CS 350 Proof 1

Team 8: Peter Inslee and 3 other classmates

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Conjecture: $f(n) \in O(h(n)) \Rightarrow 2^{f(n)} \in O(2^{h(n)})$

Disproof by Counterexample: We test the given conjecture with a choice of functions f and h satisfying the antecedent of the given implication, and failing to satisfy its consequent. In both cases we use the definition of big O notation to make our point; directly in the former case, and with a Proof by Contradiction in the latter case. Having shown for our particular choice of functions that the entailment does not hold, we conclude that the conjecture is not true in general.

1. Let $f(n) = 2lg(n)$ and let $h(n) = lg(n)$.
2. Verify the antecedent, using the definition of big O to show that $2lg(n) \in O(lg(n))$.

2.1	$\forall n \geq 1,$	$0 \leq lg(n)$	**Definition of** lg
2.2	$\forall n \geq 1,$	$0 \leq 2lg(n) \leq 3lg(n)$	**Property of** $\leq$, (since $0 \leq 2 \leq 3$)
2.3	$\forall n \geq 1,$	$0 \leq f(n) \leq 3h(n)$	**Hypothesis** 1.
2.4	$\exists c', n_0' > 0, \forall n \geq n_0',$	$0 \leq f(n) \leq c'h(n)$	**Letting** $3 = c', 1 = n_0'$
2.5		$f(n) \in O(h(n))$	**Definition of** O

3. Refute the consequent, using the definition of big O and a Proof by Contradiction demonstrating that $2^{f(n)} \notin O(2^{h(n)})$.

   (a) Suppose, (by way of contradiction), that $2^{f(n)} \in O(2^{h(n)})$.

3.1.1	$\exists c, n_0 > 0, \forall n \geq n_0,$	$0 \leq 2^{f(n)} \leq c2^{h(n)}$	**Definition of** O
3.1.2	$\exists c, n_0 > 0, \forall n \geq n_0,$	$0 \leq 2^{2lg(n)} \leq c2^{lg(n)}$	**Hypothesis** 1.
3.1.3	$\exists c, n_0 > 0, \forall n \geq n_0,$	$0 \leq 2^{lg(n^2)} \leq c2^{lg(n)}$	**Property of** lg, $(ylg(x) = lg(x^y))$
3.1.4	$\exists c, n_0 > 0, \forall n \geq n_0,$	$0 \leq n^2 \leq cn$	**Property of** lg, $(2^{lg(x)} = x^{lg(2)} = x)$
3.1.5	$\exists c, n_0 > 0, \forall n \geq n_0,$	$0 \leq n \leq c$	**Property of** $\leq$, (divide all sides by n)[1]

   (b) Note that 3.1.5 is a contradiction, as it claims that there exists some positive constant c that is an upper bound for all real numbers n (greater than or equal to some minimum value n_0). But by a property of the real numbers, we know that *no such upper bound exists*. Hence, supposition (a) must be incorrect, and its negation, given below, must be true.

3.2.1	$\neg[2^{f(n)} \in O(2^{h(n)})]$	**Proof by Contradiction**

4. Contradict the implication. Any argument of the form $A \Rightarrow B$ can be disproved by exhibiting a case in which B can be false when A is true.

4.1	$f(n) \in O(h(n))$	**From** 2.5
4.2	$\neg[2^{f(n)} \in O(2^{h(n)})]$	**From** 3.2.1
4.3	$[f(n) \in O(h(n))] \wedge \neg[2^{f(n)} \in O(2^{h(n)})]$	**Conjunction Introduction**
4.4	$f(n) \in O(h(n)) \not\Rightarrow 2^{f(n)} \in O(2^{h(n)})$	**Definition of** $\Rightarrow$

QED

[1] Since $0 < n_0 \leq n$, we may safely divide all sides by n, without fear of dividing by zero or changing the direction of the inequality.

References

[1] Anany Levitin, *Introduction to the Design and Analysis of Algorithms*, Pearson Education, Inc., publishing as Addison-Wesley, 2012 (Third Edition).

[2] Big O notation. In Wikipedia: The Free Encyclopedia. Wikimedia Foundation Inc. Encyclopedia on-line. Available from `http://en.wikipedia.org/wiki/Big_O_notation`. Internet. Retrieved 29 January 2012.

[3] Conjunction Introduction. In Wikipedia: The Free Encyclopedia. Wikimedia Foundation Inc. Encyclopedia on-line. Available from `http://en.wikipedia.org/wiki/Conjunction_Introduction`. Internet. Retrieved 29 January 2012.