APPLIED ELASTICITY

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VECTORS

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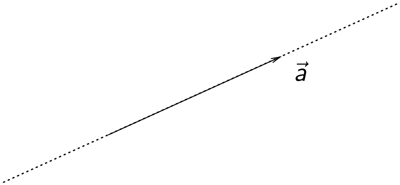
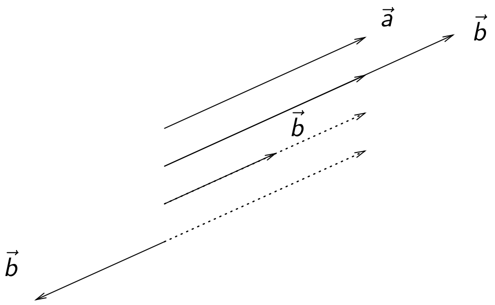
Vector



$$\vec{a} = ||\vec{a}|| \; \vec{e}$$

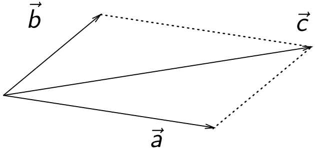
length	:	$\|\|\vec{a}\|\|$		
direction vector	:	$\vec{e}$	;	$\|\|\vec{e}\,\|\| = 1$
zero vector	:	$\vec{0}$		
unit vector	:	$\vec{e}$	;	$\|\|\vec{e}\,\|\| = 1$

Scalar multiplication



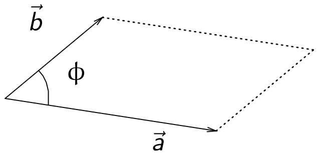
$$\vec{b} = \alpha\vec{a}$$

Sum of two vectors



$$\vec{c} = \vec{a} + \vec{b}$$

Scalar product

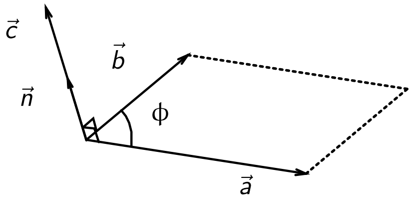


$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \; ||\vec{b}|| \; \cos(\phi)$$

properties

1. $\vec{a} \cdot \vec{a} = ||\vec{a}||^2 \geq 0$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Vector product

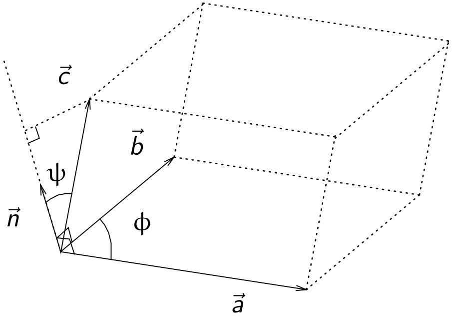


$$\vec{c} = \vec{a} * \vec{b} = \{\|\vec{a}\|\ \|\vec{b}\|\ \sin(\phi)\}\vec{n}$$
$$= [\text{area parallelogram}]\ \vec{n}$$

properties

1. $\vec{b} * \vec{a} = -\vec{a} * \vec{b}$
2. $\vec{a} * (\vec{b} * \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

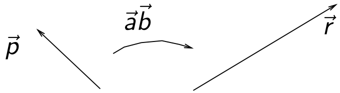
Triple product



$$\vec{a} * \vec{b} \cdot \vec{c} = \left\{ ||\vec{a}||||\vec{b}|| \sin(\phi) \right\} (\vec{n} \cdot \vec{c})$$
$$= (\text{area parrlg}) * (\text{height}) = |\text{volume}| = |V|$$

$$\begin{array}{lcll} V > 0 & \rightarrow & \{\vec{a}, \vec{b}, \vec{c}\} & \text{right handed} \\ V < 0 & \rightarrow & \{\vec{a}, \vec{b}, \vec{c}\} & \text{left handed} \\ V = 0 & \rightarrow & \{\vec{a}, \vec{b}, \vec{c}\} & \text{dependent} \end{array}$$

Tensor product



$$\vec{a}\vec{b} = \text{dyad} = \text{linear vector transformation}$$

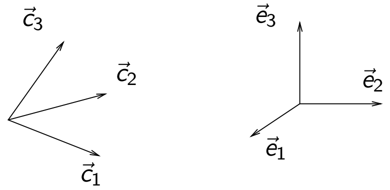
$$\vec{a}\vec{b} \cdot \vec{p} = \vec{a}(\vec{b} \cdot \vec{p}) = \vec{r}$$

$$\vec{a}\vec{b} \cdot (\alpha\vec{p} + \beta\vec{q}) = \alpha\vec{a}\vec{b} \cdot \vec{p} + \beta\vec{a}\vec{b} \cdot \vec{q} = \alpha\vec{r} + \beta\vec{s}$$

conjugated dyad $\qquad (\vec{a}\vec{b})^c = \vec{b}\vec{a} \neq \vec{a}\vec{b}$

symmetric dyad $\qquad (\vec{a}\vec{b})^c = \vec{a}\vec{b}$

Vector basis



random basis	$\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$	;	$\vec{c}_1 * \vec{c}_2 \cdot \vec{c}_3 \neq 0$
orthonormal basis	$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$		$(\delta_{ij} = \text{Kronecker delta})$

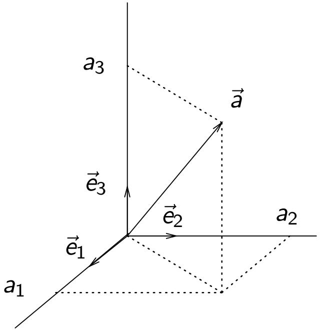
$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \quad \rightarrow \quad \vec{e}_i \cdot \vec{e}_j = 0 \quad | \quad i \neq j \quad ; \quad \vec{e}_i \cdot \vec{e}_i = 1$

right-handed basis $\quad \vec{e}_1 * \vec{e}_2 = \vec{e}_3 \quad ; \quad \vec{e}_2 * \vec{e}_3 = \vec{e}_1 \quad ; \quad \vec{e}_3 * \vec{e}_1 = \vec{e}_2$

Components of a vector

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \sum_{i=1}^{3} a_i\vec{e}_i = a_i\vec{e}_i$$

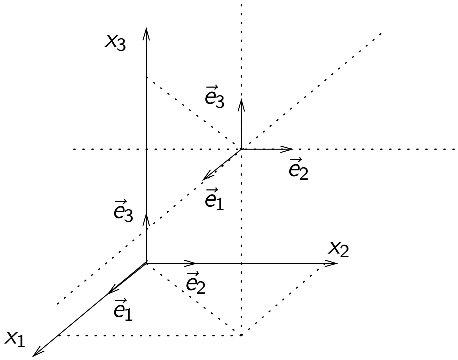
$$= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \underset{\sim}{a}^T \underset{\sim}{\vec{e}} = \underset{\sim}{\vec{e}}^T \underset{\sim}{a}$$



$$a_i = \vec{a} \cdot \vec{e}_i \qquad i = 1, 2, 3 \quad \rightarrow$$

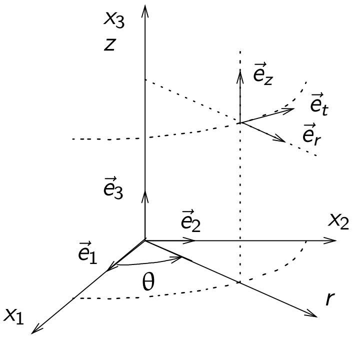
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \vec{a}\cdot\vec{e}_1 \\ \vec{a}\cdot\vec{e}_2 \\ \vec{a}\cdot\vec{e}_3 \end{bmatrix} = \vec{a} \cdot \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \vec{a} \cdot \underset{\sim}{\vec{e}}$$

Cartesian coordinate system



Cartesian coordinates	:	(x_1, x_2, x_3)	or	(x, y, z)
Cartesian basis	:	$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$	or	$\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$

Cylindrical coordinate system

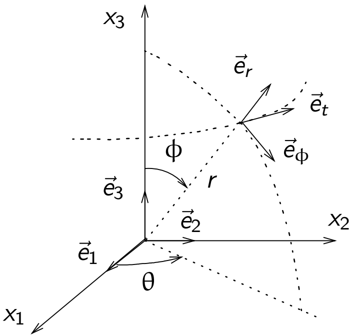


cylindrical coordinates	:	(r, θ, z)
cylindrical basis	:	$\{\vec{e}_r(\theta), \vec{e}_t(\theta), \vec{e}_z\}$

$$\vec{e}_r(\theta) = \cos(\theta)\vec{e}_1 + \sin(\theta)\vec{e}_2 \quad ; \quad \vec{e}_t(\theta) = -\sin(\theta)\vec{e}_1 + cos(\theta)\vec{e}_2 \quad ; \quad \vec{e}_z = \vec{e}_3$$

$$\frac{\partial \vec{e}_r}{\partial \theta} = -\sin(\theta)\vec{e}_1 + cos(\theta)\vec{e}_2 = \vec{e}_t \quad ; \quad \frac{\partial \vec{e}_t}{\partial \theta} = -\cos(\theta)\vec{e}_1 - \sin(\theta)\vec{e}_2 = -\vec{e}_r$$

Spherical coordinate system



spherical coordinates	:	(r, θ, ϕ)
spherical basis	:	$\{\vec{e}_r(\theta,\phi), \vec{e}_t(\theta), \vec{e}_\phi(\theta,\phi)\}$

$$
\begin{aligned}
\vec{e}_r(\theta,\phi) &= \cos(\theta)\sin(\phi)\vec{e}_1 + \sin(\theta)\sin(\phi)\vec{e}_2 + \cos(\phi)\vec{e}_3 \\
\vec{e}_t(\theta) &= -\sin(\theta)\vec{e}_1 + cos(\theta)\vec{e}_2 \\
\vec{e}_\phi(\theta,\phi) &= \cos(\theta)\cos(\phi)\vec{e}_1 + \sin(\theta)\cos(\phi)\vec{e}_2 - \sin(\phi)\vec{e}_3
\end{aligned}
$$

Spherical coordinate system

$$\frac{\partial \vec{e}_r}{\partial \theta} = -\sin(\theta)\sin(\phi)\vec{e}_1 + \cos(\theta)\sin(\phi)\vec{e}_2 = \sin(\phi)\vec{e}_t$$

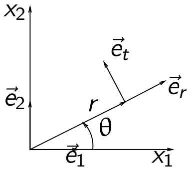
$$\frac{\partial \vec{e}_r}{\partial \phi} = \cos(\theta)\cos(\phi)\vec{e}_1 + \sin(\theta)\cos(\phi)\vec{e}_2 - \sin(\phi)\vec{e}_3 = \vec{e}_\phi$$

$$\frac{d\vec{e}_t}{d\theta} = -\cos(\theta)\vec{e}_1 - \sin(\theta)\vec{e}_2 = -\sin(\phi)\vec{e}_r - \cos(\phi)\vec{e}_\phi$$

$$\frac{\partial \vec{e}_\phi}{\partial \theta} = -\sin(\theta)\cos(\phi)\vec{e}_1 + \cos(\theta)\cos(\phi)\vec{e}_2 = \cos(\phi)\vec{e}_t$$

$$\frac{\partial \vec{e}_\phi}{\partial \phi} = -\cos(\theta)\sin(\phi)\vec{e}_1 - \sin(\theta)\sin(\phi)\vec{e}_2 - \cos(\phi)\vec{e}_3 = -\vec{e}_r$$

Polar coordinates

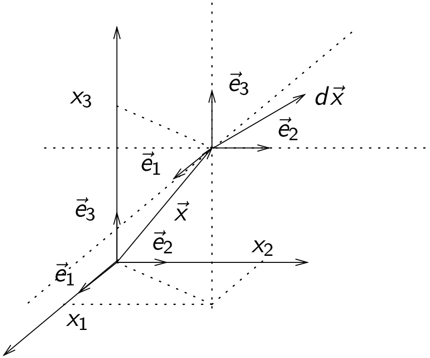


polar coordinates	:	(r, θ)
polar basis	:	$\{\vec{e}_r(\theta), \vec{e}_t(\theta)\}$

$$\vec{e}_r(\theta) = \cos(\theta)\vec{e}_1 + \sin(\theta)\vec{e}_2$$

$$\vec{e}_t(\theta) = \frac{d\vec{e}_r(\theta)}{d\theta} = -\sin(\theta)\vec{e}_1 + cos(\theta)\vec{e}_2 \quad \rightarrow \quad \frac{d\vec{e}_t(\theta)}{d\theta} = -\vec{e}_r(\theta)$$

Position vector and Cartesian components

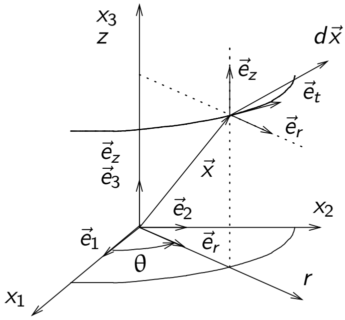


$$\vec{x} \qquad = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$$

$$\vec{x} + d\vec{x} = (x_1 + dx_1)\vec{e}_1 + (x_2 + dx_2)\vec{e}_2 + (x_3 + dx_3)\vec{e}_3$$

$$d\vec{x} = dx_1\vec{e}_1 + dx_2\vec{e}_2 + dx_3\vec{e}_3$$
$$dx_1 = d\vec{x}\cdot\vec{e}_1 \quad ; \quad dx_2 = d\vec{x}\cdot\vec{e}_2 \quad ; \quad dx_3 = d\vec{x}\cdot\vec{e}_3$$

Position vector and cylindrical components



$$
\begin{aligned}
\vec{x} &= r\vec{e}_r(\theta) + z\vec{e}_z \\
\vec{x} + d\vec{x} &= (r + dr)\vec{e}_r(\theta + d\theta) + (z + dz)\vec{e}_z = (r + dr)\left\{\vec{e}_r(\theta) + \tfrac{d\vec{e}_r}{d\theta}\, d\theta\right\} + (z + dz)\vec{e}_z \\
&= r\vec{e}_r(\theta) + z\vec{e}_z + r\vec{e}_t(\theta)d\theta + dr\vec{e}_r(\theta) + \vec{e}_t(\theta)drd\theta + dz\vec{e}_z
\end{aligned}
$$

$$
d\vec{x} = dr\,\vec{e}_r(\theta) + r\,d\theta\,\vec{e}_t(\theta) + dz\,\vec{e}_z
$$

$$
dr = d\vec{x}\cdot\vec{e}_r \quad ; \quad d\theta = \frac{1}{r}\,d\vec{x}\cdot\vec{e}_t \quad ; \quad dz = d\vec{x}\cdot\vec{e}_z
$$

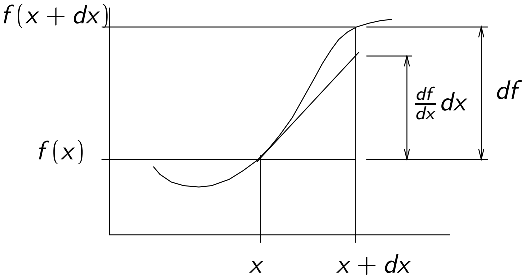
Position vector and spherical components

$$
\begin{aligned}
\vec{x} &= r\vec{e}_r(\theta,\phi) \\
\vec{x} + d\vec{x} &= (r+dr)\vec{e}_r(\theta+d\theta,\phi+d\phi) \\
&= (r+dr)\left\{\vec{e}_r(\theta,\phi) + \frac{\partial \vec{e}_r}{\partial \theta}\,d\theta + \frac{\partial \vec{e}_r}{\partial \phi}\,d\phi\right\} \\
&= r\vec{e}_r(\theta,\phi) + r\sin(\phi)\vec{e}_t(\theta)d\theta + r\vec{e}_\phi(\theta,\phi)d\phi + dr\vec{e}_r(\theta,\phi)
\end{aligned}
$$

$$
d\vec{x} = dr\,\vec{e}_r(\theta,\phi) + r\sin(\phi)\,d\theta\,\vec{e}_t(\theta) + r\,d\phi\,\vec{e}_\phi(\theta,\phi)
$$

$$
dr = d\vec{x}\cdot\vec{e}_r \quad ; \quad d\theta = \frac{1}{r\sin(\phi)}\,d\vec{x}\cdot\vec{e}_t \quad ; \quad d\phi = \frac{1}{r}\,d\vec{x}\cdot\vec{e}_\phi
$$

Variation of a scalar function $f(x)$



$$
\begin{aligned}
df &= f(x+dx) - f(x) \\
&= f(x) + \left.\frac{df}{dx}\right|_x dx + \frac{1}{2}\left.\frac{d^2 f}{dx^2}\right|_x dx^2 + \cdots - f(x) \\
&\approx \left.\frac{df}{dx}\right|_x dx
\end{aligned}
$$

Variation of a scalar function $f(x, y, z)$

$$
\begin{aligned}
df &= f(x+dx, y+dy, z+dz) - f(x,y,z) \\
&= f(x,y,z) + \left.\frac{\partial f}{\partial x}\right|_{(x,y,z)} dx + \left.\frac{\partial f}{\partial y}\right|_{(x,y,z)} dy + \left.\frac{\partial f}{\partial z}\right|_{(x,y,z)} dz + \cdots - f(x,y,z) \\
&\approx \left.\frac{\partial f}{\partial x}\right|_{(x,y,z)} dx + \left.\frac{\partial f}{\partial y}\right|_{(x,y,z)} dy + \left.\frac{\partial f}{\partial z}\right|_{(x,y,z)} dz
\end{aligned}
$$

Spatial variation of a Cartesian scalar function

$$
\begin{aligned}
da &= dx\frac{\partial a}{\partial x} + dy\frac{\partial a}{\partial y} + dz\frac{\partial a}{\partial z} \\
&= (d\vec{x}\cdot\vec{e}_x)\frac{\partial a}{\partial x} + (d\vec{x}\cdot\vec{e}_y)\frac{\partial a}{\partial y} + (d\vec{x}\cdot\vec{e}_z)\frac{\partial a}{\partial z} \\
&= d\vec{x}\cdot\left[\vec{e}_x\frac{\partial a}{\partial x} + \vec{e}_y\frac{\partial a}{\partial y} + \vec{e}_z\frac{\partial a}{\partial z}\right] \\
&= d\vec{x}\cdot(\vec{\nabla}a)
\end{aligned}
$$

gradient operator

$$
\vec{\nabla} = \left[\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right] = \underset{\sim}{\vec{e}}^T\underset{\sim}{\nabla} = \underset{\sim}{\nabla}^T\underset{\sim}{\vec{e}}
$$

Spatial variation of a cylindrical scalar function

$$
\begin{aligned}
da &= dr\,\frac{\partial a}{\partial r} + d\theta\,\frac{\partial a}{\partial \theta} + dz\,\frac{\partial a}{\partial z} \\
&= (d\vec{x}\cdot\vec{e}_r)\,\frac{\partial a}{\partial r} + (\frac{1}{r}\,d\vec{x}\cdot\vec{e}_t)\,\frac{\partial a}{\partial \theta} + (d\vec{x}\cdot\vec{e}_z)\,\frac{\partial a}{\partial z} \\
&= d\vec{x}\cdot\left[\vec{e}_r\,\frac{\partial a}{\partial r} + \frac{1}{r}\,\vec{e}_t\,\frac{\partial a}{\partial \theta} + \vec{e}_z\,\frac{\partial a}{\partial z}\right] \\
&= d\vec{x}\cdot(\vec{\nabla} a)
\end{aligned}
$$

gradient operator

$$
\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \vec{e}_z\frac{\partial}{\partial z} = \underset{\sim}{\vec{e}}^T\underset{\sim}{\nabla} = \underset{\sim}{\nabla}^T\underset{\sim}{\vec{e}}
$$

Spatial variation of a spherical scalar function

$$
\begin{aligned}
da &= dr\frac{\partial a}{\partial r} + d\theta\frac{\partial a}{\partial \theta} + d\phi\frac{\partial a}{\partial \phi} \\
&= (d\vec{x}\cdot\vec{e}_r)\frac{\partial a}{\partial r} + (\frac{1}{r\sin(\phi)}\, d\vec{x}\cdot\vec{e}_t)\frac{\partial a}{\partial \theta} + (\frac{1}{r}\, d\vec{x}\cdot\vec{e}_\phi)\frac{\partial a}{\partial \phi} \\
&= d\vec{x}\cdot \left[\vec{e}_r\frac{\partial a}{\partial r} + \frac{1}{r\sin(\phi)}\,\vec{e}_t\frac{\partial a}{\partial \theta} + \frac{1}{r}\,\vec{e}_\phi\frac{\partial a}{\partial \phi}\right] \\
&= d\vec{x}\cdot(\vec{\nabla} a)
\end{aligned}
$$

gradient operator

$$
\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\frac{1}{r\sin(\phi)}\,\frac{\partial}{\partial \theta} + \vec{e}_\phi\frac{1}{r}\,\frac{\partial}{\partial \phi} = \underset{\sim}{\vec{e}}^T\underset{\sim}{\nabla} = \underset{\sim}{\nabla}^T\underset{\sim}{\vec{e}}
$$

Spatial derivatives of a vector function

$$
\begin{array}{ll}
\text{gradient :} & \mathrm{grad}(\vec{a}) = \vec{\nabla}\vec{a} \\
\text{divergence :} & \mathrm{div}(\vec{a}) = \vec{\nabla}\cdot\vec{a} \\
\text{rotation :} & \mathrm{rot}(\vec{a}) = \vec{\nabla} * \vec{a}
\end{array}
$$

Cartesian components

$$
\begin{aligned}
\vec{\nabla}\vec{a} &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)(a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \\
&= \vec{e}_x a_{x,x}\vec{e}_x + \vec{e}_x a_{y,x}\vec{e}_y + \vec{e}_x a_{z,x}\vec{e}_z + \vec{e}_y a_{x,y}\vec{e}_x + \\
&\quad \vec{e}_y a_{y,y}\vec{e}_y + \vec{e}_y a_{z,y}\vec{e}_z + \vec{e}_z a_{x,z}\vec{e}_x + \vec{e}_z a_{y,z}\vec{e}_y + \vec{e}_z a_{z,z}\vec{e}_z \\
&= \begin{bmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{bmatrix} \\
&= \underset{\sim}{\vec{e}}^T \left(\underset{\sim}{\nabla}\, \underset{\sim}{a}^T\right) \underset{\sim}{\vec{e}}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{a} &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)\cdot(a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \\
&= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = \mathrm{tr}\left(\underset{\sim}{\nabla}\, \underset{\sim}{a}^T\right) = \mathrm{tr}\left(\vec{\nabla}\vec{a}\right)
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla} * \vec{a} &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right) * (a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z) \\
&= \left\{a_{z,y} - a_{y,z}\right\}\vec{e}_x + \left\{a_{x,z} - a_{z,x}\right\}\vec{e}_y + \left\{a_{y,x} - a_{x,y}\right\}\vec{e}_z
\end{aligned}
$$

Cylindrical components

$$
\begin{aligned}
\vec{\nabla}\vec{a} &= \{\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_z\frac{\partial}{\partial z}\}\{a_r\vec{e}_r + a_t\vec{e}_t + a_z\vec{e}_z\} \\
&= \vec{e}_r a_{r,r}\vec{e}_r + \vec{e}_r a_{t,r}\vec{e}_t + \vec{e}_r a_{z,r}\vec{e}_z + \\
&\quad \vec{e}_t\frac{1}{r}a_{r,t}\vec{e}_r + \vec{e}_t\frac{1}{r}a_{t,t}\vec{e}_t + \vec{e}_t\frac{1}{r}a_{z,t}\vec{e}_z + \vec{e}_t\frac{1}{r}a_r\vec{e}_t - \vec{e}_t\frac{1}{r}a_t\vec{e}_r \\
&\quad \vec{e}_z a_{r,z}\vec{e}_r + \vec{e}_z a_{t,z}\vec{e}_t + \vec{e}_z a_{z,z}\vec{e}_z \\
&= \underset{\sim}{\vec{e}}^T\left\{\left(\underset{\sim}{\nabla}\underset{\sim}{a}^T\right)\underset{\sim}{\vec{e}} + \begin{bmatrix} 0 \\ \frac{1}{r}\,\vec{e}_t a_r - \frac{1}{r}\,\vec{e}_r a_t \\ 0 \end{bmatrix}\}\right\} = \underset{\sim}{\vec{e}}^T\left\{\left(\underset{\sim}{\nabla}\underset{\sim}{a}^T\right)\underset{\sim}{\vec{e}} + \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{r}\,a_t & \frac{1}{r}\,a_r & 0 \\ 0 & 0 & 0 \end{bmatrix}\underset{\sim}{\vec{e}}\right\} \\
&= \underset{\sim}{\vec{e}}^T(\underset{\sim}{\nabla}\underset{\sim}{a}^T)\underset{\sim}{\vec{e}} + \underset{\sim}{\vec{e}}^T\,\underline{h}\,\underset{\sim}{\vec{e}}
\end{aligned}
$$

$$
\vec{\nabla}\cdot\vec{a} = \mathrm{tr}(\underset{\sim}{\nabla}\underset{\sim}{a}^T) + \mathrm{tr}(\underline{h}) = a_{r,r} + \frac{1}{r}\,a_{t,t} + a_{z,z} + \frac{1}{r}\,a_r
$$

$$
\begin{aligned}
\vec{\nabla}*\vec{a} &= \left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\frac{1}{r}\,\frac{\partial}{\partial\theta} + \vec{e}_z\frac{\partial}{\partial z}\right)*(a_r\vec{e}_r + a_t\vec{e}_t + a_z\vec{e}_z) \\
&= \vec{e}_r*\left\{a_{r,r}\vec{e}_r + a_{t,r}\vec{e}_t + a_{z,r}\vec{e}_z\right\} + \vec{e}_t*\frac{1}{r}\left\{a_{r,t}\vec{e}_r + a_r\vec{e}_t + a_{t,t}\vec{e}_t - a_t\vec{e}_r + a_{z,t}\vec{e}_z\right\} + \\
&\quad \vec{e}_z*\left\{a_{r,z}\vec{e}_r + a_{t,z}\vec{e}_t + a_{z,z}\vec{e}_z\right\} \\
&= a_{t,r}\vec{e}_z - a_{z,r}\vec{e}_t + \frac{1}{r}\left\{-a_{r,t}\vec{e}_z + a_t\vec{e}_z + a_{z,t}\vec{e}_r\right\} + a_{r,z}\vec{e}_t - a_{t,z}\vec{e}_r \\
&= \left[\frac{1}{r}a_{z,t} - a_{t,z}\right]\vec{e}_r + \left[a_{r,z} - a_{z,r}\right]\vec{e}_t + \left[a_{t,r} - \frac{1}{r}a_{r,t} + \frac{1}{r}a_t\right]\vec{e}_z
\end{aligned}
$$

Laplace operator

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$$

Cartesian
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

cylindrical
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

spherical
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \left(\frac{1}{r\sin(\phi)}\right)^2 \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2\tan(\phi)}\frac{\partial}{\partial \phi}$$

TENSORS

back to index

2nd-order tensor

tensor = projection *vector* $\longrightarrow$ *vector*

$$\mathbf{A} \cdot \vec{p} = \vec{q}$$

tensor = linear projection $\qquad \mathbf{A} \cdot (\alpha \vec{m} + \beta \vec{n}) = \alpha \mathbf{A} \cdot \vec{m} + \beta \mathbf{A} \cdot \vec{n}$

representation $\qquad \mathbf{A} = \alpha_1 \vec{a}_1 \vec{b}_1 + \alpha_2 \vec{a}_2 \vec{b}_2 + \alpha_3 \vec{a}_3 \vec{b}_3 + ..$

finite and not unique

Components of a tensor

$$
\begin{aligned}
\mathbf{A} &= \alpha_1 \vec{a}_1 \vec{b}_1 + \alpha_2 \vec{a}_2 \vec{b}_2 + \alpha_3 \vec{a}_3 \vec{b}_3 + .. \\
&= \alpha_1 (a_{11}\vec{e}_1 + a_{12}\vec{e}_2 + a_{13}\vec{e}_3)(b_{11}\vec{e}_1 + b_{12}\vec{e}_2 + b_{13}\vec{e}_3) + \\
&\quad \alpha_2 (a_{21}\vec{e}_1 + a_{22}\vec{e}_2 + a_{23}\vec{e}_3)(b_{21}\vec{e}_1 + b_{22}\vec{e}_2 + b_{23}\vec{e}_3) + \ldots \\
&= A_{11}\vec{e}_1\vec{e}_1 + A_{12}\vec{e}_1\vec{e}_2 + A_{13}\vec{e}_1\vec{e}_3 + A_{21}\vec{e}_2\vec{e}_1 + A_{22}\vec{e}_2\vec{e}_2 + A_{23}\vec{e}_2\vec{e}_3 + \\
&\quad A_{31}\vec{e}_3\vec{e}_1 + A_{32}\vec{e}_3\vec{e}_2 + A_{33}\vec{e}_3\vec{e}_3
\end{aligned}
$$

$$
= \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \underset{\sim}{\vec{e}}^T \underline{A} \, \underset{\sim}{\vec{e}}
$$

column notation

$$
\underset{\approx}{A} = \begin{bmatrix} A_{11} & A_{22} & A_{33} & A_{12} & A_{21} & A_{23} & A_{32} & A_{31} & A_{13} \end{bmatrix}^T
$$

Spatial derivatives of a tensor function

$$
\begin{aligned}
&\text{gradient :} && \text{grad}(\mathbf{A}) = \vec{\nabla}\mathbf{A} \\
&\text{divergence :} && \text{div}(\mathbf{A}) = \vec{\nabla} \cdot \mathbf{A} \\
&\text{rotation :} && \text{rot}(\mathbf{A}) = \vec{\nabla} * \mathbf{A}
\end{aligned}
$$

Divergence of a tensor in cylindrical components

$$
\begin{aligned}
\vec{\nabla} \cdot \mathbf{A} &= \vec{e}_i \cdot \nabla_i (\vec{e}_j A_{jk} \vec{e}_k) \\
&= \vec{e}_i \cdot (\nabla_i \vec{e}_j) A_{jk} \vec{e}_k + \vec{e}_i \cdot \vec{e}_j (\nabla_i A_{jk}) \vec{e}_k + \vec{e}_i \cdot \vec{e}_j A_{jk} (\nabla_i \vec{e}_k) \\
&= \vec{e}_i \cdot (\nabla_i \vec{e}_j) A_{jk} \vec{e}_k + \delta_{ij} (\nabla_i A_{jk}) \vec{e}_k + \delta_{ij} A_{jk} (\nabla_i \vec{e}_k) \\
&\qquad \nabla_i \vec{e}_j = \delta_{i2} \delta_{1j} \frac{1}{r} \vec{e}_t - \delta_{i2} \delta_{2j} \frac{1}{r} \vec{e}_r \\
&= \vec{e}_i \cdot (\delta_{i2} \delta_{1j} \frac{1}{r} \vec{e}_t - \delta_{i2} \delta_{2j} \frac{1}{r} \vec{e}_r) A_{jk} \vec{e}_k + \delta_{ij} (\nabla_i A_{jk}) \vec{e}_k + \delta_{ij} A_{jk} (\delta_{i2} \delta_{1k} \frac{1}{r} \vec{e}_t - \delta_{i2} \delta_{2k} \frac{1}{r} \vec{e}_r) \\
&= \vec{e}_i \cdot (\delta_{i2} \delta_{1j} \frac{1}{r} \vec{e}_t - \delta_{i2} \delta_{2j} \frac{1}{r} \vec{e}_r) A_{jk} \vec{e}_k + \delta_{ij} (\nabla_i A_{jk}) \vec{e}_k + (\delta_{i2} \delta_{1k} \frac{1}{r} \vec{e}_t - \delta_{i2} \delta_{2k} \frac{1}{r} \vec{e}_r) A_{jk} \delta_{ij} \\
&= \vec{e}_t \cdot (\delta_{1j} \frac{1}{r} \vec{e}_t - \delta_{2j} \frac{1}{r} \vec{e}_r) A_{jk} \vec{e}_k + (\nabla_j A_{jk}) \vec{e}_k + (\delta_{j2} \delta_{1k} \frac{1}{r} \vec{e}_t - \delta_{j2} \delta_{2k} \frac{1}{r} \vec{e}_r) A_{jk} \\
&= \delta_{1j} \frac{1}{r} A_{jk} \vec{e}_k + (\nabla_j A_{jk}) \vec{e}_k + (\delta_{j2} \delta_{1k} \frac{1}{r} \vec{e}_t - \delta_{j2} \delta_{2k} \frac{1}{r} \vec{e}_r) A_{jk} \\
&= \frac{1}{r} A_{1k} \vec{e}_k + (\nabla_j A_{jk}) \vec{e}_k + \frac{1}{r} (A_{21} \vec{e}_t - A_{22} \vec{e}_r) \\
&= (\frac{1}{r} A_{11} - \frac{1}{r} A_{22}) \vec{e}_1 + (\frac{1}{r} A_{12} + \frac{1}{r} A_{21}) \vec{e}_2 + \frac{1}{r} A_{13} \vec{e}_3 + (\nabla_j A_{jk}) \vec{e}_k \\
&= g_k \vec{e}_k + \nabla_j A_{jk} \vec{e}_k \\
&= \underset{\sim}{g}^T \underset{\sim}{\vec{e}} + (\underset{\sim}{\nabla}^T \underline{A}) \underset{\sim}{\vec{e}} \\
&= (\underset{\sim}{\nabla}^T \underline{A}) \underset{\sim}{\vec{e}} + \underset{\sim}{g}^T \underset{\sim}{\vec{e}} \qquad \text{with} \qquad \underset{\sim}{g}^T = \frac{1}{r} \left[\; (A_{11} - A_{22}) \quad (A_{12} + A_{21}) \quad A_{33} \; \right]
\end{aligned}
$$

Special tensors

dyad	:	$\vec{a}\vec{b}$		
null tensor	:	$\mathbf{O}$	$\rightarrow$	$\mathbf{O} \cdot \vec{p} = \vec{0}$
unit tensor	:	$\mathbf{I}$	$\rightarrow$	$\mathbf{I} \cdot \vec{p} = \vec{p}$
conjugated	:	$\mathbf{A}^c$	$\rightarrow$	$\mathbf{A}^c \cdot \vec{p} = \vec{p} \cdot \mathbf{A}$

null tensor $\rightarrow$ null matrix

$$\underline{O} = \underset{\sim}{\vec{e}} \cdot \mathbf{O} \cdot \underset{\sim}{\vec{e}}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

unity tensor $\rightarrow$ unity matrix

$$\underline{I} = \underset{\sim}{\vec{e}} \cdot \mathbf{I} \cdot \underset{\sim}{\vec{e}}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$$

$$\mathbf{I} = \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3 = \underset{\sim}{\vec{e}}^T \underset{\sim}{\vec{e}}$$

conjugate tensor $\rightarrow$ transpose matrix

$$\underline{A} = \underset{\sim}{\vec{e}} \cdot \mathbf{A} \cdot \underset{\sim}{\vec{e}}^T \rightarrow \underline{A}^T = \underset{\sim}{\vec{e}} \cdot \mathbf{A}^c \cdot \underset{\sim}{\vec{e}}^T$$

Manipulations

scalar multiplication $\qquad \mathbf{B} = \alpha \mathbf{A}$

summation $\qquad \mathbf{C} = \mathbf{A} + \mathbf{B}$

inner product $\qquad \mathbf{C} = \mathbf{A} \cdot \mathbf{B}$

double inner product $\qquad \mathbf{A} : \mathbf{B} = \mathbf{A}^c : \mathbf{B}^c = \text{scalar}$

NB :
$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$
$$\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} \quad ; \quad \mathbf{A}^3 = \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \quad ; \quad \text{etc.}$$

Euclidean norm

$$m = ||\mathbf{A}|| = \max_{\vec{e}} ||\mathbf{A} \cdot \vec{e}\,|| \qquad\qquad \forall \quad \vec{e} \quad \text{with} \quad ||\vec{e}|| = 1$$

properties

1. $||\mathbf{A}|| \geq 0$
2. $||\alpha\mathbf{A}|| = |\alpha|\ ||\mathbf{A}||$
3. $||\mathbf{A} \cdot \mathbf{B}|| \leq ||\mathbf{A}||\ ||\mathbf{B}||$
4. $||\mathbf{A} + \mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}||$

1st invariant

$$
\begin{aligned}
J_1(\mathbf{A}) &= \text{tr}(\mathbf{A}) \\
&= \frac{1}{\vec{c}_1 * \vec{c}_2 \cdot \vec{c}_3}[\vec{c}_1 \cdot \mathbf{A} \cdot (\vec{c}_2 * \vec{c}_3) + \text{cycl.}] \\
&= \frac{1}{\vec{e}_1 * \vec{e}_2 \cdot \vec{e}_3}\,[\vec{e}_1 \cdot \mathbf{A} \cdot (\vec{e}_2 * \vec{e}_3) + \text{cycl.}] \\
&= \vec{e}_1 \cdot \mathbf{A} \cdot \vec{e}_1 + \vec{e}_2 \cdot \mathbf{A} \cdot \vec{e}_2 + \vec{e}_3 \cdot \mathbf{A} \cdot \vec{e}_3 \\
&= A_{11} + A_{22} + A_{33} = \text{tr}(\underline{A})
\end{aligned}
$$

properties

1. $J_1(\mathbf{A}) = J_1(\mathbf{A}^c)$
2. $J_1(\mathbf{I}) = 3$
3. $J_1(\alpha\mathbf{A}) = \alpha J_1(\mathbf{A})$
4. $J_1(\mathbf{A} + \mathbf{B}) = J_1(\mathbf{A}) + J_1(\mathbf{B})$
5. $J_1(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} : \mathbf{B} \quad \rightarrow \quad J_1(\mathbf{A}) = \mathbf{A} : \mathbf{I}$

2nd invariant

$$
\begin{aligned}
J_2(\mathbf{A}) &= \tfrac{1}{2}\{\mathrm{tr}^2(\mathbf{A}) - \mathrm{tr}(\mathbf{A}^2)\} \\
&= \frac{1}{\vec{c}_1 * \vec{c}_2 \cdot \vec{c}_3}\left[\vec{c}_1 \cdot (\mathbf{A} \cdot \vec{c}_2) * (\mathbf{A} \cdot \vec{c}_3) + \text{cycl.}\right]
\end{aligned}
$$

properties

1. $J_2(\mathbf{A}) = J_2(\mathbf{A}^c)$
2. $J_2(\mathbf{I}) = 3$
3. $J_2(\alpha\mathbf{A}) = \alpha^2 J_2(\mathbf{A})$

3rd invariant

$$
\begin{aligned}
J_3(\mathbf{A}) &= \det(\mathbf{A}) \\
&= \frac{1}{\vec{c}_1 * \vec{c}_2 \cdot \vec{c}_3}[(\mathbf{A}\cdot\vec{c}_1)\cdot(\mathbf{A}\cdot\vec{c}_2) * (\mathbf{A}\cdot\vec{c}_3)] \\
&= \frac{1}{\vec{e}_1 * \vec{e}_2 \cdot \vec{e}_3}\,[(\mathbf{A}\cdot\vec{e}_1)\cdot(\mathbf{A}\cdot\vec{e}_2) * (\mathbf{A}\cdot\vec{e}_3)] \\
&= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{21}A_{32}A_{13} \\
&\qquad - (A_{13}A_{22}A_{31} + A_{12}A_{21}A_{33} + A_{23}A_{32}A_{11}) = \det(\underline{A})
\end{aligned}
$$

properties

1. $J_3(\mathbf{A}) = J_3(\mathbf{A}^c)$
2. $J_3(\mathbf{I}) = 1$
3. $J_3(\alpha\mathbf{A}) = \alpha^3 J_3(\mathbf{A})$
4. $J_3(\mathbf{A}\cdot\mathbf{B}) = J_3(\mathbf{A})J_3(\mathbf{B})$
5. $\det(\mathbf{A}) = 0 \leftrightarrow \mathbf{A}$ singular $\leftrightarrow \left[\mathbf{A}\cdot\vec{p} = \vec{0} \text{ with } \vec{p} \neq \vec{0}\right]$

   $\det(\mathbf{A}) \neq 0 \leftrightarrow \mathbf{A}$ regular $\leftrightarrow \left[\mathbf{A}\cdot\vec{p} = \vec{0} \quad \rightarrow \quad \vec{p} = \vec{0}\right]$

Eigenvalues and eigenvectors

$$\mathbf{A} \cdot \vec{n} = \lambda \, \vec{n} \qquad \text{with} \qquad \vec{n} \neq \vec{0}$$

$$\mathbf{A} \cdot \vec{n} = \lambda \vec{n} \quad \rightarrow$$
$$\mathbf{A} \cdot \vec{n} - \lambda \vec{n} = \vec{0} \quad \rightarrow$$
$$\mathbf{A} \cdot \vec{n} - \lambda \mathbf{I} \cdot \vec{n} = \vec{0} \quad \rightarrow$$
$$(\mathbf{A} - \lambda \mathbf{I}) \cdot \vec{n} = \vec{0} \quad \text{with} \quad \vec{n} \neq \vec{0} \quad \rightarrow$$
$$\mathbf{A} - \lambda \mathbf{I} \quad \text{singular} \quad \rightarrow \quad \det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad \rightarrow$$
$$\det(\underline{A} - \lambda \underline{I}) = 0 \quad \rightarrow \quad \text{characteristic equation}$$
$$\text{characteristic equation :} \quad 3 \text{ roots} \quad : \quad \lambda_1, \lambda_2, \lambda_3$$

$$\lambda_i \quad \rightarrow \quad \left. \begin{array}{l} (\underline{A} - \lambda_i \underline{I}) \underset{\sim}{n}_i = \underset{\sim}{0} \\ \|\underset{\sim}{n}_i\| = 1 \quad \rightarrow \quad n_{i1}^2 + n_{i2}^2 + n_{i3}^2 = 1 \end{array} \right\} \quad \rightarrow \quad \underset{\sim}{n}_i$$

$$\mathbf{A} = \lambda_1 \vec{n}_1 \vec{n}_1 + \lambda_2 \vec{n}_2 \vec{n}_2 + \lambda_3 \vec{n}_3 \vec{n}_3$$

Relations between invariants

Cayley-Hamilton theorem

$$\mathbf{A}^3 - J_1(\mathbf{A})\mathbf{A}^2 + J_2(\mathbf{A})\mathbf{A} - J_3(\mathbf{A})\mathbf{I} = \mathbf{O}$$

relation between invariants of $\mathbf{A}^{-1}$

$$J_1(\mathbf{A}^{-1}) = \frac{J_2(\mathbf{A})}{J_3(\mathbf{A})} \quad ; \quad J_2(\mathbf{A}^{-1}) = \frac{J_1(\mathbf{A})}{J_3(\mathbf{A})} \quad ; \quad J_3(\mathbf{A}^{-1}) = \frac{1}{J_3(\mathbf{A})}$$

Special tensors

inverse	$\mathbf{A}^{-1} \quad \rightarrow \quad \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$
deviatoric	$\mathbf{A}^d = \mathbf{A} - \frac{1}{3}\mathrm{tr}(\mathbf{A})\mathbf{I}$
symmetric	$\mathbf{A}^c = \mathbf{A}$
skew-symmetric	$\mathbf{A}^c = -\mathbf{A}$
positive definite	$\vec{a} \cdot \mathbf{A} \cdot \vec{a} > 0 \quad \forall \quad \vec{a} \neq \vec{0}$
orthogonal	$(\mathbf{A} \cdot \vec{a}) \cdot (\mathbf{A} \cdot \vec{b}) = \vec{a} \cdot \vec{b} \quad \forall \quad \vec{a}, \vec{b}$
adjugated	$(\mathbf{A} \cdot \vec{a}) * (\mathbf{A} \cdot \vec{b}) = \mathbf{A}^a \cdot (\vec{a} * \vec{b}) \quad \forall \quad \vec{a}, \vec{b}$

Inverse tensor

$$\det(\mathbf{A}) \neq 0 \quad \leftrightarrow \quad \exists! \quad \mathbf{A}^{-1} \quad | \quad \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}$$

property

$$\left.\begin{array}{l} (\mathbf{A} \cdot \mathbf{B}) \cdot \vec{a} = \vec{b} \quad \rightarrow \quad \vec{a} = (\mathbf{A} \cdot \mathbf{B})^{-1} \cdot \vec{b} \\ (\mathbf{A} \cdot \mathbf{B}) \cdot \vec{a} = \mathbf{A} \cdot (\mathbf{B} \cdot \vec{a}) = \vec{b} \quad \rightarrow \\ \qquad\qquad \mathbf{B} \cdot \vec{a} = \mathbf{A}^{-1} \cdot \vec{b} \quad \rightarrow \quad \vec{a} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} \cdot \vec{b} \end{array}\right\} \quad \rightarrow$$

$$(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$$

components (minor(A_{ij}) = determinant of sub-matrix of A_{ij})

$$A^{-1}_{ji} = \frac{1}{\det(\underline{A})} (-1)^{i+j} \text{ minor}(A_{ij})$$

Deviatoric part of a tensor

$$\mathbf{A}^d = \mathbf{A} - \tfrac{1}{3}\text{tr}(\mathbf{A})\mathbf{I}$$
$$\tfrac{1}{3}\text{tr}(\mathbf{A})\mathbf{I} = \mathbf{A}^h = \text{hydrostatic or spherical part}$$

properties

1. $(\mathbf{A} + \mathbf{B})^d = \mathbf{A}^d + \mathbf{B}^d$
2. $\text{tr}(\mathbf{A}^d) = 0$
3. eigenvalues (μ_i) and eigenvectors $(\vec{m}_i)$

$$\det(\mathbf{A}^d - \mu\mathbf{I}) = 0 \quad \rightarrow$$
$$\det(\mathbf{A} - \{\tfrac{1}{3}\text{tr}(\mathbf{A}) + \mu\}\mathbf{I}) = 0 \quad \rightarrow \quad \mu = \lambda - \tfrac{1}{3}\text{tr}(\mathbf{A})$$
$$(\mathbf{A}^d - \mu\mathbf{I}) \cdot \vec{m} = \vec{0} \quad \rightarrow$$
$$(\mathbf{A} - \{\tfrac{1}{3}\text{tr}(\mathbf{A}) + \mu\}\mathbf{I}) \cdot \vec{m} = \vec{0} \quad \rightarrow$$
$$(\mathbf{A} - \lambda\mathbf{I}) \cdot \vec{m} = \vec{0} \quad \rightarrow \quad \vec{m} = \vec{n}$$

Symmetric tensor

$$\mathbf{A}^c = \mathbf{A}$$

properties

$$\left.\begin{array}{lllll} 1. & \text{eigenvalues and eigenvectors are real} & & & \\ 2. & \lambda_i \quad \text{different} & \rightarrow & \vec{n}_i & \perp \\ 3. & \lambda_i \quad \text{not different} & \rightarrow & \vec{n}_i & \text{chosen } \perp \end{array}\right\} \quad \rightarrow$$

eigenvectors span orthonormal basis $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$

$$\sigma_i \vec{n}_i = \boldsymbol{\sigma} \cdot \vec{n}_i \quad \rightarrow \quad \vec{n}_i = \frac{1}{\sigma_i} \boldsymbol{\sigma} \cdot \vec{n}_i \quad \rightarrow$$

$$\vec{n}_i \cdot \vec{n}_j = \frac{1}{\sigma_i} \vec{n}_j \cdot \boldsymbol{\sigma} \cdot \vec{n}_i = \frac{1}{\sigma_j} \vec{n}_i \cdot \boldsymbol{\sigma} \cdot \vec{n}_j \quad \rightarrow \quad \vec{n}_i \cdot \boldsymbol{\sigma} \cdot \vec{n}_j = 0 \quad \rightarrow \quad \vec{n}_i \cdot \vec{n}_j = 0$$

spectral representation of $\mathbf{A}$

$$\begin{aligned} \mathbf{A} &= \mathbf{A} \cdot \mathbf{I} = \mathbf{A} \cdot (\vec{n}_1 \vec{n}_1 + \vec{n}_2 \vec{n}_2 + \vec{n}_3 \vec{n}_3) \\ &= \lambda_1 \vec{n}_1 \vec{n}_1 + \lambda_2 \vec{n}_2 \vec{n}_2 + \lambda_3 \vec{n}_3 \vec{n}_3 \end{aligned}$$

Functions of symmetric tensor

$$
\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{\lambda_1}\vec{n}_1\vec{n}_1 + \frac{1}{\lambda_2}\vec{n}_2\vec{n}_2 + \frac{1}{\lambda_3}\vec{n}_3\vec{n}_3 + \\
\sqrt{\mathbf{A}} &= \sqrt{\lambda_1}\vec{n}_1\vec{n}_1 + \sqrt{\lambda_2}\vec{n}_2\vec{n}_2 + \sqrt{\lambda_3}\vec{n}_3\vec{n}_3 \\
\ln \mathbf{A} &= \ln \lambda_1 \vec{n}_1\vec{n}_1 + \ln \lambda_2 \vec{n}_2\vec{n}_2 + \ln \lambda_3 \vec{n}_3\vec{n}_3 \\
\sin \mathbf{A} &= \sin(\lambda_1)\vec{n}_1\vec{n}_1 + \sin(\lambda_2)\vec{n}_2\vec{n}_2 + \sin(\lambda_3)\vec{n}_3\vec{n}_3 \\
J_1(\mathbf{A}) &= \mathrm{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3 \\
J_2(\mathbf{A}) &= \tfrac{1}{2}\{\mathrm{tr}^2(\mathbf{A}) - \mathrm{tr}(\mathbf{A}\cdot\mathbf{A})\} \\
&= \tfrac{1}{2}\{(\lambda_1 + \lambda_2 + \lambda_3)^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\} \\
J_2(\mathbf{A}) &= \det(\mathbf{A}) = \lambda_1\lambda_2\lambda_3
\end{aligned}
$$

Skew-symmetric tensor

$$\mathbf{A}^c = -\,\mathbf{A}$$

properties

1. $$\mathbf{A}:\mathbf{B} = \mathrm{tr}(\mathbf{A}\cdot\mathbf{B}) = \mathrm{tr}(\mathbf{A}^c\cdot\mathbf{B}^c) = \mathbf{A}^c:\mathbf{B}^c$$
$$\left.\begin{array}{l} \mathbf{A}^c = -\mathbf{A} \quad \rightarrow \quad \mathbf{A}:\mathbf{B} = -\mathbf{A}:\mathbf{B}^c \\ \mathbf{B}^c = \mathbf{B} \quad \rightarrow \quad \mathbf{A}:\mathbf{B} = -\mathbf{A}:\mathbf{B} \end{array}\right\} \quad \rightarrow \quad \mathbf{A}:\mathbf{B} = 0$$

2. $$\mathbf{B} = \mathbf{I} \quad \rightarrow \quad \mathrm{tr}(\mathbf{A}) = \mathbf{A}:\mathbf{I} = 0$$

3. $$\mathbf{A}\cdot\vec{q} = \vec{p} \quad \rightarrow \quad \vec{q}\cdot\mathbf{A}\cdot\vec{q} = \vec{q}\cdot\mathbf{A}^c\cdot\vec{q} = -\,\vec{q}\cdot\mathbf{A}\cdot\vec{q} \quad \rightarrow$$
$$\vec{q}\cdot\mathbf{A}\cdot\vec{q} = 0 \quad \rightarrow \quad \vec{q}\cdot\vec{p} = 0 \quad \rightarrow \quad \vec{q}\perp\vec{p} \quad \rightarrow$$
$$\exists! \quad \vec{\omega} \quad \text{such that} \quad \mathbf{A}\cdot\vec{q} = \vec{p} = \vec{\omega} * \vec{q}$$
ω = axial vector

Axial vector

$$\mathbf{A} \cdot \vec{q} = \underset{\sim}{\vec{e}}^T \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underset{\sim}{\vec{e}}^T \begin{bmatrix} A_{11}q_1 + A_{12}q_2 + A_{13}q_3 \\ A_{21}q_1 + A_{22}q_2 + A_{23}q_3 \\ A_{31}q_1 + A_{32}q_2 + A_{33}q_3 \end{bmatrix}$$

$$\begin{aligned} \vec{\omega} * \vec{q} &= (\omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3) * (q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3) \\ &= \omega_1 q_2(\vec{e}_3) + \omega_1 q_3(-\vec{e}_2) + \omega_2 q_1(-\vec{e}_3) + \omega_2 q_3(\vec{e}_1) + \\ &\quad \omega_3 q_1(\vec{e}_2) + \omega_3 q_2(-\vec{e}_1) \\ &= \underset{\sim}{\vec{e}}^T \begin{bmatrix} \omega_2 q_3 - \omega_3 q_2 \\ \omega_3 q_1 - \omega_1 q_3 \\ \omega_1 q_2 - \omega_2 q_1 \end{bmatrix} = \underset{\sim}{\vec{e}}^T \underset{\sim}{q} \qquad \rightarrow \end{aligned}$$

$$\underline{A} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Positive definite tensor

$$\vec{a} \cdot \mathbf{A} \cdot \vec{a} > 0 \qquad \forall \quad \vec{a} \neq \vec{0}$$

properties

1. $\mathbf{A}$ cannot be skew-symmetric, because :

$$\left.\begin{array}{l} \vec{a} \cdot \mathbf{A} \cdot \vec{a} = \vec{a} \cdot \mathbf{A}^c \cdot \vec{a} \quad \rightarrow \\ \vec{a} \cdot (\mathbf{A} - \mathbf{A}^c) \cdot \vec{a} = 0 \\ \mathbf{A} \quad \text{skew-symm.} \quad \rightarrow \quad \mathbf{A}^c = -\mathbf{A} \end{array}\right\} \quad \rightarrow \quad \vec{a} \cdot \mathbf{A} \cdot \vec{a} = 0 \quad \forall \quad \vec{a}$$

2. $\mathbf{A} = \mathbf{A}^c \quad \rightarrow \quad \vec{n}_i \cdot \mathbf{A} \cdot \vec{n}_i = \lambda_i > 0 \quad \rightarrow$

   all eigenvalues positive $\quad \rightarrow \quad$ regular

Orthogonal tensor

$$(\mathbf{A} \cdot \vec{a}) \cdot (\mathbf{A} \cdot \vec{b}) = \vec{a} \cdot \vec{b} \qquad \forall \quad \vec{a}, \vec{b}$$

properties

1. $(\mathbf{A} \cdot \vec{v}) \cdot (\mathbf{A} \cdot \vec{v}) = ||\mathbf{A} \cdot \vec{v}||^2 = \vec{v} \cdot \vec{v} = ||\vec{v}||^2 \quad \rightarrow \quad ||\mathbf{A} \cdot \vec{v}|| = ||\vec{v}||$
2. $\vec{a} \cdot \mathbf{A}^c \cdot \mathbf{A} \cdot \vec{b} = \vec{a} \cdot \vec{b} \quad \rightarrow \quad \mathbf{A} \cdot \mathbf{A}^c = \mathbf{I} \quad \rightarrow \quad \mathbf{A}^c = \mathbf{A}^{-1}$
3. $\det(\mathbf{A} \cdot \mathbf{A}^c) = \det(\mathbf{A})^2 = \det(\mathbf{I}) = 1 \quad \rightarrow$
   $\det(\mathbf{A}) = \pm 1 \quad \rightarrow \quad \mathbf{A} \quad \text{regular}$
   $\det(\mathbf{A}) = 1 \rightarrow \text{rotation} \qquad \det(\mathbf{A}) = -1 \rightarrow \text{mirroring}$

Rotation of a vector base

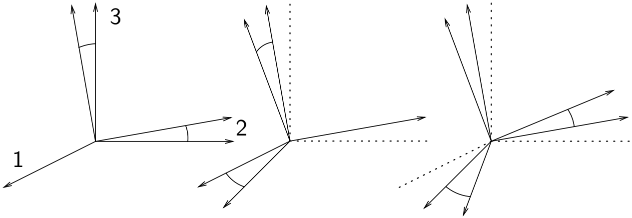
$$\left.\begin{array}{l}\vec{n}_1 = \mathbf{Q}\cdot\vec{m}_1\\ \vec{n}_2 = \mathbf{Q}\cdot\vec{m}_2\\ \vec{n}_3 = \mathbf{Q}\cdot\vec{m}_3\end{array}\right\} \quad\rightarrow\quad \left.\begin{array}{l}\vec{n}_1\vec{m}_1 = \mathbf{Q}\cdot\vec{m}_1\vec{m}_1\\ \vec{n}_2\vec{m}_2 = \mathbf{Q}\cdot\vec{m}_2\vec{m}_2\\ \vec{n}_3\vec{m}_3 = \mathbf{Q}\cdot\vec{m}_3\vec{m}_3\end{array}\right\} \quad\rightarrow\quad \mathbf{Q} = \underset{\sim}{\vec{n}}^T\,\underset{\sim}{\vec{m}}$$

$$\left.\begin{array}{l}\underline{Q}^{(n)} = \underset{\sim}{\vec{n}}\cdot\mathbf{Q}\cdot\underset{\sim}{\vec{n}}^T = (\underset{\sim}{\vec{n}}\cdot\underset{\sim}{\vec{n}}^T)\underset{\sim}{\vec{m}}\cdot\underset{\sim}{\vec{n}}^T = \underset{\sim}{\vec{m}}\cdot\underset{\sim}{\vec{n}}^T\\[2ex] \underline{Q}^{(m)} = \underset{\sim}{\vec{m}}\cdot\mathbf{Q}\cdot\underset{\sim}{\vec{m}}^T = \underset{\sim}{\vec{m}}\cdot\underset{\sim}{\vec{n}}^T(\underset{\sim}{\vec{m}}\cdot\underset{\sim}{\vec{m}}^T) = \underset{\sim}{\vec{m}}\cdot\underset{\sim}{\vec{n}}^T\end{array}\right\} \quad\rightarrow$$

$$\underline{Q}^{(n)} = \underline{Q}^{(m)} = \underline{Q}$$

$$\underset{\sim}{\vec{m}} = \underline{Q}\,\underset{\sim}{\vec{n}} \quad\rightarrow\quad \underset{\sim}{\vec{n}} = \underline{Q}^T\,\underset{\sim}{\vec{m}}$$

Rotation about three axes



$$\left.\begin{aligned}
\vec{\varepsilon}_1^{\,(1)} &= \vec{e}_1 \\
\vec{\varepsilon}_2^{\,(1)} &= c^{(1)}\vec{e}_2 + s^{(1)}\vec{e}_3 \\
\vec{\varepsilon}_3^{\,(1)} &= -s^{(1)}\vec{e}_2 + c^{(1)}\vec{e}_3
\end{aligned}\right\} \qquad \underline{Q}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^{(1)} & -s^{(1)} \\ 0 & s^{(1)} & c^{(1)} \end{bmatrix}$$

$$\left.\begin{aligned}
\vec{\varepsilon}_1^{\,(2)} &= c^{(2)}\vec{\varepsilon}_1^{\,(1)} - s^{(2)}\vec{\varepsilon}_3^{\,(1)} \\
\vec{\varepsilon}_2^{\,(2)} &= \vec{\varepsilon}_2^{\,(1)} \\
\vec{\varepsilon}_3^{\,(2)} &= s^{(2)}\vec{\varepsilon}_1^{\,(1)} + c^{(2)}\vec{\varepsilon}_3^{\,(1)}
\end{aligned}\right\} \qquad \underline{Q}_2 = \begin{bmatrix} c^{(2)} & 0 & s^{(2)} \\ 0 & 1 & 0 \\ -s^{(2)} & 0 & c^{(2)} \end{bmatrix}$$

$$\left.\begin{aligned}
\vec{\varepsilon}_1^{\,(3)} &= c^{(3)}\vec{\varepsilon}_1^{\,(2)} + s^{(3)}\vec{\varepsilon}_2^{\,(2)} \\
\vec{\varepsilon}_2^{\,(3)} &= -s^{(3)}\vec{\varepsilon}_1^{\,2)} + c^{(3)}\vec{\varepsilon}_2^{\,(2)} \\
\vec{\varepsilon}_3^{\,(3)} &= \vec{\varepsilon}_3^{\,(2)}
\end{aligned}\right\} \qquad \underline{Q}_3 = \begin{bmatrix} c^{(3)} & -s^{(3)} & 0 \\ s^{(3)} & c^{(3)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation matrix

$$\left.\begin{aligned}
\vec{\underset{\sim}{\varepsilon}}^{\,(1)} &= \underline{Q}_1{}^T \vec{\underset{\sim}{e}} \\
\vec{\underset{\sim}{\varepsilon}}^{\,(2)} &= \underline{Q}_2{}^T \vec{\underset{\sim}{\varepsilon}}^{\,(1)} \\
\vec{\underset{\sim}{\varepsilon}}^{\,(3)} &= \underline{Q}_3{}^T \vec{\underset{\sim}{\varepsilon}}^{\,(2)} = \vec{\underset{\sim}{\varepsilon}}
\end{aligned}\right\} \rightarrow \quad
\begin{aligned}
&\vec{\underset{\sim}{\varepsilon}} = \underline{Q}_3{}^T \underline{Q}_2{}^T \underline{Q}_1{}^T \vec{\underset{\sim}{e}} = \underline{Q}^T \vec{\underset{\sim}{e}} \\
&\vec{\underset{\sim}{e}} = \underline{Q}\, \vec{\underset{\sim}{\varepsilon}}
\end{aligned}$$

$$\underline{Q} = \begin{bmatrix}
c^{(2)}c^{(3)} & -c^{(2)}s^{(3)} & s^{(2)} \\
c^{(1)}s^{(3)} + s^{(1)}s^{(2)}c^{(3)} & c^{(1)}c^{(3)} - s^{(1)}s^{(2)}s^{(3)} & -s^{(1)}c^{(2)} \\
s^{(1)}s^{(3)} - c^{(1)}s^{(2)}c^{(3)} & s^{(1)}c^{(3)} + c^{(1)}s^{(2)}s^{(3)} & c^{(1)}c^{(2)}
\end{bmatrix}$$

$$\begin{aligned}
\mathbf{A} &= \vec{\underset{\sim}{e}}^T \underline{A}\, \vec{\underset{\sim}{e}} = \vec{\underset{\sim}{\varepsilon}}^T \underline{A}^*\, \vec{\underset{\sim}{\varepsilon}} \quad \rightarrow \\
\underline{A}^* &= \vec{\underset{\sim}{\varepsilon}} \cdot \vec{\underset{\sim}{e}}^T \underline{A}\, \vec{\underset{\sim}{e}} \cdot \vec{\underset{\sim}{\varepsilon}}^T = \underline{Q}^T \underline{A}\, \underline{Q} \\
\underset{\approx}{A}^* &= \underline{T}\, \underset{\approx}{A}
\end{aligned}$$

Adjugated tensor

$$(\mathbf{A} \cdot \vec{a}) * (\mathbf{A} \cdot \vec{b}) = \mathbf{A}^a \cdot (\vec{a} * \vec{b}) \quad \forall \quad \vec{a}, \vec{b}$$

property $\qquad \mathbf{A}^c \cdot \mathbf{A}^a = \det(\mathbf{A})\mathbf{I}$

Fourth-order tensor

tensor $=$ projection *tensor* $\longrightarrow$ *tensor*

$$^4\mathbf{A} : \mathbf{B} = \mathbf{C}$$

tensor $=$ linear projection $\quad ^4\mathbf{A} : (\alpha \mathbf{M} + \beta \mathbf{N}) = \alpha\, ^4\mathbf{A} : \mathbf{M} + \beta\, ^4\mathbf{A} : \mathbf{N}$

representation $\quad ^4\mathbf{A} = \alpha_1 \vec{a}_1 \vec{b}_1 \vec{c}_1 \vec{d}_1 + \alpha_2 \vec{a}_2 \vec{b}_2 \vec{c}_2 \vec{d}_2 + \alpha_3 \vec{a}_3 \vec{b}_3 \vec{c}_3 \vec{d}_3 + ..$

finite and not unique

components $\quad ^4\mathbf{A} = \vec{e}_i \vec{e}_j A_{ijkl} \vec{e}_k \vec{e}_l$

Conjugated fourth-order tensor

fourth-order tensor	:	${}^4\mathbf{A} = \vec{a}\,\vec{b}\,\vec{c}\,\vec{d}$
total conjugate	:	${}^4\mathbf{A}^c = \vec{d}\,\vec{c}\,\vec{b}\,\vec{a}$
right conjugate	:	${}^4\mathbf{A}^{rc} = \vec{a}\,\vec{b}\,\vec{d}\,\vec{c}$
left conjugate	:	${}^4\mathbf{A}^{lc} = \vec{b}\,\vec{a}\,\vec{c}\,\vec{d}$
middle conjugate	:	${}^4\mathbf{A}^{mc} = \vec{a}\,\vec{c}\,\vec{b}\,\vec{d}$
outer conjugate	:	${}^4\mathbf{A}^{oc} = \vec{b}\,\vec{a}\,\vec{d}\,\vec{c}$

symmetries

total	${}^4\mathbf{A} = {}^4\mathbf{A}^c$	;	$\mathbf{B} : {}^4\mathbf{A} : \mathbf{C} = \mathbf{C}^c : {}^4\mathbf{A} : \mathbf{B}^c$	$\forall \quad \mathbf{B}, \mathbf{C}$
right	${}^4\mathbf{A} = {}^4\mathbf{A}^{rc}$	;	${}^4\mathbf{A} : \mathbf{B} = {}^4\mathbf{A} : \mathbf{B}^c$	$\forall \quad \mathbf{B}$
left	${}^4\mathbf{A} = {}^4\mathbf{A}^{lc}$	;	$\mathbf{B} : {}^4\mathbf{A} = \mathbf{B}^c : {}^4\mathbf{A}$	$\forall \quad \mathbf{B}$
middle	${}^4\mathbf{A} = {}^4\mathbf{A}^{mc}$			
outer	${}^4\mathbf{A} = {}^4\mathbf{A}^{oc}$			

Fourth-order unity tensor

$$^4\mathbf{I} : \mathbf{B} = \mathbf{B} \qquad \forall \quad \mathbf{B}$$

components

$$\begin{aligned} ^4\mathbf{I} &= \vec{e}_1\vec{e}_1\vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_1\vec{e}_1\vec{e}_2 + \vec{e}_3\vec{e}_1\vec{e}_1\vec{e}_3 + \ldots \\ &= \vec{e}_i\vec{e}_j\vec{e}_j\vec{e}_i = \vec{e}_i\vec{e}_j\delta_{il}\delta_{jk}\vec{e}_k\vec{e}_l \end{aligned}$$

not left- or right symmetric

$$^4\mathbf{I} : \mathbf{B} = \mathbf{B} \neq \mathbf{B}^c = {}^4\mathbf{I} : \mathbf{B}^c$$

$$\mathbf{B} : {}^4\mathbf{I} = \mathbf{B} \neq \mathbf{B}^c = \mathbf{B}^c : {}^4\mathbf{I}$$

symmetric unity tensor

$$\begin{aligned} ^4\mathbf{I}^s &= \tfrac{1}{2}\left({}^4\mathbf{I} + {}^4\mathbf{I}^{rc} \right) \\ &= \tfrac{1}{2}\,\vec{e}_i\vec{e}_j(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})\vec{e}_k\vec{e}_l \end{aligned}$$

Products

$$^4\mathbf{A} \cdot \mathbf{B} = {}^4\mathbf{C} \qquad \rightarrow \qquad A_{ijkm} B_{ml} = C_{ijkl}$$

$$^4\mathbf{A} : \mathbf{B} = \mathbf{C} \qquad \rightarrow \qquad A_{ijkl} B_{lk} = C_{ij}$$

$$^4\mathbf{A} : \mathbf{B} \neq \mathbf{B} : {}^4\mathbf{A}$$

$$^4\mathbf{A} : {}^4\mathbf{B} = {}^4\mathbf{C} \qquad \rightarrow \qquad A_{ijmn} B_{nmkl} = C_{ijkl}$$

$$^4\mathbf{A} : {}^4\mathbf{B} \neq {}^4\mathbf{B} : {}^4\mathbf{A}$$

rules

$$^4\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) = ({}^4\mathbf{A} \cdot \mathbf{B}) : \mathbf{C}$$

$$\mathbf{A} \cdot \mathbf{B} + \mathbf{B}^c \cdot \mathbf{A}^c = {}^4\mathbf{I}^s : (\mathbf{A} \cdot \mathbf{B}) = ({}^4\mathbf{I}^s \cdot \mathbf{A}) : \mathbf{B}$$

KINEMATICS

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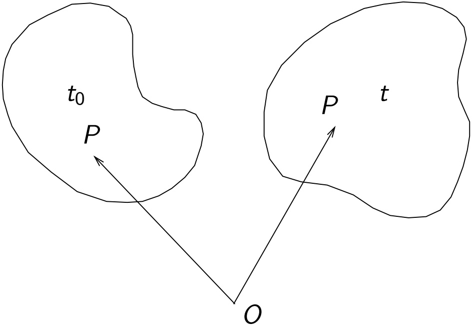
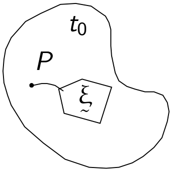
Deformation of a three-dimensional continuum

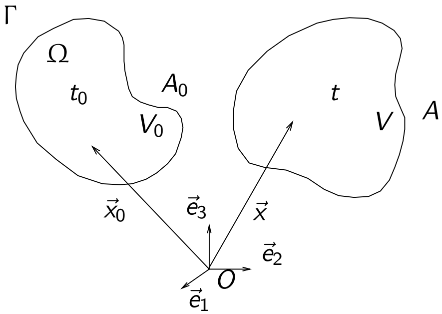


Material coordinates



$$\underset{\sim}{\xi}^T = \left[\begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \end{array} \right]$$

Position vectors



undeformed configuration (t_0) $\quad$ $\vec{x}_0 = \vec{\chi}(\underset{\sim}{\xi}, t_0) = x_{01}\vec{e}_1 + x_{02}\vec{e}_2 + x_{03}\vec{e}_3$

deformed configuration (t) $\quad$ $\vec{x} = \vec{\chi}(\underset{\sim}{\xi}, t) = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$

Euler-Lagrange

Euler : "observer" is fixed in space

$$a = \mathcal{A}_E(\vec{x}, t)$$

$$da = a_Q - a_P = \mathcal{A}_E(\vec{x} + d\vec{x}, t) - \mathcal{A}_E(\vec{x}, t) = d\vec{x} \cdot \left. (\vec{\nabla} a) \right|_t$$

$$\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3}$$

Lagrange : "observer" follows the material

$$a = \mathcal{A}_L(\vec{x}_0, t)$$

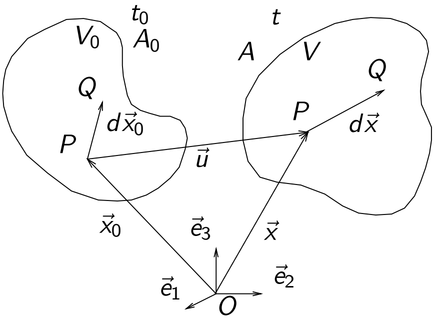
$$da = a_Q - a_P = \mathcal{A}_L(\vec{x}_0 + d\vec{x}_0, t) - \mathcal{A}_L(\vec{x}_0, t) = d\vec{x}_0 \cdot \left. (\vec{\nabla}_0 a) \right|_t$$

$$\vec{\nabla}_0 = \vec{e}_1 \frac{\partial}{\partial x_{01}} + \vec{e}_2 \frac{\partial}{\partial x_{02}} + \vec{e}_3 \frac{\partial}{\partial x_{03}}$$

position vectors

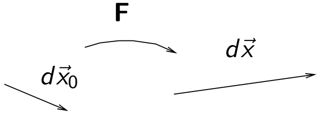
$$\vec{\nabla}\vec{x} = \mathbf{I} \qquad ; \qquad \vec{\nabla}_0\vec{x}_0 = \mathbf{I}$$

Deformation



displacement : $\vec{u} = \vec{x} - \vec{x}_0 = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$

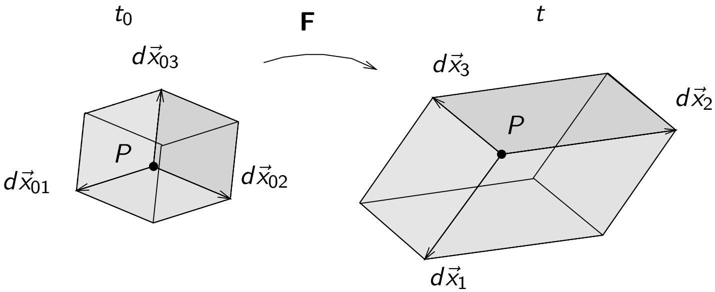
Deformation tensor



$$\begin{aligned}
d\vec{x} &= \mathbf{F}\cdot d\vec{x}_0 \\
&= \vec{X}(\vec{x}_0 + d\vec{x}_0, \mathbf{t}) - \vec{X}(\vec{x}_0, \mathbf{t}) = d\vec{x}_0 \cdot \left(\vec{\nabla}_0 \vec{x}\right) \\
&= \left(\vec{\nabla}_0 \vec{x}\right)^c \cdot d\vec{x}_0 = \mathbf{F}\cdot d\vec{x}_0
\end{aligned}$$

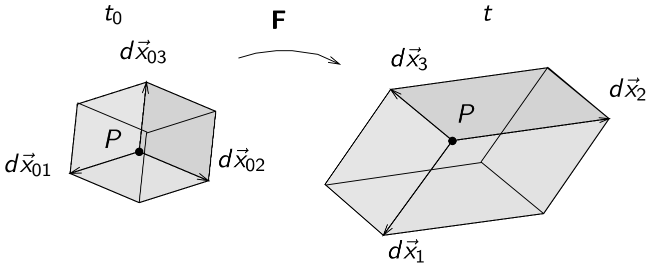
$$\mathbf{F} = \left(\vec{\nabla}_0 \vec{x}\right)^c = \left[\left(\vec{\nabla}_0 \vec{x}_0\right)^c + \left(\vec{\nabla}_0 \vec{u}\right)^c\right] = \mathbf{I} + \left(\vec{\nabla}_0 \vec{u}\right)^c$$

Deformation tensor



$$d\vec{x}_1 = \mathbf{F} \cdot d\vec{x}_{01} \quad ; \quad d\vec{x}_2 = \mathbf{F} \cdot d\vec{x}_{02} \quad ; \quad d\vec{x}_3 = \mathbf{F} \cdot d\vec{x}_{03}$$

Volume change



$$
\begin{aligned}
dV &= d\vec{x}_1 * d\vec{x}_2 \cdot d\vec{x}_3 \\
&= (\mathbf{F} \cdot d\vec{x}_{01}) * (\mathbf{F} \cdot d\vec{x}_{02}) \cdot (\mathbf{F} \cdot d\vec{x}_{03}) \\
&= \det(\mathbf{F})\,(d\vec{x}_{01} * d\vec{x}_{02} \cdot d\vec{x}_{03}) \\
&= \det(\mathbf{F})\, dV_0 \\
&= J\, dV_0
\end{aligned}
$$

Area change

$$
\begin{aligned}
dA\,\vec{n} &= d\vec{x}_1 * d\vec{x}_2 = (\mathbf{F}\cdot d\vec{x}_{01}) * (\mathbf{F}\cdot d\vec{x}_{02}) \\
dA\,\vec{n}\cdot(\mathbf{F}\cdot d\vec{x}_{03}) &= (\mathbf{F}\cdot d\vec{x}_{01}) * (\mathbf{F}\cdot d\vec{x}_{02})\cdot(\mathbf{F}\cdot d\vec{x}_{03}) \\
&= \det(\mathbf{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot d\vec{x}_{03} \qquad \forall \quad d\vec{x}_{03} \quad \longrightarrow \\
dA\,\vec{n}\cdot\mathbf{F} &= \det(\mathbf{F})(d\vec{x}_{01} * d\vec{x}_{02}) \\
dA\,\vec{n} &= \det(\mathbf{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot\mathbf{F}^{-1} \\
&= \det(\mathbf{F})dA_0\,\vec{n}_0\cdot\mathbf{F}^{-1} \\
&= dA_0\,\vec{n}_0\cdot\left(\mathbf{F}^{-1}\det(\mathbf{F})\right)
\end{aligned}
$$

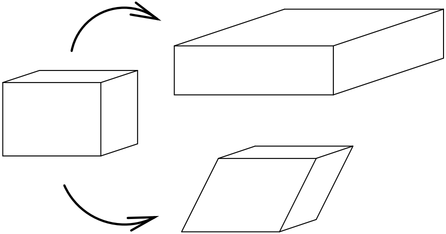
Inverse deformation

$$J = \frac{dV}{dV_0} = \det(\mathbf{F}) > 0 \quad \rightarrow \quad \mathbf{F} \text{ regular} \quad \rightarrow \quad d\vec{x}_0 = \mathbf{F}^{-1} \cdot d\vec{x}$$

relation between gradient operators

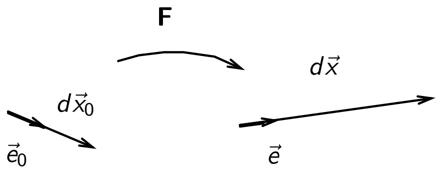
$$\mathbf{I} = \mathbf{F}^{-T} \cdot \mathbf{F}^{T} \rightarrow \left(\vec{\nabla}\vec{x}\right) = \mathbf{F}^{-T} \cdot \left(\vec{\nabla}_0\vec{x}\right) \quad \rightarrow \quad \vec{\nabla} = \mathbf{F}^{-T} \cdot \vec{\nabla}_0$$

Homogeneous deformation



$$\vec{\nabla}_0 \vec{x} = \mathbf{F}^c = \text{uniform tensor} \quad \rightarrow$$
$$\vec{x} = (\vec{x}_0 \cdot \mathbf{F}^c) + \vec{t} = \mathbf{F} \cdot \vec{x}_0 + \vec{t}$$

Elongation

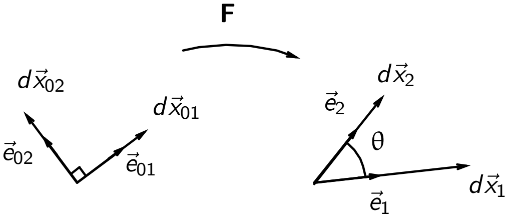


elongation factor in initial $\vec{e}_0$-direction

$$\lambda^2(\vec{e}_{01}) = \frac{d\vec{x}_1 \cdot d\vec{x}_1}{d\vec{x}_{01} \cdot d\vec{x}_{01}} = \frac{d\vec{x}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\vec{x}_{01}}{d\vec{x}_{01} \cdot d\vec{x}_{01}} = \frac{\|d\vec{x}_{01}\|^2}{\|d\vec{x}_{01}\|^2} \left(\vec{e}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \vec{e}_{01}\right)$$

$$= \vec{e}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \vec{e}_{01} = \vec{e}_{01} \cdot \mathbf{C} \cdot \vec{e}_{01}$$

Shear



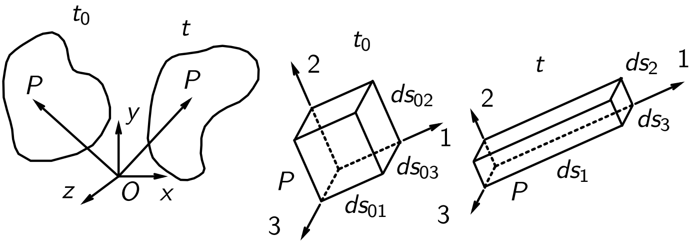
shear of initial $(\vec{e}_{01},\vec{e}_{02})$-directions

$$\gamma(\vec{e}_{01},\vec{e}_{02}) = \sin\left(\frac{\pi}{2}-\theta\right) = \cos(\theta) = \frac{d\vec{x}_1 \cdot d\vec{x}_2}{\|d\vec{x}_1\|\|d\vec{x}_2\|} = \frac{d\vec{x}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\vec{x}_{02}}{\|d\vec{x}_1\|\|d\vec{x}_2\|}$$

$$= \frac{\|d\vec{x}_{01}\|\|d\vec{x}_{02}\|(\vec{e}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \vec{e}_{02})}{\lambda(\vec{e}_{01})\|d\vec{x}_{01}\|\lambda(\vec{e}_{02})\|d\vec{x}_{02}\|} = \frac{\vec{e}_{01} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})}$$

$$= \frac{\vec{e}_{01} \cdot \mathbf{C} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})}$$

Principal directions of deformation



$$\lambda_1 = \frac{ds_1}{ds_{01}} \quad ; \quad \lambda_2 = \frac{ds_2}{ds_{02}} \quad ; \quad \lambda_3 = \frac{ds_3}{ds_{03}} \quad ; \quad \gamma_{12} = \gamma_{23} = \gamma_{31} = 0$$

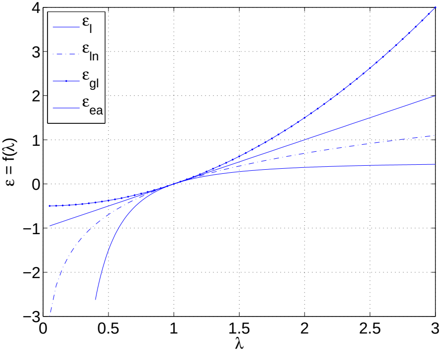
$$J = \frac{dV}{dV_0} = \frac{ds_1\, ds_2\, ds_3}{ds_{01}\, ds_{02}\, ds_{03}} = \lambda_1 \lambda_2 \lambda_3$$

Strains

$$\varepsilon = f(\lambda)$$

- $f(\lambda = 1) = 0$
- $\lim_{\lambda \longrightarrow 1} f(\lambda) = \lambda - 1$
- $f(\lambda)$ monotonic increasing
- $f(\lambda)$ differentiable

linear	$\varepsilon_l = \lambda - 1$
logarithmic	$\varepsilon_{ln} = \ln(\lambda)$
Green-Lagrange	$\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$
Euler-Almansi	$\varepsilon_{ea} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$



Strain tensor

$$\frac{1}{2}\left\{\lambda^2(\vec{e}_{01})-1\right\} = \vec{e}_{01}\cdot\left\{\frac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{I}\right)\right\}\cdot\vec{e}_{01} = \vec{e}_{01}\cdot\mathbf{E}\cdot\vec{e}_{01}$$

$$\gamma(\vec{e}_{01},\vec{e}_{02}) = \frac{\vec{e}_{01}\cdot(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{I})\cdot\vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})} = \left[\frac{2}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})}\right]\vec{e}_{01}\cdot\mathbf{E}\cdot\vec{e}_{02}$$

$$\left.\begin{array}{l} \mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{I}\right) \\ \mathbf{F} = \left(\vec{\nabla}_0\vec{x}\right)^T = \mathbf{I}+\left(\vec{\nabla}_0\vec{u}\right)^T \end{array}\right\} \quad \rightarrow$$

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}\left[\left\{\mathbf{I}+\left(\vec{\nabla}_0\vec{u}\right)\right\}\cdot\left\{\mathbf{I}+\left(\vec{\nabla}_0\vec{u}\right)^T\right\}-\mathbf{I}\right] \\ &= \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T+\left(\vec{\nabla}_0\vec{u}\right)+\left(\vec{\nabla}_0\vec{u}\right)\cdot\left(\vec{\nabla}_0\vec{u}\right)^T\right] \quad \rightarrow \quad \underline{E} \end{aligned}$$

SMALL (LINEAR) DEFORMATION

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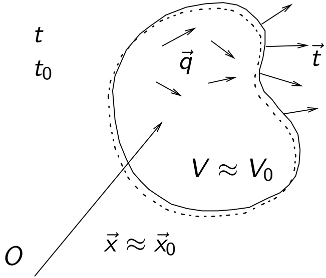
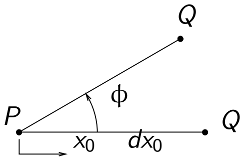
Linear deformation



$$\left.\begin{array}{l} \mathbf{E} = \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T + \left(\vec{\nabla}_0\vec{u}\right) + \left(\vec{\nabla}_0\vec{u}\right)\cdot\left(\vec{\nabla}_0\vec{u}\right)^T\right] \\ \text{small deformation} \quad \rightarrow \quad \left(\vec{\nabla}_0\vec{u}\right)^T = \mathbf{F} - \mathbf{I} \approx \mathbf{O} \end{array}\right\} \rightarrow$$

$$\mathbf{E} \approx \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T + \left(\vec{\nabla}_0\vec{u}\right)\right] \approx \frac{1}{2}\left[\left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right)\right] = \varepsilon \qquad \text{symm!}$$

Rigid rotation



$$\left.\begin{array}{l} u = u_Q = -[dx_0 - dx_0 \cos(\phi)] = [\cos(\phi) - 1]dx_0 \\ \\ v = v_Q = [\sin(\phi)]dx_0 \end{array}\right\} \quad \rightarrow$$

$$\frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \quad ; \quad \frac{\partial v}{\partial x_0} = \sin(\phi) \quad \rightarrow$$

$$\varepsilon_{gl} = \frac{\partial u}{\partial x_0} + \frac{1}{2}\left(\frac{\partial u}{\partial x_0}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x_0}\right)^2 = 0$$

$$\varepsilon_l = \frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \neq 0 \quad !!$$

Elongational, shear and volume strain

elong. strain

$$\frac{1}{2}\left(\lambda^2(\vec{e}_{01})-1\right) \;=\; \vec{e}_{01}\cdot\mathbf{E}\cdot\vec{e}_{01}$$
$$\downarrow$$
$$\lambda(\vec{e}_{01})-1 \;=\; \vec{e}_{01}\cdot\boldsymbol{\varepsilon}\cdot\vec{e}_{01}$$

shear strain

$$\gamma(\vec{e}_{01},\vec{e}_{02})=\sin\left(\frac{\pi}{2}-\theta\right) \;=\; \left(\frac{2}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})}\right)\vec{e}_{01}\cdot\mathbf{E}\cdot\vec{e}_{02}$$
$$\downarrow$$
$$\frac{\pi}{2}-\theta \;=\; 2\,\vec{e}_{01}\cdot\boldsymbol{\varepsilon}\cdot\vec{e}_{02}$$

volume change

$$J=\frac{dV}{dV_0} \;=\; \lambda_1\lambda_2\lambda_3=(\varepsilon_1+1)(\varepsilon_2+1)(\varepsilon_2+1)$$
$$\downarrow$$
$$J \;=\; \varepsilon_1+\varepsilon_2+\varepsilon_3+1=\mathrm{tr}(\boldsymbol{\varepsilon})+1$$

volume strain

$$J-1 \;=\; \mathrm{tr}(\boldsymbol{\varepsilon})$$

Linear strain matrix

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \qquad \text{with} \qquad \left\{ \begin{array}{l} \varepsilon_{21} = \varepsilon_{12} \\ \varepsilon_{32} = \varepsilon_{23} \\ \varepsilon_{31} = \varepsilon_{13} \end{array} \right.$$

principal strain matrix

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

spectral form

$$\boldsymbol{\varepsilon} = \varepsilon_1 \vec{n}_1 \vec{n}_1 + \varepsilon_2 \vec{n}_2 \vec{n}_2 + \varepsilon_3 \vec{n}_3 \vec{n}_3$$

Linear strain : Cartesian components

gradient operator	$\vec{\nabla} = \vec{e}_x \dfrac{\partial}{\partial x} + \vec{e}_y \dfrac{\partial}{\partial y} + \vec{e}_z \dfrac{\partial}{\partial z}$
displacement vector	$\vec{u} = u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z$
linear strain tensor	$\boldsymbol{\varepsilon} = \frac{1}{2}\left\{(\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u})\right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$

$$
\begin{aligned}
\underline{\varepsilon} &= \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2u_{x,x} & u_{x,y} + u_{y,x} & u_{x,z} + u_{z,x} \\ u_{y,x} + u_{x,y} & 2u_{y,y} & u_{y,z} + u_{z,y} \\ u_{z,x} + u_{x,z} & u_{z,y} + u_{y,z} & 2u_{z,z} \end{bmatrix}
\end{aligned}
$$

Linear strain : cylindrical components

gradient operator $\vec{\nabla} = \vec{e}_r \dfrac{\partial}{\partial r} + \vec{e}_t \dfrac{1}{r} \dfrac{\partial}{\partial \theta} + \vec{e}_z \dfrac{\partial}{\partial z}$

displacement vector $\vec{u} = u_r \vec{e}_r(\theta) + u_t \vec{e}_t(\theta) + u_z \vec{e}_z$

linear strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2} \left\{ (\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u}) \right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$

$$
\begin{aligned}
\underline{\varepsilon} &= \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{rt} & \varepsilon_{rz} \\ \varepsilon_{tr} & \varepsilon_{tt} & \varepsilon_{tz} \\ \varepsilon_{zr} & \varepsilon_{zt} & \varepsilon_{zz} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ u_{z,r} + u_{r,z} & \frac{1}{r}u_{z,t} + u_{t,z} & 2u_{z,z} \end{bmatrix}
\end{aligned}
$$

Compatibility relations

$$
\begin{aligned}
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} &= 2\,\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} &= 2\,\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \\
\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} &= 2\,\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \\
\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial x^2} &= \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial z} \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial y^2} &= \frac{\partial^2 \varepsilon_{yx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial x} \\
\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial z^2} &= \frac{\partial^2 \varepsilon_{zy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial y}
\end{aligned}
$$

$$
\frac{1}{r^2}\,\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \varepsilon_{tt}}{\partial r^2} - \frac{2}{r}\,\frac{\partial^2 \varepsilon_{rt}}{\partial r \partial \theta} - \frac{1}{r}\,\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\,\frac{\partial \varepsilon_{tt}}{\partial r} - \frac{2}{r^2}\,\frac{\partial \varepsilon_{rt}}{\partial \theta} = 0
$$

Planar deformation

planar deformation $\quad u_1 = u_1(x_1, x_2)\ ;\ u_2 = u_2(x_1, x_2)\ ;\ u_3 = u_3(x_1, x_2, x_3)$

Planar deformation

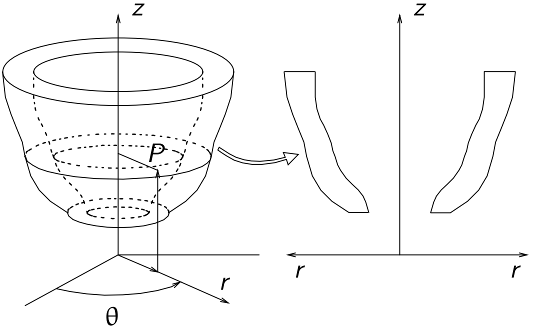
planar deformation $\quad u_1 = u_1(x_1, x_2)\ ;\ u_2 = u_2(x_1, x_2)\ ;\ u_3 = u_3(x_1, x_2, x_3)$

plane strain $\quad u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = 0$

$$\varepsilon_{33} = 0 \quad ; \quad \gamma_{13} = \gamma_{23} = 0$$

compatibility $\quad : \quad \varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}$

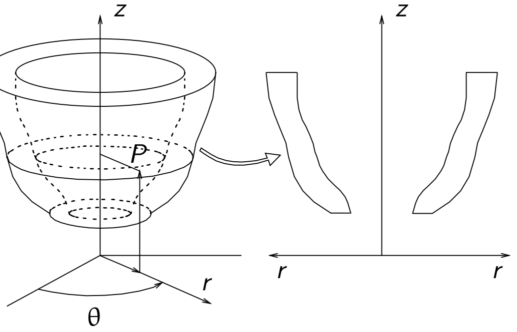
Axi-symmetric deformation



$$\frac{\partial}{\partial \theta}(\ ) = 0 \qquad \rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_t(r,z)\vec{e}_t(\theta) + u_z(r,z)\vec{e}_z$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & -\frac{1}{r}(u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ -\frac{1}{r}(u_t) + u_{t,r} & 2\frac{1}{r}(u_r) & u_{t,z} \\ u_{z,r} + u_{r,z} & u_{t,z} & 2u_{z,z} \end{bmatrix}$$

Axi-symmetric deformation with $u_t = 0$



$$\frac{\partial}{\partial \theta}(\ ) = 0 \text{ and } u_t = 0 \qquad \rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_z(r,z)\vec{e}_z$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & 0 & u_{r,z} + u_{z,r} \\ 0 & 2\frac{1}{r}(u_r) & 0 \\ u_{z,r} + u_{r,z} & 0 & 2u_{z,z} \end{bmatrix}$$

Axi-symmetric plane strain

plane strain deformation

$$\left.\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta) \\ u_z = 0 \end{array}\right\} \quad \rightarrow \quad \varepsilon_{zz} = \gamma_{rz} = \gamma_{tz} = 0$$

linear strain matrix

$$\underline{\varepsilon} = \frac{1}{2}\left[\begin{array}{ccc} 2u_{r,r} & u_{t,r} - \frac{1}{r}(u_t) & 0 \\ u_{t,r} - \frac{1}{r}(u_t) & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

plane strain deformation with $u_t = 0$

$$\left.\begin{array}{l} u_r = u_r(r) \\ u_z = 0 \end{array}\right\} \quad \rightarrow \quad \underline{\varepsilon} = \frac{1}{2}\left[\begin{array}{ccc} 2u_{r,r} & 0 & 0 \\ 0 & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

STRESS

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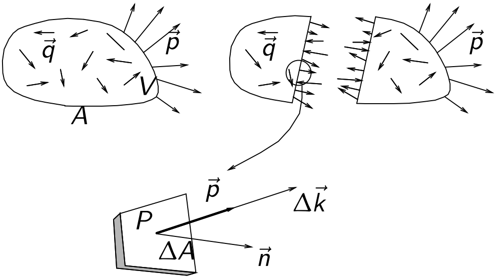
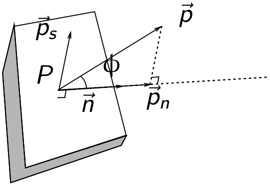
Stress vector



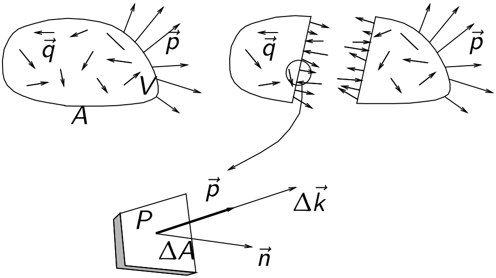
$$\vec{p} = \lim_{\Delta A \longrightarrow 0} \frac{\Delta \vec{k}}{\Delta A}$$

Normal stress and shear stress



normal stress	:	$p_n = \vec{p} \cdot \vec{n}$
tensile stress	:	positive ($\phi < \frac{\pi}{2}$)
compression stress	:	negative ($\phi > \frac{\pi}{2}$)
normal stress vector	:	$\vec{p}_n = p_n \vec{n}$
shear stress vector	:	$\vec{p}_s = \vec{p} - \vec{p}_n$
shear stress	:	$p_s = \lVert\vec{p}_s\rVert = \sqrt{\lVert\vec{p}\rVert^2 - p_n^2}$

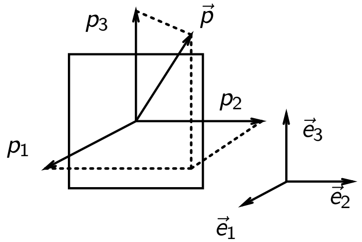
Cauchy stress tensor



Theorem of Cauchy :

$\exists!$ tensor $\boldsymbol{\sigma}$ such that : $\boxed{\vec{p} = \boldsymbol{\sigma} \cdot \vec{n}}$

Cauchy stress matrix

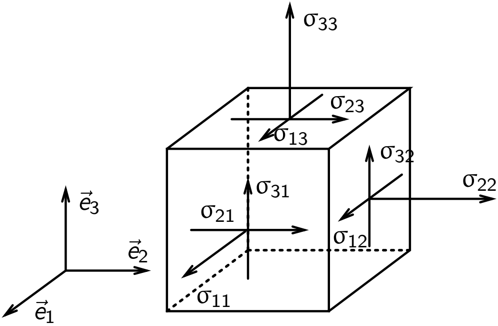


$$\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} \quad \longrightarrow \quad \underset{\sim}{\vec{e}}^T \underset{\sim}{p} = \underset{\sim}{\vec{e}}^T \underline{\sigma} \; \underset{\sim}{\vec{e}} \cdot \underset{\sim}{\vec{e}}^T \underset{\sim}{n} = \underset{\sim}{\vec{e}}^T \underline{\sigma} \; \underset{\sim}{n}$$

$$\vec{n} = \vec{e}_1 \quad \longrightarrow$$

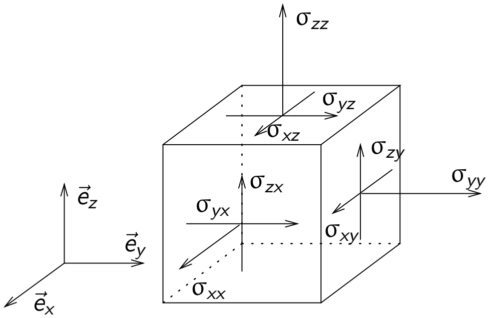
$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}$$

Stress cube



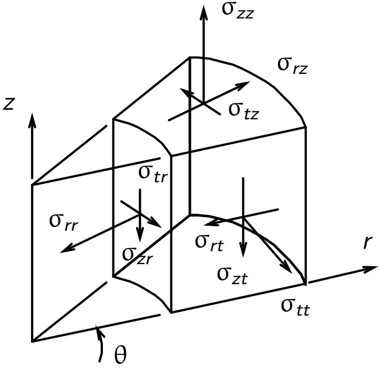
$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Cartesian components



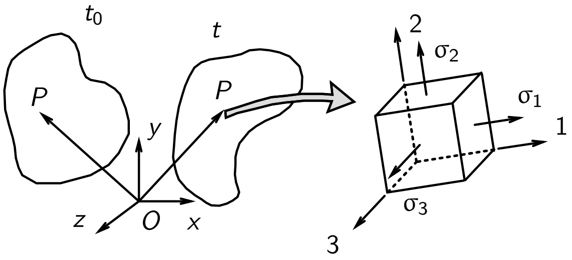
$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Cylindrical components



$$\underline{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{rt} & \sigma_{rz} \\ \sigma_{tr} & \sigma_{tt} & \sigma_{tz} \\ \sigma_{zr} & \sigma_{zt} & \sigma_{zz} \end{bmatrix}$$

Principal stresses and directions



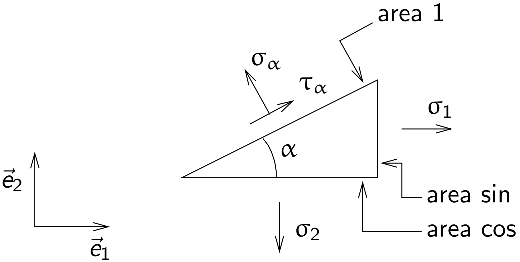
spectral form

$$\left.\begin{array}{l} \boldsymbol{\sigma} \cdot \vec{n}_1 = \sigma_1 \vec{n}_1 \\ \boldsymbol{\sigma} \cdot \vec{n}_2 = \sigma_2 \vec{n}_2 \\ \boldsymbol{\sigma} \cdot \vec{n}_3 = \sigma_3 \vec{n}_3 \end{array}\right\} \rightarrow \boldsymbol{\sigma} = \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3$$

principal stress matrix

$$\underline{\sigma}_P = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Stress transformation



$$\boldsymbol{\sigma} = \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2$$
$$\vec{n} = -\sin(\alpha)\vec{e}_1 + \cos(\alpha)\vec{e}_2$$
$$\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} = -\sigma_1 \sin(\alpha)\vec{e}_1 + \sigma_2 \cos(\alpha)\vec{e}_2$$
$$\sigma_\alpha = \sigma_1 \sin^2(\alpha) + \sigma_2 \cos^2(\alpha)$$
$$\tau_\alpha = (\sigma_2 - \sigma_1) \sin(\alpha) \cos(\alpha)$$

Mohr's circles of stress

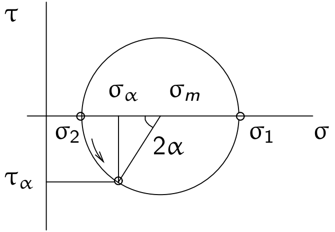
$$\sigma_\alpha = \sigma_1 \sin^2(\alpha) + \sigma_2 \cos^2(\alpha) = \sigma_1(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\alpha)) + \sigma_2(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\alpha))$$
$$= \tfrac{1}{2}(\sigma_1 + \sigma_2) - \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos(2\alpha) \quad \rightarrow$$

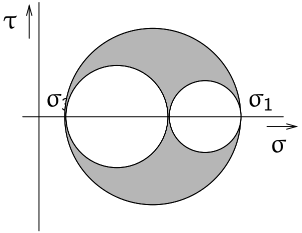
$$(1) \qquad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2 \cos^2(2\alpha)$$

$$\tau_\alpha = -\cos(\alpha)\sin(\alpha)\sigma_1 + \cos(\alpha)\sin(\alpha)\sigma_2 = \tfrac{1}{2}(\sigma_2 - \sigma_1)\sin(2\alpha) \quad \rightarrow$$

$$(2) \qquad \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_2 - \sigma_1)\right\}^2 \sin^2(2\alpha)$$

$$(1) + (2) \quad \rightarrow \quad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 + \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2$$





Mohr's circles of stress

inside σ_1, σ_3-circle

$$\{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_3)\}^2 + \tau^2 = \sigma^2 + \tau^2 = \|\vec{p}\|^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n}$$
$$= n_1^2 \alpha^2 + n_2^2 \beta^2 + n_3^2 \alpha^2$$
$$\text{with } \beta^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right)^2 \leq \alpha^2 = \left(\sigma_1 - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \leq \alpha^2$$

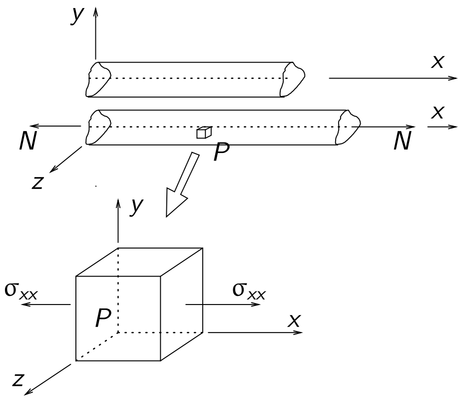
outside σ_2, σ_3-circle

$$\{\sigma - \tfrac{1}{2}(\sigma_3 + \sigma_2)\}^2 + \tau^2 = \sigma^2 + \tau^2 = \|\vec{p}\|^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n}$$
$$= n_1^2 \beta^2 + n_2^2 \alpha^2 + n_3^2 \alpha^2$$
$$\text{with } \beta^2 = \left(\sigma_1 - \tfrac{1}{2}(\sigma_3 + \sigma_2)\right)^2 \geq \alpha^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_3 + \sigma_2)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \geq \alpha^2$$

outside σ_1, σ_2-circle

$$\{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_2)\}^2 + \tau^2 = \sigma^2 + \tau^2 = \|\vec{p}\|^2 = \vec{p} \cdot \vec{p} = \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n}$$
$$= n_1^2 \alpha^2 + n_2^2 \alpha^2 + n_3^2 \beta^2$$
$$\text{with } \beta^2 = \left(\sigma_3 - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right)^2 \geq \alpha^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \geq \alpha^2$$

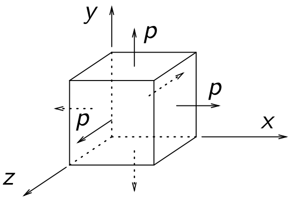
Uni-axial stress



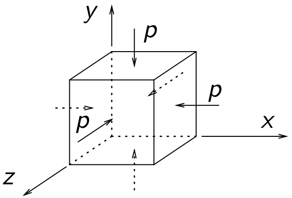
true or Cauchy stress $\quad \sigma = \dfrac{N}{A} = \sigma_{xx} \quad \rightarrow \quad \boldsymbol{\sigma} = \sigma_{xx}\vec{e}_x\vec{e}_x$

engineering stress $\quad \sigma_n = \dfrac{N}{A_0}$

Hydrostatic stress

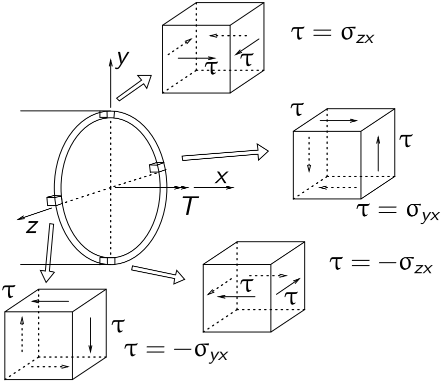


$$\sigma_{xx} = p$$
$$\sigma_{yy} = p$$
$$\sigma_{zz} = p$$



$$\sigma_{xx} = -p$$
$$\sigma_{yy} = -p$$
$$\sigma_{zz} = -p$$

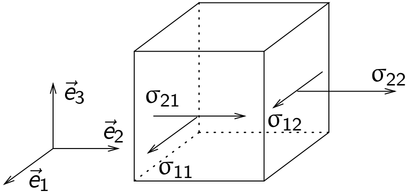
Shear stress



$$\tau = \frac{T}{2\pi R^2 t}$$

$$\boldsymbol{\sigma} = \tau(\vec{e}_i\vec{e}_j + \vec{e}_j\vec{e}_i) \text{ with } i \neq j$$

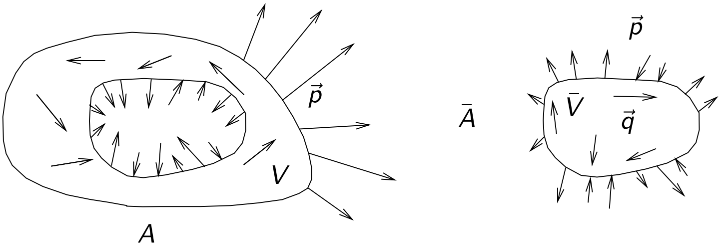
Plane stress



$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \quad \rightarrow \quad \boldsymbol{\sigma} \cdot \vec{e}_3 = \vec{0} \quad \rightarrow$$

relevant stresses : $\sigma_{11}, \sigma_{22}, \sigma_{12}$

Resulting force on arbitrary material volume

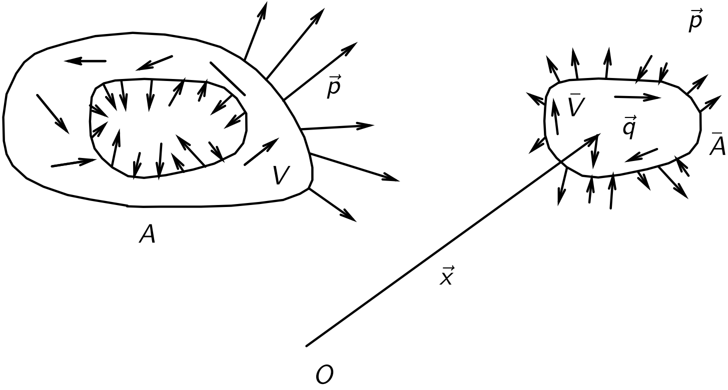


$$\vec{K} = \int_{\bar{V}} \rho \vec{q} \, dV + \int_{\bar{A}} \vec{p} \, dA = \int_{\bar{V}} \rho \vec{q} \, dV + \int_{\bar{A}} \vec{n} \cdot \boldsymbol{\sigma}^T \, dA$$

Gauss theorem : $\int_{\bar{A}} \vec{n} \cdot (\,) \, dA = \int_{\bar{V}} \vec{\nabla} \cdot (\,) \, dV \quad \rightarrow$

$$\vec{K} = \int_{\bar{V}} [\rho \vec{q} + \vec{\nabla} \cdot \boldsymbol{\sigma}^T] \, dV$$

Resulting moment on arbitrary material volume



$$\vec{M}_O = \int_{\bar{V}} \vec{x} * \rho \vec{q} \, dV + \int_{\bar{A}} \vec{x} * \vec{p} \, dA$$

BALANCE LAWS

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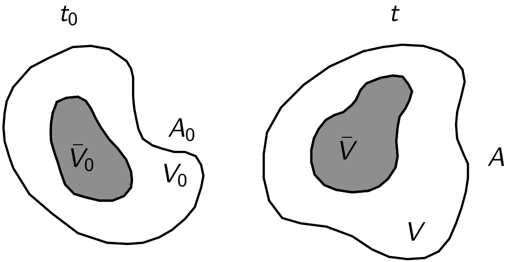
Mass balance

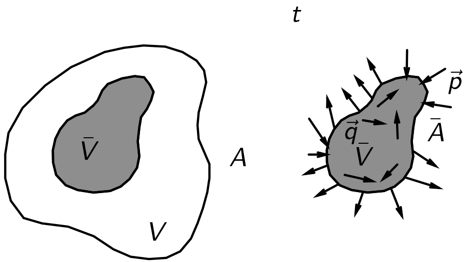


$$\int_{\bar{V}} \rho \, dV = \int_{\bar{V}_0} \rho_0 \, dV_0 \quad \forall \; \bar{V} \quad \rightarrow \quad \int_{\bar{V}_0} (\rho J - \rho_0) \, dV_0 = 0 \quad \forall \; \bar{V}_0 \quad \rightarrow$$

$$\rho J = \rho_0 \qquad \forall \quad \vec{x} \in V(t)$$

$$dM = dM_0 \quad \rightarrow \quad \rho dV = \rho_0 dV_0 \quad \rightarrow \quad \rho J = \rho_0 \quad \rightarrow \quad \dot{\rho} J + \rho \dot{J} = 0$$

Balance of momentum : global



$$\vec{K} = \frac{D\vec{i}}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \rho\vec{v}\, dV = \frac{D}{Dt}\int_{\bar{V}_0} \rho\vec{v}J\, dV_0 = \int_{\bar{V}_0} \frac{D}{Dt}\left(\rho\vec{v}J\right)\, dV_0 \qquad \forall \quad \bar{V}_0$$

$$= \int_{\bar{V}_0} \left(\dot{\rho}\vec{v}J + \rho\dot{\vec{v}}J + \rho\vec{v}\dot{J}\right)\, dV_0 \qquad \forall \quad \bar{V}_0$$

$$\text{mass balance} \quad : \quad \dot{\rho}J + \rho\dot{J} = 0 \quad \rightarrow$$

$$= \int_{\bar{V}_0} \rho\dot{\vec{v}}J\, dV_0 = \int_{\bar{V}} \rho\dot{\vec{v}}\, dV \qquad \forall \quad \bar{V}$$

Balance of momentum : local

$$\int_{\bar{V}} \left(\rho \vec{q} + \vec{\nabla} \cdot \boldsymbol{\sigma}^T \right) \, dV = \int_{\bar{V}} \rho \dot{\vec{v}} \, dV \qquad \forall \quad \bar{V} \quad \rightarrow$$

$$\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \rho \dot{\vec{v}} = \rho \frac{\delta \vec{v}}{\delta t} + \rho \vec{v} \cdot \left(\vec{\nabla} \vec{v} \right) \qquad \forall \quad \vec{x} \in V(t)$$

stationary $\left(\dfrac{\delta \vec{v}}{\delta t} = 0 \right)$ $\qquad \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \rho \vec{v} \cdot \left(\vec{\nabla} \vec{v} \right)$

static : equilibrium equation $\qquad \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \vec{0}$

Equilibrium equations : Cartesian components

$$
\begin{aligned}
\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} &= \vec{0} \\
\boldsymbol{\sigma} &= \boldsymbol{\sigma}^c
\end{aligned}
$$

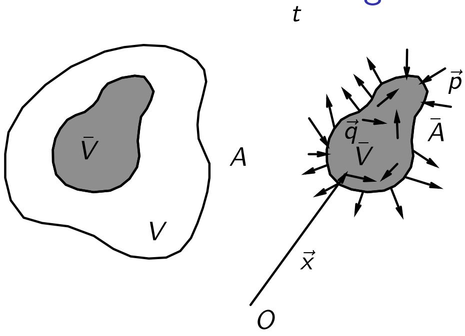
$$
\begin{aligned}
\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x &= 0 \\
\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y &= 0 \\
\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z &= 0
\end{aligned}
$$

Equilibrium equations : cylindrical components

$$\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \vec{0}$$
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^c$$

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = 0$$
$$\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t = 0$$
$$\sigma_{zr,r} + \frac{1}{r}\sigma_{zt,t} + \frac{1}{r}\sigma_{zr} + \sigma_{zz,z} + \rho q_z = 0$$

Balance of moment of momentum : global



$$\vec{M}_O = \frac{D\vec{L}_O}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \vec{x} * \rho\vec{v}\, dV = \frac{D}{Dt}\int_{\bar{V}_0} \vec{x} * \rho\vec{v}J\, dV_0 = \int_{\bar{V}_0} \frac{D}{Dt}\left(\vec{x} * \rho\vec{v}J\right)\, dV_0$$

$$= \int_{\bar{V}_0} \left(\dot{\vec{x}} * \rho\vec{v}J + \vec{x} * \dot{\rho}\vec{v}J + \vec{x} * \rho\dot{\vec{v}}J + \vec{x} * \rho\vec{v}\dot{J}\right)\, dV_0 \qquad \forall \quad \bar{V}_0$$

$$\text{mass balance} \quad : \quad \dot{\rho}J + \rho\dot{J} = 0$$

$$= \int_{\bar{V}_0} \vec{x} * \rho\dot{\vec{v}}J\, dV_0 = \int_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \qquad \forall \quad \bar{V}$$

Balance of moment of momentum : local

$$\int_{\bar{V}} \vec{x} * \rho \vec{q} \, dV + \int_{\bar{A}} \vec{x} * \vec{p} \, dA = \int_{\bar{V}} \vec{x} * \rho \dot{\vec{v}} \, dV \qquad \forall \quad \bar{V}$$

Transformation of surface integral with $\vec{x} * \vec{p} = {}^3\boldsymbol{\epsilon} : (\vec{x} \, \vec{p})$

$$\begin{aligned}
\int_{\bar{A}} \vec{x} * \vec{p} \, dA &= \int_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x} \, \vec{p}) \, dA = \int_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x} \, \boldsymbol{\sigma} \cdot \vec{n}) \, dA = \int_{\bar{A}} \vec{n} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x} \boldsymbol{\sigma})\}^c \, dA \\
&= \int_{\bar{V}} \vec{\nabla} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x} \boldsymbol{\sigma})\}^c \, dV \\
&= \int_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c) \vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} \cdot (\vec{\nabla} \cdot \vec{x}) : {}^3\boldsymbol{\epsilon}^c \right] \, dV \\
&= \int_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c) \vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} : {}^3\boldsymbol{\epsilon}^c \right] \, dV \\
&= \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c + \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) \, dV
\end{aligned}$$

Balance of moment of momentum : local

$$\int_{\bar{V}} \vec{x} * \rho \vec{q} \, dV + \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c \, dV + \int_{\bar{V}} \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) \, dV = \int_{\bar{V}} \vec{x} * \rho \dot{\vec{v}} \, dV \quad \forall \quad \bar{V} \quad \rightarrow$$

$$\int_{\bar{V}} \vec{x} * \left[\rho \vec{q} + (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) - \rho \dot{\vec{v}} \right] \, dV + \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c \, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow$$

$$\int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c \, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow \quad {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c = \vec{0} \quad \forall \quad \vec{x} \in \bar{V}$$

$$\epsilon_{ijk} = -1|0|1 \quad \rightarrow \quad \begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow$$

$$\boldsymbol{\sigma}^c = \boldsymbol{\sigma} \qquad\qquad \forall \quad \vec{x} \in V(t)$$

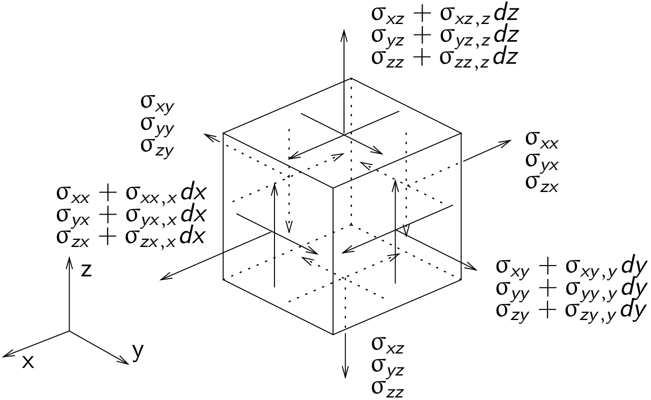
Cartesian and cylindrical components

$$\underline{\sigma} = \underline{\sigma}^T \qquad \rightarrow$$

Cartesian : $\sigma_{xy} = \sigma_{yx} \quad ; \quad \sigma_{yz} = \sigma_{zy} \quad ; \quad \sigma_{zx} = \sigma_{xz}$

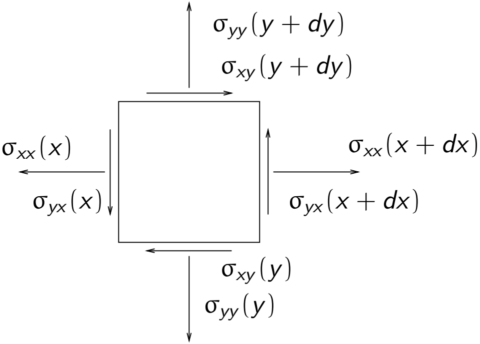
cylindrical : $\sigma_{rt} = \sigma_{tr} \quad ; \quad \sigma_{tz} = \sigma_{zt} \quad ; \quad \sigma_{zr} = \sigma_{rz}$

Equilibrium of forces : Cartesian



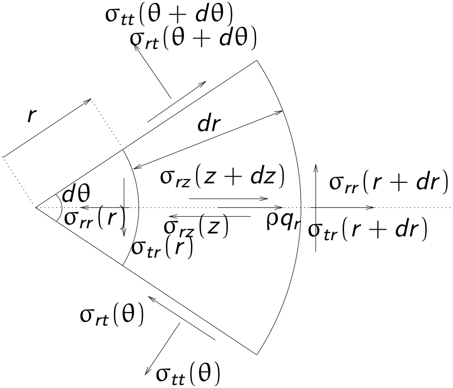
$$(\sigma_{xx} + \sigma_{xx,x}dx)dydz + (\sigma_{xy} + \sigma_{xy,y}dy)dxdz + (\sigma_{xz} + \sigma_{xz,z}dz)dxdy -$$
$$(\sigma_{xx})dydz - (\sigma_{xy})dxdz - (\sigma_{xz})dxdy + \rho q_x dxdydz = 0$$

Equilibrium of moments : Cartesian



$$\sigma_{yx}dydz\,\tfrac{1}{2}dx + \sigma_{yx}dydz\,\tfrac{1}{2}dx + \sigma_{yx,x}dxdydz\tfrac{1}{2}dx$$
$$-\,\sigma_{xy}dxdz\,\tfrac{1}{2}dy - \sigma_{xy}dxdz\,\tfrac{1}{2}dy - \sigma_{xy,x}dxdydz\,\tfrac{1}{2}dy = 0$$
$$\sigma_{yx} - \sigma_{xy} = 0 \qquad \rightarrow \qquad \sigma_{yx} = \sigma_{xy}$$

Equilibrium of forces : cylindrical

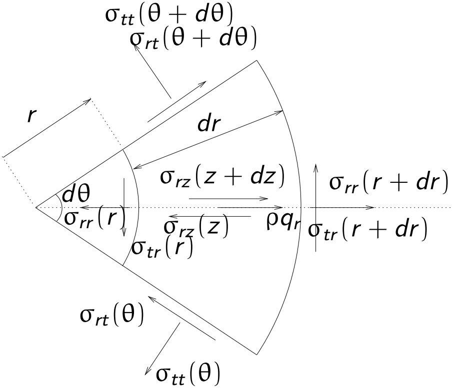


$$
\begin{aligned}
&-\sigma_{rr}(r) r d\theta dz - \sigma_{rz}(z) r dr d\theta \\
&\quad - \sigma_{rt}(\theta) dr dz - \sigma_{tt}(\theta) dr \tfrac{1}{2} d\theta dz \\
&\quad + \sigma_{rr}(r + dr)(r + dr) d\theta dz \\
&\quad + \sigma_{rz}(z + dz) r dr d\theta \\
&\quad + \sigma_{rt}(\theta + d\theta) dr dz \\
&\quad - \sigma_{tt}(\theta + d\theta) dr \tfrac{1}{2} d\theta dz \\
&\quad + \rho q_r r dr d\theta dz = 0
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_{rr,r} r dr d\theta dz + \sigma_{rr} dr d\theta dz + \sigma_{rz,z} r dr d\theta dz + \sigma_{rt,t} dr d\theta dz \\
&\quad - \sigma_{tt}(\theta) dr d\theta dz + \rho q_r dr d\theta dz = 0
\end{aligned}
$$

$$
\sigma_{rr,r} + \frac{1}{r}\sigma_{rr} + \sigma_{rz,z} + \frac{1}{r}\sigma_{rt,t} - \frac{1}{r}\sigma_{tt} + \rho q_r = 0
$$

Equilibrium of moments : cylindrical



$$
\begin{aligned}
&\sigma_{tr}(r) r d\theta dz \tfrac{1}{2} dr \\
&\quad + \sigma_{tr}(r + dr)(r + dr) d\theta dz \tfrac{1}{2} dr \\
&\quad - \sigma_{rt}(\theta) dr dz \tfrac{1}{2} r d\theta \\
&\quad - \sigma_{rt}(\theta + d\theta) dr dz \tfrac{1}{2} r d\theta = 0
\end{aligned}
$$

$$
\sigma_{tr} r dr d\theta dz - \sigma_{rt} r dr d\theta dz = 0 \qquad \rightarrow \qquad \sigma_{tr} = \sigma_{rt}
$$

LINEAR ELASTIC MATERIAL

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Linear elastic material

tensor notation	$\boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\varepsilon}$	
index notation	$\sigma_{ij} = C_{ijkl}\,\varepsilon_{lk}$	; $\quad i,j,k,l \in \{1,2,3\}$
matrix notation	$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon}$	

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

Symmetry

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} =
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\
C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\
C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\
C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\
C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\
C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\
C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\
C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\
C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}
$$

specific energy $\quad W = \frac{1}{2} \underset{\approx}{\varepsilon}^T \underline{\underline{C}}\, \underset{\approx}{\varepsilon} \quad \rightarrow$

symmetry $\quad \boxed{\underline{\underline{C}} = \underline{\underline{C}}^T}$

Symmetric stresses

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

Symmetric strains

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

$$\boxed{\varepsilon_{ij} = \varepsilon_{ji}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & [C_{1121} + C_{1112}] & [C_{1132} + C_{1123}] & [C_{1113} + C_{1131}] \\ C_{2211} & C_{2222} & C_{2233} & [C_{2221} + C_{2212}] & [C_{2232} + C_{2223}] & [C_{2213} + C_{2231}] \\ C_{3311} & C_{3322} & C_{3333} & [C_{3321} + C_{3312}] & [C_{3332} + C_{3323}] & [C_{3313} + C_{3331}] \\ C_{1211} & C_{1222} & C_{1233} & [C_{1221} + C_{1212}] & [C_{1232} + C_{1223}] & [C_{1213} + C_{1231}] \\ C_{2311} & C_{2322} & C_{2333} & [C_{2321} + C_{2312}] & [C_{2332} + C_{2323}] & [C_{2313} + C_{2331}] \\ C_{3111} & C_{3122} & C_{3133} & [C_{3121} + C_{3112}] & [C_{3132} + C_{3123}] & [C_{3113} + C_{3131}] \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Symmetric material parameters

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & [C_{1121}+C_{1112}] & [C_{1132}+C_{1123}] & [C_{1113}+C_{1131}] \\ C_{2211} & C_{2222} & C_{2233} & [C_{2221}+C_{2212}] & [C_{2232}+C_{2223}] & [C_{2213}+C_{2231}] \\ C_{3311} & C_{3322} & C_{3333} & [C_{3321}+C_{3312}] & [C_{3332}+C_{3323}] & [C_{3313}+C_{3331}] \\ C_{1211} & C_{1222} & C_{1233} & [C_{1221}+C_{1212}] & [C_{1232}+C_{1223}] & [C_{1213}+C_{1231}] \\ C_{2311} & C_{2322} & C_{2333} & [C_{2321}+C_{2312}] & [C_{2332}+C_{2323}] & [C_{2313}+C_{2331}] \\ C_{3111} & C_{3122} & C_{3133} & [C_{3121}+C_{3112}] & [C_{3132}+C_{3123}] & [C_{3113}+C_{3131}] \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\boxed{C_{ijkl} = C_{ijlk}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Shear strain

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\boxed{2\varepsilon_{ij} = \gamma_{ij}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

Material symmetry

monoclinic $\rightarrow$ orthotropic $\rightarrow$ quadratic $\rightarrow$ transversal isotropic $\rightarrow$ cubic $\rightarrow$ isotropic

MATERIAL SYMMETRY

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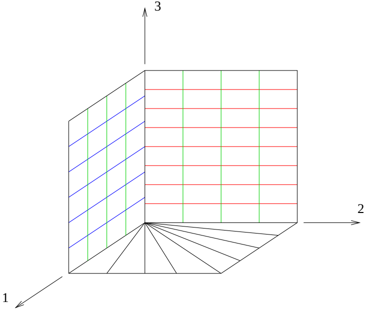
Triclinic : no symmetry

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

21	material parameters

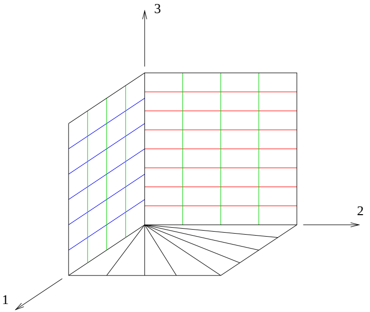
Monoclinic : 1 symmetry plane (here 12)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$



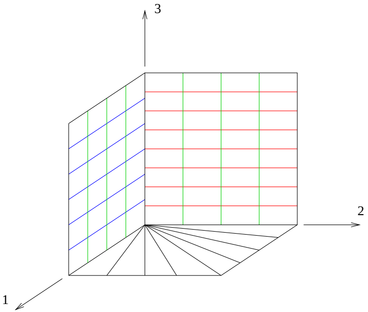
Monoclinic : tensile test

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} =
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\
C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\
C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\
C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\
C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\
C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

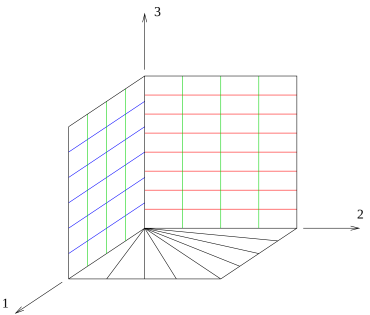


Monoclinic : tensile test

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ 0 & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ 0 & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$



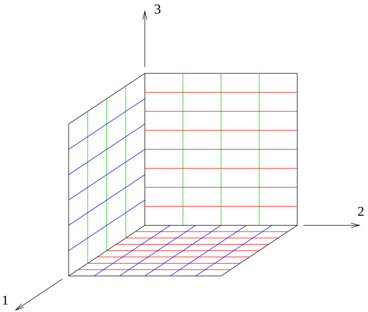
Monoclinic : 1 symmetry plane (here 12)



$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & 0 & 0 \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2332} & C_{2313} \\ 0 & 0 & 0 & 0 & C_{3132} & C_{3113} \end{bmatrix}$$

$\boxed{13 \quad \text{material parameters}}$

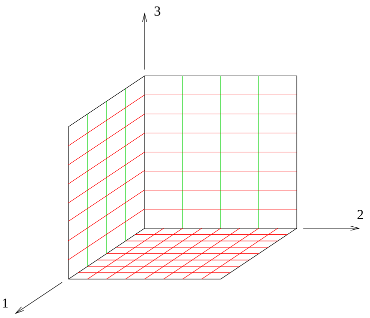
Orthotropic : 3 symmetry planes (12, 23, 31)



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix}$$

9 material parameters

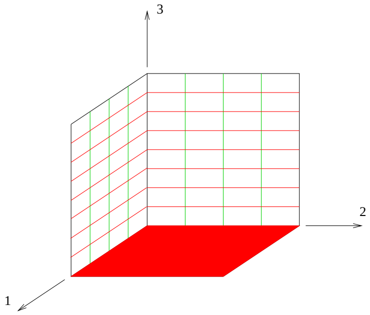
Quadratic : 2 isotropic directions (here 1 and 2)



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

6 material parameters

Transversal isotropic : 1 isotropic plane (here 12)



$$\begin{bmatrix} A & Q & R & & & \\ Q & A & R & & & \\ R & R & C & & & \\ & & & K & & \\ & & & & L & \\ & & & & & L \end{bmatrix}$$

Transversal isotropic : shear test in 12-plane

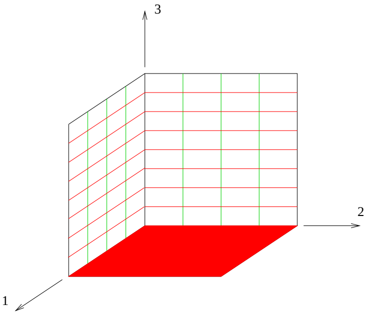
$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \quad \rightarrow \quad \det(\underline{\sigma} - \sigma \underline{I}) = 0 \quad \rightarrow \quad \begin{cases} \sigma_1 = \tau \\ \sigma_2 = -\tau \end{cases}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} \rightarrow \det(\underline{\varepsilon} - \varepsilon \underline{I}) = 0 \quad \rightarrow \quad \begin{cases} \varepsilon_1 = \frac{1}{2}\gamma \\ \varepsilon_2 = -\frac{1}{2}\gamma \end{cases}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} A & Q \\ Q & A \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \rightarrow \quad \begin{array}{l} \sigma_1 = A\varepsilon_1 + Q\varepsilon_2 = \quad \tau = \quad K\gamma \\ \sigma_2 = Q\varepsilon_1 + A\varepsilon_2 = -\tau = -K\gamma \end{array} \quad \rightarrow$$

$$\left. \begin{array}{l} (A - Q)(\varepsilon_1 - \varepsilon_2) = 2K\gamma \\ \\ \varepsilon_1 = \frac{1}{2}\gamma \quad ; \quad \varepsilon_1 = -\frac{1}{2}\gamma \end{array} \right\} \quad \rightarrow \quad \boxed{K = \frac{1}{2}(A - Q)}$$

Transversal isotropic

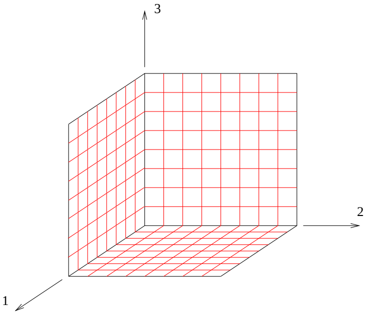


$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$K = \frac{1}{2}(A - Q)$$

5	material parameters

Cubic : 3 isotropic directions (here 1, 2 and 3)

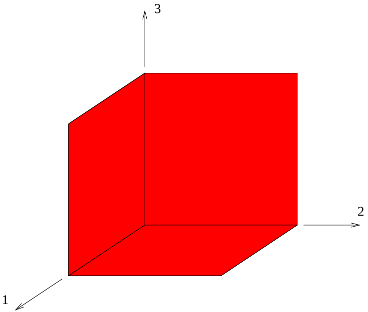


$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$L \neq \tfrac{1}{2}(A - Q)$$

$\boxed{\text{3 \quad material parameters}}$

Isotropic



$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$L = \frac{1}{2}(A - Q)$$

2	material parameters

LINEAR ELASTIC ISOTROPIC MATERIAL

ENGINEERING PARAMETERS

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Stiffness

$$\underset{\sim}{\sigma} = \underline{\underline{C}}\,\underset{\sim}{\varepsilon}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

Compliance

$$\underset{\sim}{\underset{\sim}{\varepsilon}} = \underline{\underline{C}}^{-1}\, \underset{\sim}{\underset{\sim}{\sigma}} = \underline{\underline{S}}\, \underset{\sim}{\underset{\sim}{\sigma}}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} A^2 - Q^2 & Q^2 - AQ & Q^2 - AQ & 0 & 0 & 0 \\ Q^2 - AQ & A^2 - Q^2 & Q^2 - AQ & 0 & 0 & 0 \\ Q^2 - AQ & Q^2 - AQ & A^2 - Q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & D/L & 0 & 0 \\ 0 & 0 & 0 & 0 & D/L & 0 \\ 0 & 0 & 0 & 0 & 0 & D/L \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\text{with} \quad D = \det(\underline{\underline{C}}) = A^3 + 2Q^3 - 3AQ^2$$

$$= \begin{bmatrix} a & q & q & 0 & 0 & 0 \\ q & a & q & 0 & 0 & 0 \\ q & q & a & 0 & 0 & 0 \\ 0 & 0 & 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & l \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

Tensile test

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon & \varepsilon_d & \varepsilon_d & 0 & 0 & 0 \end{bmatrix} \; ; \; \underset{\approx}{\sigma}^T = \begin{bmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\left.\begin{array}{l} \sigma = A\varepsilon + 2Q\varepsilon_d \\ 0 = Q\varepsilon + (A+Q)\varepsilon_d \rightarrow \varepsilon_d = -\dfrac{Q}{A+Q}\,\varepsilon \end{array}\right\} \rightarrow \sigma = \dfrac{A^2 + AQ - 2Q^2}{A+Q}\,\varepsilon \quad \right\} \rightarrow$$

$$\varepsilon_d = -\nu\varepsilon \qquad ; \qquad \sigma = E\varepsilon$$

$$A = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \qquad Q = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad L = \frac{E}{2(1+\nu)}$$

Shear test

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \qquad L = \frac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \; ; \; \underset{\approx}{\sigma}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \tau \end{bmatrix}$$

$$\tau = L\gamma = \frac{E}{2(1+\nu)}\gamma = G\gamma$$

Volume change

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} J - 1 \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} &= \frac{1-2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &= \frac{3(1-2\nu)}{E}\,\tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{K}\,\tfrac{1}{3}\mathrm{tr}(\underline{\sigma}) \end{aligned}$$

Isotropic compliance and stiffness matrix

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}
$$

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}
= \alpha
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu)
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}
$$

$$
\text{with} \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}
$$

LINEAR ELASTIC ISOTROPIC MATERIAL

TENSORIAL FORM

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Column/matrix notation of Hooke's law

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\text{with} \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}$$

Isotropic stiffness matrix

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \frac{E}{(1+\nu)} \begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$= \frac{E}{(1+\nu)} \left[\begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} - 1 & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} - 1 & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$= \left[\frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \right.$$

$$\left. \frac{E}{(1+\nu)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

Isotropic stiffness tensor

$$\boldsymbol{\sigma} = \left[\frac{E\nu}{(1+\nu)(1-2\nu)} \right] \mathbf{I}\, \mathrm{tr}(\boldsymbol{\varepsilon}) + \left[\frac{E}{(1+\nu)} \right] \boldsymbol{\varepsilon}$$

$$= Q\, \mathrm{tr}(\boldsymbol{\varepsilon})\, \mathbf{I} + 2L\boldsymbol{\varepsilon}$$

$$= c_0\, \mathrm{tr}(\boldsymbol{\varepsilon})\, \mathbf{I} + c_1\, \boldsymbol{\varepsilon}$$

$$= \left[c_0 \mathbf{I}\mathbf{I} + c_1\, {}^4\mathbf{I}^s \right] : \boldsymbol{\varepsilon} \qquad \text{with} \qquad {}^4\mathbf{I}^s = \tfrac{1}{2}\left({}^4\mathbf{I} + {}^4\mathbf{I}^{rc} \right)$$

$$= {}^4\mathbf{C} : \boldsymbol{\varepsilon}$$

Stiffness and compliance tensor

$$
\begin{aligned}
\boldsymbol{\sigma} &= {}^4\mathbf{C} : \boldsymbol{\varepsilon} \\
&= \left[c_0 \mathbf{I}\mathbf{I} + c_1\, {}^4\mathbf{I}^s \right] : \boldsymbol{\varepsilon} \\
& \quad \text{with} \quad {}^4\mathbf{I}^s = \tfrac{1}{2}\left({}^4\mathbf{I} + {}^4\mathbf{I}^{rc} \right) \\
&= c_0 \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + c_1 \boldsymbol{\varepsilon} \\
&= c_0 \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + c_1 \left\{ \boldsymbol{\varepsilon}^d + \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} \right\} \\
&= (c_0 + \tfrac{1}{3}c_1)\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + c_1 \boldsymbol{\varepsilon}^d \\
&= (3c_0 + c_1)\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + c_1 \boldsymbol{\varepsilon}^d \\
&= (3c_0 + c_1)\boldsymbol{\varepsilon}^h + c_1 \boldsymbol{\varepsilon}^d \\
&= \boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^d \\
&= \frac{1}{3c_0 + c_1}\, \boldsymbol{\sigma}^h + \frac{1}{c_1}\, \boldsymbol{\sigma}^d \\
&= \frac{1}{3c_0 + c_1}\, \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} + \frac{1}{c_1} \left\{ \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} \right\} \\
&= -\frac{c_0}{(3c_0 + c_1)c_1}\, \mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} + \frac{1}{c_1}\, \boldsymbol{\sigma} \\
&= \left[-\frac{c_0}{(3c_0 + c_1)c_1}\mathbf{I}\mathbf{I} + \frac{1}{c_1}\, {}^4\mathbf{I}^s \right] : \boldsymbol{\sigma} \\
&= \left[\gamma_0 \mathbf{I}\mathbf{I} + \gamma_1\, {}^4\mathbf{I}^s \right] : \boldsymbol{\sigma} \\
&= {}^4\mathbf{S} : \boldsymbol{\sigma}
\end{aligned}
$$

$$
c_0 = \frac{\nu E}{(1+\nu)(1-2\nu)} = Q \quad ; \quad c_1 = \frac{E}{1+\nu} = 2L
$$

$$
\gamma_0 = -\frac{c_0}{(3c_0 + c_1)c_1} = -\frac{\nu}{E} = q \quad ; \quad \gamma_1 = \frac{1}{c_1} = \frac{1+\nu}{E} = \tfrac{1}{2}l
$$

Stiffness and compliance components

$$
\begin{aligned}
\boldsymbol{\sigma} &= \left[c_0 \mathbf{II} + c_1\, {}^4\mathbf{I}^s\right] : \boldsymbol{\varepsilon} \\
\sigma_{ij} &= \left[c_0 \delta_{ij}\delta_{kl} + c_1 \tfrac{1}{2}\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right)\right] \varepsilon_{lk} \\
&= c_0 \delta_{ij}\varepsilon_{kk} + c_1 \varepsilon_{ij} \\
&= c_1 \left(\varepsilon_{ij} + \frac{c_0}{c_1}\delta_{ij}\varepsilon_{kk}\right) \\
&= \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \left[-\frac{c_0}{(3c_0+c_1)c_1}\mathbf{II} + \frac{1}{c_1}\, {}^4\mathbf{I}^s\right] : \boldsymbol{\sigma} \\
\varepsilon_{ij} &= \left[-\frac{c_0}{(3c_0+c_1)c_1}\delta_{ij}\delta_{kl} + \right. \\
&\qquad \left. \frac{1}{c_1}\tfrac{1}{2}\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right)\right] \sigma_{lk} \\
&= -\frac{c_0}{(3c_0+c_1)c_1}\delta_{ij}\sigma_{kk} + \frac{1}{c_1}\sigma_{ij} \\
&= \frac{1}{c_1}\left(\sigma_{ij} - \frac{c_0}{3c_0+c_1}\delta_{ij}\sigma_{kk}\right) \\
&= \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\delta_{ij}\sigma_{kk}\right)
\end{aligned}
$$

Specific elastic energy

$$W = \tfrac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\sigma} : {}^4\mathbf{S} : \boldsymbol{\sigma} = \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : {}^4\mathbf{S} : (\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d)$$

$$= \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : \left(\gamma_0 \mathbf{I}\mathbf{I} + \gamma_1 {}^4\mathbf{I}^s\right) : (\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d)$$

$$\gamma_0 \mathbf{I}\left[\mathbf{I} : \boldsymbol{\sigma}^h\right] = \gamma_0 \mathbf{I}\left[\mathbf{I} : \mathbf{I}\,\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\right] = \gamma_0 \mathbf{I}\left[\mathrm{tr}(\boldsymbol{\sigma})\right] = 3\gamma_0 \boldsymbol{\sigma}^h$$

$$\gamma_0 \mathbf{I}\left[\mathbf{I} : \boldsymbol{\sigma}^d\right] = \gamma_0\,\mathbf{I}\left[\mathrm{tr}(\boldsymbol{\sigma}^d)\right] = \gamma_0\,\mathbf{I}\left[0\right] = 0$$

$$= \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : (3\gamma_0 \boldsymbol{\sigma}^h + \gamma_1 \boldsymbol{\sigma}^h + \gamma_1 \boldsymbol{\sigma}^d)$$

$$\boldsymbol{\sigma}^h : \boldsymbol{\sigma}^h = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} : \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} = \tfrac{1}{9}\mathrm{tr}^2(\boldsymbol{\sigma})(3) = \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma})$$

$$\boldsymbol{\sigma}^h : \boldsymbol{\sigma}^d = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} : [\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}] = \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) - \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) = 0$$

$$= \left[\tfrac{1}{2}(\gamma_0 + \tfrac{1}{3}\gamma_1)\right] \mathrm{tr}^2(\boldsymbol{\sigma}) + \left[\tfrac{1}{2}\gamma_1\right] \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d$$

$$= \left[\tfrac{1}{2}\frac{1-2\nu}{3E}\right] \mathrm{tr}^2(\boldsymbol{\sigma}) + \left[\tfrac{1}{2}\frac{1+\nu}{E}\right] \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{18K}\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d$$

$$= W^h + W^d$$

PLANAR DEFORMATION

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Orthotropic material

Stiffness

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & & & \\ Q & B & S & & & \\ R & S & C & & & \\ & & & K & & \\ & & & & L & \\ & & & & & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

Compliance

$$\underset{\approx}{\varepsilon} = \underline{\underline{C}}^{-1}\,\underset{\approx}{\sigma} = \underline{\underline{S}}\,\underset{\approx}{\sigma}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & & & \\ q & b & s & & & \\ r & s & c & & & \\ & & & k & & \\ & & & & l & \\ & & & & & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

Plane strain (from $\underline{\underline{C}}$)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & & & \\ Q & B & S & & & \\ R & S & C & & & \\ & & & K & & \\ & & & & L & \\ & & & & & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \left\{ \begin{array}{l} \varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0 \\ \\ \sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22} \end{array} \right.$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{Q^2 - BA} \begin{bmatrix} -B & Q & 0 \\ Q & -A & 0 \\ 0 & 0 & \frac{Q^2 - BA}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}_\varepsilon \, \underset{\approx}{\varepsilon} \qquad ; \qquad \underset{\approx}{\varepsilon} = \underline{\underline{S}}_\varepsilon \, \underset{\approx}{\sigma}$$

Plane strain (from $\underline{\underline{S}}$)

$$\varepsilon_{33} = 0 = r\sigma_{11} + s\sigma_{22} + c\sigma_{33} \quad \rightarrow \quad \sigma_{33} = -\frac{r}{c}\sigma_{11} - \frac{s}{c}\sigma_{22}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} - \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \begin{bmatrix} \frac{r}{c} & \frac{s}{c} & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} ac - r^2 & qc - rs & 0 \\ qc - sr & bc - s^2 & 0 \\ 0 & 0 & kc \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\underset{\approx}{\varepsilon} = \underline{\underline{S}}_\varepsilon \, \underset{\approx}{\sigma} \qquad ; \qquad \underset{\approx}{\sigma} = \underline{\underline{C}}_\varepsilon \, \underset{\approx}{\varepsilon}$$

Plane stress (from $\underline{\underline{S}}$)

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & & & \\ q & b & s & & & \\ r & s & c & & & \\ & & & k & & \\ & & & & l & \\ & & & & & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} \qquad \left\{ \begin{array}{l} \sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \\ \\ \varepsilon_{33} = r\sigma_{11} + s\sigma_{22} \end{array} \right.$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{q^2 - ba} \begin{bmatrix} -b & q & 0 \\ q & -a & 0 \\ 0 & 0 & \frac{q^2 - ba}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$\underset{\approx}{\varepsilon} = \underline{\underline{S}}_\sigma \, \underset{\approx}{\sigma} \qquad ; \qquad \underset{\approx}{\sigma} = \underline{\underline{C}}_\sigma \, \underset{\approx}{\varepsilon}$$

Plane stress (from $\underline{\underline{C}}$)

$$\sigma_{33} = 0 = R\varepsilon_{11} + S\varepsilon_{22} + C\varepsilon_{33} \quad \rightarrow \quad \varepsilon_{33} = -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \begin{bmatrix} R \\ S \\ 0 \end{bmatrix} \begin{bmatrix} \frac{R}{C} & \frac{S}{C} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \frac{1}{C} \begin{bmatrix} AC - R^2 & QC - RS & 0 \\ QC - SR & BC - S^2 & 0 \\ 0 & 0 & KC \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}_\sigma \, \underset{\approx}{\varepsilon} \quad ; \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}_\sigma \, \underset{\approx}{\sigma}$$

Plane strain/stress

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix}$$

Isotropic material

Compliance

$$\underset{\sim}{\underset{\sim}{\varepsilon}} = \underline{\underline{S}}\, \underset{\sim}{\underset{\sim}{\sigma}}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & 2(1+\nu) & & \\ & & & & 2(1+\nu) & \\ & & & & & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

Plane stress (from $\underline{\underline{S}}$)

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \quad \rightarrow \quad \varepsilon_{33} = -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right)$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_{\sigma} \, \underset{\approx}{\sigma}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_{\sigma} \, \underset{\approx}{\varepsilon}$$

Plane strain (from $\underline{\underline{S}}$)

$$\varepsilon_{33} = 0 = \frac{1}{E}\left(-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}\right) \quad \rightarrow \quad \sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E}\left\{ \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} -\nu \\ -\nu \\ 0 \end{bmatrix} \sigma_{33} \right\}$$

$$= \frac{1}{E}\left\{ \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} -\nu \\ -\nu \\ 0 \end{bmatrix} \begin{bmatrix} \nu & \nu & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \right\}$$

$$= \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_{\varepsilon}\, \underset{\approx}{\sigma}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \underline{\underline{C}}_{\varepsilon}\, \underset{\approx}{\sigma}$$

THERMO-ELASTICITY

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Thermoelasticity

Anisotropic

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_T = {}^4\mathbf{S} : \boldsymbol{\sigma} + \boldsymbol{A}\Delta T \quad \rightarrow$$

$$\underset{\approx}{\varepsilon} = \underset{\approx}{\varepsilon}_m + \underset{\approx}{\varepsilon}_T = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \underline{\underline{A}}\,\Delta T$$

$$\boldsymbol{\sigma} = {}^4\mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{A}\Delta T) \quad \rightarrow$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\left(\underset{\approx}{\varepsilon} - \underline{\underline{A}}\,\Delta T\right)$$

Isotropic

$$\boldsymbol{\varepsilon} = {}^4\mathbf{S} : \boldsymbol{\sigma} + \alpha\Delta T\mathbf{I} \quad \rightarrow \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \alpha\Delta T\underset{\approx}{I}$$

$$\boldsymbol{\sigma} = {}^4\mathbf{C} : (\boldsymbol{\varepsilon} - \alpha\Delta T\mathbf{I}) \quad \rightarrow \quad \underset{\approx}{\sigma} = \underline{\underline{C}}(\underset{\approx}{\varepsilon} - \alpha\Delta T\,\underset{\approx}{I})$$

Orthotropic thermo-elasticity

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} A+Q+R \\ Q+B+S \\ R+S+C \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Plane stress

$$
\begin{aligned}
\varepsilon_{33} &= -\frac{R}{C}\,\varepsilon_{11} - \frac{S}{C}\,\varepsilon_{22} + \frac{1}{C}\,(R+S+C)\,\alpha\,\Delta T && \text{(from } \underline{\underline{C}}\text{)} \\
&= r\sigma_{11} + s\sigma_{22} + \alpha\Delta T && \text{(from } \underline{\underline{S}}\text{)}
\end{aligned}
$$

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} A_\sigma + Q_\sigma \\ B_\sigma + Q_\sigma \\ 0 \end{bmatrix}
$$

Plane strain

$$\begin{aligned}\sigma_{33} &= R\varepsilon_{11} + S\varepsilon_{22} - \alpha(R + S + C)\,\Delta T && \text{(from } \underline{C}\text{)} \\ &= -\frac{r}{c}\,\sigma_{11} - \frac{s}{c}\,\sigma_{22} - \frac{\alpha}{c}\,\Delta T && \text{(from } \underline{\underline{S}}\text{)}\end{aligned}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha\Delta T \begin{bmatrix} A + Q + R \\ B + Q + S \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 + q_\varepsilon S + a_\varepsilon R \\ 1 + q_\varepsilon R + b_\varepsilon S \\ 0 \end{bmatrix}$$

Planar general

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} \Theta_{p1} \\ \Theta_{p2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} \theta_{p1} \\ \theta_{p2} \\ 0 \end{bmatrix}$$

ELASTIC LIMIT

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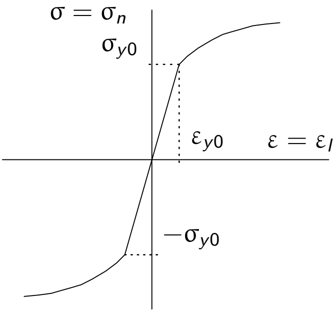
Elastic limit criteria

failure mode	**mechanism**
plastic yielding	crystallographic slip (metals)
brittle fracture	(sudden) breakage of bonds
progressive damage	micro-cracks $\rightarrow$ growth $\rightarrow$ coalescence
fatigue	damage/fracture under cyclic loading
dynamic failure	vibration $\rightarrow$ resonance
thermal failure	creep / melting
elastic instabilities	buckling $\rightarrow$ plastic deformation

1D yield

$$f(\sigma) = \sigma^2 - \sigma_{y0}^2 = 0 \quad \rightarrow$$
$$g(\sigma) = \sigma^2 = \sigma_{y0}^2 = g_t$$
$$g_t = \text{limit in tensile test}$$



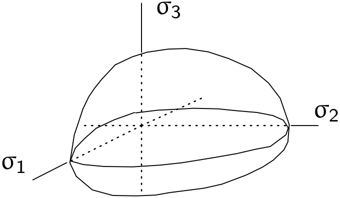
3D yield

$$f(\boldsymbol{\sigma}) = 0 \qquad \rightarrow \quad g(\boldsymbol{\sigma}) = g_t$$

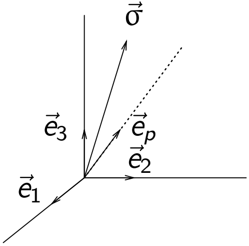
yield surface in 6D stress space

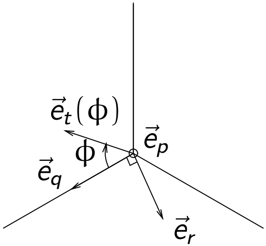
$$f(\sigma_1, \sigma_2, \sigma_3) = 0 \quad \rightarrow \quad g(\sigma_1, \sigma_2, \sigma_3) = g_t$$

yield surface in 3D principal stress space



Principal stress space





hydrostatic axis $\quad \vec{e}_p = \frac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \quad$ with $\quad \|\vec{e}_p\| = 1$

plane $\perp$ hydrostatic axis

$$\vec{e}_q^{\,*} = \vec{e}_1 - (\vec{e}_p \cdot \vec{e}_1)\vec{e}_p = \vec{e}_1 - \frac{1}{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) = \frac{1}{3}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3)$$

$$\vec{e}_q = \frac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3)$$

$$\vec{e}_r = \vec{e}_p * \vec{e}_q = \frac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) * \frac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) = \frac{1}{2}\sqrt{2}(\vec{e}_2 - \vec{e}_3)$$

vector in Π-plane $\quad \vec{e}_t(\phi) = \cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r$

Principal stress space

$$
\begin{aligned}
\vec{\sigma} &= \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3 = \vec{\sigma}^h + \vec{\sigma}^d \\
\vec{\sigma}^h &= (\vec{\sigma} \cdot \vec{e}_p)\vec{e}_p = \sigma^h \vec{e}_p = \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3)\vec{e}_p = \sqrt{3}\sigma_m \vec{e}_p \\
\sigma^h &= \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3) \\
\vec{\sigma}^d &= \vec{\sigma} - (\vec{\sigma} \cdot \vec{e}_p)\vec{e}_p \\
&= \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3 - \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3)\tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \\
&= \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3 - \\
&\qquad \tfrac{1}{3}(\sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_1 + \sigma_3 \vec{e}_1 + \sigma_1 \vec{e}_2 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_2 + \sigma_1 \vec{e}_3 + \sigma_2 \vec{e}_3 + \sigma_3 \vec{e}_3) \\
&= \tfrac{1}{3}\{(2\sigma_1 - \sigma_2 - \sigma_3)\vec{e}_1 + (-\sigma_1 + 2\sigma_2 - \sigma_3)\vec{e}_2 + (-\sigma_1 - \sigma_2 + 2\sigma_3)\vec{e}_3\} \\
\sigma^d &= \|\vec{\sigma}^d\| = \sqrt{\vec{\sigma}^d \cdot \vec{\sigma}^d} \\
&= \tfrac{1}{3}\sqrt{(2\sigma_1 - \sigma_2 - \sigma_3)^2 + (-\sigma_1 + 2\sigma_2 - \sigma_3)^2 + (-\sigma_1 - \sigma_2 + 2\sigma_3)^2} \\
&= \sqrt{\tfrac{2}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)} \\
&= \sqrt{\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d}
\end{aligned}
$$

Principal stress space

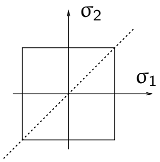
$$
\begin{aligned}
\vec{\sigma} &= \vec{\sigma}^h + \vec{\sigma}^d = \sigma^h \vec{e}_p + \sigma^d \vec{e}_t(\phi) \\
&= \sigma^h \vec{e}_p + \sigma^d \{\cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r\} \\
&= \sigma^h \tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) + \sigma^d \{\cos(\phi)\tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) - \sin(\phi)\tfrac{1}{2}\sqrt{2}(\vec{e}_2 - \vec{e}_3)\} \\
&= \{\tfrac{1}{3}\sqrt{3}\,\sigma^h + \tfrac{1}{3}\sqrt{6}\,\sigma^d \cos(\phi)\}\vec{e}_1 + \\
&\quad\ \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d \cos(\phi) - \tfrac{1}{2}\sqrt{2}\,\sigma^d \sin(\phi)\}\vec{e}_2 + \\
&\quad\ \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d \cos(\phi) + \tfrac{1}{2}\sqrt{2}\,\sigma^d \sin(\phi)\}\vec{e}_3 \\
&= \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 + \sigma_3 \vec{e}_3
\end{aligned}
$$

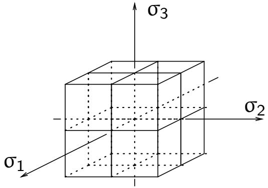
Maximum stress/strain

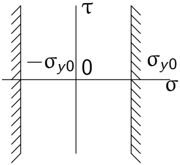
$$\boxed{\sigma_{ij} = \sigma_{max} \quad | \quad \varepsilon_{ij} = \varepsilon_{max} \quad ; \quad \{i,j\} = \{1,2,3\}}$$ (orthotropic materials)

Rankine

$$\boxed{|\sigma_{max}| = max(|\sigma_i| \ ; \ i = 1, 2, 3) = \sigma_{max,t} = \sigma_{y0}}$$ (brittle materials; cast iron)



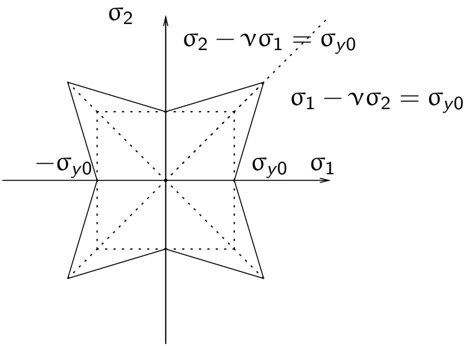




Saint Venant

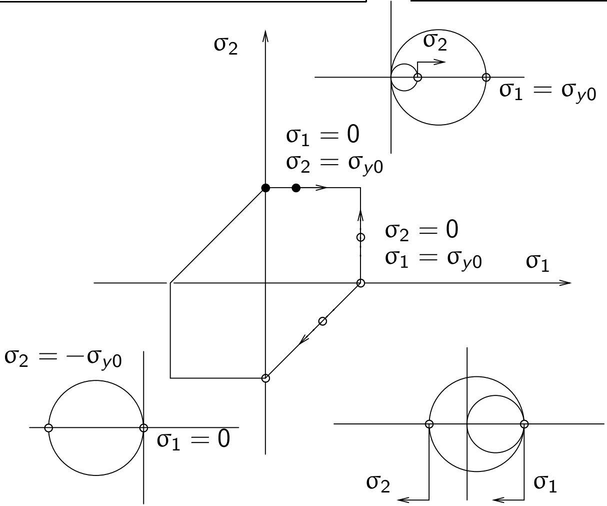
$$\varepsilon_{max} = max(|\varepsilon_i| \ ; \ i = 1,2,3) = \varepsilon_{max_t} = \varepsilon_{y0} = \frac{\sigma_{y0}}{E}$$

(brittle materials; cast iro



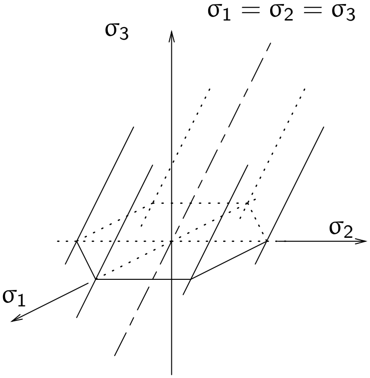
Tresca : 2D yield contour

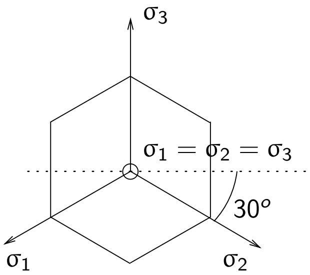
$$\boxed{\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \tau_{max,t} = \frac{1}{2}\sigma_{y0}} \longrightarrow \boxed{\bar{\sigma}_{TR} = \sigma_{max} - \sigma_{min} = \sigma_{y0}}$$



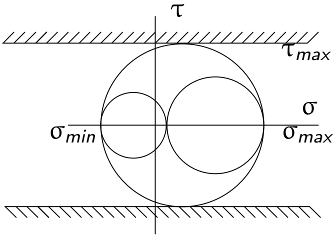
Tresca : 3D yield surface

Mohr $\quad\rightarrow\quad$ invariant for hydrostatic stress $\quad\rightarrow$
yield surface // hydrostatic axis
$\Pi -$ plane $\perp$ hydrostatic axis





Tresca : st-plane



Von Mises

$$\boxed{W^d = W_t^d}$$

$$\begin{aligned}
W^d &= \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{4G}\left\{\boldsymbol{\sigma} : \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma})\right\} \quad \left(= \; -\frac{1}{2G}\,J_2(\boldsymbol{\sigma}^d)\right) \\
&= \frac{1}{4G}\,(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{12G}\,(\sigma_1 + \sigma_2 + \sigma_3)^2 \\
&= \frac{1}{4G}\,\tfrac{1}{3}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\}
\end{aligned}$$

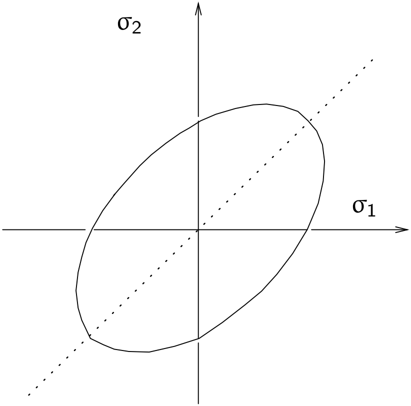
$$W_t^d = \frac{1}{4G}\,\tfrac{1}{3}\,\left\{(\sigma - 0)^2 + (0 - 0)^2 + (0 - \sigma)^2\right\} = \frac{1}{4G}\,\tfrac{1}{3}2\sigma^2 = \frac{1}{4G}\,\tfrac{1}{3}\,2{\sigma_{y0}}^2$$

$$\bar{\sigma}_{VM} = \sqrt{\tfrac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}} = \sigma_{y0}$$

Von Mises : Cartesian stress components

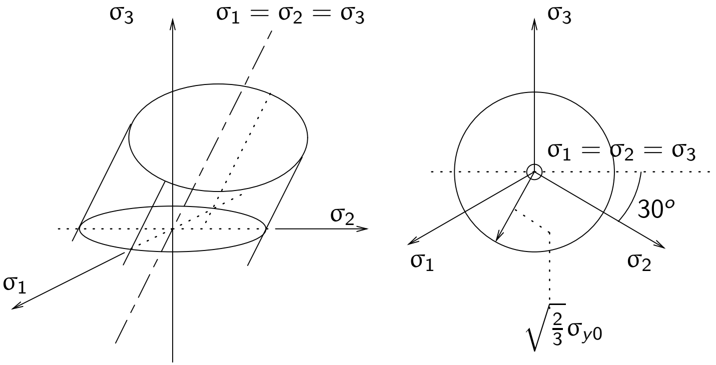
$$
\begin{aligned}
\bar{\sigma}_{VM}^2 &= \tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = 3J_2 \\
&= \tfrac{3}{2}\mathrm{tr}(\underline{\sigma}^d \underline{\sigma}^d) \quad \text{with } \underline{\sigma}^d = \underline{\sigma} - \tfrac{1}{3}\mathrm{tr}(\underline{\sigma})\underline{I} \\
&= \tfrac{3}{2}\Big\{ \left(\tfrac{2}{3}\sigma_{xx} - \tfrac{1}{3}\sigma_{yy} - \tfrac{1}{3}\sigma_{zz}\right)^2 + \sigma_{xy}^2 + \sigma_{xz}^2 + \\
&\qquad\quad \left(\tfrac{2}{3}\sigma_{yy} - \tfrac{1}{3}\sigma_{zz} - \tfrac{1}{3}\sigma_{xx}\right)^2 + \sigma_{yz}^2 + \sigma_{yx}^2 + \\
&\qquad\quad \left(\tfrac{2}{3}\sigma_{zz} - \tfrac{1}{3}\sigma_{xx} - \tfrac{1}{3}\sigma_{yy}\right)^2 + \sigma_{zx}^2 + \sigma_{zy}^2 \Big\} \\
&= \left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2\right) - \left(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}\right) + 2\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right) \\
&= \sigma_{y0}^2
\end{aligned}
$$

Von Mises : 2D yield surface



Von Mises : 3D yield surface

$$\boxed{\tfrac{1}{2}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}=\sigma_{y0}^2}$$



Beltrami-Haigh

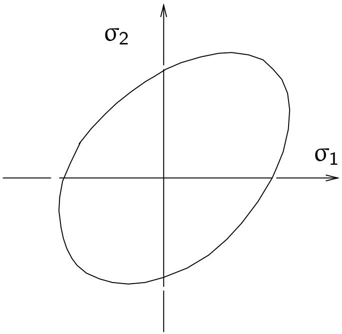
$$\boxed{W = W_t}$$

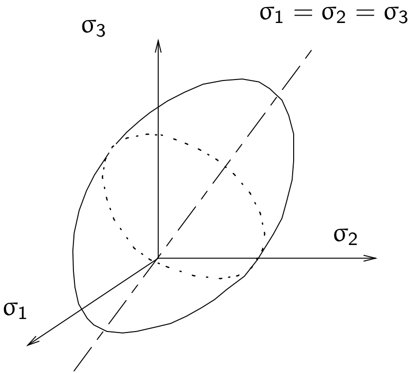
$$\begin{aligned} W &= W^h + W^d = \frac{1}{18K}\,\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d \\ &= \left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 + \frac{1}{4G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \end{aligned}$$

$$W_t = \left(\frac{1}{18K} - \frac{1}{12G}\right)\sigma^2 + \frac{1}{4G}\,\sigma^2 = \frac{1}{2E}\,\sigma^2 = \frac{1}{2E}\,{\sigma_{y0}}^2$$

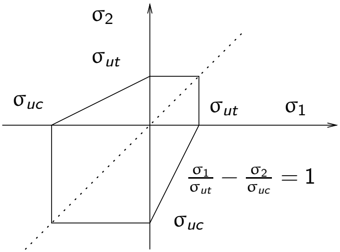
$$2E\left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 + \frac{2E}{4G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) = \sigma_{y0}^2$$

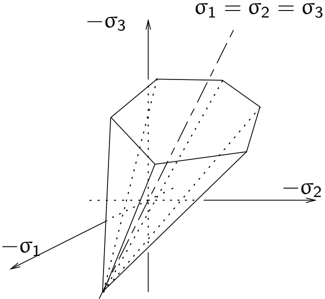
Beltrami-Haigh : 2D/3D yield surface

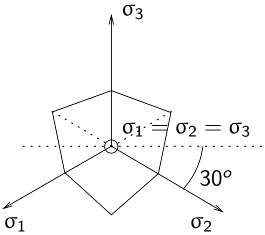




Mohr-Coulomb : 2D/3D yield surface

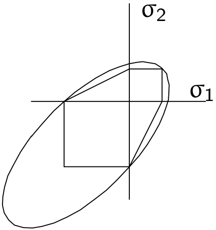


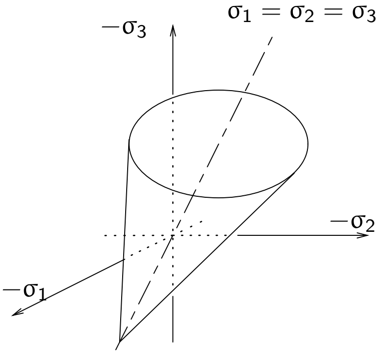


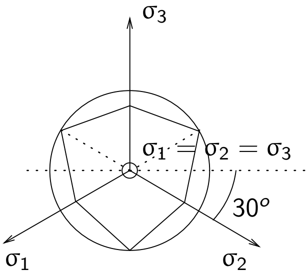


Drucker-Prager

$$\sqrt{\frac{2}{3}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} + \frac{6\sin(\phi)}{3-\sin(\phi)}\,p = \frac{6\cos(\phi)}{3-\sin(\phi)}\,C$$







Other yield criteria

parabolic Drucker-Prager $\left(3J_2 + \sqrt{3}\,\beta\,\sigma_{y0} J_1\right)^{\frac{1}{2}} = \sigma_{y0}$

Buyokozturk $\left(3J_2 + \sqrt{3}\,\beta\,\sigma_{y0} J_1 - 0.2 J_1^2\right)^{\frac{1}{2}} = \sigma_{y0}$

Hill $\dfrac{\sigma_{11}^2}{X^2} - \dfrac{\sigma_{11}\sigma_{22}}{XY} + \dfrac{\sigma_{22}^2}{Y^2} + \dfrac{\sigma_{12}^2}{S^2}$

Hoffman

$$\left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_{11} + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_{22} + \left(\frac{1}{X_t X_c}\right)\sigma_{11}^2 + \left(\frac{1}{Y_t Y_c}\right)\sigma_{22}^2 + \left(\frac{1}{S^2}\right)\sigma_{12}^2 - \left(\frac{1}{X_t X_c}\right)\sigma_{11}\sigma_{22} = 0$$

Tsai-Wu

$$\left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_{11} + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_{22} + \left(\frac{1}{X_t X_c}\right)\sigma_{11}^2 + \left(\frac{1}{Y_t Y_c}\right)\sigma_{22}^2 + \left(\frac{1}{S^2}\right)\sigma_{12}^2 + 2F_{12}\;\sigma_{11}\sigma_{22} = 0$$

with $\quad F_{12}^2 > \dfrac{1}{X_t X_c}\;\dfrac{1}{Y_t Y_c}$

GOVERNING EQUATIONS

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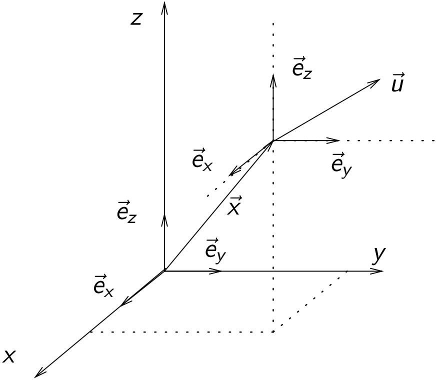
Vector/tensor equations

gradient operator	:	$\vec{\nabla} = \underset{\sim}{\nabla}^T \underset{\sim}{\vec{e}}$
position	:	$\vec{x} = \underset{\sim}{x}^T \underset{\sim}{\vec{e}}$
displacement	:	$\vec{u} = \underset{\sim}{u}^T \underset{\sim}{\vec{e}}$
strain	:	$\boldsymbol{\varepsilon} = \frac{1}{2}\left\{\left(\vec{\nabla}\vec{u}\right)^c + \left(\vec{\nabla}\vec{u}\right)\right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$
compatibility	:	$\nabla^2\{\mathrm{tr}(\boldsymbol{\varepsilon})\} - \vec{\nabla}\cdot(\vec{\nabla}\cdot\boldsymbol{\varepsilon})^c = 0$
stress	:	$\boldsymbol{\sigma} = \underset{\sim}{\vec{e}}^T \underline{\sigma}\, \underset{\sim}{\vec{e}}$
balance laws	:	$\vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \rho\ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^c$
material law	:	$\boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\varepsilon} \quad ; \quad \boldsymbol{\varepsilon} = {}^4\mathbf{C}^{-1} : \boldsymbol{\sigma} = {}^4\mathbf{S} : \boldsymbol{\sigma}$

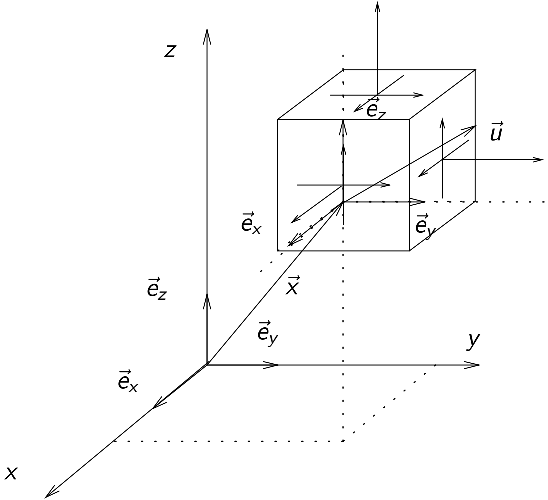
Cartesian components



$$\vec{x} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$
$$\underset{\sim}{x}^T = \left[\begin{array}{ccc} x & y & z \end{array}\right]$$

$$\underset{\sim}{\nabla}^T = \left[\begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array}\right]$$

Cartesian stress components



Cartesian components

$$\underset{\sim}{x}^T = \begin{bmatrix} x & y & z \end{bmatrix} \quad ; \quad \underset{\sim}{\nabla}^T = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \quad ; \quad \underset{\sim}{u}^T = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}$$

$$\underline{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{x,x} & u_{x,y} + u_{y,x} & u_{x,z} + u_{z,x} \\ \cdots & 2u_{y,y} & u_{y,z} + u_{z,y} \\ \cdots & \cdots & 2u_{z,z} \end{bmatrix}$$

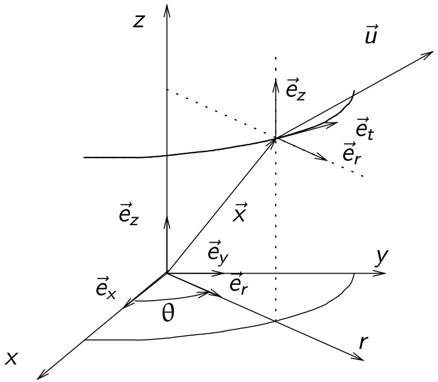
$$2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$
$$\varepsilon_{xx,yz} + \varepsilon_{yz,xx} - \varepsilon_{zx,xy} - \varepsilon_{xy,xz} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \varepsilon_{xy} & \varepsilon_{yz} & \varepsilon_{zx} \end{bmatrix} \qquad (\varepsilon_{zz})$$
$$\underset{\approx}{\sigma}^T = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{bmatrix} \qquad (\sigma_{zz})$$

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x = \rho \ddot{u}_x \qquad (\sigma_{xy} = \sigma_{yx})$$
$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y = \rho \ddot{u}_y \qquad (\sigma_{yz} = \sigma_{zy})$$
$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z = \rho \ddot{u}_z \qquad (\sigma_{zx} = \sigma_{xz})$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon} \qquad ; \qquad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}$$

Cylindrical components



$$\vec{x} = r\vec{e}_r(\theta) + z\vec{e}_z$$

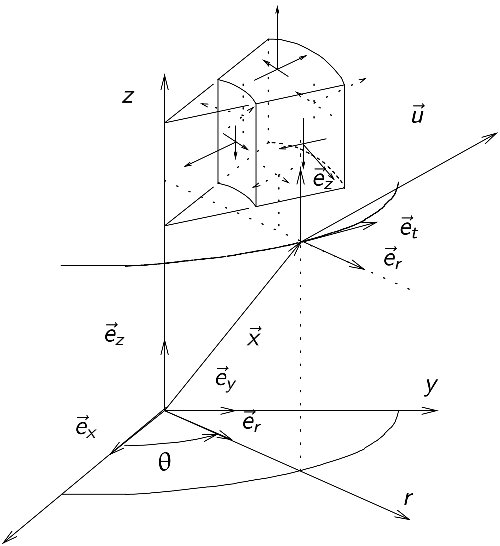
$$\underset{\sim}{x}^T = \begin{bmatrix} r & \theta & z \end{bmatrix}$$

$$x = r\cos(\theta) \ ; \ y = r\sin(\theta)$$

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_t \qquad \frac{d\vec{e}_t}{d\theta} = -\vec{e}_r$$

$$\underset{\sim}{\nabla}^T = \begin{bmatrix} \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \end{bmatrix}$$

Cylindrical stress components



Cylindrical components

$$\underset{\sim}{x}^T = [\; r \quad \theta \quad z \;] \quad ; \quad \underset{\sim}{\nabla}^T = \left[\; \frac{\partial}{\partial r} \quad \frac{1}{r}\frac{\partial}{\partial \theta} \quad \frac{\partial}{\partial z} \;\right] \quad ; \quad \underset{\sim}{u}^T = [\; u_r \quad u_t \quad u_z \;]$$

$$\underline{\varepsilon} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \cdots & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ \cdots & \cdots & 2u_{z,z} \end{bmatrix}$$

$$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$
$$\varepsilon_{rr,tz} + \varepsilon_{tz,rr} - \varepsilon_{zr,rt} - \varepsilon_{rt,rz} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$

$$\underset{\approx}{\varepsilon}^T = [\; \varepsilon_{rr} \quad \varepsilon_{tt} \quad \varepsilon_{zz} \quad \varepsilon_{rt} \quad \varepsilon_{tz} \quad \varepsilon_{zr} \;] \qquad (\varepsilon_{zz})$$
$$\underset{\approx}{\sigma}^T = [\; \sigma_{rr} \quad \sigma_{tt} \quad \sigma_{zz} \quad \sigma_{rt} \quad \sigma_{tz} \quad \sigma_{zr} \;] \qquad (\sigma_{zz})$$

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = \rho \ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$

$$\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t = \rho \ddot{u}_t \qquad (\sigma_{tz} = \sigma_{zt})$$

$$\sigma_{zr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}\sigma_{zr} + \sigma_{rz,z} + \rho q_z = \rho \ddot{u}_z \qquad (\sigma_{zr} = \sigma_{rz})$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon} \quad ; \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}$$

Material law

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2K & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L & 0 \\ 0 & 0 & 0 & 0 & 0 & 2M \end{bmatrix} \rightarrow \underline{\underline{S}} = \underline{\underline{C}}^{-1} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}k & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}l & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}m \end{bmatrix}$$

quadratic	$B = A\ ;\ S = R\ ;\ M = L;$
transversal isotropic	$B = A\ ;\ S = R\ ;\ M = L\ ;\ K = \frac{1}{2}(A - Q)$
cubic	$C = B = A\ ;\ S = R = Q\ ;\ M = L = K \neq \frac{1}{2}(A - Q)$
isotropic	$C = B = A\ ;\ S = R = Q\ ;\ M = L = K = \frac{1}{2}(A - Q)$

Hooke's law for planar states

plane strain

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0 \quad ; \quad \sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

$$\underline{S} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{C} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

plane stress

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \quad ; \quad \varepsilon_{33} = \frac{-\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22})$$

$$\underline{S} = \frac{1+\nu}{E} \begin{bmatrix} \frac{1}{1+\nu} & \frac{-\nu}{1+\nu} & 0 \\ \frac{-\nu}{1+\nu} & \frac{1}{1+\nu} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{C} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & \frac{1}{1-\nu} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

axi-symmetry : $\frac{\partial}{\partial \theta} = 0 \qquad + \qquad u_t = 0$

Planar deformation : Cartesian

$$\left.\begin{array}{lcl} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_x = u_x(x,y) \\ u_y = u_y(x,y) \end{array}\right.$$

$$\underset{\sim}{\varepsilon}^T = \left[\begin{array}{ccc} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{xy} \end{array}\right] = \left[\begin{array}{ccc} u_{x,x} & u_{y,y} & \frac{1}{2}(u_{x,y} + u_{y,x}) \end{array}\right]$$

$$2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0$$

$$\underset{\sim}{\sigma}^T = \left[\begin{array}{ccc} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{array}\right]$$

$$\begin{array}{l} \sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \\ \sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y \end{array} \qquad (\sigma_{xy} = \sigma_{yx})$$

$$\underline{\underline{C}}_p = \left[\begin{array}{ccc} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{array}\right] \quad ; \quad \underline{\underline{S}}_p = \left[\begin{array}{ccc} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{array}\right]$$

Planar deformation : cylindrical

$$\left.\begin{array}{lll}\text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0\end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta)\end{array}\right.$$

$$\underset{\approx}{\varepsilon}^T = \left[\ \varepsilon_{rr} \quad \varepsilon_{tt} \quad \varepsilon_{rt}\ \right] = \left[\ u_{r,r} \quad \frac{1}{r}(u_r + u_{t,t}) \quad \frac{1}{2}\left(\frac{1}{r}(u_{r,t} - u_t) + u_{t,r}\right)\ \right]$$

$$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0$$

$$\underset{\approx}{\sigma}^T = \left[\ \sigma_{rr} \quad \sigma_{tt} \quad \sigma_{rt}\ \right]$$

$$\sigma_{rr,r} + \frac{1}{r}\,\sigma_{rt,t} + \frac{1}{r}\,(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$

$$\sigma_{tr,r} + \frac{1}{r}\,\sigma_{tt,t} + \frac{1}{r}\,(\sigma_{tr} + \sigma_{rt}) + \rho q_t = \rho \ddot{u}_t$$

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{bmatrix}$$

Planar deformation : axi-symmetric + $u_t = 0$

$$\left.\begin{array}{lll} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r) \\ u_t = 0 \end{array}\right.$$

$$\underset{\approx}{\varepsilon}^T = \left[\ \varepsilon_{rr} \quad \varepsilon_{tt}\ \right] = \left[\ u_{r,r} \quad \frac{1}{r}(u_r)\ \right]$$

$$\varepsilon_{rr} = u_{r,r} = (r\varepsilon_{tt})_{,r} = \varepsilon_{tt} + r\varepsilon_{tt,r}$$

$$\underset{\approx}{\sigma}^T = \left[\ \sigma_{rr} \quad \sigma_{tt}\ \right]$$

$$\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r$$

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p \\ Q_p & B_p \end{bmatrix} \qquad ; \qquad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p \\ q_p & b_p \end{bmatrix}$$

SOLUTION STRATEGIES

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Governing equations for unknowns

unknown variables

displacements $\quad \vec{u} = \vec{u}(\vec{x}) \quad \rightarrow \quad \mathbf{F} = \left(\vec{\nabla}_0 \vec{x}\right)^C \quad \rightarrow \quad \mathbf{E}\ ,\ \boldsymbol{\varepsilon}$

stresses $\quad \boldsymbol{\sigma} \quad \rightarrow \quad g(\boldsymbol{\sigma}) = g(\sigma_1, \sigma_2, \sigma_3) = g_t$

equations

compatibility $\quad \nabla^2\{\mathrm{tr}(\boldsymbol{\varepsilon})\} - \vec{\nabla}\cdot(\vec{\nabla}\cdot\boldsymbol{\varepsilon})^c = 0$

equilibrium $\quad \vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \rho\ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$

material law $\quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{F}) \quad \rightarrow \quad \boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\varepsilon} \quad \rightarrow \quad \boldsymbol{\varepsilon} = {}^4\mathbf{S} : \boldsymbol{\sigma}$

Governing equations for unknowns

unknown variables

displacements $\quad \vec{u} = \vec{u}(\vec{x}) \quad \rightarrow \quad \mathbf{F} = \left(\vec{\nabla}_0 \vec{x}\right)^C \quad \rightarrow \quad \mathbf{E} \;,\; \varepsilon$

stresses $\quad \boldsymbol{\sigma} \quad \rightarrow \quad g(\boldsymbol{\sigma}) = g(\sigma_1, \sigma_2, \sigma_3) = g_t$

equations

compatibility $\quad \nabla^2 \{\mathrm{tr}(\varepsilon)\} - \vec{\nabla} \cdot (\vec{\nabla} \cdot \varepsilon)^c = 0$

equilibrium $\quad \vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \rho \ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$

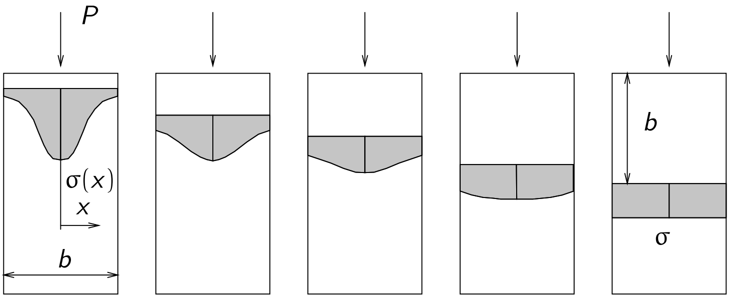
material law $\quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{F}) \quad \rightarrow \quad \boldsymbol{\sigma} = {}^4\mathbf{C} : \varepsilon \quad \rightarrow \quad \varepsilon = {}^4\mathbf{S} : \boldsymbol{\sigma}$

boundary conditions

displacement $\quad \vec{u} = \vec{u}_p \quad \forall \quad \vec{x} \in A_u$

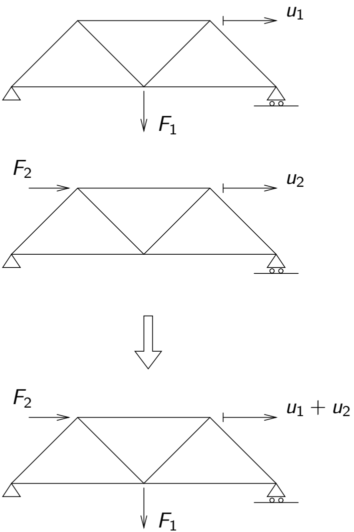
edge load $\quad \vec{p} = \vec{n} \cdot \boldsymbol{\sigma} = \vec{p}_p \quad \forall \quad \vec{x} \in A_p$

Saint-Venant's principle



$$P = \int_A \sigma(x)\, dA = \sigma\, A \qquad ; \qquad A = b * t$$

Superposition



Solution : displacement method

$$\left.\begin{array}{l}\vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \vec{0} \\[2ex] \boldsymbol{\sigma} = {}^4\mathbf{C} : \boldsymbol{\varepsilon}\end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l}\vec{\nabla}\cdot\left({}^4\mathbf{C} : \boldsymbol{\varepsilon}\right)^c + \rho\vec{q} = \vec{0} \\[2ex] \boldsymbol{\varepsilon} = \frac{1}{2}\left\{\left(\vec{\nabla}\vec{u}\right)^c + \left(\vec{\nabla}\vec{u}\right)\right\}\end{array}\right\} \quad \rightarrow$$

$$\vec{\nabla}\cdot\left\{{}^4\mathbf{C} : \left(\vec{\nabla}\vec{u}\right)\right\}^c + \rho\vec{q} = \vec{0} \quad \rightarrow \quad \vec{u} \quad \rightarrow \quad \boldsymbol{\varepsilon} \quad \rightarrow \quad \boldsymbol{\sigma}$$

Planar, Cartesian : Navier equations

$$\left.\begin{array}{l}\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \qquad ; \qquad \sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y \\[2ex] \sigma_{xx} = A_p \varepsilon_{xx} + Q_p \varepsilon_{yy} \\ \sigma_{yy} = Q_p \varepsilon_{xx} + B_p \varepsilon_{yy} \\ \sigma_{xy} = 2K \varepsilon_{xy}\end{array}\right\}$$

$$\left.\begin{array}{l}A_p \varepsilon_{xx,x} + Q_p \varepsilon_{yy,x} + 2K \varepsilon_{xy,y} + \rho q_x = \rho \ddot{u}_x \\ 2K \varepsilon_{xy,x} + Q_p \varepsilon_{xx,y} + B_p \varepsilon_{yy,y} + \rho q_y = \rho \ddot{u}_y\end{array}\right\}$$

$$\left.\begin{array}{l}A_p u_{x,xx} + Q_p u_{y,yx} + K(u_{x,yy} + u_{y,xy}) + \rho q_x = \rho \ddot{u}_x \\ K(u_{x,yx} + u_{y,xx}) + Q_p u_{x,xy} + B_p u_{y,yy} + \rho q_y = \rho \ddot{u}_y\end{array}\right\}$$

$$\left.\begin{array}{l}A_p u_{x,xx} + K u_{x,yy} + (Q_p + K) u_{y,yx} + \rho q_x = \rho \ddot{u}_x \\ K u_{y,xx} + B_p u_{y,yy} + (Q_p + K) u_{x,xy} + \rho q_y = \rho \ddot{u}_y\end{array}\right\}$$

Planar, axi-symmetric with $u_t = 0$

displacements $\quad u_r = u_r(r) \quad ; \quad u_z = u_z(r,z)$

strains $\quad \varepsilon_{rr} = u_{r,r} \; ; \; \varepsilon_{tt} = \frac{1}{r} u_r \; ; \; \varepsilon_{zz} = u_{z,z}$

stresses $\quad \sigma_{tz} = 0 \quad ; \quad \sigma_{rz} \approx 0 \quad ; \quad \sigma_{tr} = 0$

eq. of motion $\quad \sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r$

displacement method

$$\left.\begin{array}{l} \sigma_{rr} = A_p \varepsilon_{rr} + Q_p \varepsilon_{tt} - \Theta_{p1} \alpha \Delta T \\ \sigma_{tt} = Q_p \varepsilon_{rr} + B_p \varepsilon_{tt} - \Theta_{p2} \alpha \Delta T \end{array}\right\} \quad \rightarrow \quad \text{eq. of motion} \quad \rightarrow$$

$$u_{r,rr} + \frac{1}{r} u_{r,r} - \zeta^2 \frac{1}{r^2} u_r = f(r) \qquad \text{with} \qquad \zeta = \sqrt{\frac{B_p}{A_p}}$$

$$\text{with} \qquad f(r) = \frac{\rho}{A_p} (\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p} \alpha (\Delta T)_{,r} + \frac{\Theta_{p1} - \Theta_{p2}}{A_p} \frac{1}{r} \alpha \Delta T$$

Planar, axi-symmetric with $u_t = 0$, isotropic

displacements $\qquad u_r = u_r(r) \quad ; \quad u_z = u_z(r,z)$

strains $\qquad \varepsilon_{rr} = u_{r,r} \; ; \; \varepsilon_{tt} = \frac{1}{r} u_r \; ; \; \varepsilon_{zz} = u_{z,z}$

stresses $\qquad \sigma_{tz} = 0 \quad ; \quad \sigma_{rz} \approx 0 \quad ; \quad \sigma_{tr} = 0$

eq. of motion $\qquad \sigma_{rr,r} + \dfrac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r$

displacement method

$$\left.\begin{array}{l} \sigma_{rr} = A_p \varepsilon_{rr} + Q_p \varepsilon_{tt} - \Theta_{p1} \alpha \Delta T \\ \sigma_{tt} = Q_p \varepsilon_{rr} + A_p \varepsilon_{tt} - \Theta_{p2} \alpha \Delta T \end{array}\right\} \quad \rightarrow \quad \text{eq. of motion} \quad \rightarrow$$

$$u_{r,rr} + \frac{1}{r} u_{r,r} - \frac{1}{r^2} u_r = f(r)$$

$$\text{with} \qquad f(r) = \frac{\rho}{A_p} \, (\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p} \, \alpha(\Delta T)_{,r}$$

WEIGHTED RESIDUAL FORMULATION

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Weighted residual formulation for 3D deformation

equilibrium equation $\qquad \vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \vec{0} \qquad \forall \; \vec{x} \in V$

approximation $\rightarrow$ residual $\qquad \vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \vec{\Delta}(\vec{x}) \neq \vec{0} \qquad \forall \; \vec{x} \in V$

weighted residual

$$\int_V \vec{w}(\vec{x}) \cdot \vec{\Delta}(\vec{x}) \, dV = \int_V \vec{w} \cdot \left[\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} \right] \, dV$$

$\vec{w}(\vec{x})$ = weighting function

equivalent problem formulation

$$\int_V \vec{w} \cdot \left[\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} \right] \, dV = 0 \qquad \forall \quad \vec{w}(\vec{x}) \quad \leftrightarrow \quad \vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \vec{0} \qquad \forall \; \vec{x} \in V$$

Weak formulation

$$\left.\begin{array}{l}\displaystyle\int_V \vec{w}\cdot\left[\vec{\nabla}\cdot\boldsymbol{\sigma}^c+\rho\vec{q}\right]\,dV=0\\ \vec{\nabla}\cdot(\boldsymbol{\sigma}^c\cdot\vec{w})=(\vec{\nabla}\vec{w})^c:\boldsymbol{\sigma}^c+\vec{w}\cdot(\vec{\nabla}\cdot\boldsymbol{\sigma}^c)\end{array}\right\}\quad\rightarrow$$

$$\left.\begin{array}{l}\displaystyle\int_V \left[\vec{\nabla}\cdot(\boldsymbol{\sigma}^c\cdot\vec{w})-(\vec{\nabla}\vec{w})^c:\boldsymbol{\sigma}^c+\vec{w}\cdot\rho\vec{q}\right]\,dV=0\qquad\forall\ \vec{w}\\ \text{Gauss / Stokes}\ :\ \displaystyle\int_V \vec{\nabla}\cdot(\boldsymbol{\sigma}^c\cdot\vec{w})=\int_V \vec{n}\cdot\boldsymbol{\sigma}^c\cdot\vec{w}\,dA=\int_A \vec{w}\cdot\vec{p}\,dA\end{array}\right\}\quad\rightarrow$$

$$\int_V (\vec{\nabla}\vec{w})^c:\boldsymbol{\sigma}\,dV=\int_V \vec{w}\cdot\rho\vec{q}\,dV+\int_A \vec{w}\cdot\vec{p}\,dA\qquad\forall\ \vec{w}$$

$$\int_V (\vec{\nabla}\vec{w})^c:\boldsymbol{\sigma}\,dV=f_e(\vec{w})\qquad\forall\ \vec{w}$$

Weighted residual formulation : linear

$$\int_{V_0} (\vec{\nabla}_0 \vec{w})^c : \boldsymbol{\sigma} \, dV_0 = \int_{V_0} \vec{w} \cdot \rho \vec{q} \, dV_0 + \int_{A_0} \vec{w} \cdot \vec{p} \, dA_0 = f_{e0}(\vec{w}) \qquad \forall \; \vec{w}$$

$$\begin{aligned} \boldsymbol{\sigma} &= {}^4\mathbf{C} : \boldsymbol{\varepsilon} \\ &= {}^4\mathbf{C} : \tfrac{1}{2} \left\{ (\vec{\nabla}_0 \vec{u}) + (\vec{\nabla}_0 \vec{u})^c \right\} \\ &= {}^4\mathbf{C} : (\vec{\nabla}_0 \vec{u}) \end{aligned}$$

$$\int_{V_0} (\vec{\nabla}_0 \vec{w})^c : {}^4\mathbf{C} : (\vec{\nabla}_0 \vec{u}) \, dV_0 = \int_{V_0} \vec{w} \cdot \rho \vec{q} \, dV_0 + \int_{A_0} \vec{w} \cdot \vec{p} \, dA_0 = f_{e0}(\vec{w}) \qquad \forall \; \vec{w}$$

FINITE ELEMENT METHOD

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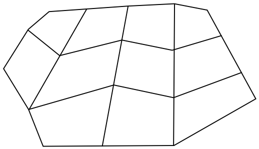
Discretisation

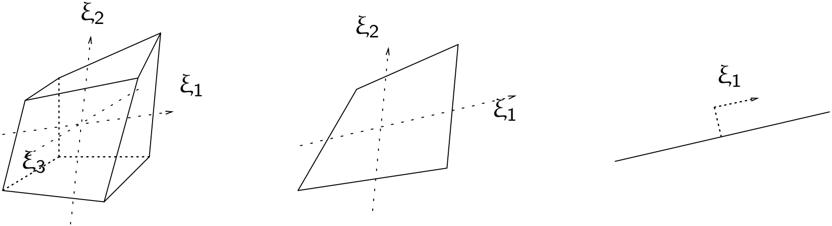
$$\int_V (\vec{\nabla}\vec{w})^c : {}^4\mathbf{C} : (\vec{\nabla}\vec{u})\, dV = \int_V \vec{w}\cdot\rho\vec{q}\, dV + \int_A \vec{w}\cdot\vec{p}\, dS \qquad \forall\ \vec{w}$$

discretisation



$$\sum_e \int_{V^e} (\vec{\nabla}\vec{w})^c : {}^4\mathbf{C} : (\vec{\nabla}\vec{u})\, dV^e = \sum_e \int_{V^e} \vec{w}\cdot\rho\vec{q}\, dV^e + \sum_{e_A} \int_{A^e} \vec{w}\cdot\vec{p}\, dA^e \qquad \forall\ \vec{w}$$

Isoparametric elements



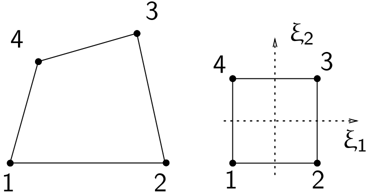
isoparametric (local) coordinates

$$(\xi_1, \xi_2, \xi_3) \quad ; \quad -1 \leq \xi_i \leq 1 \quad i = 1, 2, 3$$

Jacobian matrix

$$\underline{J} = \left(\underset{\sim}{\nabla}_{\xi} \underset{\sim}{x}^T \right)^T \quad dV^e = \det(\underline{J}) \, d\xi_1 d\xi_2 d\xi_3$$

Interpolation : 4-node linear element



$$\vec{u} = N^1(\underset{\sim}{\xi})\,\vec{u}^1 + N^2(\underset{\sim}{\xi})\,\vec{u}^2 + N^3(\underset{\sim}{\xi})\,\vec{u}^3 + N^4(\underset{\sim}{\xi})\,\vec{u}^4$$

interpolation functions

$$N^1 = \tfrac{1}{4}(\xi_1 - 1)(\xi_2 - 1) \qquad ; \qquad N^2 = -\tfrac{1}{4}(\xi_2 + 1)(\xi_2 - 1)$$
$$N^3 = \tfrac{1}{4}(\xi_1 + 1)(\xi_2 + 1) \qquad ; \qquad N^4 = -\tfrac{1}{4}(\xi_1 - 1)(\xi_2 + 1)$$

Galerkin

$$\vec{w} = N^1(\underset{\sim}{\xi})\,\vec{w}^1 + N^2(\underset{\sim}{\xi})\,\vec{w}^2 + N^3(\underset{\sim}{\xi})\,\vec{w}^3 + N^4(\underset{\sim}{\xi})\,\vec{w}^4$$

Interpolation

$$\vec{u} = N^1(\underset{\sim}{\xi})\,\vec{u}^1 + N^2(\underset{\sim}{\xi})\,\vec{u}^2 + N^3(\underset{\sim}{\xi})\,\vec{u}^3 + N^4(\underset{\sim}{\xi})\,\vec{u}^4 = \underset{\sim}{N}^T(\underset{\sim}{\xi})\underset{\sim}{\vec{u}}^e \quad \rightarrow$$
$$\vec{\nabla}\vec{u} = (\vec{\nabla}\underset{\sim}{N})^T\underset{\sim}{\vec{u}}^e = \underset{\sim}{\vec{B}}^T\,\underset{\sim}{\vec{u}}^e$$

$$\vec{w} = N^1(\underset{\sim}{\xi})\,\vec{w}^1 + N^2(\underset{\sim}{\xi})\,\vec{w}^2 + N^3(\underset{\sim}{\xi})\,\vec{w}^3 + N^4(\underset{\sim}{\xi})\,\vec{w}^4 = \underset{\sim}{N}^T(\underset{\sim}{\xi})\underset{\sim}{\vec{w}}^e \quad \rightarrow$$
$$\vec{\nabla}\vec{w} = (\vec{\nabla}\underset{\sim}{N})^T\underset{\sim}{\vec{w}}^e = \underset{\sim}{\vec{B}}^T\,\underset{\sim}{\vec{w}}^e$$

$$\int\limits_{V^e}(\underset{\sim}{\vec{B}}^T\,\underset{\sim}{\vec{w}}^e)^T : {}^4\mathbf{C} : (\underset{\sim}{\vec{B}}^T\,\underset{\sim}{\vec{u}}^e)\,dV^e = \int\limits_{V^e}\underset{\sim}{\vec{w}}^{e^T}\,\underset{\sim}{N}\cdot\rho\vec{q}\,dV^e + \int\limits_{A^e}\underset{\sim}{\vec{w}}^{e^T}\,\underset{\sim}{N}\cdot\vec{p}\,dA^e$$
$$\underset{\sim}{\vec{w}}^{e^T}\cdot\left[\int\limits_{V^e}\underset{\sim}{\vec{B}}\cdot{}^4\mathbf{C}\cdot\underset{\sim}{\vec{B}}^T\,dV^e\right]\cdot\underset{\sim}{\vec{u}}^e = \underset{\sim}{\vec{w}}^{e^T}\cdot\left[\int\limits_{V^e}\underset{\sim}{N}\,\rho\vec{q}\,dV^e\right] + \underset{\sim}{\vec{w}}^{e^T}\cdot\left[\int\limits_{A^e}\underset{\sim}{N}\,\vec{p}\,dA^e\right]$$

$$\underset{\sim}{\vec{w}}^{e^T}\cdot\underline{\mathbf{K}}^e\cdot\underset{\sim}{\vec{u}}^e = \underset{\sim}{\vec{w}}^{e^T}\cdot\underset{\sim}{\vec{f}}^e_e$$

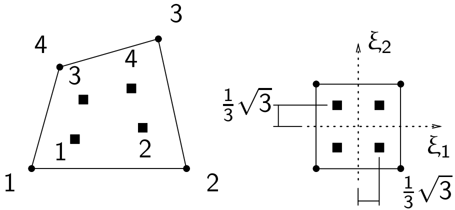
$\underline{\mathbf{K}}^e$: element stiffness matrix

$\underset{\sim}{\vec{f}}^e_e$: external nodal forces

Integration

$$\int_{V^e} g(x_1, x_2, x_3)\, dV^e = \int_{\xi_1=-1}^{1} \int_{\xi_2=-1}^{1} \int_{\xi_3=-1}^{1} f(\xi_1, \xi_2, \xi_3)\, d\xi_1 d\xi_2 d\xi_3$$

$$= \sum_{ip=1}^{nip} c^{ip} f(\xi_1^{ip}, \xi_2^{ip}, \xi_3^{ip})$$



$$\int_{V^e} g(x_1, x_2)\, dV^e = \int_{\xi_1=-1}^{1} \int_{\xi_2=-1}^{1} f(\xi_1, \xi_2)\, d\xi_1 d\xi_2 = \sum_{ip=1}^{4} c^{ip} f(\xi_1^{ip}, \xi_2^{ip})$$

Assembling

$$\sum_e \tilde{\vec{w}}^{e^T} \cdot \underline{\mathbf{K}}^e \cdot \tilde{\vec{u}}^e = \sum_e \tilde{\vec{w}}^{e^T} \cdot \tilde{\vec{f}}_e^e \quad \rightarrow$$

$$\tilde{\vec{w}}^T \cdot \underline{\mathbf{K}} \cdot \tilde{\vec{u}} = \tilde{\vec{w}}^T \cdot \tilde{\vec{f}}_e \quad \forall \; \tilde{\vec{w}} \qquad \rightarrow$$

$$\underline{\mathbf{K}} \cdot \tilde{\vec{u}} = \tilde{\vec{f}}_e \quad \rightarrow \quad \tilde{\vec{u}} = \underline{\mathbf{K}}^{-1} \cdot \tilde{\vec{f}}_e$$

Boundary conditions

rigid translation

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots \\ k_{21} & k_{22} & k_{23} & \cdots \\ k_{31} & k_{32} & k_{33} & \cdots \\ \vdots & \vdots & \vdots & . \end{bmatrix} \begin{bmatrix} a \\ a \\ a \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

stiffness matrix is singular $\quad\rightarrow\quad$ determinant is zero $\quad\rightarrow\quad$ exit 2004

prevent rigid body movement with BC's
other BC's : prescribed displacements / loads / temperature

ANALYTICAL SOLUTIONS

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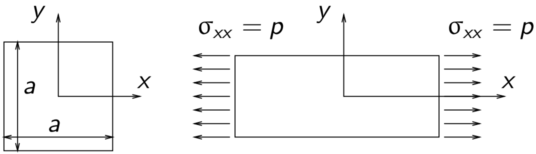
Cartesian, planar

$$A_p u_{x,xx} + K u_{x,yy} + (Q_p + K) u_{y,yx} + \rho q_x = 0$$

$$K u_{y,xx} + B_p u_{y,yy} + (Q_p + K) u_{x,xy} + \rho q_y = 0$$

Tensile test



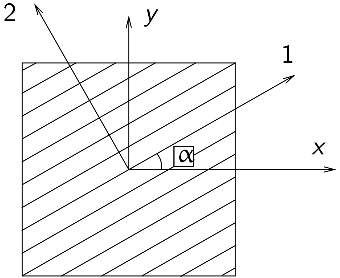
$$\varepsilon_{xx} = \frac{1}{E}\,\sigma_{xx} = \frac{p}{E} \quad \rightarrow \quad u_x = \frac{p}{E}\,x + c \quad ; \quad u_x(x=0) = 0 \rightarrow c = 0$$

$$u_x = \frac{p}{E}\,x \quad \rightarrow \quad u_x(x=a) = \frac{p}{E}\,a$$

$$\varepsilon_{yy} = -\nu\,\varepsilon_{xx} = -\nu\frac{p}{E} \quad \rightarrow \quad u_y = -\nu\frac{p}{E}y + c \quad ; \quad u_y(y=0) = 0 \quad \rightarrow \quad c = 0$$

$$u_y = -\nu\frac{p}{E}y \quad ; \quad u_y(y=a/2) = -\nu\frac{p}{E}\frac{a}{2}$$

Orthotropic plate



$$\left[\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array}\right] = \frac{1}{1-\nu_{12}\nu_{21}} \left[\begin{array}{ccc} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & (1-\nu_{12}\nu_{21})G_{12} \end{array}\right] \left[\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{array}\right] \quad \rightarrow$$

$$\underset{\sim}{\sigma}^* = \underline{\underline{C}}^* \underset{\sim}{\varepsilon}^*$$

Transformation from material to global coordinate system

$$\underset{\sim}{\sigma}^* = \left[\begin{array}{ccc} \sigma_{11} & \sigma_{22} & \sigma_{12} \end{array}\right]^T \qquad \underset{\sim}{\varepsilon}^* = \left[\begin{array}{ccc} \varepsilon_{11} & \varepsilon_{22} & \gamma_{12} \end{array}\right]^T$$

$$\underset{\sim}{\sigma} = \left[\begin{array}{ccc} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{array}\right]^T \qquad \underset{\sim}{\varepsilon} = \left[\begin{array}{ccc} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{array}\right]^T$$

$$\underline{T}_\sigma = \left[\begin{array}{ccc} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{array}\right] \qquad \underline{T}_\sigma^{-1} = \left[\begin{array}{ccc} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{array}\right]$$

$$\underline{T}_\varepsilon = \left[\begin{array}{ccc} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{array}\right] \qquad \underline{T}_\varepsilon^{-1} = \left[\begin{array}{ccc} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{array}\right]$$

$$\underset{\sim}{\sigma}^* = \underline{T}_\sigma \, \underset{\sim}{\sigma} \qquad \underset{\sim}{\varepsilon}^* = \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon}$$

$$\begin{array}{lclcl} \underset{\sim}{\sigma}^* = \underline{\underline{C}}^* \, \underset{\sim}{\varepsilon}^* & \rightarrow & \underline{T}_\sigma \underset{\sim}{\sigma} = \underline{\underline{C}}^* \, \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon} & \rightarrow & \underset{\sim}{\sigma} = \underline{T}_\sigma^{-1} \, \underline{\underline{C}}^* \, \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon} = \underline{\underline{C}} \, \underset{\sim}{\varepsilon} \\ \underset{\sim}{\varepsilon}^* = \underline{\underline{S}}^* \, \underset{\sim}{\sigma}^* & \rightarrow & \underline{T}_\varepsilon \underset{\sim}{\varepsilon} = \underline{\underline{S}}^* \, \underline{T}_\sigma \, \underset{\sim}{\sigma} & \rightarrow & \underset{\sim}{\varepsilon} = \underline{T}_\varepsilon^{-1} \, \underline{\underline{S}}^* \, \underline{T}_\sigma \, \underset{\sim}{\sigma} = \underline{\underline{S}} \, \underset{\sim}{\sigma} \end{array}$$

Axi-symmetric, planar, $u_t = 0$

$$u_{r,rr} + \frac{1}{r}\, u_{r,r} - \zeta^2\, \frac{1}{r^2}\, u_r = f(r)$$

$$\text{with} \qquad \zeta = \sqrt{\frac{B_p}{A_p}}$$

$$\text{and} \qquad f(r) = \frac{\rho}{A_p}\, (\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p}\, \alpha(\Delta T)_{,r} + \frac{\Theta_{p1} - \Theta_{p2}}{A_p}\, \frac{1}{r}\, \alpha \Delta T$$

general solution

$$\begin{aligned}
&\hat{u}_r = r^{\lambda} \quad \rightarrow \quad \hat{u}_{r,r} = \lambda\, r^{\lambda-1} \quad \rightarrow \quad \hat{u}_{r,rr} = \lambda(\lambda-1)\, r^{\lambda-2} \quad \rightarrow \\
&\left[\lambda(\lambda-1) + \lambda - \zeta^2\right] r^{\lambda-2} = 0 \quad \rightarrow \\
&\lambda^2 = \zeta^2 \quad \rightarrow \quad \lambda = \pm\zeta \quad \rightarrow \quad \hat{u}_r = c_1\, r^{\zeta} + c_2 r^{-\zeta} \\
&u_r = c_1\, r^{\zeta} + c_2 r^{-\zeta} + \bar{u}_r
\end{aligned}$$

Orthotropic material

general solution $\qquad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} + \bar{u}_r$

$$\varepsilon_{rr} = c_1 \zeta r^{\zeta-1} - c_2 \zeta r^{-\zeta-1} + \bar{u}_{r,r}$$
$$\varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1} + \frac{\bar{u}_r}{r}$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} + A_p\bar{u}_{r,r} + Q_p\frac{\bar{u}_r}{r} - (\Theta_{p1})\alpha\Delta T$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} + Q_p\bar{u}_{r,r} + B_p\frac{\bar{u}_r}{r} - (\Theta_{p2})\alpha\Delta T$$

Isotropic material

general solution $\qquad u_r = c_1 r + \dfrac{c_2}{r} + \bar{u}_r$

$$
\begin{aligned}
\varepsilon_{rr} &= c_1 - c_2 r^{-2} + \bar{u}_{r,r} \\
\varepsilon_{tt} &= c_1 + c_2 r^{-2} + \frac{\bar{u}_r}{r}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{rr} &= (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} + A_p \bar{u}_{r,r} + Q_p \frac{\bar{u}_r}{r} - (\Theta_{p1})\alpha\Delta T \\
\sigma_{tt} &= (Q_p + A_p)c_1 - (Q_p - A_p)\frac{c_2}{r^2} + Q_p \bar{u}_{r,r} + A_p \frac{\bar{u}_r}{r} - (\Theta_{p2})\alpha\Delta T
\end{aligned}
$$

Cylinder

$a \le r \le b$, plane stress, axisymm, $u_t = 0$, $\ddot{u}_r = 0$, $q_r = 0$

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{tt} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{tt} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} u_{r,r} \\ \frac{1}{r} u_r \end{bmatrix} \; ; \quad \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{rr} + \sigma_{tt})$$

$$\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) = 0 \quad \rightarrow \quad \boxed{r^2 u_{r,rr} + r u_{r,r} - u_r = 0}$$

general solution $\quad u_r = c_1 r + \dfrac{c_2}{r} \; ; \quad \sigma_{rr} = .. \; ; \quad \sigma_{tt} = .. \; ; \quad \varepsilon_{zz} = -\dfrac{2\nu(1+\nu)}{1-\nu^2} c_1$

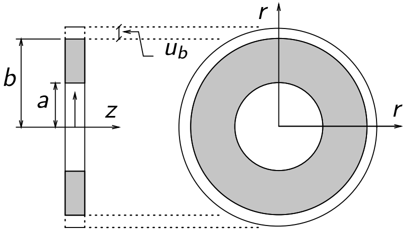
BC's $\quad \sigma_{rr}(r=a) = -p_i \quad ; \quad \sigma_{rr}(r=b) = -p_e$

open cyl. $\quad \sigma_{zz} = 0$

closed cyl. $\quad \sigma_{zz} = \dfrac{p_i a^2 - p_e b^2}{b^2 - a^2} \quad$ (ax.eq.) $\quad \rightarrow \quad u_r = u_r(\text{open}) - \dfrac{\nu}{E}\sigma_{zz} r$

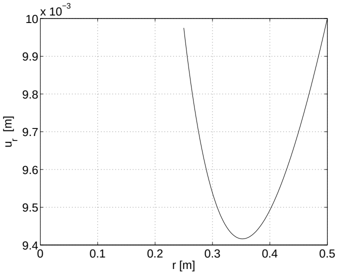
plane strain $\quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{tt})$

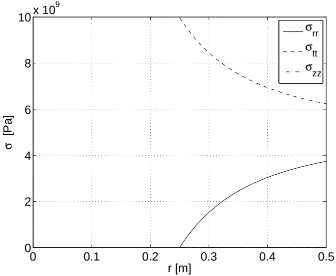
Prescribed edge displacement



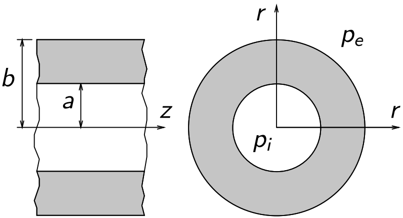
$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$u_r(r = b) = u_b$$
$$\sigma_{rr}(r = a) = 0$$
$$c_1, c_2 \ : \ \triangleright$$

$$|\, u_b = 0.01 \text{ m} \,|\, a = 0.25 \text{ m} \,|\, b = 0.5 \text{ m} \,|\, h = 0.05 \text{ m} \,|\, E = 250 \text{ GPa} \,|\, \nu = 0.33 \,|$$





Edge load



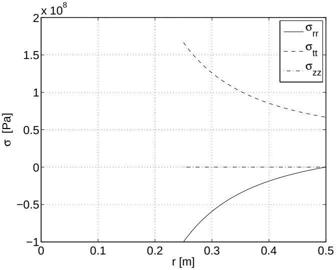
$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$\sigma_{rr}(r = a) = -p_i$$
$$\sigma_{rr}(r = b) = -p_e$$
$$c_1, c_2 \;:\; \triangleright$$

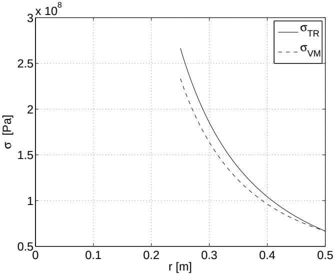
$$\sigma_{rr} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} - \frac{a^2 b^2 (p_i - p_e)}{b^2 - a^2} \frac{1}{r^2}$$
$$\sigma_{tt} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} + \frac{a^2 b^2 (p_i - p_e)}{b^2 - a^2} \frac{1}{r^2}$$

$$\sigma_{TR} = 2\tau_{max} = \max\left[\,|\sigma_{rr} - \sigma_{tt}|, |\sigma_{tt} - \sigma_{zz}|, |\sigma_{zz} - \sigma_{rr}|\,\right]$$
$$\sigma_{VM} = \sqrt{\tfrac{1}{2}\left\{(\sigma_{rr} - \sigma_{tt})^2 + (\sigma_{tt} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2\right\}}$$

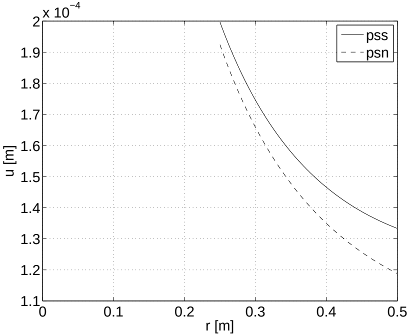
Open cylinder

$| \, p_i = 100 \text{ MPa} \, | \, a = 0.25 \text{ m} \, | \, b = 0.5 \text{ m} \, | \, h = 0.5 \text{ m} \, | \, E = 250 \text{ GPa} \, | \, \nu = 0.33 \, |$





Open cylinder

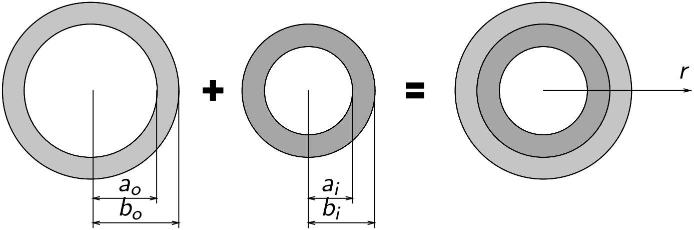


Closed cylinder

$$\text{axial equilibrium} \qquad \sigma_{zz} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} \quad \rightarrow$$

$$u_{ra} = \varepsilon_{tta} r = -\frac{\nu}{E}\,\sigma_{zz} r$$

Shrink-fit compound pressurized cylinder



$$\sigma_{rr_i} = \frac{-p_c b_i^2}{b_i^2 - a_i^2} + \frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2) r^2} \quad ; \quad \sigma_{tt_i} = \frac{-p_c b_i^2}{b_i^2 - a_i^2} - \frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2) r^2}$$

$$\sigma_{rr_o} = \frac{p_c a_o^2}{b_o^2 - a_o^2} - \frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2) r^2} \quad ; \quad \sigma_{tt_o} = \frac{p_c a_o^2}{b_o^2 - a_o^2} + \frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2) r^2}$$

Shrink-fit compound pressurized cylinder

$$u_{r_i}(r=a_i) = -\frac{2}{E}\frac{p_c a_i b_i^2}{b_i^2 - a_i^2} \qquad ; \qquad u_{r_o}(r=b_o) = \frac{2}{E}\frac{p_c a_o^2 b_o}{b_o^2 - a_o^2}$$

$$u_{r_o}(r=a_o) = \frac{1-\nu}{E}\frac{p_c a_o^2}{b_o^2 - a_o^2}\,a_o + \frac{1+\nu}{E}\frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2)}\frac{1}{a_o}$$

$$u_{r_i}(r=b_i) = -\frac{1-\nu}{E}\frac{p_c b_i^2}{b_i^2 - a_i^2}\,b_i - \frac{1+\nu}{E}\frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2)}\frac{1}{b_i}$$

$$r_i = a_i + u_i(r=a_i) \quad ; \quad r_o = b_o + u_o(r=b_o)$$

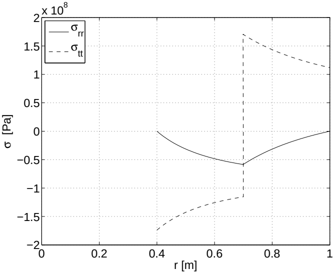
contact pressure

$$r_c = b_i + u_{r_i}(r=b_i) = a_o + u_{r_o}(r=a_o) \quad \rightarrow$$

$$p_c = \frac{E(b_i - a_o)(b_o^2 - a_o^2)(b_i^2 - a_i^2)}{a_o(b_i^2 - a_i^2)\{(b_0^2 + a_o^2) + \nu(b_o^2 - a_o^2)\} + b_i(b_o^2 - a_o^2)\{(b_i^2 + a_i^2) - \nu(b_i^2 - a_i^2)\}}$$

Example

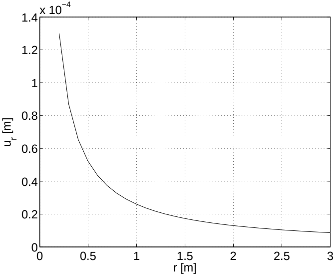
$$|\, a_i = 0.4 \text{ m} \,|\, b_i = 0.7 \text{ m} \,|\, a_o = 0.699 \text{ m} \,|\, b_o = 1 \text{ m} \,|\, E = 200 \text{ GPa} \,|\, \nu = 0.3 \,|$$

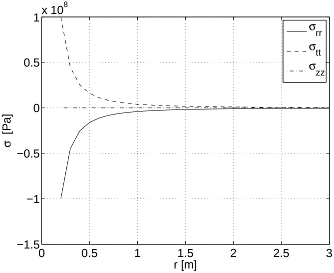


Pressurized hole in infinite medium

$$b \to \infty \quad ; \quad p_i = p \quad ; \quad p_e = 0 \qquad \to \qquad \sigma_{rr} = -\frac{pa^2}{r^2} \quad ; \quad \sigma_{tt} = \frac{pa^2}{r^2}$$

$$|\, p_i = 100 \text{ MPa} \,|\, a = 0.2 \text{ m} \,|\, b = 20 \text{ m} \,|\, h = 0.5 \text{ m} \,|\, E = 200 \text{ GPa} \,|\, \nu = 0.3 \,|$$



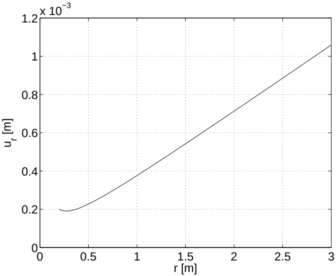


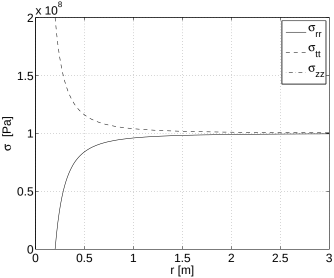
Stress-free hole in bi-axially loaded infinite medium

$$b \rightarrow \infty \; ; \; p_i = 0 \; ; \; p_e = -T \quad \rightarrow \quad \sigma_{rr} = T\left(1 - \frac{a^2}{r^2}\right) \; ; \; \sigma_{tt} = T\left(1 + \frac{a^2}{r^2}\right)$$

stress concentration factor $$K_t = \frac{\sigma_{max}}{T} = \frac{\sigma_{tt}(r = a)}{T} = \frac{2T}{T} = 2$$

$$|\, p_e = -100 \text{ MPa} \,|\, a = 0.2 \text{ m} \,|\, b = 20 \text{ m} \,|\, h = 0.5 \text{ m} \,|\, E = 200 \text{ GPa} \,|\, \nu = 0.3 \,|$$





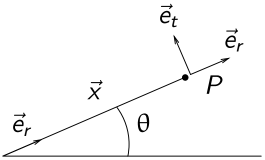
Centrifugal load

$$\vec{x} = r\vec{e}_r(\theta) + z\vec{e}_z \qquad \text{with} \qquad \dot{z} = 0$$

$$\dot{\vec{x}} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r = \dot{r}\vec{e}_r + r\,\frac{d\vec{e}_r}{d\theta}\,\omega = \dot{r}\vec{e}_r + r\omega\vec{e}_t$$

$$\ddot{\vec{x}} = \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + \dot{r}\omega\vec{e}_t + r\dot{\omega}\vec{e}_t + r\omega\dot{\vec{e}}_t = \left(\ddot{r} - r\omega^2\right)\vec{e}_r + \left(2\dot{r}\omega + r\dot{\omega}\right)\vec{e}_t$$

constant r and ω $\quad\rightarrow\quad$ $\ddot{\vec{x}} = -r\omega^2\vec{e}_r = \ddot{u}_r\vec{e}_r$



Rotating disc

$a \leq r \leq b$; plane stress, axisymm, $u_t =$ rigid rot., $\ddot{u}_r(r) = -\omega^2 r$, $q_r = 0$

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{tt} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{tt} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} u_{r,r} \\ \frac{1}{r} u_r \end{bmatrix} \quad ; \quad \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{rr} + \sigma_{tt})$$

$$\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) = \rho \ddot{u}_r \quad \rightarrow \quad \boxed{r^2 u_{r,rr} + r u_{r,r} - u_r = \frac{1-\nu^2}{E} r^2 \rho \ddot{u}_r = A r^3}$$

$$A = -\frac{1-\nu^2}{E} \rho \omega^2$$

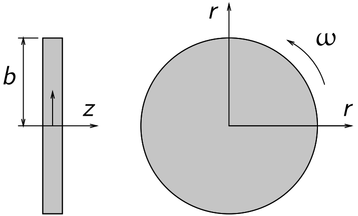
general solution $\qquad u_r = c_1 r + \dfrac{c_2}{r} + \dfrac{1}{8} A r^3$

BC's $\qquad \rightarrow \quad c_1, c_2$

BC's solid cyl. $(a = 0)$ $\qquad u_r(r = 0) \neq \infty \rightarrow c_2 = 0 \quad ; \quad \sigma_{rr}(r = b) = 0$

BC's central hole cyl. $\qquad \sigma_{rr}(r = a) = 0 \quad ; \quad \sigma_{rr}(r = b) = 0$

Solid disc

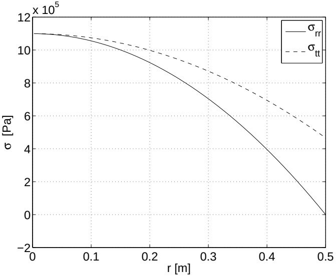


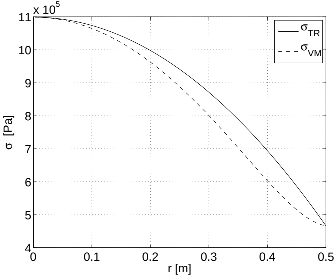
$u_r(r=0) \neq \infty$

$\sigma_{rr}(r=b)=0$

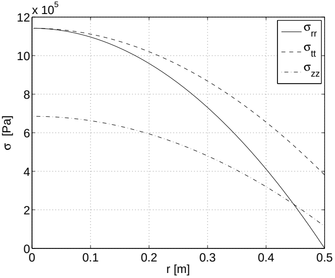
c_1, c_2 : $\triangleright$

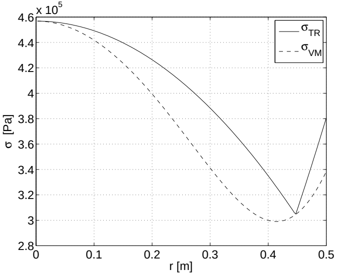
$|\ \omega = 6\ \text{c/s}\ \ |\ a = 0\ \text{m}\ |\ b = 0.5\ \text{m}\ |\ t = 0.05\ \text{m}\ |\ \rho = 7500\ \text{kg/m}^3\ |$
$|\ E = 200\ \text{GPa}\ |\ \nu = 0.3\ |$



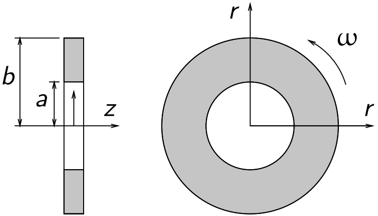


Solid disc



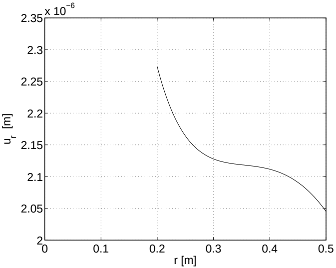


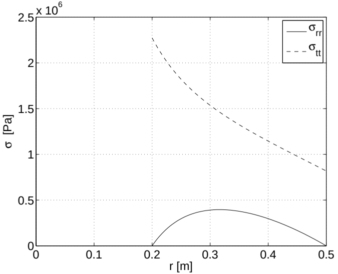
Disc with central hole



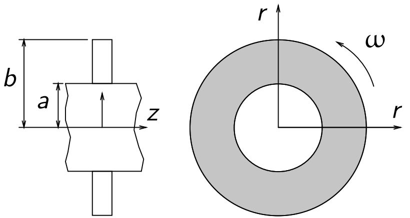
$$\sigma_{rr}(r=a)=0$$
$$\sigma_{rr}(r=b)=0$$
$$c_1, c_2 \;:\; \triangleright$$

| $\omega = 6$ c/s | $a = 0.2$ m | $b = 0.5$ m | $t = 0.05$ m | $\rho = 7500$ kg/m^3 |
| $E = 200$ GPa | $\nu = 0.3$ |



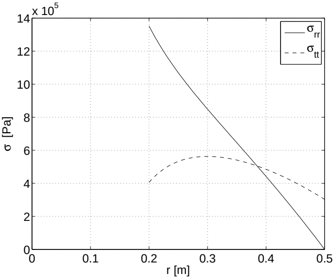


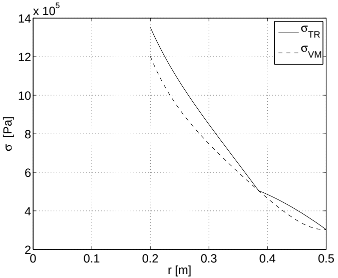
Disc fixed on rigid axis



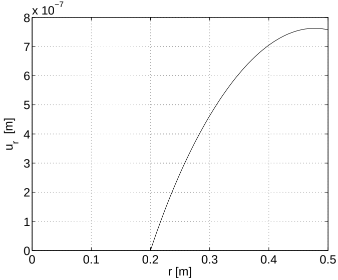
$$u_r(r = a) = 0$$
$$\sigma_{rr}(r = b) = 0$$
$$c_1, c_2 \ : \ \triangleright$$

$| \ \omega = 6 \text{ c/s} \ | \ a = 0.2 \text{ m} \ | \ b = 0.5 \text{ m} \ | \ t = 0.05 \text{ m} \ | \ \rho = 7500 \text{ kg/m}^3 \ | \ E = 200 \text{ GPa} \ | \ \nu =$





Disc fixed on rigid axis



Rotating disc with variable thickness

equilibrium $$\frac{\partial(t(r)r\sigma_{rr})}{\partial r} - t(r)\sigma_{tt} = -\rho\omega^2 t(r) r^2 \quad \text{with } t(r) = \frac{t_a}{2}\,\frac{a}{r}$$

general solution stresses

$$\sigma_{rr} = \frac{2c_1}{at_a}\, r^{d_1} + \frac{2c_2}{at_a}\, r^{d_2} - \frac{3+\nu}{5+\nu}\rho\omega^2 r^2$$

$$\sigma_{tt} = \frac{2c_1}{at_a}\, d_1 r^{d_1} + \frac{2c_2}{at_a}\, d_2 r^{d_2} - \frac{1+3\nu}{5+\nu}\rho\omega^2 r^2$$

with $$d_1 = -\tfrac{1}{2} + \sqrt{\tfrac{5}{4}+\nu} \qquad ; \qquad d_2 = -\tfrac{1}{2} - \sqrt{\tfrac{5}{4}+\nu}$$

boundary conditions $\sigma_{rr}(r=a) = \sigma_{rr}(r=b) = 0 \quad \rightarrow$

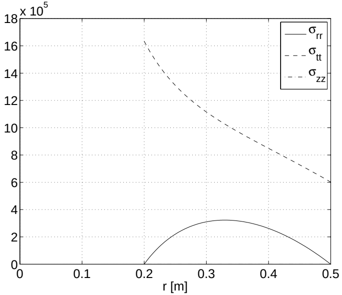
$$\frac{2c_1}{at_a} = \frac{3+\nu}{5+\nu}\,\rho\omega^2\, a^{-d_1}\left[a^2 - a^{d_2}\left(\frac{b^2 - a^{-d_1} b^{d_1} a^2}{b^{d_2} - a^{d_2} a^{-d_1} b^{d_1}}\right)\right]$$

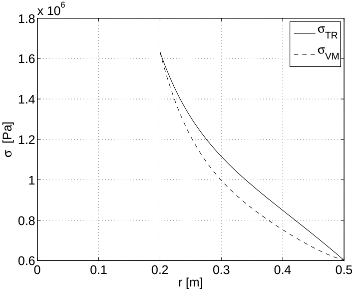
$$\frac{2c_2}{at_a} = \frac{3+\nu}{5+\nu}\,\rho\omega^2\left(\frac{b^2 - a^{-d_1} b^{d_1} a^2}{b^{d_2} - a^{d_2} a^{-d_1} b^{d_1}}\right)$$

Disc with variable thickness

isotropic	plane stress	$\omega = 6$ c/s

$a = 0.2$ m	$b = 0.5$ m	$t_a = 0.05$ m	$\rho = 7500$ kg/m^3	$E = 200$ GPa	$\nu = 0.3$





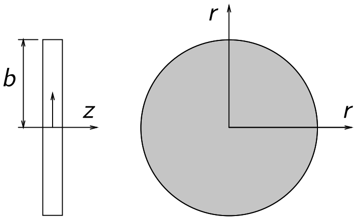
Thermal load

external load $\qquad f(r) = \dfrac{\Theta_{p1}}{A_p}\,\alpha(\Delta T)_{,r}$

radial gradient

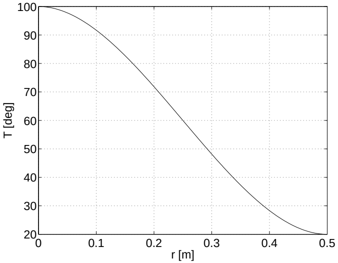
$$
\begin{aligned}
\Delta T(r) &= a_0 + a_1 r + a_2 r^2 + a_3 r^3 \quad \rightarrow \\
f(r) &= \frac{\Theta_{p1}}{A_p}\,\alpha\left(a_1 + 2a_2 r + 3a_3 r^2\right) \quad \rightarrow \\
\bar{u}_r(r) &= \frac{\Theta_{p1}}{A_p}\,\alpha\left(\tfrac{1}{3}a_1 r^2 + \tfrac{1}{4}a_2 r^3 + \tfrac{1}{5}a_3 r^4\right)
\end{aligned}
$$

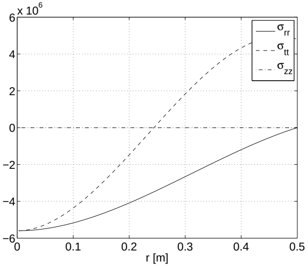
Solid disc, free outer edge



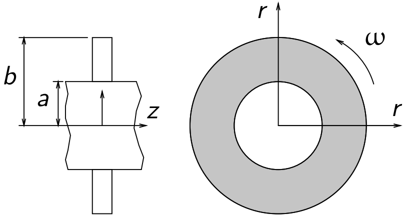
$$u_r(r=0) \neq \infty$$
$$\sigma_{rr}(r=b) = 0$$
$$c_1, c_2 \;:\; \triangleright$$

$$|\, \underset{\sim}{a}^T = [100\ 20\ 0\ 0] \,|\, a = 0 \text{ m} \,|\, b = 0.5 \text{ m} \,|\, E = 200 \text{ GPa} \,|\, \nu = 0.3 \,|\, \alpha = 10^{-6}\ 1/^o\text{C} \,|$$





Disc on a rigid axis

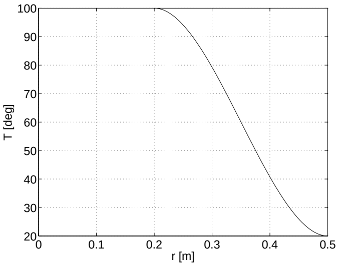


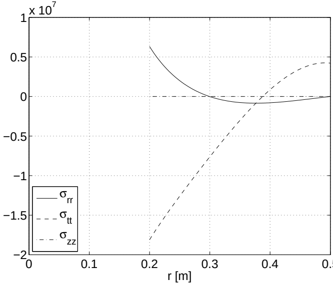
$u_r(r = a) = 0$

$\sigma_{rr}(r = b) = 0$

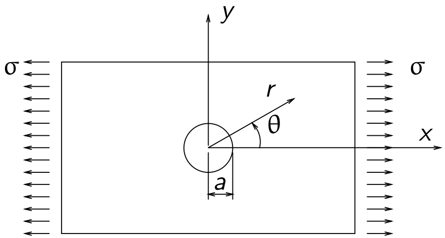
$c_1 , c_2 \ : \ \triangleright$

$|$ isotropic $|$ plane stress $|$ $\underset{\sim}{a}^T = [100\ 20\ 0\ 0]$
$|\ a = 0.2$ m $|\ b = 0.5$ m $|\ E = 200$ GPa $|\ \nu = 0.3\ |\ \alpha = 10^{-6}\ 1/^oC\ |$





Large thin plate with central hole



$$\sigma_{rr} = \frac{\sigma}{2}\left[\left(1-\frac{a^2}{r^2}\right)+\left(1+3\,\frac{a^4}{r^4}-4\,\frac{a^2}{r^2}\right)\cos(2\theta)\right]$$

$$\sigma_{tt} = \frac{\sigma}{2}\left[\left(1+\frac{a^2}{r^2}\right)-\left(1+3\,\frac{a^4}{r^4}\right)\cos(2\theta)\right]$$

$$\sigma_{rt} = -\frac{\sigma}{2}\left[1-3\,\frac{a^4}{r^4}+2\,\frac{a^2}{r^2}\right]\sin(2\theta)$$

Large thin plate with central hole

stress concentrations

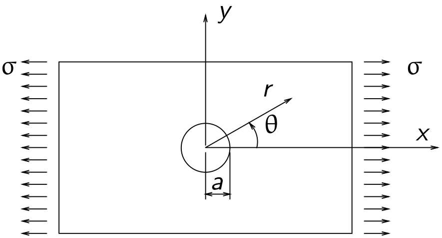
$$\sigma_{tt}(r = a, \theta = \tfrac{\pi}{2}) = 3\sigma \quad ; \quad \sigma_{tt}(r = a, \theta = 0) = -\sigma$$

stress concentration factor $\qquad K_t = \dfrac{\sigma_{max}}{\sigma} = 3$

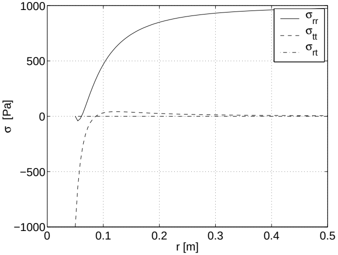
stress at larger r

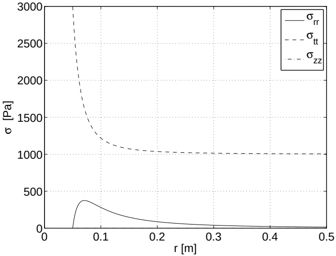
$$\begin{aligned}
\sigma_{rr} &= \frac{\sigma}{2}\left[1 + \cos(2\theta)\right] = \sigma\ \cos^2(\theta) \\
\sigma_{tt} &= \frac{\sigma}{2}\left[1 - \cos(2\theta)\right] = \sigma\left[1 - \cos^2(\theta)\right] = \sigma \sin^2(\theta) \\
\sigma_{rt} &= -\frac{\sigma}{2}\ \sin(2\theta) = -\sigma\ \sin(\theta)\cos(\theta)
\end{aligned}$$

Large thin plate with central hole



$$\left| \, a = 0.05 \text{ m} \, \right| \, \sigma = 1000 \text{ Pa} \, \Big|$$





EXAMPLES : INTEGRATION CONSTANTS

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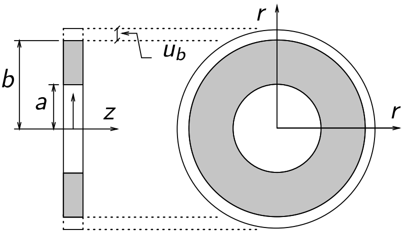
Disc, edge displacement



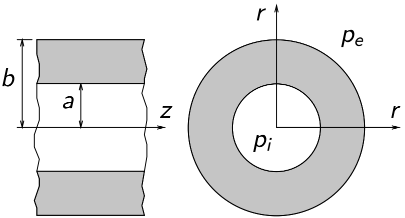
$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$u_r(r=b) = u_b$$
$$\sigma_{rr}(r=a) = 0;$$

$\triangleleft$

$$c_1 = \frac{(A_p\zeta - Q_p)b^{\zeta}\, u_b}{(A_p\zeta + Q_p)a^{2\zeta} + (A_p\zeta - Q_p)b^{2\zeta}} \quad ; \quad c_2 = \frac{(A_p\zeta + Q_p)b^{\zeta}a^{2\zeta}\, u_b}{(A_p\zeta + Q_p)a^{2\zeta} + (A_p\zeta - Q_p)b^{2\zeta}}$$

$$c_1 = \frac{(A_p - Q_p)b}{(A_p + Q_p)a^2 + (A_p - Q_p)b^2}\, u_b \quad ; \quad c_2 = \frac{(A_p + Q_p)ba^2}{(A_p + Q_p)a^2 + (A_p - Q_p)b^2}\, u_b$$

Disc/cylinder, edge load

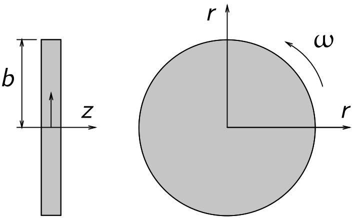


$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$\sigma_{rr}(r = a) = -p_i$$
$$\sigma_{rr}(r = b) = -p_e$$
$$\triangleleft$$

$$c_1 = \frac{1}{A_p\zeta + Q_p} \frac{a^{\zeta+1}p_i - b^{\zeta+1}p_e}{b^{2\zeta} - a^{2\zeta}} \quad ; \quad c_2 = \frac{1}{A_p\zeta - Q_p} \frac{a^{\zeta+1}b^{2\zeta}p_i - b^{\zeta+1}a^{2\zeta}p_e}{b^{2\zeta} - a^{2\zeta}}$$

$$c_1 = \frac{1}{A_p + Q_p} \frac{1}{b^2 - a^2} (p_i a^2 - p_e b^2) \quad ; \quad c_2 = \frac{1}{A_p - Q_p} \frac{a^2 b^2}{b^2 - a^2} (p_i - p_e)$$

Rotating solid disc



$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$

$$u_r(r=0) \neq \infty$$

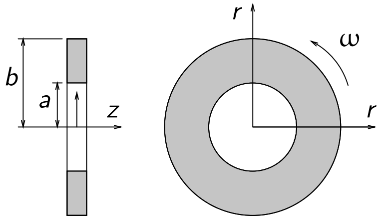
$$\sigma_{rr}(r=b) = 0$$

$\triangleleft$

$$c_2 = 0 \qquad ; \qquad c_1 = \frac{3A_p + Q_p}{A_p(A_p\zeta + Q_p)}\,\beta\, b^{-\zeta+3}$$

$$c_2 = 0 \qquad ; \qquad c_1 = \frac{(3A_p + Q_p)}{A_p(A_p + Q_p)}\,\beta\, b^{2}$$

Rotating disc with central hole



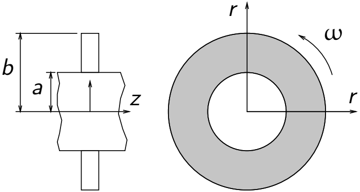
$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$
$$\sigma_{rr}(r=a) = 0$$
$$\sigma_{rr}(r=b) = 0$$

◁

$$c_1 = \frac{3A_p + Q_p}{A_p(A_p\zeta + Q_p)}\left(\frac{b^{\zeta+3} - a^{\zeta+3}}{b^{2\zeta} - a^{2\zeta}}\right)\beta$$
$$c_2 = \frac{3A_p + Q_p}{A_p(A_p\zeta - Q_p)}\left(\frac{a^{2\zeta-2}b^{\zeta+1} - a^{\zeta+1}b^{2\zeta-2}}{b^{2\zeta} - a^{2\zeta}}\right)(a^2b^2)\beta$$

$$c_1 = \frac{(3A_p + Q_p)}{A_p(A_p + Q_p)}(a^2 + b^2)\beta \quad ; \quad c_2 = \frac{(3A_p + Q_p)}{A_p(A_p - Q_p)}(a^2b^2)\beta$$

Rotating disc fixed on rigid axis



$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$

$$u_r(r = a) = 0$$

$$\sigma_{rr}(r = b) = 0$$

◁

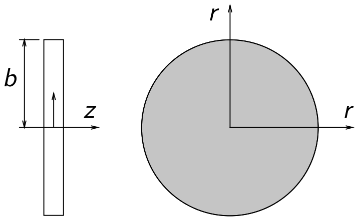
$$c_1 = \frac{\beta}{(A_p\zeta + Q_p)b^{\zeta+1}a^{-\zeta+1} + (A_p\zeta - Q_p)b^{-\zeta+1}a^{\zeta+1}} \left\{ \frac{3A_p + Q_p}{A_p}\, b^4 a^{-\zeta+1} + \frac{A_p\zeta - Q_p}{A_p}\, b^{-\zeta+1} a^4 \right\}$$

$$c_2 = \frac{\beta}{(A_p\zeta + Q_p)b^{\zeta+1}a^{-\zeta+1} + (A_p\zeta - Q_p)b^{-\zeta+1}a^{\zeta+1}} \left\{ \frac{A_p\zeta + Q_p}{A_p}\, b^{\zeta+1} a^4 - \frac{3A_p + Q_p}{A_p}\, b^4 a^{\zeta+1} \right\}$$

$$c_1 = \frac{\beta}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2} \left\{ \frac{3A_p + Q_p}{A_p}\, b^4 + \frac{A_p - Q_p}{A_p}\, a^4 \right\}$$

$$c_2 = \frac{\beta}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2} \left\{ \frac{A_p + Q_p}{A_p}\, a^4 b^2 - \frac{3A_p + Q_p}{A_p}\, a^2 b^4 \right\}$$

Solid disc with radial temperature gradient



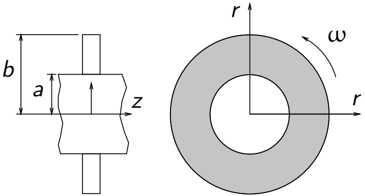
$$u_r(r=0) \neq \infty$$

$$\sigma_{rr}(r=b) = 0$$

$\triangleleft$

$$c_2 = 0 \qquad ; \qquad c_1 = \alpha \left\{ a_0 + \frac{(A_p - Q_p)}{A_p} \left(\tfrac{1}{3} a_1 b + \tfrac{1}{4} a_2 b^2 + \tfrac{1}{5} a_3 b^3 \right) \right\}$$

Disc on a rigid axis with radial temperature gradient



$u_r(r = a) = 0$

$\sigma_{rr}(r = b) = 0$

$\triangleleft$

$$c_1 = \frac{-\alpha(A_p + Q_p)}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2}$$

$$\left\{ \frac{(A_p - Q_p)}{A_p} a^2 \left(\tfrac{1}{3}a_1 a + \tfrac{1}{4}a_2 a^2 + \tfrac{1}{5}a_3 a^3\right) - b^2 a_0 - \frac{(A_p - Q_p)}{A_p} b^2 \left(\tfrac{1}{3}a_1 b + \tfrac{1}{4}a_2 b^2 + \tfrac{1}{5}a_3 b^3\right) \right\}$$

$$c_2 = \frac{-\alpha(A_p + Q_p)a^2 b^2}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2}$$

$$\left\{ \frac{(A_p + Q_p)}{A_p} \left(\tfrac{1}{3}a_1 a + \tfrac{1}{4}a_2 a^2 + \tfrac{1}{5}a_3 a^3\right) + a_0 + \frac{(A_p - Q_p)}{A_p} \left(\tfrac{1}{3}a_1 b + \tfrac{1}{4}a_2 b^2 + \tfrac{1}{5}a_3 b^3\right) \right\}$$

NUMERICAL SOLUTIONS

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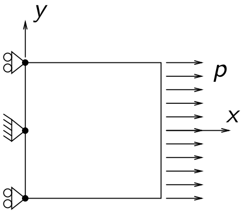
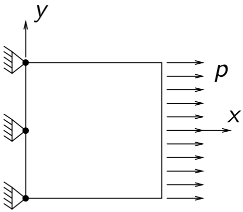
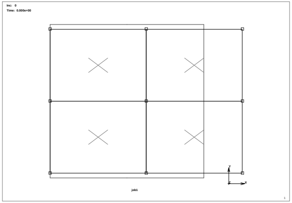
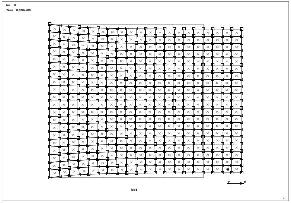
Tensile test





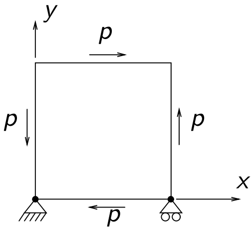
$$| \, p = 100 \text{ MPa} \, | \, a = 0.5 \text{ m} \, | \, h = 0.05 \text{ m} \, | \, E = 200 \text{ GPa} \, | \, \nu = 0.25 \, |$$

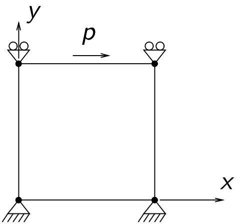




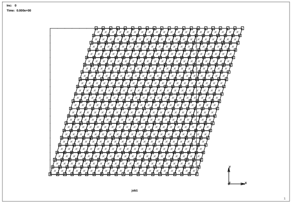
$$u_x(x = a) = 0.25 \times 10^{-3} \text{ m} \quad ; \quad u_y(y = a/2) = -0.3125 \times 10^{-4}$$

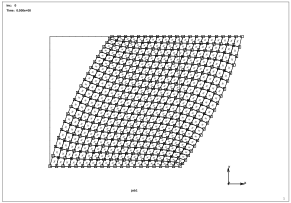
Shear test



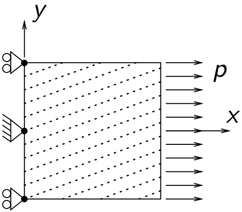


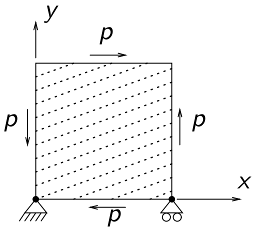
isotropic	plane stress	$p = 100$ MPa	$a = 0.5$ m	$h = 0.05$ m
$E = 200$ GPa	$\nu = 0.25$			



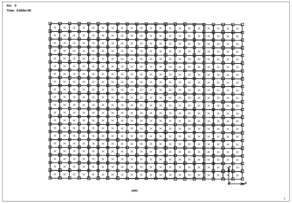


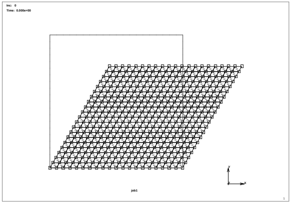
Orthotropic plate





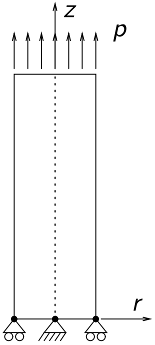
orthotropic	plane stress	$p = 100$ MPa	$a = 0.5$ m	$h = 0.05$ m	$\alpha = 20^o$
$E_{11} = 200$ GPa	$E_{22} = 50$ GPa	$E_{33} = 50$ GPa			
$\nu_{12} = 0.4$	$\nu_{23} = 0.25$	$\nu_{31} = 0.25$			
$G_{12} = 100$ GPa	$G_{23} = 20$ GPa	$G_{31} = 20$ GPa			

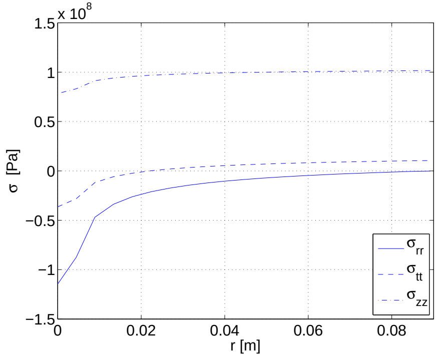




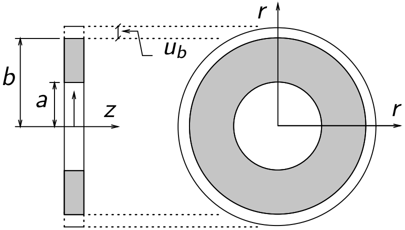
Axi-symmetric, orthotropic, $u_t = 0$

$$
\begin{array}{|l|l|l|l|}
E_{11} = 200 \text{ GPa} & E_{22} = 50 \text{ GPa} & E_{33} = 50 \text{ GPa} & \\
\nu_{12} = 0.4 & \nu_{23} = 0.25 & \nu_{31} = 0.25 & \\
G_{12} = 100 \text{ GPa} & G_{23} = 20 \text{ GPa} & G_{31} = 20 \text{ GPa} & p = 100 \text{ MPa}
\end{array}
$$

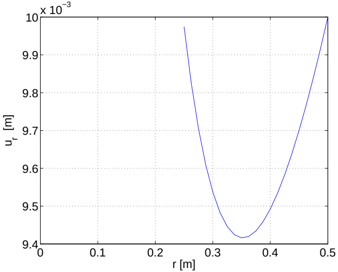


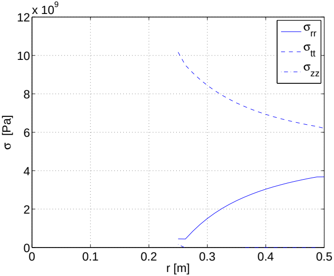


Prescribed edge displacement

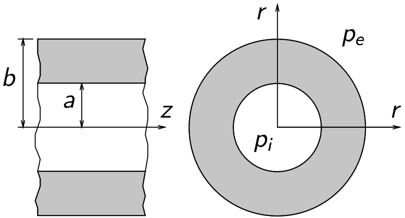


$$| \; u_b = 0.01 \text{ m} \; | \; a = 0.25 \text{ m} \; | \; b = 0.5 \text{ m} \; | \; h = 0.05 \text{ m} \; | \; E = 250 \text{ GPa} \; | \; \nu = 0.33 \; |$$

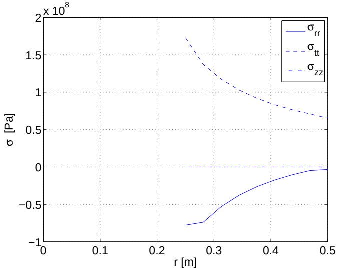


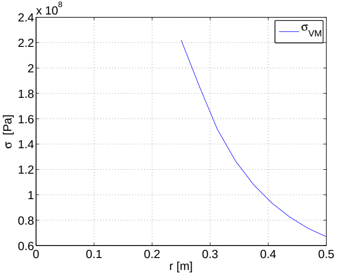


Edge load

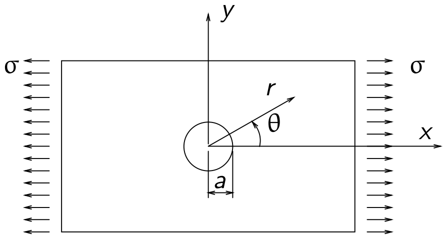


$$| \; p_i = 100 \text{ MPa} \; | \; a = 0.25 \text{ m} \; | \; b = 0.5 \text{ m} \; | \; h = 0.5 \text{ m} \; | \; E = 250 \text{ GPa} \; | \; \nu = 0.33 \; |$$





Large thin plate with a central hole



$$|\ a = 0.05 \text{ m}\ |\ \sigma = 1000 \text{ Pa}\ |$$