

# Introduction to Machine Learning

## Summary

Philip Hartout

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## 1 Linear Regression

Objective, approximate:

$$\begin{aligned}
 f(x) &= w_1x_1 + \dots + w_dx_d + w_0 \\
 &= \sum_{i=1}^d w_ix_i + w_0 \\
 &= \mathbf{w}^T \mathbf{x} + w_0
 \end{aligned}$$

$\forall \mathbf{x}, \mathbf{w} \in \mathbb{R}^d$ . This expression can be further compressed to the homogeneous representation where  $\forall \tilde{\mathbf{x}}, \tilde{\mathbf{w}} \in \mathbb{R}^{d+1}$ , i.e.  $\tilde{x}_{d+1} = 1$ . We have w.l.o.g.:

$$f(x) = \mathbf{w}^T \mathbf{x}$$

Quantify errors using residuals:

$$\begin{aligned}
 r_i &= y_i - f(x_i) \\
 &= y_i - \mathbf{w}^T \mathbf{x}_i
 \end{aligned}$$

We can use squared residuals and sum over all residuals to get the cost:

$$\hat{R}(w) = \sum_{i=1}^n r_i^2 \tag{1}$$

$$= \sum_{i=1}^n (y_i - \mathbf{w}^T \mathbf{x}_i)^2 \tag{2}$$

Optimization objective to find optimal weight vector  $\mathbf{w}$  with least squares is the following:

$$\mathbf{w} = \arg \min_{\mathbf{w}} \sum_{i=1}^n (y_i - \mathbf{w}^T \mathbf{x}_i)^2$$

## 1.1 Closed form solution

This can be solved in closed form:

$$\hat{\mathbf{w}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

where:

$$X = \begin{bmatrix} X_{1,1} & \dots & X_{1,d} \\ \vdots & \ddots & \vdots \\ X_{n,1} & \dots & X_{n,d} \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

## 1.2 Optimization

### 1.2.1 Requirements

Requires a convex objective function.

**Definition 1.1** (Convexity). *A function is convex iff  $\forall \mathbf{x}, \mathbf{x}', \lambda \in [0, 1]$  it holds that  $f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{x}') \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{x}')$*

Note that the least squares objective function defined in 1 is convex.

### 1.2.2 Gradient descent

We start with an arbitrary  $w_0 \in \mathbb{R}^d$ , then for  $t = 0, 1, 2, \dots$  we perform the following operation:

$$w_{t+1} = w_t - \eta_t \nabla \hat{R}(w_t)$$

where  $\eta_t$  is the learning rate.

Under mild assumptions, if the step size is sufficiently small, the gradient descent procedure converges to a stationary point, where the gradient is zero. For convex objectives, it therefore finds the optimal solution. In the case of the squared loss and a constant step size (e.g. 0.5), the algorithm converges at linear rate. If you look at the difference in empirical value at iteration  $t$  and compare that with the optimal value, then the gap is going to shrink at linear rate. If we look for a solution within a margin  $\epsilon$ , it is found in  $\mathcal{O}(\ln(\frac{1}{\epsilon}))$  iterations. The fact that the objective function converges at linear rate can be formally described as follows:

$$\exists t_0 \forall t \geq t_0, \exists \alpha < 1 \text{ s.t. } (\hat{R}(w_{t+1}) - \hat{R}(\hat{w})) \leq \alpha (\hat{R}(w_t) - \hat{R}(\hat{w}))$$

where  $\hat{w}$  is the optimal value for the hyperparameters.

For computing the gradient, we recall that:

$$\nabla \hat{R}(\hat{w}) = \left[ \frac{\partial}{\partial w_1} \hat{R}(w) \quad \dots \quad \frac{\partial}{\partial w_d} \hat{R}(w) \right]$$

In one dimension, we have that:

$$\begin{aligned}
 \nabla \hat{R}(w) &= \frac{d}{dw} \hat{R}(w) = \frac{d}{dw} \sum_{i=1}^n (y_i - w \cdot x_i)^2 \\
 &= \sum_{i=1}^n \frac{d}{dw} (y_i - w \cdot x_i)^2 \\
 &= 2(y_i - w \cdot x_i) \cdot (-x_i) \\
 &= \sum_{i=1}^n 2(y_i - w \cdot x_i) \cdot (-x_i) \\
 &= -2 \sum_{i=1}^n r_i x_i.
 \end{aligned}$$

In  $d$ -dimension, we have that:

$$\nabla \hat{R}(w) = -2 \sum_{i=1}^n r_i x_i,$$

where  $r_i \in \mathbb{R}$  and  $x_i \in \mathbb{R}^d$

### 1.2.3 Adaptive step size for gradient descent

The step size can be updates adaptively, via either:

#### 1. Line search:

Suppose at iteration  $t$ , we have  $w_t, g_t = \nabla \hat{R}(w_t)$ . We then define:

$$y_t^* = \arg \min_{y \in [0, \infty)} \hat{R}(w_t) - \eta g_t$$

#### 2. Bold driver heuristic:

- If the function decreases, increase the step size.

$$\text{If } \hat{R}(w_{t+1}) < \hat{R}(w_t) : \eta_{t+1} \leftarrow \eta_t \cdot c_{acc}$$

where  $c_{acc} > 1$

- If the function increases, decrease the step size.

$$\text{If } \hat{R}(w_{t+1}) > \hat{R}(w_t) : \eta_{t+1} \leftarrow \eta_t \cdot c_{dec}$$

where  $c_{dec} < 1$ .

### 1.2.4 Tradeoff between gradient descent and closed form

Several reasons:

- Computational complexity:

$$\hat{w} = (X^T X)^{-1} (X^T y)$$

$(X^T X)$  can be computed in  $\mathcal{O}(nd^2)$ ,  $(X^T X)^{-1}$  can be computed in  $\mathcal{O}(d^3)$ .

By comparison, for gradient descent calculating  $\nabla \hat{R}(w) = \sum_{i=1}^n (y_i - w^T x_i) x_i$  can be computed in  $\mathcal{O}(nd)$ , where  $n = \ln(\frac{1}{\epsilon})$

- the problem may not require an optimal solution.
- many problems do not admit a closed form solution.

### 1.3 other loss functions

Least squares is part of a general case of the following general loss function, which is convex for  $p \geq 1$ .

$$l_p(r) = |r|^p \quad (3)$$

Least squares is where  $p = 2$ .

## 2 Probability (interlude)

### 2.1 Gaussians

The p.d.f. of a Gaussian distribution is given by:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x-\mu}{2\sigma^2}\right) \quad (4)$$

The p.d.f. of a multivariate Gaussian distribution is given by:

$$\frac{1}{2\pi\sqrt{|\sigma|}} \exp\left(-\frac{1}{2}(x-\mu)^T \sigma^{-1}(x-\mu)\right) \quad (5)$$

where:

$$\sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} \text{ and } \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \quad (6)$$

### 2.2 Expectations

Expected value of a random variable can be calculated as follows:

$$\mathbb{E} = \begin{cases} \sum_x xp(x) & \text{if } X \text{ is discrete} \\ \int xp(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

Expectations respect linear properties, i.e. let  $X, Y$  be random variable and  $a, b \in \mathbb{R}$ , then we have  $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ .

## 3 Generalization and model validation

### 3.1 Fitting nonlinear functions via linear regression

Using nonlinear features of our data (basis functions), we can fit nonlinear functions via linear regression. Then, the model takes on the form:

$$f(\mathbf{x}) = \sum_{i=1}^d w_i \phi(\mathbf{x}) \quad (7)$$

where  $\mathbf{x} \in \mathbb{R}^d$ ,  $x \mapsto \tilde{x} = \phi(\mathbf{x}) \in \mathbb{R}^d$  and  $w \in \mathbb{R}^d$ .

- 1 dim.:  $\phi(\mathbf{x}) = [1, x, x^2, \dots, x^k]$
- 2 dim.:  $\phi(\mathbf{x}) = [1, x_1, x_2, x_1^2, x_2^2, \dots, x_1^k, x_2^k]$
- p dim.:  $\phi(\mathbf{x})$  vector of all monomials in  $x_1, \dots, x_p$  of degree up to  $k$ .

## 3.2 Achieving generalization

### 3.2.1 Independence and identical distribution

A fundamental assumption needs to be met: the dataset is generated from an independently and identically distributed from some unknown distribution  $P$ , i.e:

$$(x_i, y_i) \sim P(\mathbf{X}, Y).$$

The i.i.d. assumption is invalid when:

- we deal with time series data
- spatially correlated data
- correlated noise

If violated, we can still use ML but the interpretation of the results needs to be carefully analyzed. The most important thing is to choose the train/test split to assess the desired generalization properties of the trained model.

## 3.3 Expected error and generalization error

Once the iid assumption is verified, our goal is then to minimize the expected error (true risk) under  $P$ , i.e.:

$$\begin{aligned} R(\mathbf{w}) &= \int P(\mathbf{x}, y)(y - \mathbf{w}^T \mathbf{x})^2 dxdy \\ &= \mathbb{E}[(y - \mathbf{w}^T \mathbf{x})^2] \end{aligned}$$

The true risk can be estimated by the empirical risk on a sample dataset  $D$ :

$$\hat{R}_D(\mathbf{w}) = \frac{1}{|D|} \sum_{\mathbf{x}, y \in D} (y - \mathbf{w}^T \mathbf{x})^2$$

The reason behind this approximation is because of the law of large numbers

**Definition 3.1** (Law of large numbers).  $\hat{R}_D(\mathbf{w}) \rightarrow R_D(\mathbf{w})$  for any fixed  $\mathbf{w}$  as  $|D| \rightarrow \infty$ .

$$\hat{\mathbf{w}}_D = \arg \min_{\mathbf{w}} \hat{R}_D(\mathbf{w}) \tag{8}$$

$$\mathbf{w}^* = \arg \min_{\mathbf{w}} \hat{R}(\mathbf{w}) \tag{9}$$

We don't want to minimize the empirical risk given in equation 8 but the true risk given in equation 9, which are similar as the amount points in the dataset increases.

## 3.4 Uniform convergence

For learning via empirical risk minimization, uniform convergence is required, i.e.:

$$\sup_{\mathbf{w}} |R(\mathbf{w}) - \hat{R}_D(\mathbf{w})| \rightarrow 0 \text{ as } |D| \rightarrow \infty$$

Note that this is not implied by the law of large numbers alone, but depends on model class. It holds for instance for squared loss on data distributions with bounded support. Statistical learning theory is required to define these properties.

### 3.5 Evaluation of performance on training data

In general it holds that:

$$\mathbb{E}_D[\hat{R}_D(\hat{\mathbf{w}}_D)] \leq \mathbb{E}_D[R_D(\hat{\mathbf{w}}_D)]$$

*Proof.*

$$\begin{aligned} \mathbb{E}[\hat{R}_D(\hat{\mathbf{w}}_D)] &= \mathbb{E}_D[\min_{\mathbf{w}} \hat{R}_D(\mathbf{w})] && \text{(ERM)} \\ &\leq \min_{\mathbf{w}} \mathbb{E}_D[\hat{R}_D(\mathbf{w})] && \text{(Jensen's inequality)} \\ &= \min_{\mathbf{w}} \mathbb{E}_D\left[\frac{1}{|D|} \sum_{i=1}^{|D|} (y_i - wx_i)^2\right] && \text{(Definition of } \hat{R}_D(.)) \\ &= \min_{\mathbf{w}} \mathbb{E}_D\left[\frac{1}{|D|} \sum_{i=1}^{|D|} (y_i - wx_i)^2\right] && \text{(linear expectations)} \\ &= \min_{\mathbf{w}} R(\mathbf{w}) \leq \mathbb{E}[R(\hat{\mathbf{w}}_D)] \end{aligned}$$

□

Thus, we obtain an overly optimistic estimate. A more realistic evaluation would be to use a separate test set from the same distribution  $P$ . Then:

- Optimize  $w$  on training set:

$$\hat{\mathbf{w}}_{D_{\text{train}}} = \arg \min_{\mathbf{w}} \hat{R}_{\text{train}}(\mathbf{w})$$

- Evaluate on test set:

$$\hat{R}_{\text{test}}(\hat{\mathbf{w}}) = \frac{1}{|D_{\text{test}}|} \sum_{\mathbf{x}, y \in D_{\text{test}}} (y - \hat{\mathbf{w}}^T \mathbf{x})^2$$

- Then:

$$\mathbb{E}_{D_{\text{train}}, D_{\text{test}}}[\hat{R}_{D_{\text{test}}}(\hat{\mathbf{w}}_{D_{\text{train}}})] = \mathbb{E}_{D_{\text{train}}}[R(\hat{\mathbf{w}}_{D_{\text{train}}})]$$

*Proof.* Let  $D_{\text{train}} = D$ ,  $D_{\text{test}} = V$  and  $D, V \sim P$ . Then:

$$\begin{aligned} \mathbb{E}_{D, V}[\hat{R}_V(\hat{\mathbf{w}}_D)] &= \mathbb{E}_D[\mathbb{E}_V[\hat{R}_V(\hat{\mathbf{w}}_D)]] && \text{independence of } D, V \\ &= \mathbb{E}_D[\mathbb{E}_V\left[\frac{1}{|V|} \sum_{i=1}^{|V|} (y_i - \hat{\mathbf{w}}_D^T x_i)^2\right]] && \text{(Definition of } \hat{R}_D(.)) \\ &= \mathbb{E}_D\left[\frac{1}{|V|} \sum_{i=1}^{|V|} \mathbb{E}_{x_i, y_i} (y_i - \hat{\mathbf{w}}_D^T x_i)^2\right] && \text{since } (x_i, y_i) \perp D \\ &= \mathbb{E}_D[R(\hat{\mathbf{w}}_D)] \end{aligned}$$

□

### 3.6 Evaluation for model selection

For each candidate model  $m$ , we repeat the following procedure for  $i = 1 : k$ :

- We split the same dataset into training and validation sets:

$$D = D_{\text{train}}^{(i)} \uplus D_{\text{val}}^{(i)}$$

- We train the model:

$$\hat{\mathbf{w}}_{[i, m]} = \arg \min_{\mathbf{w}} \hat{R}_{\text{train}}^{(i)}(\mathbf{w})$$

- Then we estimate the error:

$$\hat{R}_m^{(i)} = \hat{R}_{\text{val}}^{(i)}(\hat{\mathbf{w}}_i)$$

Finally, select the model:

$$\hat{m} = \arg \min_m \frac{1}{k} \sum_{i=1}^k \hat{R}_m^{(i)}$$

### 3.7 Splitting the data for model selection

This splitting can be done randomly through Monte Carlo cross-validation.

- Pick training set of given size uniformly at random
- Validate on remaining points
- Estimate prediction error by averaging the validation error over multiple random trials.

It can also be achieved through  $k$ -fold cross-validation, which is the default choice.

- Partition the data into  $k$  folds
- Train on  $k - 1$  folds, evaluating on remaining fold.
- Estimate prediction error by averaging the validation error obtained while varying the validation fold.

Note that the cross-validation error is almost unbiased for large enough  $k$ . The following should be considered to pick  $k$ :

- Too small:
  - Risk of overfitting on test set
  - Using too little data for training
  - Risk of underfitting to training set
- Too large:
  - In general, leads to better performance.  $k = n$  is perfectly fine, specific instance called leave-one-out cross-validation
  - Higher computational complexity.

In practice,  $k = 5$  or  $k = 10$  is often used and works well.

### 3.8 Best practice for evaluating models in supervised learning

Follow the following steps:

- Split data set into training and test set
- Never look at test set when fitting the model. For example, use  $k$ -fold cross-validation on training set
- Report final accuracy on test set, but never optimize on it.

Note that this procedure only works if the data is i.i.d. I.e. one should be careful if there are temporal trends or other dependencies.

## 4 Regularization

We want to avoid having overly complex models when minimizing the loss function. This can be achieved through regularization, which encourages small weights via penalty functions, which are called regularizers.



## 4.1 Ridge regression

This is a regularized optimization problem:

$$\min_{\mathbf{w}} \frac{1}{n} \sum_{i=1}^n (y_i - \mathbf{w}^T \mathbf{x}_i)^2 + \lambda \|\mathbf{w}\|_2^2 = \sum_{j=1}^d \mathbf{w}_j^2 \quad \forall \lambda \geq 0$$

### 4.1.1 Closed form solution

This can be optimized using the closed form solution or gradient descent. The closed form for Ridge regression is:

$$\hat{\mathbf{w}} = (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{X}^T \mathbf{y}$$

where  $\mathbf{I} \in \mathbb{R}^{d \times d}$  is the identity matrix.

### 4.1.2 Gradient descent

$$\nabla \left( \frac{1}{n} \sum_{i=1}^n (y_i - \mathbf{w}^T \mathbf{x}_i)^2 + \lambda \|\mathbf{w}\|_2^2 \right) = \nabla_{\mathbf{w}} \hat{R}(\mathbf{w}) + \lambda \nabla_{\mathbf{w}} \|\mathbf{w}\|_2^2$$

One step of the gradient descent is therefore performed as follows:

$$\begin{aligned} \mathbf{w}_{t+1} &\leftarrow \mathbf{w}_t - \eta_t (\nabla_{\mathbf{w}} \hat{R}(\mathbf{w}_t) + 2\lambda \mathbf{w}_t) \\ &= (1 - 2\lambda \eta_t) \mathbf{w}_t - \eta_t \nabla_{\mathbf{w}} \hat{R}(\mathbf{w}_t) \end{aligned}$$

Choosing the regularization parameter is done through cross-validation. Typically, the choice is between values of  $\lambda$  values which are logarithmically spaced.

### 4.1.3 Generalization of a tradeoff in ML

A lot of supervised learning problems can be written in this way:

$$\min_{\mathbf{w}} \hat{R}(\mathbf{w}) + \lambda C(\mathbf{w}).$$

It's possible to control complexity by varying regularization parameter  $\lambda$ .

## 4.2 Renormalizing data through standardization

This process ensures that each feature has zero mean and unit variance:

$$\tilde{x}_{i,j} = \frac{(x_{i,j}) - \hat{\mu}_j}{\hat{\sigma}_j}$$

where  $x_{i,j}$  is the value of the  $j^{\text{th}}$  feature of the  $i^{\text{th}}$  data point:

$$\hat{\mu}_j = \frac{1}{n} \sum_{i=1}^n x_{i,j} \quad \sigma_j^2 = \frac{1}{n} \sum_{i=1}^n (x_{i,j} - \hat{\mu}_j)^2$$

## 5 Classification

**Definition 5.1** (Classification). *Classification is an instance of supervised learning where  $Y$  is discrete (categorical). We want to assign data points  $X$  (documents, queries, images, user visits) a label  $Y$  (spam/not spam, topic such as sports, politics, entertainment, click/no-click etc).*

The input of the model is labeled data set with positive and negative examples, see Figure 1. The output is a decision rule, i.e. a hypothesis. Given a dataset  $D = \{(x_1, y_1), \dots (x_n, y_n)\}$ , we have:

$$y \approx h_{\mathbf{w}}(x) = \text{sign}(\mathbf{w}^T x)$$

Linear classification works well in high-dimensional settings when using the right features; prediction is typically very efficient despite linear classification seeming very restrictive at first.

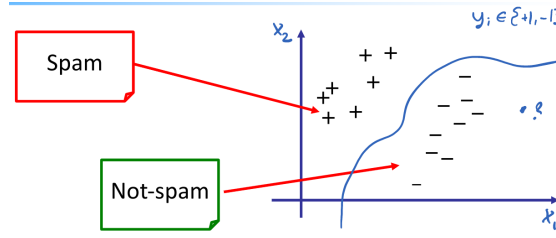


Figure 1: Illustration of binary classification

## 5.1 Finding linear separators

Writing the search for a classifier can be seen as an optimization problem: we seek the set of weights  $\mathbf{w}$  that minimizes the number of mistakes, i.e.:

$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w} \in \mathbb{R}} \sum_{i=1}^n [y_i \neq \text{sign}(w^T x_i)]$$

$$= \begin{cases} 1 & \text{if } y_i \neq \text{sign}(w^T x_i) \\ 0 & \text{otherwise.} \end{cases}$$

The goal is then to optimize the following function:

$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w} \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n [y_i \neq \text{sign}(w^T x_i)]$$

$$= \frac{1}{n} \sum_{i=1}^n \mathbb{I}(w; x_i, y_i)$$

Note that this poses as challenge as it is not convex or even differentiable. Therefore we need to replace this loss by a tractable loss function for the sake of optimization/model fitting. When evaluating a model, we then use the original cost/performance function. The function we can use to optimize in this case is the surrogate loss:

$$l_P(\mathbf{w}; y_i, x_i) = \max(0, -y_i \mathbf{w}^T x_i)$$

which is also referred to as the perceptron loss.

The gradient of the perceptron loss function can be computed as follows:

$$R(\mathbf{w}) = \sum_{i=1}^n \max(0, -y_i \mathbf{w}^T x_i)$$

$$\nabla R(\mathbf{w}) = \sum_{i=1}^n \nabla_{\mathbf{w}} \max(0, -y_i \mathbf{w}^T x_i)$$

$$= \begin{cases} 0 & \text{if } y_i \mathbf{w}^T x_i \geq 0 \text{ i.e. correctly classified} \\ -y_i x_i & \text{otherwise} \end{cases}$$

So we have the following update rule:

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \eta_t \sum_{i: (x_i, y_i) \text{ incorrectly classified by } \mathbf{w}} x_i y_i$$

## 5.2 Stochastic gradient descent

Computing the gradient requires summing over all data, which is inefficient for large datasets. Additionally, our initial estimates are likely very wrong and we can get a good unbiased gradient estimate by evaluating the gradient on few points. In the worst case, we can evaluate only one randomly chosen point, which is a procedure called stochastic gradient descent. It consists of the following steps:

1. Start at an arbitrary  $\mathbf{w}_0 \in \mathbb{R}^d$
2. For  $t = 1, 2, \dots$  do:
  - Pick data point  $(\mathbf{x}', y') \in D$  from training set uniformly at random (with replacement), and set:

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \eta_t \nabla \uparrow(\mathbf{w}_t; \mathbf{x}', y')$$

Where  $\eta_t$  is called the learning rate. Guaranteed to converge under mild conditions, if:

$$\sum_t \eta_t = \infty \text{ and } \sum_t \eta_t^2 < \infty$$

for instance  $\eta_t = \frac{1}{t}$  and  $\eta_t = \min(c, \frac{c'}{t})$ .

The perceptron algorithm is just stochastic gradient descent on the perceptron loss function  $\uparrow_P$  with learning rate 1.

**Theorem 1** (Perceptron algorithm). *If the data is linearly separable, the perceptron will obtain a linear separator.*

The variance of the gradient estimate can be reduced by averaging over the gradients w.r.t. multiple randomly selected points, which are called minibatches. Adaptive learning rates can be additionally applied. There exist various approaches for adaptively tuning the learning rate. Often times, these even use a different learning rate per feature. Examples of adaptive learning rate algorithms include AdaGrad, RMSProp, Adam, ...

### 5.3 Hinge loss vs. perceptron loss

The Hinge loss encourages the margin of the classifier, and is defined as follows:

$$\uparrow_H(\mathbf{w}; \mathbf{x}, y) = \max\{0, 1 - y\mathbf{w}^T \mathbf{x}\}$$

### 5.4 Support vector machines

The optimization objective for the support vector machine is defined as minimizing Hinge loss while also adding another regularization term. There are several lines that need to be considered in the max. margin linear classification, which are summarised in figure 2

Support vector machines are widely used, very effective linear classifiers. They behave almost like a perceptron. The only differences include:

- Optimize slightly different, shifted loss (hinge loss)
- They regularize the weights

It can be optimized using a stochastic gradient descent. A safe choice for the learning rate is:

$$\eta_t = \frac{1}{\lambda t}$$

#### 5.4.1 Stochastic gradient descent for support vector machines

Let's recall the objective function:

$$\hat{R}(\mathbf{w}) = \sum_{i=1}^n \uparrow_H(\mathbf{w}_i; x_i, y_i) + \lambda \|\mathbf{w}_2\|^2$$

This requires taking care of the regularizer, as follows:

$$\hat{R}(\mathbf{w}) = \sum_{i=1}^n \left( \uparrow_H(\mathbf{w}_i; x_i, y_i) + \frac{\lambda}{n} \|\mathbf{w}_2\|^2 \right) = f_i(\mathbf{w})$$

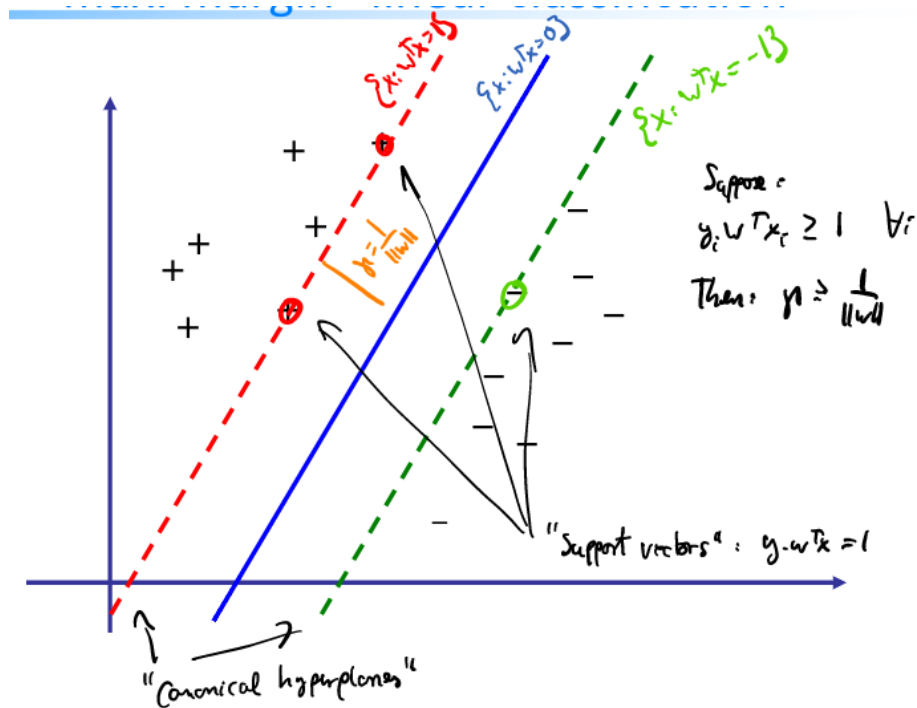


Figure 2: Important elements and their defining equations for support vector machines.

So, in order to estimate the gradient  $\nabla_{\mathbf{w}} \hat{R}(\mathbf{w})$  which is equivalent to:

$$\nabla_{\mathbf{w}} \hat{R}(\mathbf{w}) = \sum_{i=1}^n \nabla_{\mathbf{w}} f_i(\mathbf{w})$$

where:

$$\nabla_{\mathbf{w}} f_i(\mathbf{w}) = \nabla \downarrow_H(\mathbf{w} + \frac{\lambda}{n} \nabla \|\mathbf{w}\|_2^2)$$

so the gradient of the regularization term is just  $2\mathbf{w}$ . For the Hinge loss:

$$\nabla \downarrow_H = \nabla \max(0, 1 - y_i \mathbf{w}^T x_i) = \begin{cases} 0 & \text{if } y_i \mathbf{w}^T x_i \geq 1 \\ -y_i x_i & \text{otherwise.} \end{cases}$$

Therefore the entire update rule for stochastic gradient descent for SVM is:

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t \left( 1 - \eta_t \frac{2\lambda}{n} \right) + [y_i \mathbf{w}^T x_i]$$

The regularization parameter can be picked via cross-validation just like in linear regression. Instead of using the Hinge loss for validation, the target performance metric needs to be used.

## 5.5 Key takeaways

The key takeaways are:

- The perceptron is an algorithm for linear classification
- It applies SGD on the perceptron loss
- Mini-batches exploit parallelism and reduce variance compared to a single sample
- The perceptron loss is a convex surrogate function for the 0-1 misclassification loss

- It is guaranteed to produce a feasible solution if the data is separable
- SGD is much more generally applicable
- SVMs are closely related to Perceptron, they use a hinge loss and regularization.

Summary so far:

- List of representations/features: linear hypotheses, nonlinear hypotheses with nonlinear feature transforms.
- Model/Objective: loss function (squared loss, 0/1 loss, perceptron loss, Hinge loss) + regularization ( $L^2$  norm)
- Method: exact solution, gradient descent, mini-batch SGD, convex programming.
- Evaluation metric: MSE, accuracy
- Model selection: k-fold cross-validation, Monte Carlo CV.

## 6 Feature selection

Reasons why we don't want to work with all potentially available features:

- interpretability: understand which features are most important
- generalizationL simpler models may generalize better
- storage computation and cost: if we select the most important features we don't need to store, sum and acquire data for unused features.

The naive way to select features is to try all subsets and pick the best features via cross-validation. Greedy feature selection, which is a general purpose approach, consists of greedily add or remove features to maximize cross-validated prediction accuracy and mutual information or other notions of informativeness not discussed here. It can be used for any method, not only linear regression or classifiers.

### 6.1 General greedy approach

Consider the set of features  $V = \{1, \dots, d\}$ . We then define the cost function for scoring subsets  $S$  of  $V$ .  $\hat{L}(S)$  is the cross-validation error using features in  $S$  only. More precisely:

$$\mathbf{x}_i = [x_{i,1}, \dots, x_{i,d}] \rightarrow \mathbf{x}_{S,i} = [x_{i,j}, \dots, x_{i,j_k}]$$

where  $S$  is defined as:

$$S = \{j, \dots, j_k\}, k = |k|$$

We then train the model on  $\{x_{j,1}, y_1, \dots, x_{j,n}, y_n\}$ , in order to obtain an estimate based on the weights  $\hat{\mathbf{w}}_S$  with associated loss  $\hat{L}(S)$ , which represents the cross-validated performance of the weight estimates  $\hat{\mathbf{w}}_S$ .

### 6.2 Greedy forward selection

Start with  $S = \emptyset$  and  $E_0 = \infty$ , then for  $i = 1 : d$ , we find the best element to add, i.e.:

$$s_i = \arg \min_{j \in V \setminus S} \hat{L}(S \cup \{j\})$$

then we compute the error:

$$E_i = \hat{L}(S \cup \{s_i\})$$

if  $E_i > E_{i-1}$  break, else set  $S \leftarrow S \cup \{s_i\}$

### 6.3 Greedy backward selection

Start with  $S = V$  and  $E_{d+1} = \infty$ , then for  $i = d : -1 : 1$ , we find the best element to remove, i.e.:

$$s_i = \arg \min_{j \in S \setminus \{j\}} \hat{L}(S \setminus \{j\})$$

then we compute the error:

$$E_i = \hat{L}(S \setminus \{s_i\})$$

if  $E_i > E_{i-1}$  break, else set  $S \leftarrow S \setminus \{s_i\}$

### 6.4 Advantages and drawbacks of forward vs. backward feature selection

	Forward Feature Selection	Backward Feature Selection
Method Advantages	Faster (if few relevant features)	Can handle “dependent” features
Method drawback	Computational cost	Computational cost
	Suboptimal	Suboptimal

We want a method that simultaneously solves the learning and the feature selection problem via a single optimization step. So far we have only done optimization via sparsity: i.e. explicitly select a subset of features. This is equivalent to constraining  $\mathbf{w}$  to be sparse, i.e. contain at most  $k$  non-zero entries. Alternatively, we can penalize the number of nonzero entries:

$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w}} \sum_{i=1}^n (y_i - \mathbf{w}^T \mathbf{x}_i)^2 + \lambda \|\mathbf{w}\|_0$$

However, this is a difficult combinatorial optimization problem. The key idea then is to replace  $\|\mathbf{w}\|_0$  by a more tractable term.

The idea here is to use  $L_1$  as a surrogate for  $L_0$ , where:

$$\|\mathbf{w}\|_1 = \sum_{i=1}^d |w_i|$$

and we use  $\|\mathbf{w}\|_1$  instead of  $\|\mathbf{w}\|_0$ .

### 6.5 Lasso regression

In ridge regression, we use  $\|\mathbf{w}\|_2^2$  to control the weights. In Lasso, we replace  $\|\mathbf{w}\|_2^2$  by  $\|\mathbf{w}\|_1$ , hence leading to the following L1-regularized regression:

$$\min_{\mathbf{w}} \lambda \|\mathbf{w}\|_1 + \sum_{i=1}^n (y_i - \mathbf{w}^T \mathbf{x}_i)^2$$

This alternative penalty encourages coefficients to be exactly 0, which entails an automatic feature selection.

The regularization parameter can be picked using cross-validation.

#### 6.5.1 L1 regularization in SVM

The sparsity trick can be applied to SVMs as well:

$$\min_{\mathbf{w}} \|\mathbf{w}\|_1 + \sum_{i=1}^n \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$

This alternative penalty encourages coefficients to be exactly 0, which ignores those features as well, just like in Lasso regression.

## 6.6 Solving L1-regularized problems

The L1-norm is convex. Combined with convex losses, we can obtain convex optimization problems, which include Lasso and L1-SVM. Those problems can, in principle, be solved using stochastic gradient descent. Convergence is, however, usually slow, and we rarely obtain exact 0 entries. Recent work in convex optimization deals with solving such problems very efficiently using proximal methods.

Method	Greedy	L1-regularization
Advantages	Applies to any prediction method	Faster (training and feature selection happen jointly)
Disadvantages	Slower (need to train many models)	Only works for linear model

## 7 Non-linear prediction with kernels

### 7.1 Revisiting the perceptron/SVM

There is a fundamental observation to be made here where the optimal hyperplane lies in the span of the data.

$$\hat{\mathbf{w}} = \sum_{i=1}^n \alpha_i y_i \mathbf{x}_i$$

that means that the output of any of the models discussed so far can be rewritten as a linear combination of the feature inputs that we have seen so far. Proving this loosely can be done by stating that SGD starts from 0 and constructs such a representation. A more abstract proof follows from the representer theorem.

### 7.2 Reformulating the perceptron

In order to make the objective function only depend on the inner product of pairs of data point and work implicitly in high-dimensional spaces as long as we can do inner products efficiently, we need to reformulate the optimization problem in terms of  $\alpha$  instead of  $\mathbf{w}$ . Therefore, we do the following:

$$\begin{aligned}
 \hat{\mathbf{w}} &\in \arg \min_{\mathbf{w} \in \mathbb{R}^d} \sum_{i=1}^n \max(0, -y_i \mathbf{w}^T x_i) \quad \text{Note: we make the following ansatz: } \hat{\mathbf{w}} = \sum_{j=1}^n \alpha_j y_j x_j \\
 &= \sum_{i=1}^n \max(0, -y_i (\sum_{j=1}^n \alpha_j y_j x_j)^T x_i) \\
 &= \sum_{i=1}^n \max(0, -y_i \sum_{j=1}^n \alpha_j y_j (x_j^T x_i)) \\
 \hat{\alpha} &\in \arg \min_{\alpha \in \mathbb{R}^n} \sum_{i=1}^n \max(0, -c)
 \end{aligned}$$

Often, computing  $k(\mathbf{x}, \mathbf{x}')$  can be computed much more efficiently than  $\phi(\mathbf{x})^T \phi(\mathbf{x}')$ . For the polynomial kernel of degree 2, the computational complexity of computing the explicit feature map of the kernel function is in  $\mathcal{O}(d^2)$  whereas it is in  $\mathcal{O}(d)$  for the kernel computation.

The perceptron can then be reformulated as follows:

---

**Algorithm 1:** Kernelized perceptron

---

**Result:** Trained vector  $\hat{\alpha}$  used for prediction

---

```

1  $\alpha_0 \leftarrow 0$ ;
2 for  $t = 1, \dots$  do
3   Sample  $(\mathbf{x}_i, y_i) \sim D$ ;
4   if  $y_i \sum_{j=1}^n \alpha_j y_j k(\mathbf{x}_j^T \mathbf{x}_i) > 0$  then
5      $\alpha_{t+1} \leftarrow \alpha_t$ ;
6   else
7      $\alpha_{t+1} \leftarrow \alpha_t$ ;
8      $\alpha_{t+1,i} \leftarrow \alpha_{t+1,i} + \eta_t$ ;
9   end
10 end

```

---

For a new point, we can predict:

$$\hat{y} = \text{sign}\left(\sum_{j=1}^n \alpha_j y_j k(\mathbf{x}_j, \mathbf{x})\right)$$

### 7.3 The kernel trick

Non-linear decision boundaries can be found by using non-linear transformations of the feature vectors followed by linear classification. An important aspect to keep in mind when doing these feature transformation is the dimensionality of the data that is being used. We need, for instance,  $\mathcal{O}(d^k)$  dimensions to represent multivariate polynomials of degree  $k$  on  $d$  features. The challenge then becomes to efficiently implicitly operate in such high-dimensional feature spaces without ever explicitly computing the transformation.

**Definition 7.1** (The kernel trick). *The kernel trick consists in expressing a problem such that it only depends on inner products, which can then be replaced by kernels.*

An example of such a kernel can be applied to the perceptron loss:

$$\begin{aligned} \hat{\alpha} &= \arg \min_{\alpha_{1:n}} \frac{1}{n} \sum_{i=1}^n \max \left\{ 0 - \sum_{j=1}^n \alpha_j y_i y_j x_i^T x_j \right\} \\ \Leftrightarrow \hat{\alpha} &= \arg \min_{\alpha_{1:n}} \frac{1}{n} \sum_{i=1}^n \max \left\{ 0 - \sum_{j=1}^n \alpha_j y_i y_j k(\mathbf{x}_j, \mathbf{x}_i) \right\} \end{aligned}$$

### 7.4 The kernelized perceptron

At training time, the problem can be solved as follows:

- Initialize  $\alpha_1 = \dots = \alpha_n = 0$
- For  $t = 1, 2, \dots$ 
  - Pick data point  $(x_i, y_i)$  uniformly at random
  - Predict:

$$\hat{y} = \text{sign}\left(\sum_{j=1}^n \alpha_j y_j k(\mathbf{x}_j, \mathbf{x}_i)\right)$$

- If  $\hat{y} \neq y_i$  set  $\alpha \leftarrow \alpha_i + \eta_t$

At the time of prediction, for a new point  $x$ , we predict:

$$\hat{y} = \text{sign}\left(\sum_{j=1}^n \alpha_j y_j k(\mathbf{x}_j, \mathbf{x}_i)\right)$$



## 7.5 Kernel functions

**Definition 7.2** (Kernel functions). *Given a data space  $X$ , a kernel is a function  $k : X \times X \rightarrow \mathbb{R}$  satisfying the following properties:*

- **Symmetry:** for any pair of vectors  $\mathbf{x}, \mathbf{x}' \in X$  it must hold that:

$$k(\mathbf{x}, \mathbf{x}') = k(\mathbf{x}', \mathbf{x})$$

- **Positive semi-definiteness:** for any  $n$ , any set  $S = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subseteq X$ , the kernel (Gram) matrix defined as:

$$K = \begin{bmatrix} k(\mathbf{x}_1, \mathbf{x}_1) & \dots & k(\mathbf{x}_1, \mathbf{x}_n) \\ \vdots & & \vdots \\ k(\mathbf{x}_n, \mathbf{x}_1) & \dots & k(\mathbf{x}_n, \mathbf{x}_n) \end{bmatrix}$$

must be positive semi-definite.

A matrix is positive semidefinite iff:

- i  $\forall \mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T M \mathbf{x} \geq 0$
- ii All eigenvalues of  $M \geq 0$

Suppose the data space  $X = \{1, \dots, n\}$  is finite, and we are given a p.s.d. matrix  $\mathbf{K} \in \mathbb{R}^{n \times n}$ , then we can always construct a feature map:

$$\phi : Z \rightarrow \mathbb{R}^n$$

such that  $\mathbf{K}_{i,j} = \phi(i)^T \phi(j)$ .

*Proof.*  $\mathbf{K}$  is p.s.d.  $\Rightarrow \mathbf{K} = UDU^T$  where  $D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$  and  $\lambda_i \geq 0 \forall i$ . We then define a

matrix  $D = D^{1/2} T D^{1/2}$  where  $D^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sqrt{\lambda_n} \end{bmatrix}$  and  $\lambda_i \geq 0 \forall i$ . This gives us the following:

$\mathbf{K} = U D^{1/2} D^{1/2} U^T = \phi^T \phi$  where  $\phi = [\phi_1] \dots [\phi_n]$ . Now it holds that taking  $k(i, j) = \mathbf{K}_{i,j} = \phi_i^T \phi_j$  which means that that  $\phi : X \rightarrow \mathbb{R}^n$  and  $\phi : i \rightarrow \phi_i$  which is a constructed valid feature map. It shows that for finite data spaces  $X$ , positive definiteness of the function is also a sufficient condition for it being a valid kernel.  $\square$

More generally:

**Theorem 2** (Mercer's theorem). *Let  $X$  be a compact subset of  $\mathbb{R}^n$  and  $k : X \times X \rightarrow \mathbb{R}$  a kernel function. Then one can expand  $k$  in a uniformly convergent series of bounded functions  $\phi_i$  s.t.*

$$k(x, x') = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \phi_i(x').$$

## 7.6 Examples of kernels

- Linear kernel:  $k(\mathbf{x}, \mathbf{x}') = \mathbf{x}^T \mathbf{x}'$
- Polynomial kernel:  $k(\mathbf{x}, \mathbf{x}') = (\mathbf{x}^T \mathbf{x}' + 1)^d$
- Gaussian (RBF, squared exponential kernel):  $k(\mathbf{x}, \mathbf{x}') = \exp(-\|\mathbf{x} - \mathbf{x}'\|_2^2 / h^2)$  where  $h^2$  is the bandwidth/length scale parameter.
- Laplacian kernel:  $k(\mathbf{x}, \mathbf{x}') = \exp(-\|\mathbf{x} - \mathbf{x}'\|_1 / h)$

## 7.7 Examples of non-kernels

- $k(\mathbf{x}, \mathbf{x}') = \sin(\mathbf{x})\cos(\mathbf{x}')$ . It is not symmetric. Take for instance  $x = 0$  and  $x' = \frac{\pi}{2}$
- $k(\mathbf{x}, \mathbf{x}') = \mathbf{x}^T M \mathbf{x}' \forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^d, M \in \mathbb{R}^{d \times d}$

*Proof.* If  $M$  is symmetric:  $k(\mathbf{x}, \mathbf{x}') = \mathbf{x}^T M^T \mathbf{x}' = \mathbf{x}'^T M^T \mathbf{x} = k(\mathbf{x}', \mathbf{x})$ . If  $M$  is not symmetric,  $k$  in general is not symmetric. If  $M$  is not positive semi-definite, then the kernel function is not definite, e.g. for the situation where  $M = -1$  for the normal dot product. If  $M$  is positive semi-definite:  $M = U D^{\frac{1}{2}} D^{\frac{1}{2}} T U^T = V^T V$  for  $V = (U D^{\frac{1}{2}})^T$  then:  $k(\mathbf{x}, \mathbf{x}') = \mathbf{x}^T M \mathbf{x}' = \mathbf{x}^T V^T V \mathbf{x}' = (V \mathbf{x})^T (V \mathbf{x}') = \phi(\mathbf{x})^T \phi(\mathbf{x}')$  for  $\phi(\mathbf{x}) = V \mathbf{x}$   $\square$

## 7.8 Effect of kernel on function class

Given a kernel  $k$ , predictors for kernelized classification have the form:

$$\hat{y} = \text{sign}\left(\sum_{j=1}^n \alpha_j y_j k(\mathbf{x}_j, \mathbf{x})\right)$$

Grafically, it looks like in figure 3.

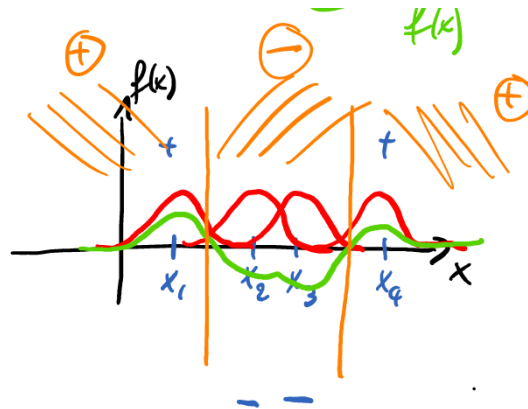


Figure 3: This figure shows that the green line is nothing but a scaled version of the individual Gaussian distributions and the associated decision function sign.

## 7.9 Graphical representations of kernels

The graphical representation of a sample gaussian and exponential kernel can be found in figures

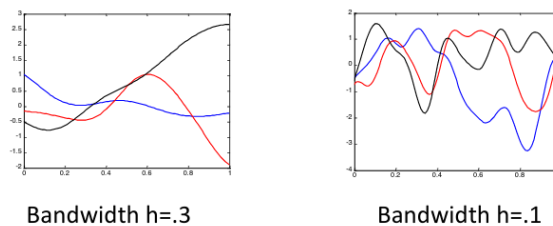


Figure 4: Graphical representation of the Gaussian kernel.

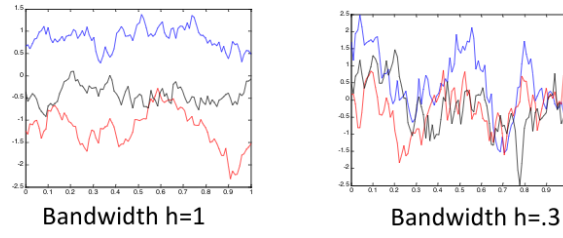


Figure 5: Graphical representation of the Laplacian kernel.

## 7.10 Objects where kernels can be used

The kernels can be defined on a variety of objects:

- Sequence kernels
- Graph kernels
- Diffusion kernels
- Kernels on probability distributions

Graph kernels can be used for measuring similarity between graphs by comparing random walks on both graphs. They can also be used to measure similarity among nodes in a graph via diffusion kernels not defined here.

## 7.11 Kernel engineering

Suppose we have two kernels:

$$k_1 : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \quad k_2 : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$$

Then the following functions are valid kernels:

- $k(\mathbf{x}, \mathbf{x}') = k_1(\mathbf{x}, \mathbf{x}') + k_2(\mathbf{x}, \mathbf{x}')$
- $k(\mathbf{x}, \mathbf{x}') = k_1(\mathbf{x}, \mathbf{x}')k_2(\mathbf{x}, \mathbf{x}')$
- $k(\mathbf{x}, \mathbf{x}') = ck_1(\mathbf{x}, \mathbf{x}')$  for  $c > 0$
- $k(\mathbf{x}, \mathbf{x}') = f(k_1(\mathbf{x}, \mathbf{x}'))$  where  $f$  is a polynomial with positive coefficients or the exponential function.

## 7.12 The ANOVA kernel

$$k(\mathbf{x}, \mathbf{x}') = \sum_{j=1}^d k_j(x_j, x'_j)$$

where:

$$\mathbf{x}, \mathbf{x}' \in \mathbb{R}^d \quad k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \quad k_j : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

The functions modelled by this kernel can be shown by showing what forms  $f(x)$  takes:

$$\begin{aligned}
 f(x) &= \sum_{i=1}^n \alpha_i y_i k(x^{(i)}, \mathbf{x}) \\
 &= \sum_{i=1}^n \alpha_i y_i \sum_{j=1}^d k_j(x^{(i)}_j, \mathbf{x}_j) \\
 &= \sum_{j=1}^d \sum_{i=1}^n \alpha_i y_i k_j(x^{(i)}_j, \mathbf{x}_j) \\
 &= \sum_{j=1}^d f_j(\mathbf{x}_j)
 \end{aligned}$$

which means that the function  $f$  decomposes into functions that depend on individual coordinates only, i.e. it is additive decomposition of the function. This property can be useful for high dimensional domains.

### 7.13 Modelling pairwise data

Suppose we have the following two kernels:

$$k((x, z), (x', z')) = k_x(x, x') \cdot k_z(z, z') \quad (10)$$

$$k((x, z), (x', z')) = k_x(x, x') + k_z(z, z') \quad (11)$$

$$(12)$$

They can be used to represent the following data:

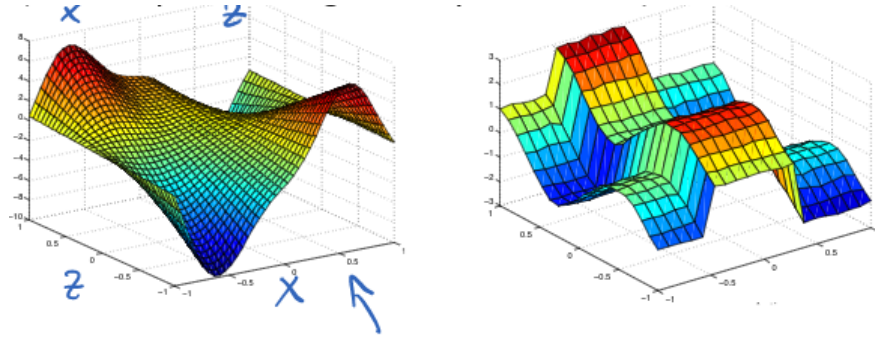


Figure 6: The left figure can be used to represent a situation where multiplying the kernel could be useful and the right figure can be used to represent a situation where adding the kernel could be useful.

### 7.14 Kernels as similarity functions

Kernels can be used as similarity measures. For instance, consider the Gaussian kernel  $k(\mathbf{x}, \mathbf{x}') = \exp(-\|\mathbf{x} - \mathbf{x}'\|^2 / h^2)$ . If a point  $\mathbf{x}$  is close to  $\mathbf{x}'$ , then the value of  $k(\mathbf{x}, \mathbf{x}') \approx 1$ , else it is closer to 0.

### 7.15 Comparing k-NN to kernel perceptron

The prediction for each point in  $k$ -NN is provided by:

$$y = \text{sign}\left(\sum_{i=1}^n y_i [\mathbf{x} \text{ among } k \text{ nearest neighbors of } \mathbf{x}]\right)$$

Method	$k$ -NN	Kernelized perceptron
Advantages	No training required	Optimized weights can lead to improved performance, can capture global trends with suitable kernels
Disadvantages	Depends on all data	Depends on wrongly classified examples only Training requires optimisation

As we can see, it compares to the loss of the perceptron:

$$y = \text{sign}\left(\sum_{i=1}^n y_i \alpha_i k(\mathbf{x}_i, \mathbf{x})\right)$$

Note: choose  $k$  in  $k$ -NN using cross-validation.

## 7.16 Deriving non-parametric models from parametric ones

Parametric models have a finite set of parameters. Examples of such models include linear regression, linear perceptron, etc ... Nonparametric models grow in complexity with the size of the data. They have the potential to be much more expressive but also more computationally complex. Examples of such models include the kernelized perceptron,  $k$ -NN, etc... Kernels provide a principled way of deriving non-parametric models from parametric ones.

## 7.17 Kernelized SVM

The SVM optimization step can be kernelized as follows:

$$\begin{aligned}
\hat{\mathbf{w}} &= \arg \min_{\mathbf{w}} \frac{1}{n} \sum_{i=1}^n \max \{0, 1 - y_i \mathbf{w}^T \mathbf{x}_i\} + \lambda \|\mathbf{w}\|_2^2 & \text{assuming } \mathbf{w} = \sum_{j=1}^n \alpha_j y_j x_j \\
&= \max \left\{ 0, 1 - y_i \left( \sum_{j=1}^n \alpha_j y_j x_j \right)^T x_i \right\} \\
&= \max \left\{ 0, 1 - y_i \sum_{j=1}^n \alpha_j y_j (x_j^T x_i) \right\} \\
&= \max \left\{ 0, 1 - y_i \sum_{j=1}^n \alpha_j y_j k(x_j, x_i) \right\} \\
&= \max \{0, 1 - y_i \alpha^T k_i\}
\end{aligned}$$

where  $k_i = [y_1 k(\mathbf{x}_i, \mathbf{x}_1), \dots, y_n k(\mathbf{x}_i, \mathbf{x}_n)]^T$  and  $\alpha = [\alpha_1, \dots, \alpha_n]^T$ .

The regularizer can be kernelized as follows:

$$\begin{aligned}
&\lambda \|\mathbf{w}\|_2^2 \\
&= \lambda \mathbf{w} \mathbf{w}^T = \lambda \left( \sum_{i=1}^n \alpha_i y_i x_i \right)^T \left( \sum_{j=1}^n \alpha_j y_j x_j \right) = \lambda \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j x_j^T x_i \\
&= \lambda \alpha^T D_y \mathbf{K} D_y \alpha
\end{aligned}$$

$$\text{where } D_y = \begin{bmatrix} y_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & y_n \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} k(x_1, x_1) & \dots & k(x_1, x_n) \\ \vdots & \ddots & \vdots \\ k(x_n, x_1) & \dots & k(x_n, x_n) \end{bmatrix}$$

### 7.18 Kernelizing linear regression

Original parametric linear regression optimization problem is stated as follows:

$$\hat{\mathbf{w}} = \arg \min_{\mathbf{w}} \frac{1}{n} \sum_{i=1}^n (\mathbf{w}^T \mathbf{x}_i - y_i)^2 + \lambda \|\mathbf{w}\|_2^2$$

We kernelize linear regression in two parts:

$$\begin{aligned} &= \sum_{i=1}^n (\mathbf{w}^T \mathbf{x}_i - y_i)^2 \\ &= \sum_{i=1}^n ((\sum_{j=1}^n \alpha_j x_j x_i)^T \mathbf{x}_i - y_i)^2 \\ &= \sum_{i=1}^n (\sum_{j=1}^n \alpha_j (x_j^T x_i) \mathbf{x}_i - y_i)^2 \\ &= \sum_{i=1}^n (\sum_{j=1}^n \alpha_j (x_j^T x_i) \mathbf{x}_i - y_i)^2 \\ &= (\alpha^T \mathbf{k}_i - y_i)^2 \end{aligned}$$

and

$$= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j x_i^T x_j = \alpha^T \mathbf{K} \alpha$$

hence we have:

$$\begin{aligned} \hat{\alpha} &= \arg \min_{\alpha \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^n (\alpha^T \mathbf{k}_i - y_i)^2 + \lambda \alpha^T \mathbf{K} \alpha \\ \hat{\alpha} &= \arg \min_{\alpha \in \mathbb{R}^n} \frac{1}{n} \|\alpha^T \mathbf{K} - \mathbf{y}\|_2^2 + \lambda \alpha^T \mathbf{K} \alpha \end{aligned}$$

which has a closed-form solution:

$$\hat{\alpha} = (\mathbf{K} + n\lambda \mathbf{I})^{-1} \mathbf{y}$$

For prediction, given a data point  $\mathbf{x}$ , we predict the response  $y$  as:

$$\hat{y} = \sum_{i=1}^n \hat{\alpha}_i k(\mathbf{x}_i, \mathbf{x})$$