1.1 ***Problem:***

A city office wants to decide on location for the Critical Infrastructures (hospitals). The city office have current list of hospital locations and possible proposed location. They want to decide on where to locate the new hospitals, how many in numbers, what capacity is required and how is the demand allocation after the establishment of new hospitals.

The city office further want to consider the disruptions events to identify the reliable locations. They are more concern with indirect failure of a hospital rather than the direct failure. Indirect failure refers to the failure or inability of a hospitals to serve its patients due to failure of its connected supporting stations. Infrastructures including power, water, communications services and emergency services are the supporting services for the hospitals. Form the past data of disruptions and disasters the failure probability for each infrastructures types are known and the city office wants to incorporate that probability to see its impact on total expected cost on long run.

The primary interest of the office is to get the minimum total cost. However, they want to consider the impact of hospital failure on the patients and assign a penalty cost if patients remain unserved.

1.2 ***Mathematical formulation***

Sets:-

I = Set of Demand Nodes indexed by i

J = Set of Candidate facility indexed by j (includes existing and proposed facility)

J_E = Set of Candidate that already exist ($J_E \subset J$)

M = Set of different layers of Supporting Stations indexed by m

R = Set of all possible rank for the supporting stations indexed by r

Parameters:-

f_j = Initial setup cost to establish a facility at candidate location $j \in J$

v_j = variable cost to establish a facility at candidate location $j \in J$

θ_i = Demand at each location $i \in I$

d_{ij} = Transportation cost for customers in location i to get service from facility j

q = Number of Facility to open

p_m = Probability of failure of $m \in M$ layer of infrastructure

π_m = Expected penalty cost incurred due to failure of layer $m \in M$ infrastructure

δ_{jm} = Self sustaining index for a candidate $j \in J$ if supporting station $m \in M$ fails

c_j = capacity for existing candidate $j \in J_E$

c_{min} = Minimum capacity for candidate

c_{max} = Maximum capacity for candidate

c_{jmr} = per capita cost of operation for facility $j \in J$ to use service from r ranked supporting stations node from layer $m \in m$

Variables:-

$$x_j = \begin{cases} 1, \text{if candidate facility is open at location } j \in J; \\ 0, \text{otherwise} \end{cases}$$

y_{ij} = Demand served in location $i \in I$ from candidate facility $j \in J$

w_j = additional capacity of the facility $j \in J$

k_j = capacity of the facility $j \in J$

Objective function:

Minimize

$$z = \sum_{j \in J} v_j w_j + \sum_{j \in J} f_i x_j \, (\forall\, j \notin J_E) + \sum_{i \in I} \sum_{j \in J} d_{ij} y_{ij} + \sum_{i \in I} \sum_{j \in J} \left(\sum_{m \in M} \sum_{r \in R} p_m^{r-1} (1 - p_m) c_{jmr} \right) y_{ij}$$

$$+ \sum_{i \in I} \sum_{j \in J} \sum_{m \in M} p^R \pi_m (1 - \delta_{jm})\, y_{ij} \qquad (1)$$

(Establishment cost + Transportation cost + Operation cost + Penalty cost)

Constraints Subject To:

$$y_{ij} \le B\, x_j \qquad \forall\, i \in I\,; \qquad (2)$$

$$\sum_j y_{ij} \ge \theta_i \qquad \forall\, i \in I\,; \qquad (3)$$

$$\sum_i y_{ij} \le c_j \qquad \forall\, j \in J \text{ and } j \in J_E\,; \qquad (4)$$

$$\sum_i y_{ij} \le k_j \qquad \forall\, j \in J \text{ and } j \notin J_E\,; \qquad (5)$$

$$k_j - c_j \le w_j \qquad \forall\, j \in J \text{ and } j \in J_E\,; \qquad (6)$$

$$k_j - c_{min} \le w_j \qquad \forall\, j \in J \text{ and } j \notin J_E\,; \qquad (7)$$

$$k_j \le c_{max} \qquad \forall\, j \in J \text{ and } j \notin J_E\,; \qquad (8)$$

$$x_j \in \{0,1\} \qquad \forall j \in J\,; \qquad (9)$$

$$y_{ij}, k_j, w_j \ge 0 \qquad (10)$$

1.3 ***Explanation of Model and Constraints:***

For the first phase, all possible candidate location $j \in J$ are identified along with the different types of supporting stations layer $m \in M$. There are $k_m \in K_M$ nodes in each layer $m \in M$. These supporting station nodes are then ranked through $r \in R$ for each candidate facility. All the supporting stations connected to the candidate facility j are then listed with their respective rank r. This list gives all the supporting station nodes connected to candidate facility, their rank, and their expected cost. The cost is represented as c_{jmr} that refers to the expected per unit operation cost for facility j to get service from r ranked supporting stations of type m.

Objective function

The objective function is composed of 4 major cost components.

1. Establishment costs:

Establishment costs includes all the fixed costs to establish the infrastructure and variable costs for the additional increase in capacity of the infrastructure. The fixed cost is independent of the size and capacity of the established structure while variable cost will increase for additional capacity after certain minimum capacity limit(c_{min}). It is assumed that the CI can have c_{min} capacity with the fixed cost and additional capacity are charged based on variable costs.

2. Transportation Costs:

The transportation costs include all the expected expenses for service delivery. This is directly proportional to the distance between the candidate infrastructure and demand location.

3. Operating Costs:

The operating costs represent the expected costs of providing services from the supporting stations when the facility is in operation. Supporting stations are subject to failure and the no service cost is imposed if the supporting stations fails. With the given probability of failure P_m, the probability of the r-rank supporting stations to be in operation can be calculated from equation (11).

4. Penalty Costs:

In case if all the connected supporting stations fails, the facility will not be able to accommodate all demands and is assumed to function partially with reduced capacity. The capacity reduction depends upon the available resources within the CI to self-sustain. This is given by an Index for self-sustaining capacity(δ_m). It is mathematically represented by the ratio of the demand served after the loss of service (m) to total demand capacity of the CI. The product of self-sustaining, the probability of failure of the SS, the penalty cost assigned for each unserved demand and annual demand allocation to the CI give the total expected annual penalty costs.

The objective function is the sum of establishment costs, operating costs, transportation costs and penalty costs. The transportation costs is calculated based on distance between the two nodes. For this modeling purpose the spherical linear earth distance is considered. The distance is calculated based on "The Haversine formula". The operating cost represents the expected cost of service from the supporting stations when the critical facility is in operation. The objective function is bounded by the following set of constraints.

- Multiplying the binary variable x_j with the big number B, Constraint (2) insures that the demand can be fulfilled by the open facility only.

- Constraint (3) makes sure that the flow of service meets the demand in all demand locations.
- Constraint (4 & 5) limit the flow of services from the candidate facilities to their capacity limits.
- Constraint (6 & 7) gives the additional capacity to be installed.
- Constraints (8) limits the capacity of the opened facility
- While constraint (9) and constraint (10) are binary and non-negativity constraints respectively.