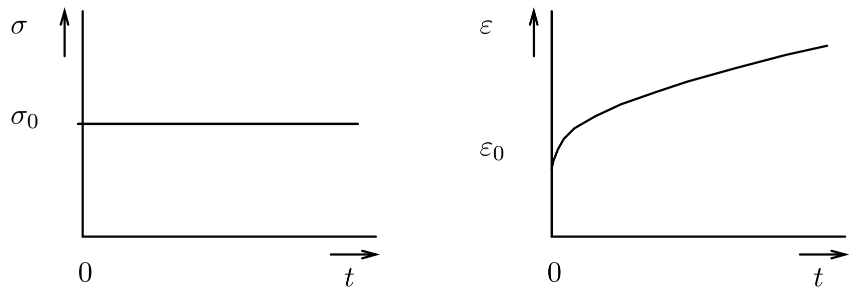


Fig. 76 : *Creep strain response to unit stress step*

6.3.2 Relaxation

The stress response to a strain step excitation at $t = 0$ having an amplitude ε_0 equals $\varepsilon_0 E(t)$. The relaxation function $E(t)$ can be measured and experiments have revealed the following properties :

- $\dot{E}(t) \leq 0 \qquad \forall \quad t \geq 0$
- $\ddot{E}(t) > 0 \qquad \forall \quad t \geq 0$
- $\displaystyle\int_{t=0}^{\infty} \dot{E}(t)\, dt \geq 0 \qquad \rightarrow \qquad \lim_{t\rightarrow\infty} \dot{E}(t) = 0$

The measured stress response can be used to fit a proposed model for $E(t)$.

strain step $\qquad \varepsilon(t) = \varepsilon_0 H(t,0) \qquad \rightarrow \qquad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0)$

stress response

$$\boxed{\sigma(t) = \int_{\tau=0^-}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau = \int_{\tau=0^-}^{t} E(t-\tau)\varepsilon_0\delta(\tau,0)\, d\tau = \varepsilon_0 E(t)}$$

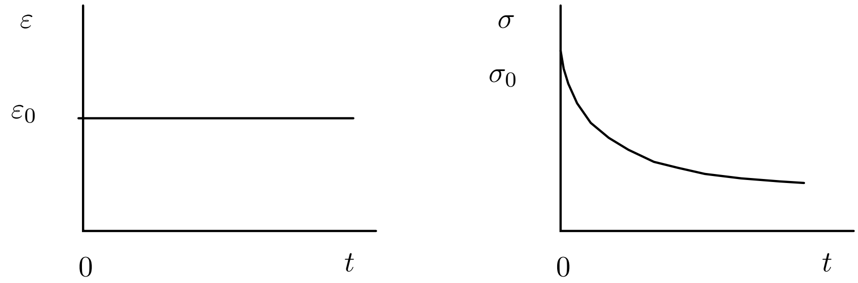


Fig. 77 : *Stress relaxation response to unit strain step*

6.4 Harmonic strain excitation

For the experimental characterization of viscoelastic materials, harmonic excitation is of great importance.

We consider first a tensile test where the strain is prescribed harmonically with an angular frequency ω [rad s^{-1}].

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$

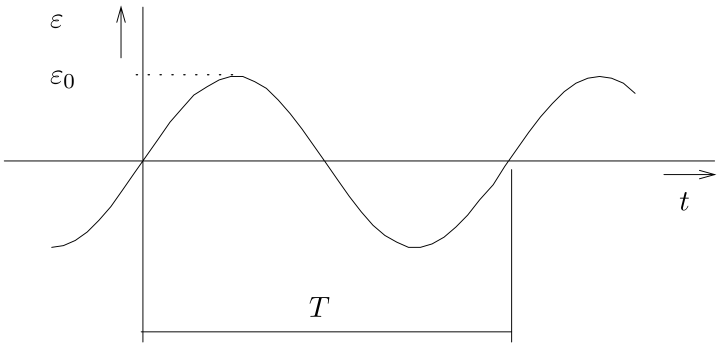


Fig. 78 : *Harmonic strain excitation*

amplitude	ε_0
period and frequency	$T = \dfrac{2\pi}{\omega}$ [s^{-1}] ; $f = \dfrac{1}{T}$

6.4.1 Stress response

The stress response to the harmonic strain excitation can be calculated using the Boltzmann integral. Evaluating the integral involves transformation to another integration variable.

The result reveals two important viscoelastic material parameters : the *storage modulus* E' and the *loss modulus* E'', which both are a function of the angular frequency ω. They can be measured with a Dynamic Mechanical Analysis (DMA) experiment and transformed into a relaxation function $E(t)$. This experiment is more easy to perform and more accurate than the direct measurement of $E(t)$ in a relaxation experiment.

$$\begin{aligned}
\sigma(t) &= \int_{\tau=-\infty}^{t} E(t-\tau)\varepsilon_0\omega\cos(\omega\tau)\,d\tau = \varepsilon_0\omega\int_{\xi=-\infty}^{t} E(t-\tau)\cos(\omega\tau)\,d\tau \\
&\qquad t-\tau=s \quad\rightarrow\quad \tau=t-s \quad\rightarrow\quad d\tau=-ds \\
&= \varepsilon_0\omega\int_{s=0}^{\infty} E(s)\cos\{\omega(t-s)\}\,ds \\
&\qquad \cos(\omega t-\omega s)=\cos(\omega t)\cos(\omega s)+\sin(\omega t)\sin(\omega s) \\
&= \varepsilon_0\left[\omega\int_{s=0}^{\infty} E(s)\sin(\omega s)\,ds\right]\sin(\omega t)+\varepsilon_0\left[\omega\int_{s=0}^{\infty} E(s)\cos(\omega s)\,ds\right]\cos(\omega t) \\
&= \varepsilon_0 E'\sin(\omega t)+\varepsilon_0 E''\cos(\omega t)
\end{aligned}$$

$$\boxed{\begin{array}{lcll}
E'(\omega) & = & \omega\displaystyle\int_{s=0}^{\infty} E(s)\sin(\omega s)\,ds & : \text{ storage modulus} \\
E''(\omega) & = & \omega\displaystyle\int_{s=0}^{\infty} E(s)\cos(\omega s)\,ds & : \text{ loss modulus}
\end{array}}$$

6.4.2 Energy dissipation

The dissipated energy per unit of volume during one period of the harmonic strain excitation can be calculated. This dissipated energy must always be positive. As shown below, it must be concluded that the loss modulus E'' is also positive.

Referring to the calculated stress response, we can conclude that the stress at time $t = 0$ where the strain was taken to be $\varepsilon = 0$, has a positive value. We thus have proved something which we already knew from experiments : there is a phase difference between strain and stress and the stress shows a gain w.r.t. the strain.

$$\begin{aligned}
U_d &= \int_{\varepsilon(0)}^{\varepsilon(T)} \sigma\,d\varepsilon = \int_{t=0}^{T} \sigma\dot{\varepsilon}\,dt \\
&= \int_{t=0}^{T} \left\{\varepsilon_0 E'\sin(\omega t)+\varepsilon_0 E''\cos(\omega t)\right\}\left\{\varepsilon_0\omega\cos(\omega t)\right\}\,dt \\
&= \int_{t=0}^{T} \varepsilon_0^2\omega\left\{E'\sin(\omega t)\cos(\omega t)+E''\cos^2(\omega t)\right\}\,dt \\
&= \int_{t=0}^{T} \varepsilon_0^2\omega\left\{\tfrac{1}{2}E'\sin(2\omega t)+\tfrac{1}{2}E''+\tfrac{1}{2}E''\cos(2\omega t)\right\}\,dt \\
&= \tfrac{1}{2}\varepsilon_0^2\omega\left[-E'\frac{1}{2\omega}\cos(2\omega t)+E''t+E''\frac{1}{2\omega}\sin(2\omega t)\right]_0^{T=\frac{2\pi}{\omega}} \\
&= \tfrac{1}{2}\varepsilon_0^2\omega\left[-E'\frac{1}{2\omega}+E'\frac{1}{2\omega}+E''\frac{2\pi}{\omega}\right] \\
&= \pi\varepsilon_0^2 E''>0 \quad\Rightarrow\quad E''>0 \quad\rightarrow
\end{aligned}$$

$$\sigma(t=0)=\varepsilon_0 E''>0$$

6.4.3 Phase difference

The phase difference between stress and strain results in a so-called hysteresis loop, when a stress-strain diagram is drawn. The area enclosed by the hysteresis loop is a measure for the dissipated energy per unit of volume during one cycle.

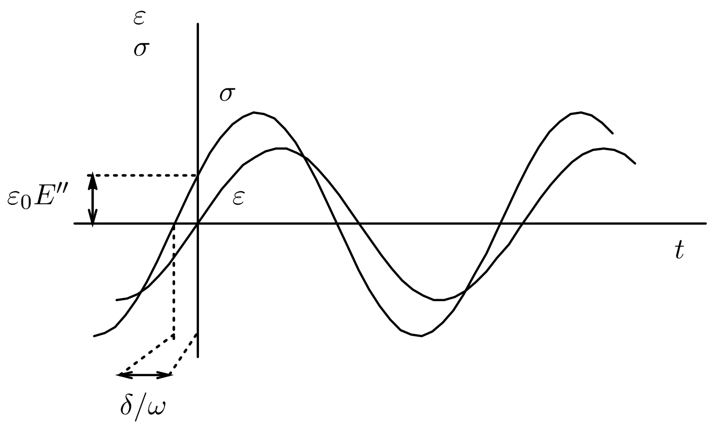


Fig. 79 : *Harmonic strain excitation and stress response*

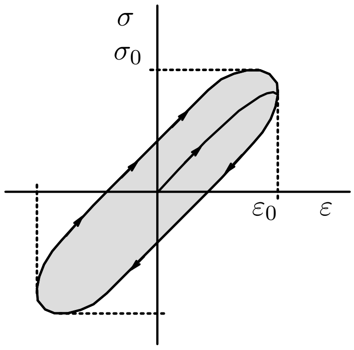


Fig. 80 : *Hysteresis of stress ans strain*

6.4.4 Relation between E', E'' and δ

Writing the stress response with two different relations, results in relations between E', E'' and δ. The amplitude σ_0 of the stress response can also be calculated.

$$
\begin{aligned}
\sigma(t) &= \sigma_0 \sin(\omega t + \delta) \\
&= \sigma_0 \cos(\delta) \sin(\omega t) + \sigma_0 \sin(\delta) \cos(\omega t) \\
\sigma(t) &= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)
\end{aligned}
$$

storage and loss modulus

$$\left.\begin{aligned} E' &= \frac{\sigma_0}{\varepsilon_0}\cos(\delta) \\ E'' &= \frac{\sigma_0}{\varepsilon_0}\sin(\delta) \end{aligned}\right\} \rightarrow \left\{\begin{aligned} &\frac{E''}{E'} = \tan(\delta) \quad \rightarrow \\ &\boxed{\delta = \arctan\left(\frac{E''}{E'}\right)} \end{aligned}\right.$$

amplitude $$\boxed{\sigma_0 = \varepsilon_0\sqrt{(E')^2 + (E'')^2}}$$

6.4.5 Measured E', E'' and $\tan(\delta)$

Typical measured values for $E'(\omega)$, $E''(\omega)$ and $\tan(\delta)$ are shown in the plots. For low and high frequencies, the loss modulus is zero, indicating that there is no dissipation and the material behaves elastically. For high frequencies, the "stiffness" E' is much higher than for low frequencies.

Storage and loss moduli can be measured accurately using DMA test equipment. From $E'(\omega)$ and $E''(\omega)$, the relaxation function $E(t)$ can be calculated using dedicated software.

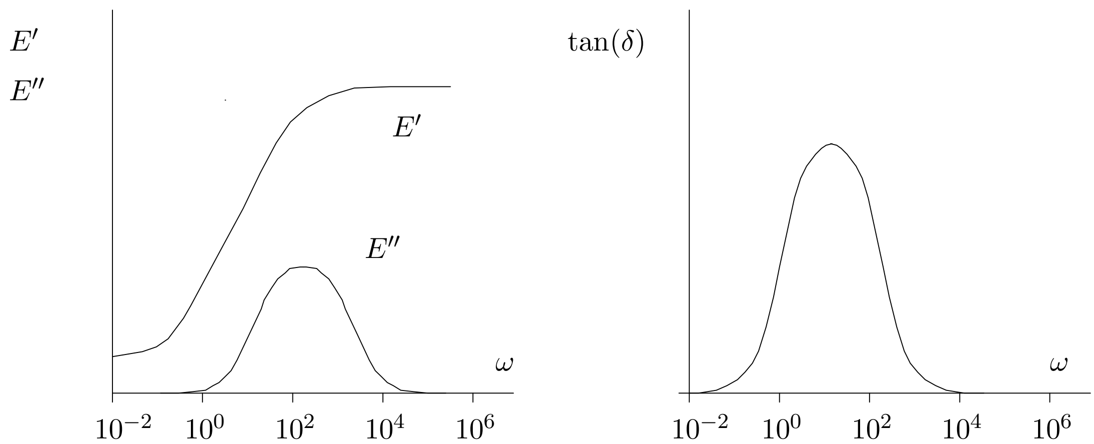


Fig. 81 : *Characteristic values of* E', E'' *and* $\tan(\delta)$

6.5 Harmonic stress excitation

The axial stress can be prescribed harmonically with an angular frequency ω [rad s^{-1}]. The strain response can be calculated with the Boltzmann integral and appears to be characterized by the *the storage compliance* $D'(\omega)$ and the *loss compliance* $D''(\omega)$. Both compliances are positive for all ω. Because $\varepsilon(t = 0) < 0$, the definition of D' includes a minus sign.

$$\sigma(t) = \sigma_0 \sin(\omega t) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

$$\begin{aligned}
\varepsilon(t) &= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\,d\tau = \int_{\tau=-\infty}^{t} D(t-\tau)\sigma_0\omega\cos(\omega\tau)\,d\tau \\
&= \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s)\sin(\omega s)\,ds\right] \sin(\omega t) + \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s)\cos(\omega s)\,ds\right] \cos(\omega t) \\
&= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}$$

$$\boxed{\begin{array}{lclcl}
D'(\omega) & = & \omega \displaystyle\int_{s=0}^{\infty} D(s)\sin(\omega s)\,ds & : & \text{storage compliance} \\
D''(\omega) & = & -\,\omega \displaystyle\int_{s=0}^{\infty} D(s)\cos(\omega s)\,ds & : & \text{loss compliance}
\end{array}}$$

6.5.1 Relation between D', D'' and δ

Writing the strain response with two different relations, results in relations between D', D'' and δ. The amplitude ε_0 of the strain response can also be calculated.

$$\begin{aligned}
\varepsilon(t) &= \varepsilon_0 \sin(\omega t - \delta) \\
&= \varepsilon_0 \cos(\delta)\sin(\omega t) - \varepsilon_0 \sin(\delta)\cos(\omega t) \\
\varepsilon(t) &= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}$$

storage and loss compliance

$$\left.\begin{array}{l}
D' = \dfrac{\varepsilon_0}{\sigma_0}\cos(\delta) \\[2ex]
D'' = \dfrac{\varepsilon_0}{\sigma_0}\sin(\delta)
\end{array}\right\} \quad \rightarrow \quad \left\{\begin{array}{l}
\dfrac{D''}{D'} = \tan(\delta) \quad \rightarrow \\[3ex]
\qquad\qquad \boxed{\delta = \arctan\left(\dfrac{D''}{D'}\right)}
\end{array}\right.$$

amplitude $\quad \boxed{\varepsilon_0 = \sigma_0 \sqrt{(D')^2 + (D'')^2}}$

6.5.2 Measured D' and D''

Typical measured values for $D'(\omega)$, $D''(\omega)$ are shown in the plots. Again it is obvious that the loss compliance is zero for both very low and very high frequencies. The storage compliance is reversely proportional to the frequency.

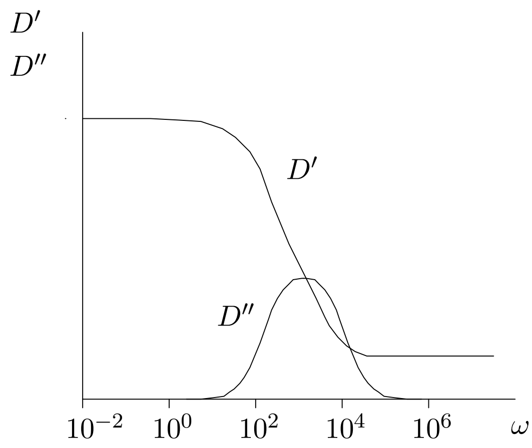


Fig. 82 : *Characteristic values of D' and D''*

6.5.3 Relation between (D', D'') and (E', E'')

There is a relation between storage and loss modulus on the one hand and storage and loss compliance on the other. Remember that there is **not** such a relation between the relaxation function and the creep function.

$$\left.\begin{array}{l} \sigma_0 = \varepsilon_0 \sqrt{(E')^2 + (E'')^2} \\ \\ \varepsilon_0 = \sigma_0 \sqrt{(D')^2 + (D'')^2} \end{array}\right\} \quad \rightarrow \quad [(E')^2 + (E'')^2][(D')^2 + (D'')^2] = 1 \tag{1}$$

$$\frac{D''}{D'} = \frac{E''}{E'} \quad \rightarrow \quad D'' = D' \frac{E''}{E'} \tag{2}$$

$$\begin{array}{lllll} \text{(1) \& (2)} & \rightarrow & D' = \dfrac{E'}{(E')^2 + (E'')^2} & ; & D'' = \dfrac{E''}{(E')^2 + (E'')^2} \\ \text{idem} & & E' = \dfrac{D'}{(D')^2 + (D'')^2} & ; & E'' = \dfrac{D''}{(D')^2 + (D'')^2} \end{array}$$

6.5.4 Complex variables

In literature on viscoelastic behavior and modeling, complex variables are often used. They can be derived easily by writing the strain excitation and the stress response as the real part of a complex number, where Euler's formula $e^{i\alpha x} = \cos(\alpha x) + i \sin(\alpha x)$ is used.

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) = \varepsilon_0 \cos(\omega t - \tfrac{\pi}{2}) = Re\left[\varepsilon_0 e^{-i\frac{\pi}{2}} e^{i\omega t}\right] = Re\left[\varepsilon^* e^{i\omega t}\right]$$
$$\sigma(t) = \sigma_0 \sin(\omega t + \delta) = \sigma_0 \cos(\omega t - \tfrac{\pi}{2} + \delta) = Re\left[\sigma_0 e^{i(\delta - \frac{\pi}{2})} e^{i\omega t}\right] = Re\left[\sigma^* e^{i\omega t}\right]$$

complex modulus and compliance

$$E^* = \frac{\sigma^*}{\varepsilon^*} = \frac{\sigma_0}{\varepsilon_0}\, e^{i\delta} = \frac{\sigma_0}{\varepsilon_0}\cos(\delta) + i\,\frac{\sigma_0}{\varepsilon_0}\sin(\delta) = E' + iE''$$
$$D^* = \frac{\varepsilon^*}{\sigma^*} = \frac{\varepsilon_0}{\sigma_0}\, e^{-i\delta} = \frac{\varepsilon_0}{\sigma_0}\cos(\delta) - i\,\frac{\varepsilon_0}{\sigma_0}\sin(\delta) = D' - iD''$$

dynamic modulus en compliance

$$E_d = |E^*| = \sqrt{(E')^2 + (E'')^2} = \frac{\sigma_0}{\varepsilon_0}$$
$$D_d = |D^*| = \sqrt{(D')^2 + (D'')^2} = \frac{\varepsilon_0}{\sigma_0}$$

6.6 Viscoelastic models

The response of a viscoelastic material is given by the Boltzmann integral and to calculate it we need the creep and/or relaxation functions $D(t)$ and $E(t)$.

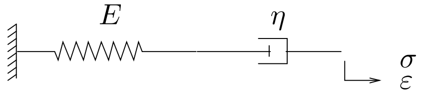
Mathematical expressions can be chosen for these functions taking into account some general requirements. The chosen functions can than be fitted onto data from creep and relaxation tests.

Instead of choosing rather arbitrary functions, they are generally derived from the behavior of one-dimensional mechanical spring-dashpot systems. Simple systems like the Maxwell, Kelvin-Voigt and Standard Solid element, are not always useful, because the lack of parameters prohibits a good fit of experimental data. In practice Generalized Maxwell or Generalized Kelvin-Voigt models are used.

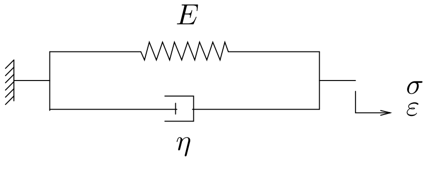
Because creep and relaxation tests may need a long experimental time period and accuracy is not high, harmonic excitation tests are carried out to determine $D'(\omega)$, $D''(\omega)$, $E'(\omega)$ and $E''(\omega)$. These parameters can be converted to $D(t)$ and $E(t)$. These experiments are generally known as D(ynamic) M(echanical) A(nalysis) or D(ynamic) M(echanical) T(hermal) A(nalysis), because time-temperature superposition is mostly used.

In the following we will study some mechanical models. Their behavior is described by a differential equation. Solving this for stress or strain excitations results in the viscoelastic material functions.

Maxwell



Kelvin-Voigt



Standard Solid

Generalized Maxwell

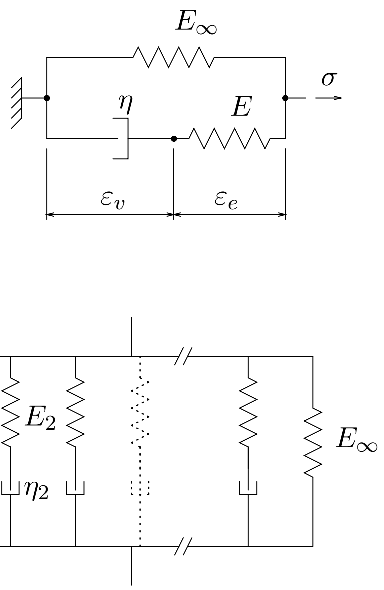


Fig. 83 : *Discrete mechanical models for viscoelastic material behavior*

6.6.1 Maxwell model

One of the simplest models to describe linear viscoelastic material behavior is the Maxwell model. It consists of a spring (modulus E) and a dashpot (viscosity η) in series.

The stress and strain in/of the Maxwell element is related by a first-order differential equation. For both stress and strain excitation, the differential equation can be solved, using appropriate initial conditions. General solutions – integrals for stress and strain – can be derived. (See Appendix)

Fig. 84 : *Maxwell model*

$$\varepsilon = \varepsilon_E + \varepsilon_\eta \quad \rightarrow \quad \dot{\varepsilon} = \dot{\varepsilon}_E + \dot{\varepsilon}_\eta = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

Maxwell : step excitations

For a step excitations of stress and strain the differential equation of the Maxwell model can be solved. The response represents the creep and relaxation functions, respectively.

$$\text{stress step} \quad : \quad \sigma(t) = \sigma_0 H(t,0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \delta(t,0) \quad \rightarrow \quad \text{creep}$$

$$\dot{\varepsilon}(t) = \frac{\sigma_0}{E}\,\delta(t,0) + \frac{\sigma_0}{\eta}$$

$$\varepsilon(t) = \frac{\sigma_0}{E} H(t,0) + \frac{\sigma_0}{\eta}\,t = \sigma_0 \left[\frac{1}{\eta} \left(t + \frac{\eta}{E} \right) \right] = \sigma_0 D(t)$$

$$\text{strain step} \quad : \quad \varepsilon(t) = \varepsilon_0 H(t,0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t,0) \quad \rightarrow \quad \text{relaxation}$$

$$\sigma(t) = \varepsilon_0 E\, e^{-\frac{E}{\eta}t} = \varepsilon_0 E\, e^{-\frac{t}{\tau_m}} = \varepsilon_0 E(t)$$

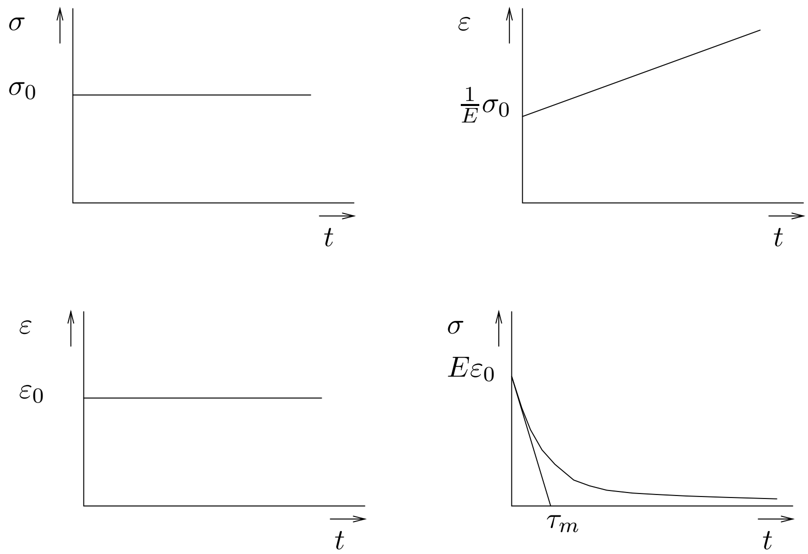


Fig. 85 : *Creep and relaxation for a Maxwell model*

Maxwell : Boltzmann integrals

For a general stress and strain excitation the differential equation of the Maxwell model can also be solved (See Appendix). These general solutions are Boltzmann integrals, which can be used to calculate strain/stress responses to stress/strain excitations.

The creep and relaxation functions of the Maxwell model are readily recognized in the integrals. Response to step excitations reveals that the Maxwell model describes viscoelastic fluid behavior, characterized by a time constant $\tau = \frac{\eta}{E}$ [s].

$$\varepsilon(t) = \frac{1}{\eta} \int_{\tau=-\infty}^{t} \left\{ (t-\tau) + \frac{\eta}{E} \right\} \dot{\sigma}(\tau)\, d\tau = \int_{\tau=-\infty}^{t} D(t-\tau) \dot{\sigma}(\tau)\, d\tau$$

$$\sigma(t) = \int_{\tau=-\infty}^{t} \left\{ E e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\varepsilon}(\tau)\, d\tau = \int_{\tau=-\infty}^{t} E(t-\tau) \dot{\varepsilon}(\tau)\, d\tau$$

Maxwell : harmonic stress excitation

The strain response of the Maxwell model to an harmonic stress excitation is readily calculated from the differential equation. Storage and loss compliances are thus determined.

harmonic stress

$$\sigma(t) = \sigma_0 \sin(\omega t) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

strain response

$$\begin{aligned}
\dot{\varepsilon}(t) &= = \frac{1}{E}\,\sigma_0 \omega \cos(\omega t) + \frac{1}{\eta}\,\sigma_0 \sin(\omega t) \\
\varepsilon(t) &= \sigma_0 \left[\frac{1}{E}\right] \sin(\omega t) - \sigma_0 \left[\frac{1}{\eta\omega}\right] \cos(\omega t) \\
&= \varepsilon_P(t) \qquad\qquad\qquad \varepsilon_H \text{ damps out} \\
&= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}$$

dynamic quantities

$$D' = \frac{1}{E} \qquad ; \qquad D'' = \frac{1}{\eta\omega} \qquad ; \qquad \delta = \arctan\left(\frac{D''}{D'}\right) = \arctan\left(\frac{E}{\eta\omega}\right)$$

Maxwell : harmonic strain excitation

With the Boltzmann integral for the Maxwell model, the stress response to an harmonic strain excitation can be calculated. Storage and loss moduli are obtained as a function of ω. Comparing these functions with measured values reveals that the Maxwell model is generally not adequate to describe viscoelastic behavior of real materials.

harmonic strain

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \qquad \rightarrow \qquad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$

stress response

$$\begin{aligned}
\sigma(t) &= \int_{\tau=-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\,d\tau \\
&= E\varepsilon_0 \omega e^{-\frac{E}{\eta}t} \int_{\tau=0}^{t} e^{\frac{E}{\eta}\tau} \cos(\omega\tau)\,d\tau \\
&= \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2+\omega^2}\,\frac{E}{\eta}\right] e^{-\frac{E}{\eta}t} + \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2+\omega^2}\,\omega\right] \sin(\omega t) + \left[\frac{E\varepsilon_0\omega}{(\frac{E}{\eta})^2+\omega^2}\,\frac{E}{\eta}\right] \cos(\omega t) \\
&= \varepsilon_0 \left[\frac{E\omega}{(\frac{E}{\eta})^2+\omega^2}\,\omega\right] \sin(\omega t) + \varepsilon_0 \left[\frac{E\omega}{(\frac{E}{\eta})^2+\omega^2}\,\frac{E}{\eta}\right] \cos(\omega t) \qquad \text{for } t \geq 0 \\
&= \varepsilon_0 \left[\frac{E\omega^2\tau_m^2}{1+\omega^2\tau_m^2}\right] \sin(\omega t) + \varepsilon_0 \left[\frac{E\omega\tau_m}{1+\omega^2\tau_m^2}\right] \cos(\omega t) \\
&= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)
\end{aligned}$$

dynamic quantities

$$E' = \frac{E\omega^2}{(\frac{E}{\eta})^2 + \omega^2} \quad ; \quad E'' = \frac{E\omega(\frac{E}{\eta})}{(\frac{E}{\eta})^2 + \omega^2} \quad ; \quad \tan(\delta) = \frac{E''}{E'} = \frac{1}{\omega\tau_m}$$

6.6.2 Kelvin-Voigt model

The Kelvin-Voigt model is a simple model for the description of linear viscoelastic material behavior. It consists of a spring (modulus E) parallel to a dashpot (viscosity η).

The stress and strain in/of the Kelvin-Voigt element is related by a first-order differential equation. For strain excitation, this equation directly describes the stress response. For stress excitation, a general integral solution of the differential equation can be derived.

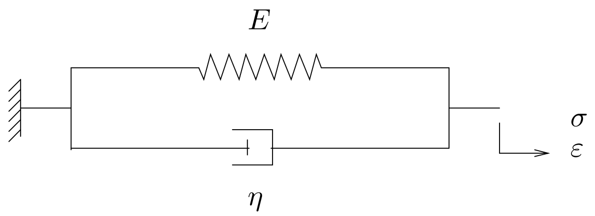


Fig. 86 : *Kelvin model*

$$\sigma = \sigma_E + \sigma_\eta = E\varepsilon + \eta\dot{\varepsilon}$$

Kelvin-Voigt : step excitations

Strain response to a step excitation of stress reveals that the Kelvin-Voigt model describes viscoelastic solid behavior, characterized by the time constant $\tau = \frac{\eta}{E}$ [s]. A stepwise strain excitation leads to infinite stress.

stress step : $\sigma(t) = \sigma_0 H(t,0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0\delta(t,0) \quad \rightarrow$ creep

$$\left.\begin{array}{l} \eta\dot{\varepsilon}(t) + E\varepsilon(t) = \sigma(t) = \sigma_0 H(t,0) \\ \varepsilon(t) = \varepsilon_H(t) + \varepsilon_P = C\, e^{-\frac{E}{\eta}t} + \frac{\sigma_0}{E} \\ \varepsilon(t=0) = 0 \end{array}\right\} \quad \rightarrow \quad C = -\frac{\sigma_0}{E}$$

$$\varepsilon(t) = \frac{\sigma_0}{E}\left[1 - e^{-\frac{E}{\eta}t}\right] = \sigma_0 D(t)$$

strain step : $\varepsilon(t) = \varepsilon_0 H(t,0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0\delta(t,0) \quad \rightarrow$ relaxation

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t)$$
$$\sigma(t) = E\varepsilon_0 + \eta\varepsilon_0\delta(t,0) = \varepsilon_0\left[E + \eta\delta(t,0)\right] = \infty$$

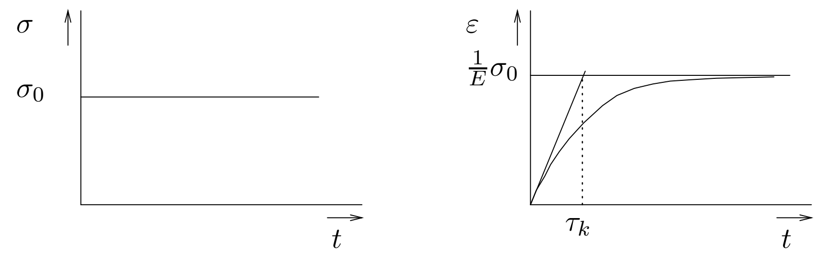


Fig. 87 : *Creep of a Kelvin model*

Kelvin-Voigt : Boltzmann integral

The general solution for the strain response to a stress excitation is given by a Boltzmann integral, in which we recognize the creep function of the Kelvin-Voigt element. For a general strain excitation the stress response can be calculated directly from the Kelvin-Voigt element equation.

$$\varepsilon(t) = \frac{1}{E} \int_{\tau=-\infty}^{t} \left\{ 1 - e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\sigma}(\tau)\, d\tau = \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau$$

Kelvin-Voigt : harmonic stress-excitation

For the Kelvin-Voigt model, the storage and loss compliance can be calculated. The Boltzmann integral with the Kelvin-Voigt creep function is used to calculate the strain response for an harmonic stress excitation.

harmonic stress

$$\sigma(t) = \sigma_0 \sin(\omega t) \qquad \rightarrow \qquad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

strain response

$$\begin{aligned}
\varepsilon(t) &= \int_{\tau=0}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau \\
&= \sigma_0 \left[\frac{1}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta^2} \right] \sin(\omega t) - \sigma_0 \left[\frac{\omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{1}{\eta} \right] \cos(\omega t) \\
&= \sigma_0 \left[\frac{1}{E(1+\omega^2\tau_k^2)} \right] \sin(\omega t) - \sigma_0 \left[\frac{\omega\tau_k}{E(1+\omega^2\tau_k^2)} \right] \cos(\omega t) \\
&= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t)
\end{aligned}$$

dynamic quantities

$$D'(\omega) = \frac{1}{\left(\frac{E}{\eta}\right)^2 + \omega^2}\,\frac{E}{\eta^2} = \frac{1}{E(1+\omega^2\tau_k^2)}$$
$$D''(\omega) = \frac{\omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2}\,\frac{1}{\eta} = \frac{\omega\tau}{E(1+\omega^2\tau_k^2)}$$
$$\tan(\delta) = \frac{D''}{D'} = \omega\tau_k \qquad \rightarrow \qquad \delta = arctan\left(\frac{\eta\omega}{E}\right)$$

6.6.3 Standard Solid model

The Standard Solid model consists of a parallel arrangement of a Maxwell element (modulus E, viscosity η) and a linear spring (modulus E_∞).

This model incorporates the Maxwell model ($E_\infty = 0$) and the Kelvin-Voigt model ($E = 0$). The stress-strain relation is described by a differential equation, which can be solved resulting in Boltzmann integrals for strain and stress.

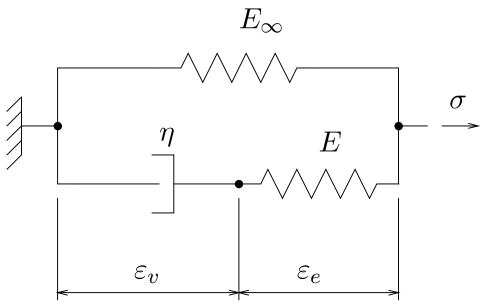


Fig. 88 : *Standard Solid model*

constitutive relations

- $\sigma = \sigma_\infty + \sigma_{ve}$
- $\dot{\varepsilon} = \dot{\varepsilon}_v + \dot{\varepsilon}_e$
- $\dot{\varepsilon}_v = \frac{1}{\eta}\,\sigma_{ve}$
- $\sigma_{ve} = E\varepsilon_e \quad \rightarrow \quad \dot{\varepsilon}_e = \frac{1}{E}\,\dot{\sigma}_{ve}$
- $\varepsilon = \frac{1}{E_\infty}\,\sigma_\infty$

constitutive equation

$$\begin{aligned}
\sigma &= \sigma_\infty + \sigma_{ve} = E_\infty\varepsilon + \eta\dot{\varepsilon}_v \\
&= E_\infty\varepsilon + \eta(\dot{\varepsilon} - \dot{\varepsilon}_e) = E_\infty\varepsilon + \eta\dot{\varepsilon} - \eta\,\frac{\dot{\sigma}_{ve}}{E} \\
&= E_\infty\varepsilon + \eta\dot{\varepsilon} - \frac{\eta}{E}\,(\dot{\sigma} - E_\infty\dot{\varepsilon}) \quad \rightarrow \\
\sigma + \frac{\eta}{E}\,\dot{\sigma} &= E_\infty\varepsilon + \frac{\eta(E+E_\infty)}{E}\,\dot{\varepsilon}
\end{aligned}$$

Standard Solid : step excitations

Solutions for the differential equation when applying a step in the stress or a step in the strain can be derived. The time constant for creep is defined as $\tau_c = \dfrac{\eta}{E} + \dfrac{\eta}{E_\infty}$ and the time constant for relaxation as $\tau_r = \dfrac{\eta}{E}$. They represent the intersection point of the tangent to the creep/relaxation curve at $t = 0$ and the asymptote for strain ($\frac{\sigma_0}{E_\infty}$) and stress ($\varepsilon_0\, E_\infty$), respectively.

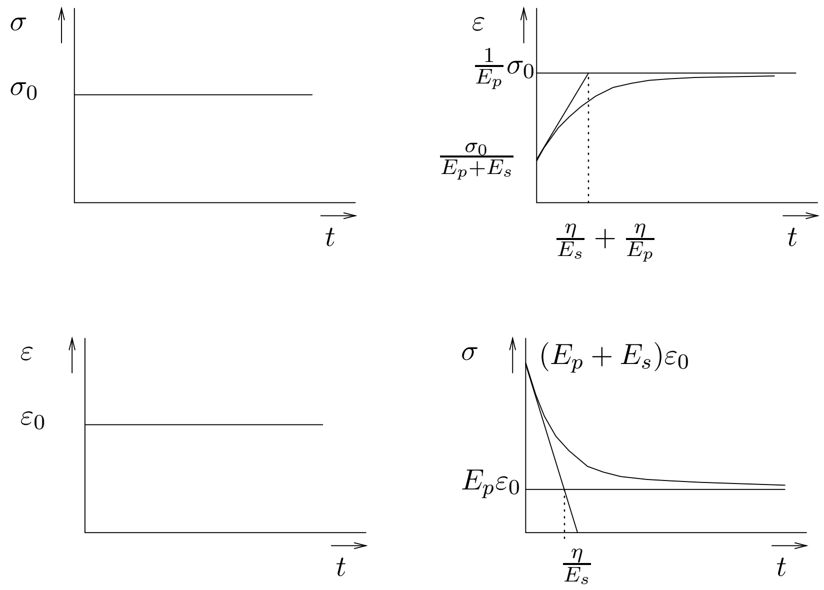


Fig. 89 : *Creep and relaxation of a Standard Solid model*

Standard Solid : Boltzmann integrals

In the Boltzmann integrals for strain and stress, the creep and relaxation functions of the Standard Solid element are readily recognized.

creep

$$\begin{aligned}
\varepsilon(t) &= \int_{\tau=-\infty}^{t} \left\{ \frac{1}{E_\infty} - \frac{E}{E_\infty(E_\infty + E)} e^{-\frac{E_\infty E}{\eta(E_\infty+E)}\,(t-\tau)} \right\} \dot{\sigma}(\tau)\, d\tau \\
&= \int_{\tau=-\infty}^{t} D(t-\tau)\dot{\sigma}(\tau)\, d\tau
\end{aligned}$$

relaxation

$$\begin{aligned}
\sigma(t) &= \int_{\tau=-\infty}^{t} \left\{ E_\infty + E e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\varepsilon}(\tau)\, d\tau \\
&= \int_{\tau=-\infty}^{t} E(t-\tau)\dot{\varepsilon}(\tau)\, d\tau
\end{aligned}$$

6.6.4 Generalized Maxwell model

Both the Maxwell and the Kelvin-Voigt models are to simple to describe the viscoelastic behavior of real materials. Combining a number of Maxwell elements in a parallel configuration, leads to the generalized Maxwell model, which mostly also has an extra parallel spring for the correct description of long-term behavior of viscoelastic solid materials. Such a model is generally used for experimental characterization of the behavior of linear viscoelastic materials in a Dynamic Mechanical (Thermal) Analysis (DM(T)A) test.

The creep function $E(t)$ is easily determined and has a number of time constants to characterize the viscoelastic material response. A model like the generalized Maxwell model is therefore also referred to as multi mode.

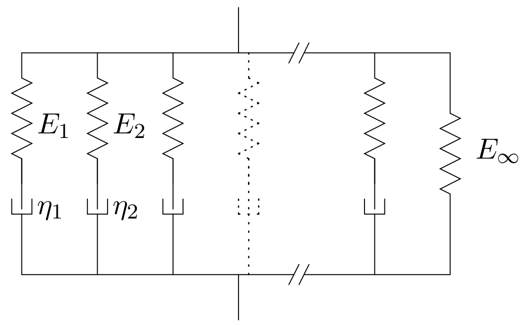


Fig. 90 : *Generalized Maxwell model*

$$E(t) = E_\infty + \sum_i E_i e^{-\frac{t}{\tau_i}} \qquad ; \qquad \tau_i = \frac{\eta_i}{E_i}$$

equilibrium modulus	$E_\infty = \lim_{t \longrightarrow \infty} E(t)$
glass modulus	$E_g = \lim_{t \longrightarrow 0} E(t) = E_\infty + \sum_i E_i$

6.6.5 Generalized Kelvin model

The generalized Kelvin model consists of a number of Kelvin-Voigt elements arranged in series. An extra spring – sometimes a dashpot – is also provided.

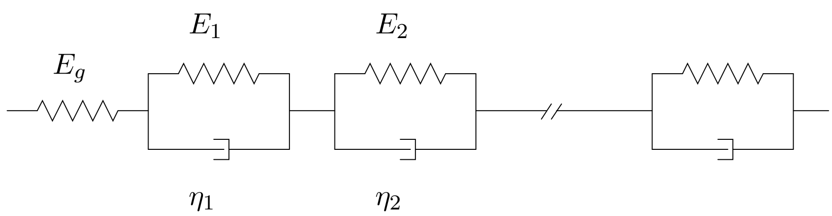


Fig. 91 : *Generalized Kelvin model*

$$D(t) = \frac{1}{E_g} + \sum_i \frac{1}{E_i}(1 - e^{-\frac{t}{\tau_i}}) \qquad ; \qquad \tau_i = \frac{\eta_i}{E_i}$$

$$= D_g + \sum_i D_i(1 - e^{-\frac{t}{\tau_i}})$$

glass compliance $\qquad D_g = \frac{1}{E_g} = \lim_{t \longrightarrow 0} D(t)$

equilibrium compliance $\qquad D_\infty = \lim_{t \longrightarrow \infty} D(t) = D_g + \sum_i D_i$

Bibliography

[1] Bathe, K.-J. *Finite Element Procedures.* Prentice Hall, New Jersey, 1996.

[2] Fenner, Roger T. *Mechanics of solids.* Blackwell Scientific Publications, 1989.

[3] Gordon, J.E. *The new science of strong materials.* Penguin Books, 1976.

[4] Gordon, J.E. *Structures.* Penguin Books, 1978.

[5] Groover, M.P. *Fundamentals of modern manufacturing. Materials, Processes, and Systems.* Prentice Hall, 1996.

[6] Hunter, S.C. *Mechanics of continuous media, 2nd edition.* Ellis Horwood Limited, 1983.

[7] Schreurs, Piet *Marc/Mentat tutorial : Staaf- en balkconstructies.*, 2009.

[8] Schreurs, Piet *Marc/Mentat tutorial : Non-linear and time-dependent material models.*, 2009.

Appendix A

General solutions for differential equations

A general solution is determined for a first order linear differential equation in the time-dependent variable $y(t)$. The right-hand side is time-dependent. When the left-hand side has no time derivative term ($a_1 = 0$), the solution is trivial.

First a solution y_H of the homogeneous differential equation is determined. Subsequently one particular solution y_P is determined. The general solution is the sum of the homogeneous and the particular solution. Finally initial conditions are used to determine the integration constants.

$$a_0 y + a_1 \dot{y} = b_0 x + b_1 \dot{x}$$

Homogeneous solution

The homogeneous solution is the solution of the homogeneous differential equation, with the right-hand side being zero. The solution is determined by *separation of variables* and appears to be an exponential function. It comprises an integration constant C, which has to be determined later, using the initial condition.

$$\begin{aligned}
&a_0 y + a_1 \dot{y} = 0 \quad \rightarrow \\
&a_0 y + a_1 \frac{dy}{dt} = 0 \quad \rightarrow \quad \frac{dy}{y} = -\frac{a_0}{a_1}\, dt \quad \rightarrow \quad \text{integration} \quad \rightarrow \\
&\int \frac{dy}{y} = -\frac{a_0}{a_1} \int dt \quad \rightarrow \quad \ln(y) = -\frac{a_0}{a_1} t + \hat{C} \quad \rightarrow \quad y_H(t) = e^{-\frac{a_0}{a_1} t + \hat{C}} = C e^{-\frac{a_0}{a_1} t}
\end{aligned}$$

Particular solution

A particular solution of the differential equation is denoted as : $y_P = g(t) y_H$. Substitution in the differential equation results in a differential equation for $g(t)$, which can be solved by integration over the time period $[s\ t]$.

$$
\begin{aligned}
&y_P = g(t) y_H \quad \rightarrow \quad a_0 y + a_1 \dot{y} = b_0 x + b_1 \dot{x} \quad \Rightarrow \\
&a_0 g y_H + a_1 \dot{g} y_H + a_1 g \dot{y}_H = b_0 x + b_1 \dot{x} \quad \rightarrow \\
&a_1 \dot{g} y_H = b_0 x + b_1 \dot{x} \quad \rightarrow \quad \dot{g} = \frac{1}{a_1 y_H} \left(b_0 x + b_1 \dot{x} \right) = \frac{1}{a_1 C} e^{\frac{a_0}{a_1} t} \left(b_0 x + b_1 \dot{x} \right) \\
&g(t) = \frac{1}{a_1 C} \int_s^t e^{\frac{a_0}{a_1} \tau} \left(b_0 x + b_1 \dot{x} \right) \, d\tau + g(s) \quad \rightarrow \\
&y_P = g(t) y_H = \frac{1}{a_1} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} \left(b_0 x + b_1 \dot{x} \right) \, d\tau + C g(s) e^{-\frac{a_0}{a_1} t}
\end{aligned}
$$

General solution

The general solution is the sum of the homogeneous and the particular solution. The initial solutiion $y(s)$ can be incorporated in the solution.

$$
\begin{aligned}
y(t) &= y_H(t) + y_P(t) \\
&= \frac{1}{a_1} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} \left(b_0 x + b_1 \dot{x} \right) \, d\tau + \{1 + g(s)\} C e^{-\frac{a_0}{a_1} t} \\
&\qquad \text{initial condition} \quad y(s) = \{1 + g(s)\} C e^{-\frac{a_0}{a_1} s} \quad \rightarrow \quad \{1 + g(s)\} = y(s) \frac{1}{C} e^{\frac{a_0}{a_1} s} \\
&= \frac{1}{a_1} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} \left(b_0 x + b_1 \dot{x} \right) \, d\tau + y(s) e^{-\frac{a_0}{a_1}(t-s)}
\end{aligned}
$$

General solution for $a_0 = 0$

When $a_0 = 0$, the general solution can be simplified and the first term in the integral can be integrated by parts.

$$
\begin{aligned}
y(t) &= \frac{1}{a_1} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} \left(b_0 x + b_1 \dot{x} \right) \, d\tau + y(s) e^{-\frac{a_0}{a_1}(t-s)} \\
&= \frac{1}{a_1} \int_s^t \left(b_0 x + b_1 \dot{x} \right) \, d\tau + y(s) \\
&\qquad \int_s^t x \, d\tau = \int_s^t (t - \tau) \dot{x} \, d\tau + (t - s) x(s) \\
&= \frac{b_0}{a_1} \int_s^t (t - \tau) \dot{x} \, d\tau + \frac{b_0}{a_1} (t - s) x(s) + \frac{b_1}{a_1} \int_s^t \dot{x} \, d\tau + y(s) \\
&= \int_s^t \left\{ \frac{b_0}{a_1} (t - \tau) + \frac{b_1}{a_1} \right\} \dot{x} \, d\tau + \frac{b_0}{a_1} (t - s) x(s) + y(s)
\end{aligned}
$$

General solution for $a_0 \neq 0$

When $a_0 \neq 0$ the first term is also integrated by parts.

$$y(t) = \frac{1}{a_1} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} \left(b_0 x + b_1 \dot{x}\right)\, d\tau + y(s) e^{-\frac{a_0}{a_1}(t-s)}$$

$$\frac{1}{a_1} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} b_0 x\, d\tau$$

$$= \frac{1}{a_1} \int_s^t \frac{d}{d\tau} \left\{ \frac{a_1}{a_0} e^{-\frac{a_0}{a_1}(t-\tau)} b_0 x \right\} d\tau - \frac{1}{a_1} \int_s^t \frac{a_1}{a_0} e^{-\frac{a_0}{a_1}(t-\tau)} b_0 \dot{x}\, d\tau$$

$$= \frac{b_0}{a_0} x(t) - \frac{b_0}{a_0} e^{-\frac{a_0}{a_1}(t-s)} x(s) - \frac{b_0}{a_0} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} \dot{x}\, d\tau$$

$$= \frac{b_0}{a_0} x(t) - \frac{b_0}{a_0} e^{-\frac{a_0}{a_1}(t-s)} x(s) - \frac{b_0}{a_0} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} \dot{x}\, d\tau +$$

$$\frac{b_1}{a_1} \int_s^t e^{-\frac{a_0}{a_1}(t-\tau)} \dot{x}\, d\tau + y(s) e^{-\frac{a_0}{a_1}(t-s)}$$

$$= \int_s^t \left\{ \frac{b_0}{a_0} + \left(\frac{b_1}{a_1} - \frac{b_0}{a_0} \right) e^{-\frac{a_0}{a_1}(t-\tau)} \right\} \dot{x}\, d\tau + \frac{b_0}{a_0} \left(1 - e^{-\frac{a_0}{a_1}(t-s)} \right) x(s) + y(s) e^{-\frac{a_0}{a_1}(t-s)}$$

A.1 General solutions for spring-dashpot models

The mechanical behavior of simple spring-dashpot elements is described by a first-order linear differential equation. For strain and stress excitation, the response can be determined from the general solution.

Maxwell with strain excitation

$$\sigma + \frac{\eta}{E} \dot{\sigma} = \eta \dot{\varepsilon} \quad \rightarrow$$

$$y = \sigma \ ; \ x = \varepsilon \quad ; \quad a_0 = 1 \ ; \ a_1 = \frac{\eta}{E} \ ; \ b_0 = 0 \ ; \ b_1 = \eta$$

$$\sigma(t) = \int_s^t \left\{ E e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\varepsilon}(\tau)\, d\tau + \sigma(s) e^{-\frac{E}{\eta}(t-s)}$$

Maxwell with stress excitation

$$\eta \dot{\varepsilon} = \sigma + \frac{\eta}{E} \dot{\sigma} \quad \rightarrow$$

$$y = \varepsilon \ ; \ x = \sigma \quad ; \quad a_0 = 0 \ ; \ a_1 = \eta \ ; \ b_0 = 1 \ ; \ b_1 = \frac{\eta}{E}$$

$$\varepsilon(t) = \int_s^t \left\{ \frac{1}{\eta}(t-\tau) + \frac{1}{E} \right\} \dot{\sigma}(\tau)\, d\tau + \frac{1}{\eta}(t-s)\sigma(s) + \varepsilon(s)$$

Kelvin with strain excitation

$$\sigma = E\varepsilon + \eta\dot{\varepsilon} \quad \rightarrow$$
$$y = \sigma\ ;\ x = \varepsilon \quad ; \quad a_0 = 1\ ;\ a_1 = 0\ ;\ b_0 = E\ ;\ b_1 = \eta$$

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t)$$

Kelvin with stress excitation

$$E\varepsilon + \eta\dot{\varepsilon} = \sigma \quad \rightarrow$$
$$y = \varepsilon\ ;\ x = \sigma \quad ; \quad a_0 = E\ ;\ a_1 = \eta\ ;\ b_0 = 1\ ;\ b_1 = 0$$

$$\varepsilon(t) = \frac{1}{E}\int_s^t \left\{1 - e^{-\frac{E}{\eta}(t-\tau)}\right\}\dot{\sigma}(\tau)\,d\tau + \frac{1}{E}\left(1 - e^{-\frac{E}{\eta}(t-s)}\right)\sigma(s) + \varepsilon(s)e^{-\frac{E}{\eta}(t-s)}$$

Standard Solid with strain excitation

$$\sigma + \frac{\eta}{E}\dot{\sigma} = E_\infty\varepsilon + \frac{\eta}{E}(E_\infty + E)\dot{\varepsilon} \quad \rightarrow$$
$$y = \sigma\ ;\ x = \varepsilon \quad ; \quad a_0 = 1\ ;\ a_1 = \frac{\eta}{E}\ ;\ b_0 = E_\infty\ ;\ b_1 = \frac{\eta}{E}(E_\infty + E)$$

$$\sigma(t) = \int_s^t \left\{E_\infty + Ee^{-\frac{E}{\eta}(t-\tau)}\right\}\dot{\varepsilon}(\tau)\,d\tau + E_\infty\left(1 - e^{-\frac{E}{\eta}(t-s)}\right)\varepsilon(s) + \sigma(s)e^{-\frac{E}{\eta}(t-s)}$$

Standard Solid with stress excitation

$$E_\infty\varepsilon + \frac{\eta}{E}(E_\infty + E)\dot{\varepsilon} = \sigma + \frac{\eta}{E}\dot{\sigma} \quad \rightarrow$$
$$y = \varepsilon\ ;\ x = \sigma \quad ; \quad a_0 = E_\infty\ ;\ a_1 = \frac{\eta}{E}(E_\infty + E)\ ;\ b_0 = 1\ ;\ b_1 = \frac{\eta}{E}$$

$$\varepsilon(t) = \int_s^t \frac{1}{E_\infty}\left\{1 - \frac{E}{E_\infty + E}\, e^{-\frac{E_\infty E}{\eta(E_\infty + E)}\,(t-\tau)}\right\}\dot{\sigma}(\tau)\,d\tau +$$
$$\frac{1}{E_\infty}\left(1 - e^{-\frac{E_\infty E}{\eta(E_\infty + E)}(t-s)}\right)\sigma(s) + \varepsilon(s)e^{-\frac{E_\infty E}{\eta(E_\infty + E)}\,(t-s)}$$